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Research article

Piecewise pseudo almost periodic solutions of interval general BAM neural networks with mixed time-varying delays and impulsive perturbations

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Abstract: This paper is concerned with piecewise pseudo almost periodic solutions of a class of interval general BAM neural networks with mixed time-varying delays and impulsive perturbations. By adopting the exponential dichotomy of linear differential equations and the fixed point theory of contraction mapping. The sufficient conditions for the existence of piecewise pseudo almost periodic solutions of the interval general BAM neural networks with mixed time-varying delays and impulsive perturbations are obtained. By adopting differential inequality techniques and mathematical methods of induction, the global exponential stability for the piecewise pseudo almost periodic solutions of the interval general BAM neural networks with mixed time-varying delays and impulsive perturbations is discussed. An example is given to illustrate the effectiveness of the results obtained in the paper.

Keywords: piecewise pseudo almost periodic solutions; interval; general BAM neural networks; impulsive perturbations; mixed time-varying delays; existence; global exponential stability
Mathematics Subject Classification: 34A12, 34A34, 34A37, 34D09, 34D20

1. Introduction

This paper considers the existence and global exponential stability of piecewise pseudo almost periodic solutions for interval general BAM neural networks with mixed time-varying delays and impulsive perturbations. The mixed time-varying delays include leakage delays and time-varying delays:

$$\begin{cases} x_i'(t) = -a_i(t)x_i(t-\alpha_i(t)) + \sum\limits_{j=1}^{m} s_{ji}(t)f_j(x_j(t), y_j(t-\tau_{ji}(t))) + c_i(t), \\ t > 0, t \neq t_k, i = 1, 2, \cdots, m, \\ \Delta x_i(t) = x_i(t^+) - x_i(t^-) = I_k(x_i(t)), t = t_k, k \in \mathbb{Z}^+, \\ y_j'(t) = -b_j(t)y_j(t-\beta_j(t)) + \sum\limits_{i=1}^{m} t_{ij}(t)g_i(x_i(t-\delta_{ij}(t)), y_i(t)) + d_j(t), \\ t > 0, t \neq t_k, j = 1, 2, \cdots, m, \\ \Delta y_j(t) = y_j(t^+) - y_j(t^-) = J_k(y_j(t)), t = t_k, k \in \mathbb{Z}^+, \end{cases} \tag{1.1}$$

where $\mathbb{Z}^+$ is the set of nonnegative integers; the numbers of neurons in layers X and Y are denoted by m. $x_i(t)$ represents the state variables of the i-th neurons at time t. $y_j(t)$ represents the state variables of the j-th neurons at time t. $a_i(t) > 0$, $b_j(t) > 0$ are continuous functions and represent the decay rates of neurons in different layers, respectively. $s_{ji}(t)$, $t_{ij}(t)$ are the connection weights. $f_j(\cdot,\cdot)$, $g_i(\cdot,\cdot)$ represent the activation functions of the j-th and i-th units, respectively. $c_i(t)$, $d_j(t)$ represent the external inputs of the i-th neuron and the j-th neuron acting on different layers at time t, respectively. $\alpha_i(t) > 0, \beta_j(t) > 0$ are time-varying leakage delays. $\tau_{ji}(t) > 0$, $\delta_{ij}(t) > 0$ are time-varying delays respectively satisfying $1 - \tau'_{ji}(t) > 0$, $1 - \delta'_{ij}(t) > 0$. The sequence $\{t_k\}$ has no finite accumulation point. $I_k(\cdot) : \mathbb{R}^m \to \mathbb{R}$ and $J_k(\cdot) : \mathbb{R}^m \to \mathbb{R}$, $k \in \mathbb{Z}^+$.

The initial value conditions of the system (1.1) are as follows:

$$\begin{cases} x_i(\theta) = \varphi_i(\theta), \theta \in [-r_1, 0], i = 1, 2, \cdots, m, \\ y_j(\theta) = \psi_j(\theta), \theta \in [-r_2, 0], j = 1, 2, \cdots, m, \end{cases} \tag{1.2}$$

where $r_1 = \max\limits_{1\le i,j\le m} \{\sup\limits_{t\in\mathbb{R}^+} |\alpha_i(t)|, \sup\limits_{t\in\mathbb{R}^+} |\delta_{ij}(t)|\}$, $r_2 = \max\limits_{1\le i,j\le m} \{\sup\limits_{t\in\mathbb{R}^+} |\beta_j(t)|, \sup\limits_{t\in\mathbb{R}^+} |\tau_{ji}(t)|\}$, $\varphi_i(\theta) \in C^1([-r_1, 0], \mathbb{R})$, $i = 1, 2, \cdots, m$, $\psi_j(\theta) \in C^1([-r_2, 0], \mathbb{R})$ and $j = 1, 2, \cdots, m$.

BAM neural networks have more complex dynamic behavior. They are composed of two-layer nonlinear feedback networks. Neurons in different layers are interconnected, and there is no connection between neurons in the same layer [1]. It extends the single-layer self-association learning rule to the two-layer hetero-association mode. Time delays in neural networks not only decrease the transmission speed of the networks, but may also change the stability of the networks. There have been many excellent results on the stability and bifurcation behavior of delayed neural networks [2–9]. The existence and stability of solutions for neural networks with time delays have been widely explored [10–29]. In particular, the dynamic behavior of neural networks with leakage delays [10,12,13,24–25,27,29] is still a hot research topic.

Impulses are ubiquitous in the actual neural networks and the dynamic behavior of the neural networks are often affected by impulsive perturbations. And the study of impulsive control theory for neural networks has made a lot of advancements [11,15,25–40].

Compared with the activation functions $f_j(\cdot)$, $g_i(\cdot)$ in BAM neural networks [10,11,13–15,17,19,21–24], the activation functions $f_j(x_j, y_j)$, $g_i(x_j, y_j)$ in the interval general BAM neural networks [16,17,20–22,24] more clearly show the connections between different layers of neurons in the networks. Ding and Huang [16] studied the existence and global robust exponential stability of equilibrium points for the following a class of interval general BAM neural networks with constant delays by using fixed-point theory and constructing suitable Lyapunov functions:

$$\begin{cases} x_i'(t) = -a_i x_i(t) + \sum\limits_{j=1}^{m} s_{ji} f_j(x_j(t), y_j(t-\tau_{ji})) + c_i, \\ y_j'(t) = -b_j y_j(t) + \sum\limits_{i=1}^{m} t_{ij} g_i(x_i(t-\delta_{ij}), y_i(t)) + d_j. \end{cases} \tag{1.3}$$

By variable transformations, fixed point theory and constructing suitable delay differential inequalities, Xu et al. [17] studied the existence, uniqueness and global exponential stability of the equilibrium point of system (1.3) with proportional delays. Given $a_i = a_i(t)$, $b_j = b_j(t)$, $s_{ji} = s_{ji}(t)$ and $t_{ij} = t_{ij}(t)$ in the system (1.3), the authors of [20] studied the existence and global exponential stability of periodic solutions for the system (1.3) on time scales by using fixed point theory and constructing suitable Lyapunov functions. Given $a_i = a_i(t)$, $b_j = b_j(t)$, $s_{ji} = s_{ji}(t)$, $t_{ij} = t_{ij}(t)$, $c_i = c_i(t)$ and $d_j = d_j(t)$ in the system (1.3), Duan [22] studied the existence and global exponential stability of pseudo almost periodic solutions for system (1.3) by using exponential dichotomy, fixed point theory and inequality techniques. Given $a_i = a_i(t)$, $b_j = b_j(t)$, $s_{ji} = s_{ji}(t)$, $t_{ij} = t_{ij}(t)$, $c_i = c_i(t)$ and $d_j = d_j(t)$ in the system (1.3), the authors of [24] studied the existence and exponential stability of almost periodic solutions for system (1.3) with leakage delays by using exponential dichotomy, fixed point theory and inequality techniques.

However, in the existing literature, we have not found any research on the dynamic behavior of interval general BAM neural networks with impulses and leakage delays. The stability of the interval general BAM neural networks may be destroyed by external perturbations and leakage delays. Therefore, the effects of leakage delays and impulsive perturbations on the dynamic behavior of interval general BAM neural networks are worth exploring.

Motivated by the above discussions, this paper studies the existence and global exponential stability of piecewise pseudo almost periodic solutions for the interval general BAM neural network described by (1.1) with mixed time-varying delays and impulsive perturbations. The mixed time-varying delays include leakage delays and time-varying delays.

Throughout this paper, let $e_i^+ = \sup\limits_{t\in\mathbb{R}^+} |e_i(t)|$, $e_i^- = \inf\limits_{t\in\mathbb{R}^+} |e_i(t)|$, $e_{ij}^+ = \sup\limits_{t\in\mathbb{R}^+} |e_{ij}(t)|$ and $e_{ij}^- = \inf\limits_{t\in\mathbb{R}^+} |e_{ij}(t)|$. Let T be the set of the real sequences $\{t_k\}_{k\in\mathbb{Z}^+}$ such that $\underline{\sigma} = \inf\limits_{k\in\mathbb{Z}^+} t_k^1 = \inf\limits_{k\in\mathbb{Z}^+}(t_{k+1} - t_k) > 0$, so $\lim\limits_{k\to+\infty} t_k = +\infty$. For $\{t_k\}_{k\in\mathbb{Z}^+} \subset T$, assume that $\{t_k^j : t_k^j = t_{k+j} - t_k, k, j \in \mathbb{Z}^+\}$ are equipotentially almost periodic and $t_0 = 0$; it can be easily proved that the sequence $\{t_k^j\}$ satisfies $t_{k+i}^j - t_k^j = t_{k+j}^i - t_k^i$ and $t_k^j - t_k^i = t_{k+i}^{j-i}$. And suppose that the following conditions hold.

- (H_1) For u^1, v^1, u^2, $v^2 \in \mathbb{R}$, there are nonnegative constants P_j, Q_j, H_i and K_i such that $|f_j(u^1, v^1) - f_j(u^2, v^2)| \le P_j|u^1 - u^2| + Q_j|v^1 - v^2|$ and $|g_i(u^1, v^1) - g_i(u^2, v^2)| \le H_i|u^1 - u^2| + K_i|v^1 - v^2|$.
- (H_2) $a_i(t)$, $b_j(t) \in C(\mathbb{R}^+, \mathbb{R}^+)$ are almost periodic functions, and $\sup\limits_{t\in\mathbb{R}^+} |a_i(t)| = a_i^+$, $\inf\limits_{t\in\mathbb{R}^+} |a_i(t)| = a_i^-$, $\sup\limits_{t\in\mathbb{R}^+} |b_j(t)| = b_j^+$, $\inf\limits_{t\in\mathbb{R}^+} |b_j(t)| = b_j^-$. $s_{ji}(t)$, $t_{ij}(t)$, $c_i(t)$, $d_j(t) : \mathbb{R}^+ \to \mathbb{R}$ are piecewise pseudo almost periodic functions, and $\sup\limits_{t\in\mathbb{R}^+} |s_{ji}(t)| = s_{ji}^+$, $\sup\limits_{t\in\mathbb{R}^+} |t_{ij}(t)| = t_{ij}^+$, $\sup\limits_{t\in\mathbb{R}^+} |c_i(t)| = c_i^+$ and $\sup\limits_{t\in\mathbb{R}^+} |d_j(t)| = d_j^+$.
- (H_3) I_k, $J_k \in PAP(\mathbb{Z}^+, \mathbb{R}^n)$ and there are two positive constants I, J satisfying $0 < I$, $J < 1$ such that $|I_k(u^1) - I_k(u^2)| \le I|u^1 - u^2|$, $|J_k(u^1) - J_k(u^2)| \le J|u^1 - u^2|$ for u^1, $u^1 \in \mathbb{R}$, $k \in \mathbb{Z}^+$.

Remark 1.1. In the existing literature on BAM neural networks, in addition to the condition (H_1), the activation functions $f_j(\cdot)$ and $g_i(\cdot)$ are required to satisfy $f_j(0) = 0$ and $g_i(0) = 0$. In this paper, the activation functions $f_j(0,0) = 0$, $g_i(0,0) = 0$ are not required, as it only needs to meet the condition (H_1).

In this paper, first, by using the exponential dichotomy of linear differential equations and the fixed point theory for contraction mapping, the existence of piecewise pseudo almost periodic solutions for system (1.1) satisfying the initial value conditions given by (1.2) is studied. And then by using mathematical methods of induction and inequality techniques, the global exponential stability of piecewise pseudo-almost periodic solutions for system (1.1) satisfying the initial value conditions given by (1.2) will be discussed.

Remark 1.2. Compared with the systems studied in [11,16,17,20–22,24,25], the system (1.1) studied in this paper is more general, as some classical Hopfield neural networks and BAM neural networks are special cases of the system (1.1). Duan [22] studied the effects of constant delays in the activation functions on the dynamic behavior of the system, but they did not consider the effects of leakage delays and impulsive perturbations on the dynamic behavior of the system. The mixed time-varying delays studied in this paper include not only the time-varying delays in the activation functions but also the leakage delays; the influences of impulsive perturbations on the dynamic behavior of the system are also considered. This paper not only considers the norm $\|\cdot\|_\infty$ but also $\|(\cdot)'\|_\infty$. Therefore, the results obtained in [21,22,24] are special cases of the research results in this paper.

2. Preliminaries

This section mainly gives the necessary definitions, lemmas and notations to prove the existence and global exponential stability of the piecewise pseudo almost periodic solutions of the system described by (1.1)–(1.2).

$C^1(\mathbb{R},\mathbb{R}^n)$ is the set of continuous functions with a continuous derivative, and

$$\|\phi\|_1 = \max\{\max\{\|\varphi\|_\infty, \|\varphi'\|_\infty\}, \max\{\|\psi\|_\infty, \|\psi'\|_\infty\}\},$$

where $\|\cdot\|_\infty = \max\limits_{1<i<m} \sup\limits_{t\in\mathbb{R}} |\cdot|$.

$BC(\mathbb{R},\mathbb{R}^n)$ is the set of bounded continuous functions, and $(BC(\mathbb{R},\mathbb{R}^n), \|\cdot\|_\infty)$ is a Banach space.

Let $PC(\mathbb{R},\mathbb{R}^n)$ be the set of all piecewise continuous functions $\phi : \mathbb{R} \to \mathbb{R}^n$ such that ϕ is countinuous at t for $t \notin \{t_k : k \in \mathbb{Z}^+\}$ and that $\phi(t_k^+)$, $\phi(t_k^-)$ exist and $\phi(t_k^-) = \phi(t_k)$ for $k \in \mathbb{Z}^+$.

Define $L^\infty(\mathbb{Z}^+,\mathbb{R}^n) = \{u : \mathbb{Z}^+ \to \mathbb{R}^n, \|u\| = \sup\limits_{n\in\mathbb{Z}^+} |u(n)|\}$ and $PAP_0(\mathbb{Z}^+,\mathbb{R}^n) = \{u \in L^\infty(\mathbb{Z}^+,\mathbb{R}^n) : \lim\limits_{n\to\infty} \frac{1}{2n} \sum\limits_{k=-n}^{n} \|u(k)\| = 0\}$.

Definition 2.1. *[34] A function $f \in C(\mathbb{R},\mathbb{R}^n)$ is called almost periodic if for $\forall\epsilon > 0$, there is $L(\epsilon) > 0$ such that every interval of length $L(\epsilon)$ contains at least a number $\delta \in T(f,\epsilon)$ such that $|f(t+\delta)-f(t)| < \epsilon$ for any $t \in \mathbb{R}$, where $T(f,\epsilon) = \{\delta \mid |f(t+\delta) - f(t)| < \epsilon, \forall t \in \mathbb{R}\}$. The set of all almost periodic functions is represented by $AP(\mathbb{R},\mathbb{R}^n)$.*

Definition 2.2. *[35] A sequence $\{u_n\}$ is called almost periodic if for $\forall\epsilon > 0$, there is a natural number $L(\epsilon)$ such that for $k \in \mathbb{Z}$, there is at least one number q in $[k, k + L(\epsilon)]$ for which the inequality $\|u_{n+q} - u_n\| < \epsilon$ holds for all $n \in \mathbb{N}$. The set of all almost periodic sequences is represented by $AP(\mathbb{Z},\mathbb{R}^n)$.*

Definition 2.3. *[36] A function $\phi \in PC(\mathbb{R}, \mathbb{R}^n)$ is called piecewise almost periodic if the following conditions are satisfied.*

- (1) $\{t_k^j\}$, $t_k^j = t_{k+j} - t^k$, k, $j \in \mathbb{Z}$ are equipotentially almost periodic, that is to say, for any $\epsilon > 0$, there is a relatively dense set in $\mathbb{R}$ of ϵ-almost periods that are common for any of the sequences $\{t_k^j\}$.
- (2) For any $\epsilon > 0$, there is a constant $\delta(\epsilon) > 0$ such that, if t' and t'' belong to the same interval of continuity of ϕ and $|t' - t''| < \delta$, then $\|\phi(t') - \phi(t'')\| < \epsilon$.
- (3) For any $\epsilon > 0$, there is a relatively dense set $\Omega(\epsilon)$ in $\mathbb{R}$ such that, if $\rho \in \Omega(\epsilon)$, then $\|\phi(t+\rho) - \phi(t)\| < \epsilon$ for $t \in \mathbb{R}$, which satisfies $|t - t_i| > \epsilon$, $i \in \mathbb{Z}$.

Denote by $AP_T(\mathbb{R}, \mathbb{R}^n)$ the set of piecewise almost periodic functions; $UPC(\mathbb{R}, \mathbb{R}^n)$ is the set of the function $f \in PC(\mathbb{R}, \mathbb{R}^n)$ such that f satisfies the condition (2) in Definition 2.3. Let

$$PC_T^0(\mathbb{R}, \mathbb{R}^n) = \{\phi \in PC(\mathbb{R}, \mathbb{R}^n), \lim_{t \to \infty} \|\phi(t)\| = 0\},$$

$$PAP_T^0(\mathbb{R}, \mathbb{R}^n) = \{\phi \in PC(\mathbb{R}, \mathbb{R}^n), \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \|\phi\| dt = 0\}.$$

Obviously, $PC_T^0(\mathbb{R}, \mathbb{R}^n) \subset PAP_T^0(\mathbb{R}, \mathbb{R}^n)$.

Definition 2.4. *$f \in PC(\mathbb{R} \times \Omega_1 \times \Omega_2, \mathbb{R}^n)$ is said to be piecewise almost periodic in t uniformly in $(\varphi, \psi) \in \Omega_1 \times \Omega_2$ if for every compact set $K_1 \times K_2 \subseteq \Omega_1 \times \Omega_2$, $\{f(\cdot, \varphi, \psi) : (\varphi, \psi) \in K_1 \times K_2\}$ is uniformly bounded, and for $\forall \epsilon > 0$, there exists a relatively dense set $\Omega(\epsilon)$ such that $\|f(t+\tau, \varphi, \psi) - f(t, \varphi, \psi)\| \leq \epsilon$ for $(\varphi, \psi) \in K_1 \times K_2$, $\tau \in \Omega(\epsilon)$, and given $t \in \mathbb{R}$, $|t - t_i| > \epsilon$.*

Definition 2.5. *[37] Sequence $\{u_n\}_{n \in \mathbb{Z}} \in L^\infty(\mathbb{Z}, \mathbb{R}^n)$ is called pseudo almost periodic if there exist $u_n^1 \in AP(\mathbb{Z}, \mathbb{R}^n)$ and $u_n^2 \in PAP_0(\mathbb{Z}, \mathbb{R}^n)$ such that $u_n = u_n^1 + u_n^2$. The set of pseudo almost periodic sequences is denoted by $PAP(\mathbb{Z}, \mathbb{R}^n)$.*

Definition 2.6. *[38] A function $\phi \in PC(\mathbb{R}, \mathbb{R}^n)$ is known as piecewise pseudo almost periodic if it can be disintegrated into $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP_T(\mathbb{R}, \mathbb{R}^n)$ and $\phi_2 \in PAP_T^0(\mathbb{R}, \mathbb{R}^n)$. The set of piecewise pseudo almost periodic functions is denoted by $PAP_T(\mathbb{R}, \mathbb{R}^n)$.*

Lemma 2.1. *[38] Let the sequence of vector-valued functions $\{I_i\}_{i \in \mathbb{Z}}$ be pseudo almost periodic and there be a constant $L > 0$ such that $\|I_i(u) - I_i(v)\| \leq L\|u - v\|$ for all $u, v \in \Omega$, $i \in \mathbb{Z}$ if $\phi \in PAP_T(\mathbb{R}, X) \cap UPC(\mathbb{R}, X)$ such that $R(\phi) \subset \Omega$; then $I_i(\phi(t_i))$ is pseudo almost periodic.*

Lemma 2.2. *[39] If $\varphi(\cdot) \in PAP_T(\mathbb{R}, \mathbb{R}^n)$, then $\varphi(\cdot - h) \in PAP_T(\mathbb{R}, \mathbb{R}^n)$.*

Lemma 2.3. *[39] If $\varphi(\cdot), \psi(\cdot) \in PAP_T(\mathbb{R}, \mathbb{R})$, then $\varphi \times \psi \in PAP_T(\mathbb{R}, \mathbb{R})$.*

Lemma 2.4. *[39] If $f_i(\cdot) \in C(\mathbb{R}, \mathbb{R}^n)$ satisfies the Lipschitz condition, $\psi(\cdot) \in PAP_T(\mathbb{R}, \mathbb{R})$, $\psi'(\cdot) \in PAP_T(\mathbb{R}, \mathbb{R})$ and $\delta(\cdot) \in AP_T(\mathbb{R}, \mathbb{R})$ such that for all $t \in \mathbb{R}$, $1 - \delta'(t) > 0$, $f_i(\psi(\cdot - \delta(\cdot))) \in PAP_T(\mathbb{R}, \mathbb{R})$.*

Lemma 2.5. *[40] Assume that $f \in PAP_T(\mathbb{R} \times \Omega_1 \times \Omega_2, X)$ and the following conditions hold:*

- (1) $\{f(t, u, v) : t \in \mathbb{R}, u \in K_1, v \in K_2\}$ is bounded for every bounded subset $K_1 \times K_2 \subseteq \Omega_1 \times \Omega_2$.
- (2) $\{f(t, u, v)\}$ is uniformly continuous in each bounded subset of $\Omega_1 \times \Omega_2$ uniformly in $t \in \mathbb{R}$.

If $\varphi \in PAP_T(\mathbb{R}, X)$, $\psi \in PAP_T(\mathbb{R}, X)$ such that $R(\varphi) \times R(\psi) \subset \Omega_1 \times \Omega_2$, then $f(t, \varphi(\cdot), \psi(\cdot)) \in PAP_T(\mathbb{R}, X)$, where $R(\varphi)$, $R(\psi)$ are the ranges of φ, ψ, respectively.

Lemma 2.6. *[40] Let $f \in PAP_T(\mathbb{R} \times \Omega_1 \times \Omega_2, X)$, $\varphi \in PAP_T(\mathbb{R}, X)$, $\psi \in PAP_T(\mathbb{R}, X)$ and $R(\varphi) \times R(\psi) \subset \Omega_1 \times \Omega_2$. Assume that there exists a constant $L_f > 0$ such that $\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq L_f(\|u_1 - u_2\| + \|v_1 - v_2\|)$, $t \in \mathbb{R}$, u_1, $u_2 \in \Omega_1$ and v_1, $v_2 \in \Omega_2$. Then, $f(\cdot, \varphi(\cdot), \psi(\cdot)) \in PAP_T(\mathbb{R}, X)$.*

Definition 2.7. *[34] Let $x \in \mathbb{R}^n$ and $A(\cdot)$ be an $n \times n$ continuous matrix defined on $\mathbb{R}$. The linear system $x'(t) = A(t)x(t)$ is said to admit an exponential dichotomy on $\mathbb{R}$ if there exist positive constants k and α and a projection P such that the fundamental solution matrix $X(t)$ of $x'(t) = A(t)x(t)$ satisfies the following:*

$$\|X(t)PX^{-1}(s)\| \leq ke^{-\alpha(t-s)}, t \geq s,$$
$$\|X(t)(I - P)X^{-1}(s)\| \leq ke^{-\alpha(s-t)}, t \leq s.$$

Lemma 2.7. *[34] Let $a_i(\cdot)$ be an almost periodic function on $\mathbb{R}$ and $M[a_i] = \lim_{T\to+\infty} \int_t^{t+T} a_i(s)ds > 0, i = 1, 2, ..., n$. Then, the system $x'(t) = diag(-a_1(t), -a_2(t), ..., -a_n(t))x(t)$ admits an exponential dichotomy on $\mathbb{R}$.*

Lemma 2.8. *[41] Let $A(\cdot)$ be an almost periodic matrix function and $f(\cdot) \in PAP(\mathbb{R}, \mathbb{R}^n)$. If $x'(t) = A(t)x(t)$ admits an exponential dichotomy, then the pseudo almost periodic system $x'(t) = A(t)x(t) + f(t)$ has a unique pseudo almost periodic solution $x(t)$, and*

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)f(s)ds - \int_{t}^{\infty} X(t)(I - P)X^{-1}(s)f(s)ds.$$

Definition 2.8. *Let $\bar{z}(t) = (\bar{x}(t), \bar{y}(t))^T$ be the solution of system (1.1) satisfying the initial value conditions $\overline{\phi} = (\overline{\varphi}(\theta), \overline{\psi}(\theta))^T$. If there exist constants $M \geq 1$, $\eta > 0$ such that for any solution $z(t) = (x(t), y(t))^T$ of the system (1.1) satisfying the initial value conditions $\phi = (\varphi(\theta), \psi(\theta))^T$, the following inequality holds*

$$\|z(t) - \bar{z}(t)\|_1 \leq M\|\phi - \overline{\phi}\|_1 e^{-\eta t}, t \geq 0,$$

where $\|\phi - \overline{\phi}\|_1 = \max\{\max\{\|\varphi\|_\infty, \|\varphi'\|_\infty\}, \max\{\|\psi\|_\infty, \|\psi'\|_\infty\}\}$. Then, $\bar{z}(t)$ is said to be globally exponentially stable.

3. Existence of piecewise pseudo almost periodic solutions

This section establishes the conditions of the existence for piecewise pseudo almost periodic solutions of the system described by (1.1)–(1.2).

Lemma 3.1. *Let $f \in PAP_T(\mathbb{R} \times \Omega_1 \times \Omega_2, X)$, φ, $\psi \in PAP_T(\mathbb{R}, \mathbb{R}^n)$ and $R(\varphi) \times R(\psi) \subset \Omega_1 \times \Omega_2$. There are constants $P > 0$, $Q > 0$ such that for $t \in \mathbb{R}$, φ, $\overline{\varphi} \in \Omega_1$, ψ, $\overline{\psi} \in \Omega_2$, $|f(t, \varphi, \psi) - f(t, \overline{\varphi}, \overline{\psi})| \leq P|\varphi - \overline{\varphi}| + Q|\psi - \overline{\psi}|$ and $\theta(\cdot) \in C(\mathbb{R}, \mathbb{R})$. Then, $f(t, \varphi(t - \theta(\cdot)), \psi)$, $f(t, \varphi, \psi(t - \theta(\cdot))) \in PAP_T(\mathbb{R}, X)$ and $f(t, \varphi(t - \theta(\cdot)), \psi(t - \theta(\cdot))) \in PAP_T(\mathbb{R}, X)$.*

Proof. Since $\varphi, \psi \in PAP_T(\mathbb{R},\mathbb{R})$, then $\varphi = \varphi_1 + \varphi_2$, $\psi = \psi_1 + \psi_2$, where $\varphi_1, \psi_1 \in AP_T(\mathbb{R},\mathbb{R})$ and φ_2, $\psi_2 \in PAP_T^0(\mathbb{R},\mathbb{R})$. We have

$$\begin{aligned} & f(t,\varphi(t-\theta(\cdot)),\psi) = f(t,\varphi_1(t-\theta(\cdot)) + \varphi_2(t-\theta(\cdot)),\psi_1+\psi_2) \\ = \; & f(t,\varphi_1(t-\theta(\cdot)),\psi_1) + f(t,\varphi_1(t-\theta(\cdot)) + \varphi_2(t-\theta(\cdot)),\psi_1+\psi_2) - f(t,\varphi_1(t-\theta(\cdot)),\psi_1) \\ = \; & f_1 + f_2, \end{aligned}$$

where $f_1 = f(t,\varphi_1(t-\theta(\cdot)),\psi_1)$, $f_2 = f(t,\varphi_1(t-\theta(\cdot))+\varphi_2(t-\theta(\cdot)),\psi_1+\psi_2) - f(t,\varphi_1(t-\theta(\cdot)),\psi_1)$. From Lemmas 2.2–2.5, $f_1 = f(t,\varphi_1(t-\theta(\cdot)),\psi_1) \in AP_T(\mathbb{R},\mathbb{R})$ holds. Now $f_2 \in PAP_T^0(\mathbb{R},\mathbb{R})$ will be proved. Since

$$\begin{aligned} \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} |f_2| dt &\leq \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} (P|\varphi_2(t-\theta(t)| + Q|\psi_2|)dt \\ &= P\lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} |\varphi_2(t-\theta(t)|dt + Q\lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} |\psi_2| dt \\ &= 0. \end{aligned}$$

Thus $f_2 \in PAP_T^0(\mathbb{R},\mathbb{R})$. So, $f(t,\varphi(t-\theta(\cdot)),\psi) \in PAP_T(\mathbb{R},X)$. In a similar way, $f(t,\varphi,\psi(t-\theta(\cdot))) \in PAP_T(\mathbb{R},X)$ and $f(t,\varphi(t-\theta(\cdot)),\psi(t-\theta(\cdot))) \in PAP_T(\mathbb{R},X)$. The proof is complete. □

Theorem 3.1. *If conditions (H_1)–(H_3) are true. For any $\phi = (\varphi,\psi)^T \in PAP_T(\mathbb{R}^+,\mathbb{R}^{2m})$ and $\overline{\phi} = (\overline{\varphi},\overline{\psi})^T \in PAP_T(\mathbb{R}^+,\mathbb{R}^{2m})$, define the operator*

$$\Phi_\phi^1(t) = \begin{pmatrix} \int_{-\infty}^{t} e^{-\int_s^t a_1(u)du}\overline{F}_1(s,\varphi,\psi)ds \\ \cdots \\ \int_{-\infty}^{t} e^{-\int_s^t a_m(u)du}\overline{F}_m(s,\varphi,\psi)ds \\ \int_{-\infty}^{t} e^{-\int_s^t b_1(u)du}\overline{G}_1(s,\varphi,\psi)ds \\ \cdots \\ \int_{-\infty}^{t} e^{-\int_s^t b_m(u)du}\overline{G}_m(s,\varphi,\psi)ds \end{pmatrix},$$

where

$$\overline{F}_i(t,\varphi,\psi) = a_i(t)\int_{t-\alpha_i(t)}^{t} \varphi_i'(s)ds + \sum_{j=1}^{m} s_{ji}(t)f_j[\varphi_j(t),\psi_j(t-\tau_{ji}(t))] + c_i(t), i = 1,2,\cdots,m,$$

$$\overline{G}_j(t,\varphi,\psi) = b_j(t)\int_{t-\beta_j(t)}^{t} \psi_j'(s)ds + \sum_{i=1}^{m} t_{ij}(t)g_i[\varphi_i(t-\delta_{ij}(t)),\psi_i(t)] + d_j(t), j = 1,2,\cdots,m.$$

Then, $\Phi_\phi^1(\cdot)$ is a mapping from $PAP_T(\mathbb{R}^+,\mathbb{R}^{2m})$ to $PAP_T(\mathbb{R}^+,\mathbb{R}^{2m})$.

Proof. By Lemmas 2.2–2.5 and Lemma 3.1, $\overline{F}_i(t,\varphi,\psi) \in PAP_T(\mathbb{R}^+,\mathbb{R})$; thus,

$$\overline{F}_i(t,\varphi,\psi) = \overline{F}_i^1(t,\varphi,\psi) + \overline{F}_i^2(t,\varphi,\psi),$$

where $\overline{F}_i^1(t,\varphi,\psi) \in AP_T(\mathbb{R}^+,\mathbb{R})$, $\overline{F}_i^2(t,\varphi,\psi) \in PAP_T^0(\mathbb{R}^+,\mathbb{R})$.

For $i = 1, 2, ..., m$, we have

$$\begin{aligned}\int_{-\infty}^{t} e^{-\int_s^t a_i(u)du}\overline{F}_i(s,\varphi,\psi)ds &= \int_{-\infty}^{t} e^{-\int_s^t a_i(u)du}\overline{F}_i^1(s,\varphi,\psi)ds + \int_{-\infty}^{t} e^{-\int_s^t a_i(u)du}\overline{F}_i^2(s,\varphi,\psi)ds \\ &= \Psi_1(t) + \Psi_2(t).\end{aligned}$$

Step 1: Prove that $\Psi_1(t) \in UPC(\mathbb{R}^+, \mathbb{R})$. Obviously, $\Psi_1(t) \in PC(\mathbb{R}^+, \mathbb{R})$. For $t', t'' \in (t_k, t_{k+1}), k \in \mathbb{Z}^+, t'' < t'$,

$$\begin{aligned}&|\Psi_1(t') - \Psi_1(t'')| \\ = \ & |\int_{-\infty}^{t'} e^{-\int_s^{t'} a_i(u)du}\overline{F}_i^1(s,\varphi,\psi)ds - \int_{-\infty}^{t''} e^{-\int_s^{t''} a_i(u)du}\overline{F}_i^1(s,\varphi,\psi)ds| \\ \leq \ & |\int_{-\infty}^{t''} (e^{-\int_s^{t'} a_i(u)du} - e^{-\int_s^{t''} a_i(u)du})\overline{F}_i^1(s,\varphi,\psi)ds| + |\int_{t''}^{t'} e^{-\int_s^{t'} a_i(u)du}\overline{F}_i^1(s,\varphi,\psi)ds| \\ \leq \ & |e^{-\int_{t''}^{t'} a_i(u)du} - 1| \int_{-\infty}^{t''} e^{-\int_s^{t''} a_i(u)du}|\overline{F}_i^1(s,\varphi,\psi)|ds + \int_{t''}^{t'} e^{-\int_s^{t'} a_i(u)du}|\overline{F}_i^1(s,\varphi,\psi)|ds \\ \leq \ & a_i^+(t'-t'')\int_{-\infty}^{t''} e^{-a_i^-(t''-s)}|\overline{F}_i^1(s,\varphi,\psi)|ds + \int_{t''}^{t'} e^{-a_i^-(t'-s)}|\overline{F}_i^1(s,\varphi,\psi)|ds \\ \leq \ & (\frac{a_i^+}{a_i^-}(t'-t'') + \int_{t''}^{t'} e^{-a_i^-(t'-s)}ds)\|\overline{F}_i^1(s,\varphi,\psi)\|_\infty.\end{aligned}$$

For $\forall\epsilon > 0$, there is $0 < \delta < \min\{\frac{a_i^-\epsilon}{2\|\overline{F}_i^1\|_\infty a_i^+}, \frac{\epsilon}{2\|\overline{F}_i^1\|_\infty}\}$ such that for t', t' satisfying $0 < |t' - t''| < \delta$, $|\Psi_1(t') - \Psi_1(t'')| < \epsilon$ holds. So, $\Psi_1(t) \in UPC(\mathbb{R}^+, \mathbb{R})$.

Step 2: Prove that $\Psi_1(\cdot) \in AP_T(\mathbb{R}^+, \mathbb{R})$. Since $\overline{F}_i^1(t,\varphi,\psi) \in AP_T(\mathbb{R}^+, \mathbb{R})$, for $\forall\epsilon > 0$, there is a relatively dense set $\Omega(\epsilon)$; for $\tau \in \Omega(\epsilon)$, $t \in \mathbb{R}^+$, $|t - t_k| > \epsilon$, $i \in \mathbb{Z}^+$ such that $|\overline{F}_i^1(s+\tau,\varphi(s+\tau),\psi(s+\tau)) - \overline{F}_i^1(s,\varphi(s),\psi(s))| < a_i^-\epsilon$. Then

$$\begin{aligned}&|\Psi_1(t+\tau) - \Psi_1(t)| \\ = \ & |\int_{-\infty}^{t+\tau} e^{-\int_s^{t+\tau} a_i(u)du}\overline{F}_i^1(s,\varphi,\psi)ds - \int_{-\infty}^{t} e^{-\int_s^{t} a_i(u)du}\overline{F}_i^1(s,\varphi,\psi)ds| \\ \leq \ & |\int_{-\infty}^{t} e^{-\int_{s+\tau}^{t+\tau} a_i(u)du}\overline{F}_i^1(s+\tau,\varphi(s+\tau),\psi(s+\tau))ds - \int_{-\infty}^{t} e^{-\int_s^{t} a_i(u)du}\overline{F}_i^1(s,\varphi(s),\psi(s))ds| \\ \leq \ & |\int_{-\infty}^{t} e^{-a_i^-(t-s)}\overline{F}_i^1(s+\tau,\varphi(s+\tau),\psi(s+\tau))ds - \int_{-\infty}^{t} e^{-a_i^-(t-s)}\overline{F}_i^1(s,\varphi(s),\psi(s))ds| \\ \leq \ & \int_{-\infty}^{t} e^{-a_i^-(t-s)}|\overline{F}_i^1(s+\tau,\varphi(s+\tau),\psi(s+\tau)) - \overline{F}_i^1(s,\varphi(s),\psi(s))|ds \\ < \ & \epsilon.\end{aligned}$$

Consequently, $\Psi_1(\cdot) \in AP_T(\mathbb{R}, \mathbb{R})$.

Step 3: Prove that $\Psi_2(\cdot) \in PAP_T^0(\mathbb{R}^+, \mathbb{R})$. Since $\overline{F}_i^2(s-u,\varphi(s-u),\psi(s-u)), \overline{F}_i^2(s,\varphi(s),\psi(s)) \in PAP_T^0(\mathbb{R}^+, \mathbb{R})$, then

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|\Psi_2(\cdot)|dt$$

$$\begin{aligned}
&= \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|\int_{-\infty}^{t}e^{-\int_s^t a_i(u)du}\overline{F}_i^2(s,\varphi,\psi)ds|dt\\
&\leq \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|\int_{-\infty}^{t}e^{-a_i^-(t-s)}\overline{F}_i^2(s,\varphi,\psi)ds|dt\\
&\leq \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}e^{-a_i^- t}dt\int_{-\infty}^{-r}e^{a_i^- s}\|\overline{F}_i^2(s,\varphi,\psi)\|_\infty ds+\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\int_{-r}^{t}e^{-a_i^-(t-s)}|\overline{F}_i^2(s,\varphi,\psi)|dsdt\\
&\leq \lim_{r\to\infty}\frac{\|\overline{F}_i^2(s,\varphi,\psi)\|_\infty}{2ra_i^-}\int_{-r}^{r}e^{-a_i^-(t+r)}dt+\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\int_{0}^{t+r}e^{-a_i^- u}|\overline{F}_i^2(t-u,\varphi(t-u),\psi(t-u))|dudt\\
&\leq \lim_{r\to\infty}\frac{\|\overline{F}_i^2(s,\varphi,\psi)\|_\infty}{2ra_i^-}\int_{-r}^{r}e^{-a_i^-(t+r)}dt+\int_{0}^{\infty}e^{-a_i^- u}(\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|\overline{F}_i^2(t-u,\varphi(t-u),\psi(t-u))|dt)du\\
&= 0.
\end{aligned}$$

Consequently, $\Psi_2(\cdot)\in PAP_T^0(\mathbb{R}^+,\mathbb{R})$. So, $\int_{-\infty}^{t}e^{-\int_s^t a_i(u)du}\overline{F}_i(s,\varphi,\psi)ds\in PAP_T(\mathbb{R}^+,\mathbb{R})$. Similarly, for $j=1,2,...,m$, we have that $\int_{-\infty}^{t}e^{-\int_s^t b_j(u)du}\overline{G}_j(s,\varphi,\psi)ds\in PAP_T(\mathbb{R}^+,\mathbb{R})$.

Therefore $\Phi_\phi^1(\cdot)\in PAP_T(\mathbb{R}^+,\mathbb{R}^{2m})$. □

Theorem 3.2. *If conditions (H_1)–(H_3) are true, for any $\phi=(\varphi,\psi)^T\in PAP_T(\mathbb{R}^+,\mathbb{R}^{2m})$, define the operator*

$$\Phi_\phi^2(t)=\begin{pmatrix}
\sum_{t_k<t}e^{-\int_{t_k}^t a_1(u)du}I_k(\varphi_1(t_k))\\
\cdots\\
\sum_{t_k<t}e^{-\int_{t_k}^t a_m(u)du}I_k(\varphi_m(t_k))\\
\sum_{t_k<t}e^{-\int_{t_k}^t b_1(u)du}J_k(\psi_1(t_k))\\
\cdots\\
\sum_{t_k<t}e^{-\int_{t_k}^t b_m(u)du}J_k(\psi_m(t_k))
\end{pmatrix}.$$

Then, $\Phi_\phi^2(\cdot)$ is a mapping from $PAP_T(\mathbb{R}^+,\mathbb{R}^{2m})$ to $PAP_T(\mathbb{R}^+,\mathbb{R}^{2m})$.

Proof. Assume that $t\in(t_l,t_{l+1}]$, $l\in\mathbb{Z}^+$; therefore, $t-t_k=t-t_l+t_l-t_k\geq(l-k)\underline{\sigma}$, and

$$\sum_{t_k<t}e^{-a_i^-(t-t_k)}\leq\sum_{0\leq j=l-k<\infty}e^{-a_i^- j\underline{\sigma}}\leq\frac{1}{1-e^{-a_i^-\underline{\sigma}}}.$$

For any $\epsilon>0$, take the positive constant $\delta<\frac{\epsilon(1-e^{-a_i^-\underline{\sigma}})}{a_i^+\|I_k(\varphi_i(t_k))\|_\infty}$, for $t',t''\in(t_k,t_{k+1}),k\in\mathbb{Z}^+,t'<t''$, provided that $|t'-t''|<\delta$; there is $\zeta\in(t',t'')$ such that

$$\begin{aligned}
&|\sum_{t_k<t'}e^{-\int_{t_k}^{t'}a_i(u)du}I_k(\varphi_i(t_k))-\sum_{t_k<t''}e^{-\int_{t_k}^{t''}a_i(u)du}I_k(\varphi_i(t_k))|\\
\leq\ & a_i^+\sum_{t_k<\zeta}e^{-a_i^-(\zeta-t_k)}\|I_k(\varphi_i(t_k))\|_\infty|t'-t''|
\end{aligned}$$

$$\begin{aligned}
&\le \frac{a_i^+\|I_k(\varphi_i(t_k))\|_\infty}{1-e^{-a_i^-\underline{\sigma}}}|t'-t''| \\
&< \epsilon.
\end{aligned}$$

That is, $\sum\limits_{t_k<t} e^{-\int_{t_k}^t a_i(u)du} I_k(\varphi_i(t_k)) \in UPC(\mathbb{R}^+,\mathbb{R})$. By Lemma 2.1 and (H_3), $I_k(\varphi_i(t_k)) \in PAP(\mathbb{Z}^+,\mathbb{R})$. Let $I_k(\varphi_i(t_k)) = I_k^1 + I_k^2$, where $I_k^1 \in AP(\mathbb{Z}^+,\mathbb{R})$, $I_k^2 \in PAP_0(\mathbb{Z}^+,\mathbb{R})$; then,

$$\begin{aligned}
\sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)du} I_k(\varphi_i(t_k)) &= \sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)du} I_k^1(\varphi_i(t_k)) + \sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)du} I_k^2(\varphi_i(t_k)) \\
&= \overline{I_1}(\cdot) + \overline{I_2}(\cdot).
\end{aligned}$$

First, we show that $\overline{I_1}(\cdot) \in AP_T(\mathbb{R}^+,\mathbb{R})$. Since $\{t_k^j : k, j \in \mathbb{Z}^+\}$ are equipotentially almost periodic, for any $\epsilon > 0$, there is a relatively dense set $\Omega(\epsilon)$ of $\mathbb{R}^+$ and $Q(\epsilon)$ of $\mathbb{Z}^+$ such that for $t \in (t_k, t_{k+1}]$, $|t-t_k| > \epsilon$, $|t-t_{k+1}| > \epsilon$, $k \in \mathbb{Z}^+$, $\tau \in \Omega(\epsilon)$, $q \in Q(\epsilon)$, from Lemma 35 (see [35]), we have that $t+\tau > t_k+\tau+\epsilon > t_{k+q}$ and $t_{k+q+1} > t_{k+1} - \epsilon + \tau > t + \tau$; so, $t_{k+q} < t+\tau < t_{k+q+1}$. Then

$$\begin{aligned}
|\overline{I_1}(t+\tau) - \overline{I_1}(t)| &= |\sum_{t_k<t+\tau} e^{-\int_{t_k}^{t+\tau} a_i(u)du} I_k^1(\varphi_i(t_k)) - \sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)du} I_k^1(\varphi_i(t_k))| \\
&\le \sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)du} |I_{k+q}^1(\varphi_i(t_k)) - I_k^1(\varphi_i(t_k))| \\
&\le \epsilon \sum_{t_k<t} e^{-a_i^-(t-t_k)} \\
&\le \frac{\epsilon}{1-e^{-a_i^-\underline{\sigma}}}.
\end{aligned}$$

So, $\overline{I_1}(\cdot) \in AP_T(\mathbb{R}^+,\mathbb{R})$.

Next, we show that $\overline{I_2}(\cdot) \in PAP_T^0(\mathbb{R}^+,\mathbb{R})$. For $k, n \in \mathbb{Z}^+$ and $t \in (t_k, t_{k+1}]$, let

$$\chi_n(t) = \begin{cases} e^{-\int_{t_{k-n}}^t a_i(u)du} I_{k-n}^2(\varphi_i(t_{k-n})), 0 \le n \le k, \\ 0, n > k; \end{cases}$$

since

$$\lim_{t\to\infty} |e^{-\int_{t_{k-n}}^t a_i(u)du} I_{k-n}^2(\varphi_i(t_{k-n}))| \le \lim_{t\to\infty} e^{-a_i^-(t-t_{k-n})} \|I_k^2(\varphi_i(t_k))\|_\infty = 0,$$

$\lim\limits_{t\to\infty} |\chi_n(t)| = 0$; there is $\chi_n(t) \in PC_T^0(\mathbb{R},\mathbb{R}^n) \subset PAP_T^0(\mathbb{R},\mathbb{R}^n)$.

Let $S_n(t) = \sum\limits_{l=0}^n \chi_l(t)$, because

$$|s_n(t)| \le \sum_{l=0}^n e^{-a_i^-(t-t_{k-l})} \|I_k^2(\varphi_i(t_k))\|_\infty \le \sum_{l=0}^n e^{-a_i^-(t-t_k)} e^{-l\underline{\sigma} a_i^-} \|I_k^2(\varphi_i(t_k))\|_\infty.$$

Therefore, $\lim\limits_{t\to\infty} |s_n(t)| = 0$. So, $s_n(t) \in PAP_T^0(\mathbb{R},\mathbb{R}^n)$.

And because $\lim_{n\to\infty} s_n(t) = \overline{I_2}(\cdot)$ uniformly holds on $\mathbb{R}^+$, from Lemma 2.2 in [40], $\overline{I_2}(\cdot) \in PAP_T^0(\mathbb{R}^+,\mathbb{R})$ holds. Therefore, $\sum\limits_{t_k<t} e^{-\int_{t_k}^t a_i(u)du} I_k(\varphi_i(t_k)) \in PAP_T(\mathbb{R}^+,\mathbb{R})$.

In the same way, $\sum\limits_{t_k<t} e^{-\int_{t_k}^t b_j(u)du} J_k(\psi_j(t_k)) \in PAP_T(\mathbb{R}^+,\mathbb{R})$ can be obtained. Thus $\Phi_\phi^2(\cdot) \in PAP_T(\mathbb{R}^+,\mathbb{R}^{2m})$. □

From Theorems 3.1 and 3.2, the following theorem is obtained.

Theorem 3.3. *If conditions* (H_1)–(H_3) *are true, for any* $\phi = (\varphi,\psi)^T \in PAP_T(\mathbb{R}^+,\mathbb{R}^{2m})$ *and* $\phi' = (\varphi',\psi')^T \in PAP_T(\mathbb{R}^+,\mathbb{R}^{2m})$*, define the operator*

$$\Phi_\phi(t) = \Phi_\phi^1(t) + \Phi_\phi^2(t).$$

Then, $\Phi_\phi(t)$ maps $PAP_T(\mathbb{R}^+,\mathbb{R}^{2m})$ into $PAP_T(\mathbb{R}^+,\mathbb{R}^{2m})$, and $\Phi'_\phi(t)$ maps $PAP_T(\mathbb{R}^+,\mathbb{R}^{2m})$ into $PAP_T(\mathbb{R}^+,\mathbb{R}^{2m})$.

Theorem 3.4. *Assume that conditions* (H_1)–(H_3) *are true and the following condition holds:*

(H_4) $p_{ij} = \max\{\max\{\frac{\theta_i}{a_i^-} + \frac{I}{1-e^{-a_i^-\underline{\sigma}}}, \frac{\theta_i}{a_i^-}(a_i^+ + a_i^-) + \frac{Ia_i^+}{1-e^{-a_i^-\underline{\sigma}}}\}, \max\{\frac{\gamma_j}{b_j^-} + \frac{J}{1-e^{-b_j^-\underline{\sigma}}}, \frac{\gamma_i}{b_j^-}(b_j^+ + b_j^-) + \frac{Ib_j^+}{1-e^{-b_j^-\underline{\sigma}}}\}\} < 1,$

where $\theta_i = a_i^+\alpha_i^+ + \sum\limits_{j=1}^m s_{ji}^+(P_j + Q_j)$, $\gamma_j = b_j^+\beta_j^+ + \sum\limits_{i=1}^m t_{ij}^+(H_i + K_i)$, $i, j = 1, 2, \cdots, m$.

Then, there is a unique piecewise differentiable pseudo almost periodic solution of system (1.1) in the region

$$\mathbb{X}_0 = \{\phi = (\varphi_1,\varphi_2,\cdots,\varphi_m,\psi_1,\psi_2,\cdots,\psi_m)^T \mid \phi \in PAP_T(\mathbb{R}^+,\mathbb{R}^{2m}), \|\phi - \phi^0\|_1 \le \frac{pL}{1-p}\},$$

where $\phi^0 = (\varphi_1^0,\varphi_2^0,\cdots,\varphi_m^0,\psi_1^0,\psi_2^0,\cdots,\psi_m^0)^T$, and

$$\begin{aligned} L &= \max\{\max_{1\le i\le m}\{\frac{l_i}{a_i^-} + l_i', \frac{l_i}{a_i^-}(a_i^+ + a_i^-) + a_i^+ l_i'\}, \max_{1\le i\le m}\{\frac{l_j}{b_j^-} + l_j', \frac{l_j}{b_j^-}(b_j^+ + b_j^-) + b_j^+ l_j'\}\}, \\ \varphi_i^0(t) &= \int_{-\infty}^t e^{-\int_s^t a_i(u)du}(c_i(s) + \sum_{j=1}^m s_{ji}(s) f_j(0,0))ds + \sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)du} I_k(0), \\ \psi_j^0(t) &= \int_{-\infty}^t e^{-\int_s^t b_j(u)du}(d_j(s) + \sum_{i=1}^m t_{ij}(s) g_i(0,0))ds + \sum_{t_k<t} e^{-\int_{t_k}^t b_j(u)du} J_k(0). \end{aligned}$$

Proof. It is easy to show that $\mathbb{X}_0 \subset PAP_T(\mathbb{R}^+,\mathbb{R}^{2m})$ is a closed convex set. Rewrite (1.1) into the following form

$$\begin{cases} x_i'(t) = -a_i(t)x_i(t) + a_i(t)\int_{t-\alpha_i(t)}^t x_i'(s)ds + \sum\limits_{j=1}^m s_{ji}(t) f_j(x_j(t), y_j(t-\tau_{ji}(t))) + c_i(t), \\ t \ne t_k, i = 1,2,\cdots,m, \\ \Delta x_i(t) = x_i(t_k^+) - x_i(t_k^-) = I_k(x_i(t_k)), t = t_k, \\ y_j'(t) = -b_j(t)y_j(t) + b_j(t)\int_{t-\beta_j(t)}^t y_j'(s)ds + \sum\limits_{i=1}^m t_{ij}(t) g_i(x_i(t-\delta_{ij}(t)), y_i(t)) + d_j(t), \\ t \ne t_k, j = 1,2,\cdots,m, \\ \Delta y_j(t) = y_j(t_k^+) - y_j(t_k^-) = J_k(y_j(t_k)), t = t_k. \end{cases} \tag{3.1}$$

We consider the following system

$$\begin{cases} x_i'(t) = -a_i(t)x_i(t) + \overline{F}_i(t,\varphi,\psi) + I_k(\varphi_i), i = 1,2,\cdots,m, \\ y_j'(t) = -b_j(t)y_j(t) + \overline{G}_j(t,\varphi,\psi) + J_k(\psi_j), j = 1,2,\cdots,m, \end{cases} \tag{3.2}$$

where

$$\overline{F}_i(t,\varphi,\psi) = a_i(t)\int_{t-\alpha_i(t)}^{t} \varphi_i'(s)ds + \sum_{j=1}^{m} s_{ji}(t)f_j(\varphi_j(t), \psi_j(t-\tau_{ji}(t))) + c_i(t), i = 1,2,\cdots,m,$$

$$\overline{G}_j(t,\varphi,\psi) = b_j(t)\int_{t-\beta_j(t)}^{t} \psi_j'(s)ds + \sum_{i=1}^{m} t_{ij}(t)g_i(\varphi_i(t-\delta_{ij}(t)), \psi_i(t)) + d_j(t), j = 1,2,\cdots,m.$$

Since $M[a_i] > 0$, $M[b_i] > 0$, from Lemma 2.7, we have that the linear system

$$\begin{cases} x_i'(t) = -a_i(t)x_i(t), i = 1,2,\cdots,m, \\ y_j'(t) = -b_j(t)y_j(t), j = 1,2,\cdots,m \end{cases} \tag{3.3}$$

admits an exponential dichotomy on $\mathbb{R}^+$. Thus, by Lemma 2.8, we obtain that system (3.2) has a unique piecewise pseudo almost periodic solution, which is expressed as follows:

$$\Phi_\phi(t) = \begin{pmatrix} \int_{-\infty}^{t} e^{-\int_s^t a_1(u)du}\overline{F}_1(s,\varphi,\psi)ds + \sum\limits_{t_k<t} e^{-\int_{t_k}^t a_1(u)du} I_k(\varphi_1(t_k)) \\ \cdots \\ \int_{-\infty}^{t} e^{-\int_s^t a_m(u)du}\overline{F}_m(s,\varphi,\psi)ds + \sum\limits_{t_k<t} e^{-\int_{t_k}^t a_m(u)du} I_k(\varphi_m(t_k)) \\ \int_{-\infty}^{t} e^{-\int_s^t b_1(u)du}\overline{G}_1(s,\varphi,\psi)ds + \sum\limits_{t_k<t} e^{-\int_{t_k}^t b_1(u)du} J_k(\psi_1(t_k)) \\ \cdots \\ \int_{-\infty}^{t} e^{-\int_s^t b_m(u)du}\overline{G}_m(s,\varphi,\psi)ds + \sum\limits_{t_k<t} e^{-\int_{t_k}^t b_m(u)du} J_k(\psi_m(t_k)) \end{pmatrix}. \tag{3.4}$$

Let $l_i = c_i^+ + \sum\limits_{j=1}^{m} s_{ji}^+|f_j(0,0)|$, $l_i' = \frac{\sup_{k\in\mathbb{Z}^+}|I_k(0)|}{1-e^{-a_i^-\underline{\sigma}}}$, $l_j = d_j^+ + \sum\limits_{i=1}^{m} t_{ij}^+|g_i(0,0)|$ and $l_j' = \frac{\sup_{k\in\mathbb{Z}^+}|J_k(0)|}{1-e^{-b_j^-\underline{\sigma}}}$. Then,

$$\begin{aligned} |\varphi_i^0(t)| &= \left|\int_{-\infty}^{t} e^{-\int_s^t a_i(u)du}(c_i(s) + \sum_{j=1}^{m} s_{ji}(s)f_j(0,0))ds + \sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)du} I_k(0)\right| \\ &\le \int_{-\infty}^{t} e^{-a_i^-(t-s)}(c_i^+ + \sum_{j=1}^{m} s_{ji}^+|f_j(0,0)|)ds + \sum_{t_k<t} e^{-a_i^-(t-t_k)}|I_k(0)| \\ &\le \frac{l_i}{a_i^-} + l_i'. \end{aligned}$$

Similarly, $\psi_j^0(t) \le \frac{l_j}{b_j^-} + l_j'$.
In addition,

$$|(\varphi_i^0)'(t)| = |c_i(t) + \sum_{j=1}^{m} s_{ji}(t)f_j[0,0] - a_i(t)\int_{-\infty}^{t} e^{-\int_s^t a_i(u)du}(c_i(s) + \sum_{j=1}^{m} s_{ji}(s)f_j[0,0])ds$$

$$
\begin{aligned}
&-\sum_{t_k<t} a_i(t)e^{-\int_{t_k}^t a_i(u)du} I_k(0)| \\
\le\ & l_i(1+a_i^+\int_{-\infty}^t e^{-a_i^-(t-s)}ds) + a_i^+\sum_{t_k<t} e^{-a_i^-(t-t_k)} I_k(0) \\
\le\ & \frac{l_i}{a_i^-}(a_i^+ + a_i^-) + a_i^+ l_i'.
\end{aligned}
$$

Similarly, $|(\psi_j^0)'(t)| \le \frac{l_j}{b_j^-}(b_j^+ + b_j^-) + b_j^+ l_j'$.
Therefore, $\|\phi^0\|_1 \le L$. For $\phi \in \mathbb{X}_0$, $\|\phi\|_1 \le \|\phi-\phi^0\|_1 + \|\phi^0\|_1 \le \frac{L}{1-p}$.

We will prove that $\Phi_\phi : \mathbb{X}_0 \to \mathbb{X}_0$ is a contraction.

First, we prove that for any $\phi \in \mathbb{X}_0$ and $\Phi_\phi \in \mathbb{X}_0$, let

$$
\begin{aligned}
F_i(t,\varphi,\psi) &= a_i(t)\int_{t-\alpha_i(t)}^t \varphi_i'(s)ds + \sum_{j=1}^m s_{ji}(t)(f_j(\varphi_j(t),\psi_j(t-\tau_{ji}(t))) - f_j(0,0)), \\
G_j(t,\varphi,\psi) &= b_j(t)\int_{t-\beta_j(t)}^t \psi_j'(s)ds + \sum_{i=1}^m t_{ij}(t)(g_i(\varphi_i(t-\delta_{ij}(t)),\psi_i(t)) - g_i(0,0)).
\end{aligned}
$$

Then,

$$
\begin{aligned}
|F_i(t,\varphi,\psi)| &\le a_i^+\|\varphi'\|_\infty \alpha_i^+ + \sum_{j=1}^m s_{ji}^+(\left|f_j(\varphi_j(t),\psi_j(t-\tau_{ji}(t))) - f_j(0,0)\right|) \\
&\le a_i^+\|\varphi'\|_\infty \alpha_i^+ + \sum_{j=1}^m s_{ji}^+(P_j|\varphi_j(t)| + Q_j|\psi_j(t-\tau_{ji}(t))|) \\
&\le a_i^+\|\varphi'\|_\infty \alpha_i^+ + \sum_{j=1}^m s_{ji}^+(P_j\|\varphi\|_\infty + Q_j\|\psi\|_\infty) \\
&\le (a_i^+\alpha_i^+ + \sum_{j=1}^m s_{ji}^+(P_j+Q_j))\|\phi\|_1 \\
&= \theta_i\|\phi\|_1, i=1,2,\cdots,m.
\end{aligned}
$$

Similarly, $|G_j(t,\varphi,\psi)| \le (b_j^+\beta_j^+ + \sum_{i=1}^m t_{ij}^+(H_i+K_i))\|\phi\|_1 = \gamma_j\|\phi\|_1, j=1,2,\cdots,m$, where $\theta_i = a_i^+\alpha_i^+ + \sum_{j=1}^m s_{ji}^+(P_j+Q_j), \gamma_j = b_j^+\beta_j^+ + \sum_{i=1}^m t_{ij}^+(H_i+K_i), i,j=1,2,\cdots,m$.
For $i=1,2,\cdots,m$, we have

$$
\begin{aligned}
|(\Phi_\phi-\phi^0)_i(t)| &= \left|\int_{-\infty}^t e^{-\int_s^t a_i(u)du} F_i(s,\varphi,\psi)ds + \sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)du}(I_k(\varphi_i(t_k)) - I_k(0))\right| \\
&\le \theta_i\|\phi\|_1\int_{-\infty}^t e^{-(t-s)a_i^-}ds + \sum_{t_k<t} e^{-(t-t_k)a_i^-} I\|\phi\|_1 \\
&= (\frac{\theta_i}{a_i^-} + \frac{I}{1-e^{-a_i^-\underline{\sigma}}})\|\phi\|_1;
\end{aligned}
$$

similarly, for $j = 1, 2, \cdots, m$, $|(\Phi_\phi - \phi^0)_j(t)| \le (\frac{\gamma_j}{b_j^-} + \frac{J}{1-e^{-b_j^- \underline{\sigma}}})\|\phi\|_1$.

Furthermore, for $i = 1, 2, \cdots, m$, we have

$$\begin{aligned}
& \left|(\Phi_\phi - \phi^0)_i'(t)\right| \\
= & \left|\left(\int_{-\infty}^t e^{-\int_s^t a_i(u)du} F_i(s, \varphi, \psi)ds + \sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)du}(I_k(\varphi_i(t_k)) - I_k(0))\right)'\right| \\
= & \left|F_i(t, \varphi, \psi) - a_i(t)\int_{-\infty}^t e^{-\int_s^t a_i(u)du} F_i(s, \varphi, \psi)ds - \sum_{t_k<t} a_i(t)e^{-\int_{t_k}^t a_i(u)du}(I_k(\varphi_i(t_k)) - I_k(0))\right| \\
\le & \left(\frac{\theta_i}{a_i^-}(a_i^+ + a_i^-) + \frac{a_i^+ I}{1 - e^{-a_i^- \underline{\sigma}}}\right)\|\phi\|_1.
\end{aligned}$$

Similarly, for $j = 1, 2, \cdots, m$, we have

$$\left|(\Phi_\phi - \phi^0)_j'(t)\right| \le \left(\frac{\gamma_j}{b_j^-}(b_j^+ + b_j^-) + \frac{b_j^+ J}{1 - e^{-b_j^- \underline{\sigma}}}\right)\|\phi\|_1.$$

In the light of (H_4), we have that $\|\Phi_\phi - \phi^0\|_1 \le p\|\phi\|_1 < \frac{pL}{1-p}$, that is $\Phi_\phi \in \mathbb{X}_0$.

Next, we prove that Φ_ϕ is a contraction. For $\phi = (\varphi_1, \varphi_2, \cdots, \varphi_m, \psi_1, \psi_2, \cdots, \psi_m)^T, \overline{\phi} = (\overline{\varphi}_1, \overline{\varphi}_2, \cdots, \overline{\varphi}_m, \overline{\psi}_1, \overline{\psi}_2, \cdots, \overline{\psi}_m)^T \in \mathbb{X}_0$, it follows that

$$\begin{aligned}
|\overline{F}_i(t, \varphi, \psi) - \overline{F}_i(t, \overline{\varphi}, \overline{\psi})| \le & \; a_i(t)\int_{t-\alpha_i(t)}^t \left|\varphi_i'(s) - \overline{\varphi}_i'(s)\right|ds + \sum_{j=1}^m s_{ji}(t)\Big|f_j[\varphi_j(t), \psi_j(t - \tau_{ji}(t))] \\
& -f_j[\overline{\varphi}_j(t), \overline{\psi}_j(t - \tau_{ji}(t))]\Big| \\
\le & \; a_i^+|\varphi' - \overline{\varphi}'|_\infty \alpha_i^+ + \sum_{j=1}^m s_{ji}^+(P_j|\varphi - \overline{\varphi}|_\infty + Q_j|\psi - \overline{\psi}|_\infty) \\
\le & \; (a_i^+\alpha_i^+ + \sum_{j=1}^m s_{ji}^+(P_j + Q_j))\|\phi - \overline{\phi}\|_1 \\
= & \; \theta_i\|\phi - \overline{\phi}\|_1, \;\; i = 1, 2, \cdots, m,
\end{aligned}$$

and $|\overline{G}_j(t, \varphi, \psi) - \overline{G}_j(t, \overline{\varphi}, \overline{\psi})| \le (b_j^+\beta_j^+ + \sum\limits_{i=1}^m t_{ij}^+(P_i + Q_i))\|\phi - \overline{\phi}\|_1 = \gamma_j\|\phi - \overline{\phi}\|_1, \;\; j = 1, 2, \cdots, m.$
And because

$$\begin{aligned}
& \Big|\sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)du} I_k(\varphi_i(t_k)) - \sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)du} I_k(\overline{\varphi}_i(t_k))\Big| \\
\le & \sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)du}|I_k(\varphi_i(t_k)) - I_k(\overline{\varphi}_i(t_k))| \\
\le & \frac{I}{1 - e^{-a_i^- \underline{\sigma}}}\|\phi - \overline{\phi}\|_1,
\end{aligned}$$

and

$$\Big|\Big(\sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)du} I_k(\varphi_i(t_k)) - \sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)du} I_k(\overline{\varphi}_i(t_k))\Big)'\Big|$$

$$\leq \quad a_i(t)\sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)du}|I_k(\overline{\varphi}_i(t_k)) - I_k(\varphi_i(t_k))|$$
$$\leq \quad \frac{a_i^+ I}{1-e^{-a_i^-\underline{\sigma}}}\|\phi-\overline{\phi}\|_1.$$

Similarly,

$$|\sum_{t_k<t} e^{-\int_{t_k}^t b_j(u)du}J_k(\psi_j(t_k)) - \sum_{t_k<t} e^{-\int_{t_k}^t b_j(u)du}J_k(\overline{\psi}_j(t_k))| \leq \frac{I}{1-e^{-b_j^-\underline{\sigma}}}\|\phi-\overline{\phi}\|_1$$
$$|(\sum_{t_k<t} e^{-\int_{t_k}^t b_j(u)du}J_k(\psi_j(t_k)) - \sum_{t_k<t} e^{-\int_{t_k}^t b_j(u)du}J_k(\overline{\psi}_j(t_k)))'| \leq \frac{b_j^+ I}{1-e^{-b_j^-\underline{\sigma}}}\|\phi-\overline{\phi}\|_1.$$

So, for $i = 1, 2, \cdots, m$, we have

$$\begin{aligned}
&\left|\Phi_\phi - \Phi_{\overline{\phi}}\right|_i \\
\leq\ & \int_{-\infty}^t e^{-\int_s^t a_i(u)du}|\overline{F}_i(s,\varphi,\psi) - \overline{F}_i(s,\overline{\varphi},\overline{\psi})|ds + \sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)du}|I_k(\varphi_i(t_k)) - I_k(\overline{\varphi}_i(t_k))| \\
\leq\ & \frac{\theta_i}{a_i^-}\|\phi-\overline{\phi}\|_1 + \frac{I}{1-e^{-a_i^-\underline{\sigma}}}\|\phi-\overline{\phi}\|_1 \\
=\ & (\frac{\theta_i}{a_i^-} + \frac{I}{1-e^{-a_i^-\underline{\sigma}}})\|\phi-\overline{\phi}\|_1,
\end{aligned}$$

and

$$\begin{aligned}
\left|(\Phi_\phi - \Phi_{\overline{\phi}})'\right|_i \ =\ & \Big(\int_{-\infty}^t e^{-\int_s^t a_i(u)du}(\overline{F}_i(s,\varphi,\psi) - \overline{F}_i(s,\overline{\varphi},\overline{\psi}))ds \\
& + \sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)du}(I_k(\varphi_i(t_k)) - I_k(\overline{\varphi}_i(t_k)))\Big)' \\
\leq\ & |F_i(t,\varphi,\psi) - F_i(t,\overline{\varphi},\overline{\psi})| \\
& + a_i^+\int_{-\infty}^t e^{-\int_s^t a_i(u)du}|F_i(s,\varphi,\psi) - F_i(s,\overline{\varphi},\overline{\psi})|ds + \frac{a_i^+ I}{1-e^{-a_i^-\underline{\sigma}}}\|\phi-\overline{\phi}\|_1 \\
\leq\ & (\frac{\theta_i}{a_i^-}(a_i^+ + a_i^-) + \frac{a_i^+ I}{1-e^{-a_i^-\underline{\sigma}}})\|\phi-\overline{\phi}\|_1.
\end{aligned}$$

Similarly, for $j = 1, 2, \cdots, m$, we have

$$\begin{aligned}
\left|\Phi_\phi - \Phi_{\overline{\phi}}\right|_j \ \leq\ & (\frac{\gamma_j}{b_j^-} + \frac{J}{1-e^{-b_j^-\underline{\sigma}}})\|\phi-\overline{\phi}\|_1, \\
\left|(\Phi_\phi - \Phi_{\overline{\phi}})'\right|_j \ \leq\ & (\frac{\gamma_j}{b_j^-}(b_j^+ + b_j^-) + \frac{b_j^+ J}{1-e^{-b_j^-\underline{\sigma}}})\|\phi-\overline{\phi}\|_1.
\end{aligned}$$

By (H_4), we have

$$\|\Phi_\phi - \Phi_{\overline{\phi}}\|_1 \leq p_{ij}\|\phi-\overline{\phi}\|_1,$$

which implies that Φ_ϕ is a contraction. Therefore, Φ_ϕ has a fixed point in $\mathbb{X}_0$, that is, (1.1) has a unique piecewise pseudo almost periodic solution in $\mathbb{X}_0$. This completes the proof. □

4. Global exponential stability of piecewise pseudo almost periodic solutions

This section states and proves the sufficient conditions for the global exponential stability of piecewise pseudo almost periodic solutions of the system described by (1.1)–(1.2).

Theorem 4.1. *Assume that conditions* (H_1)–(H_4) *are true and the following condition holds.*

(H_5) For $i, j = 1, 2, \cdots, m$,

$$\begin{aligned} a_i^- &> \max\{a_i^+\alpha_i^+ + \sum_{j=1}^m s_{ji}^+(P_j + Q_j), (a_i^+ + a_i^-)[a_i^+\alpha_i^+ + \sum_{j=1}^m s_{ji}^+(P_j + Q_j)]\}, \\ b_j^- &> \max\{b_j^+\beta_j^+ + \sum_{i=1}^m t_{ij}^+(H_i + K_i), (b_j^+ + b_j^-)[b_j^+\beta_j^+ + \sum_{i=1}^m t_{ij}^+(H_i + K_i)]\}. \end{aligned}$$

Then, the piecewise pseudo almost periodic solutions of the system described by (1.1)–(1.2) is globally exponentially stable.

Proof. According to Theorem 3.4, the system (1.1) has at least one piecewise differentiable pseudo-almost periodic solution $(\overline{x}_1(t), \overline{x}_2(t), \cdots, \overline{x}_m(t), \overline{y}_1(t), \overline{y}_2(t), \cdots, \overline{y}_m(t))^T \in X_0$ with the initial value condition $\overline{\phi} = (\overline{\varphi}_1(\theta), \overline{\varphi}_2(\theta), \cdots, \overline{\varphi}_m(\theta), \overline{\psi}_1(\theta), \overline{\psi}_2(\theta), \cdots, \overline{\psi}_m(\theta))^T$. Let

$$(x_1(t), x_2(t), \cdots, x_m(t), y_1(t), y_2(t), \cdots, y_m(t))^T$$

be an arbitrary solution of system (1.1) with the initial value condition

$$\phi = (\varphi_1(\theta), \varphi_2(\theta), \cdots, \varphi_m(\theta), \psi_1(\theta), \psi_2(\theta), \cdots, \psi_m(\theta))^T.$$

Let $u_i(\cdot) = x_i(\cdot) - \overline{x}_i(\cdot)$, $v_j(\cdot) = y_j(\cdot) - \overline{y}_j(\cdot)$; for $t \neq t_k$ and $t > 0$, we have

$$\begin{aligned} u_i'(t) &= -a_i(t)u_i(t) + a_i(t)\int_{t-\alpha_i(t)}^t u_i'(s)ds \\ &\quad + \sum_{j=1}^m s_{ji}(t)[f_j(x_j(t), y_j(t-\tau_{ji}(t))) - f_j(\overline{x}_j(t), \overline{y}_j(t-\tau_{ji}(t)))], \end{aligned} \tag{4.1}$$

$$\begin{aligned} v_j'(t) &= -b_j(t)v_j(t) + b_j(t)\int_{t-\beta_j(t)}^t v_j(s)ds \\ &\quad + \sum_{i=1}^m t_{ij}(t)[g_i(x_i(t-\delta_{ij}(t)), y_i(t) - g_i(x_i(t-\delta_{ij}(t)), y_i(t)]. \end{aligned} \tag{4.2}$$

The initial value conditions are as follows:

$$\begin{cases} u_i(\theta) = \varphi_i(\theta) - \overline{\varphi}_i(\theta), \theta \in [-r_1, 0], i = 1, 2, \cdots, m, \\ v_j(\theta) = \psi_j(\theta) - \overline{\psi}_j(\theta), \theta \in [-r_2, 0], j = 1, 2, \cdots, m. \end{cases}$$

Multiplying $e^{\int_{t_k}^{s} a_i(\tau)d\tau}$ on both sides of (4.1) and $e^{\int_{t_k}^{s} b_j(\tau)d\tau}$ on both sides of (4.2), and integrating over $[t_k, t]$, we derive

$$\begin{aligned} u_i(t) &= u_i(t_k)e^{-\int_{t_k}^{t} a_i(\tau)d\tau} + \int_{t_k}^{t} e^{-\int_{s}^{t} a_i(\tau)d\tau}\Big\{a_i(s)\int_{s-\alpha_i(s)}^{s} u_i'(\tau)d\tau \\ &\quad + \sum_{j=1}^{m} s_{ji}(s)[f_j(x_j(s), y_j(s-\tau_{ji}(s))) - f_j(\overline{x}_j(s), \overline{y}_j(s-\tau_{ji}(s)))]\Big\}ds, \end{aligned} \tag{4.3}$$

$$\begin{aligned} v_j(t) &= v_j(t_k)e^{-\int_{t_k}^{t} b_j(\tau)d\tau} + \int_{t_k}^{t} e^{-\int_{s}^{t} b_j(\tau)d\tau}\Big\{b_j(s)\int_{s-\beta_j(s)}^{s} v_j'(\tau)d\tau \\ &\quad + \sum_{i=1}^{m} t_{ij}(s)[f_i(x_i(s-\delta_{ij}(s)), y_i(s)) - f_j(\overline{x}_j(s-\delta_{ij}(s)), \overline{y}_j(s))]\Big\}ds. \end{aligned} \tag{4.4}$$

Consider the following functions:

$$\begin{aligned} &C_i(s) = a_i^- - s - h_i(s), s \in (0, a_i^-), s \in (0, a_i^-), \\ &D_i(s) = a_i^- - s - (a_i^+ + a_i^-)h_i(s) + sh_i(s), s \in (0, a_i^-), \\ &\overline{C}_j(s) = b_j^- - s - \overline{h}_j(s), s \in (0, b_j^-), \\ &\overline{D}_j(s) = b_j^- - s - (b_j^+ + b_j^-)\overline{h}_j(s) + s\overline{h}_j(s), s \in (0, b_j^-), \end{aligned}$$

where $h_i(s) = a_i^+ \frac{e^{s\alpha_i^+}-1}{s} + \sum_{j=1}^{m} s_{ji}^+(P_j + Q_j e^{s\tau_{ji}^+})$, $\overline{h}_j(s) = b_j^+ \frac{e^{s\beta_j^+}-1}{s} + \sum_{i=1}^{m} t_{ij}^+(H_i e^{s\delta_{ij}^+} + K_i)$.

For $i, j = 1, 2, \cdots, m$, since $C_i(s)$, $D_i(s)$ are continuous on $(0, a_i^-)$, $\overline{C}_j(s)$, $\overline{D}_j(s)$ are continuous on $(0, b_j^-)$, $\lim_{s\to 0} C_i(s) > 0$, $\lim_{s\to 0} \overline{C}_j(s) > 0$, $\lim_{s\to 0} D_i(s) > 0$, $\lim_{s\to 0} \overline{D}_j(s) > 0$, $\lim_{s\to a_i^-} C_i(s) < 0$, $\lim_{s\to a_i^-} D_i(s) < 0$, $\lim_{s\to b_j^-} \overline{C}_j(s) < 0$ and $\lim_{s\to b_j^-} \overline{D}_j(s) < 0$. There exist constants s_1^i, $s_2^i \in (0, a_i^-)$ and s_3^j, $s_4^j \in (0, b_j^-)$ such that $C_i(s_1^i) = 0$, $D_i(s_2^i) = 0$, $\overline{C}_j(s_3^j) = 0$ and $\overline{D}_j(s_4^j) = 0$. There exists a constant $\eta \in (0, \min_{1\le i,j\le m}\{s_1^i, s_2^i, s_3^j, s_4^j\})$ such that $C_i(\eta) > 0$, $D_i(\eta) > 0$, $\overline{C}_j(\eta) > 0$ and $\overline{D}_j(\eta) > 0$. Obviously, there is $\eta \in (0, \min\{a_i^-, b_j^-\})$.

Take a positive sequence $\{M_k\}_{k=1}^{\infty}$ satisfying

$$\frac{1}{M_1} < \frac{1}{N_0}\min\{\frac{h_i(\eta)}{a_i^- - \eta}, \frac{\overline{h}_j(\eta)}{b_j^- - \eta}\},$$

$$\frac{1}{M_k} < \frac{1}{N_{k-1}}\min\{\frac{h_i(\eta)}{a_i^- - \eta}, \frac{\overline{h}_j(\eta)}{b_j^- - \eta}\},$$

where $N_k = M_k \max\{1 + I, 1 + J\}$, $k \ge 1$, $N_0 = \max\{1 + I, 1 + J\}$.

Let

$$w(t) = \begin{cases} w_i(t) = u_i(t), i = 1, 2, \cdots, m, \\ w_{m+j}(t) = v_j(t), j = 1, 2, \cdots, m. \end{cases}$$

Let $\Upsilon = \|\phi - \overline{\phi}\|_1$; for $i, j = 1, 2, \cdots, m$, it is obvious that there exists a positive constant $M_0 > 1$ such that the following inequalities hold

$$\|w_i(\theta)\|_1 \le M_0 \Upsilon e^{-\eta t}, \theta \in [-r_1, 0],$$
$$\|w_{m+j}(\theta)\|_1 \le M_0 \Upsilon e^{-\eta t}, \theta \in [-r_2, 0].$$

For $t = t_0^+$, we have

$$\begin{aligned} |u_i(t_0^+)| &= |x_i(t_0^+) - \overline{x}_i(t_0^+)| \le |x_i(t_0^-) - \overline{x}_i(t_0^-)| + |I_1(x_i(t_0)) - I_1(\overline{x}_i(t_0))| \\ &= |u_i(t_0^-)| + I|u_i(t_0^-)| \le (1 + I)\Upsilon, \\ |v_j(t_0^+)| &= |y_j(t_0^+) - \overline{y}_j(t_0^+)| \le |y_j(t_0^-) - y_j^*(t_0^-)| + |J_1(y_j(t_0)) - J_1(y_j^*(t_0))| \\ &= |v_j(t_0^-)| + J|v_j(t_0^-)| \le (1 + J)\Upsilon. \end{aligned}$$

Therefore,

$$\|w_l(t_0^+)\|_\infty \le N_0 \Upsilon. \tag{4.5}$$

Now we will prove, for $t \in (0, t_1]$, $l = 1, 2, \cdots, 2m$, that

$$\|w_l(t)\|_1 \le M_1 \Upsilon e^{-\eta t}. \tag{4.6}$$

In order to prove that (4.6) is true, we first prove that for any $\rho > 1$, $t \in (0, t_1]$, the following inequality holds:

$$\|w_l(t)\|_1 < M_1 \rho \Upsilon e^{-\eta t}. \tag{4.7}$$

If (4.7) is not established, there are $l_0 \in \{1, 2, \cdots, 2m\}$ and $t' \in (0, t_1]$ such that

$$\|w_{l_0}(t')\|_1 \ge M_1 \rho \Upsilon e^{-\eta t'}, \quad \|w_l(t)\|_1 < M_1 \rho \Upsilon e^{-\eta t}, \quad t \in (0, t'), \tag{4.8}$$

where $t' \in (0, t_1]$ is the smallest variable such that (4.8) holds. There is a constant $\sigma \ge 1$ such that

$$\|w_{l_0}(t')\|_1 = \sigma M_1 \rho \Upsilon e^{-\eta t'} \tag{4.9}$$

and $\|w_l(t)\|_1 < M_1 \rho \Upsilon e^{-\eta t}$, $t \in (0, t')$.

For $l_0 \in \{1, 2, \cdots, m\}$, let $i_0 = l_0$; from (H_5) and (4.3), we derive

$$\begin{aligned} |u_{i_0}(t')| \le\ & N_0 \Upsilon e^{-a_{i_0}^- t'} + \int_0^{t'} e^{-\int_s^{t'} a_{i_0}(\tau) d\tau} \Big\{ a_{i_0}^+ \int_{s-\alpha_{i_0}(s)}^{s} M_1 \rho \Upsilon e^{-\eta \tau} d\tau \\ & + \sum_{j=1}^m s_{ji_0}^+ (P_j |u_j(s)| + Q_j |v_j(s - \tau_{ji_0}(s))|) \Big\} ds \\ \le\ & N_0 \Upsilon e^{-a_{i_0}^- t'} + \int_0^{t'} e^{-\int_s^{t'} a_{i_0}(\tau) d\tau} \Big\{ a_{i_0}^+ \int_{s-\alpha_i(s)}^{s} M_1 \rho \Upsilon e^{-\eta \tau} d\tau \\ & + \sum_{j=1}^m s_{ji_0}^+ (P_j M_1 \rho \Upsilon e^{-\eta s} + Q_j M_1 \rho \Upsilon e^{-\eta (s - \tau_{ji_0}^+)}) \Big\} ds \end{aligned}$$

$$
\begin{aligned}
&\leq N_0\Upsilon e^{-a_{i_0}^- t'} + e^{-a_{i_0}^- t'} M_1\rho\Upsilon \frac{e^{(a_{i_0}^- - \eta)t'} - 1}{a_{i_0}^- - \eta} h_{i_0}(\eta) \\
&= N_0\Upsilon e^{-a_{i_0}^- t'} - M_1\rho\Upsilon \frac{e^{-a_{i_0}^- t'}}{a_{i_0}^- - \eta} h_{i_0}(\eta) + M_1\rho\Upsilon \frac{e^{-\eta t'}}{a_{i_0}^- - \eta} h_{i_0}(\eta) \\
&\leq \{\frac{N_0}{M_1} - \frac{1}{a_{i_0}^- - \eta} h_{i_0}(\eta)\} e^{-a_{i_0}^- t'} M_1\rho\Upsilon + \frac{1}{a_{i_0}^- - \eta} h_{i_0}(\eta) M_1\rho\Upsilon e^{-\eta t'} \\
&< M_1\rho\Upsilon e^{-\eta t'},
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
|u_{i_0}'(t')| &= \Big| - u_{i_0}(0) a_{i_0}(t') e^{-\int_0^{t'} a_{i_0}(\tau)d\tau} - a_{i_0}(t') \int_0^{t'} e^{-\int_s^{t'} a_{i_0}(\tau)d\tau} \Big\{ a_{i_0}(s) \int_{s-\alpha_{i_0}(s)}^{s} u_{i_0}'(\tau) d\tau \\
&\quad + \sum_{j=1}^{m} s_{ji_0}(s)[f_j(x_j(s), y_j(s - \tau_{ji_0}(s))) - f_j(\overline{x}_j(s), \overline{y}_j(s - \tau_{ji_0}(s)))] \Big\} ds \\
&\quad + e^{-\int_0^{t'} a_{i_0}(\tau)d\tau} e^{\int_0^{t'} a_{i_0}(\tau)d\tau} \Big\{ a_{i_0}(t') \int_{t'-\alpha_{i_0}(t')}^{t'} u_{i_0}'(\tau) d\tau \\
&\quad + \sum_{j=1}^{m} s_{ji_0}(t')[f_j(x_j(t'), y_j(t' - \tau_{ji_0}(t'))) - f_j(\overline{x}_j(t'), \overline{y}_j(t' - \tau_{ji_0}(t')))] \Big\} \Big| \\
&\leq |u_{i_0}(0)| a_{i_0}^+ e^{-a_{i_0}^- t'} + a_{i_0}^+ e^{-a_{i_0}^- t'} \int_0^{t'} e^{a_{i_0}^- s} \Big\{ a_{i_0}^+ \int_{s-\alpha_{i_0}(s)}^{s} |u_{i_0}'(\tau)| d\tau \\
&\quad + \sum_{j=1}^{m} s_{ji_0}^+ (P_j |u_j(s)| + Q_j |v_j(s - \tau_{ji_0}(s))|) \Big\} ds + a_i^+ \int_{t'-\alpha_{i_0}(t')}^{t'} |u_{i_0}'(\tau)| d\tau \\
&\quad + \sum_{j=1}^{m} s_{ji_0}^+ (P_j |u_j(t')| + Q_j |v_j(t' - \tau_{ji_0}(t'))|) \\
&\leq N_0\Upsilon a_{i_0}^+ e^{-a_{i_0}^- t'} + a_{i_0}^+ e^{-a_{i_0}^- t'} \int_0^{t'} e^{a_{i_0}^- s} \Big\{ a_{i_0}^+ \int_{s-\alpha_{i_0}(s)}^{s} M_1\rho\Upsilon e^{-\eta\tau} d\tau \\
&\quad + \sum_{j=1}^{m} s_{ji_0}^+ (P_j M_1\rho\Upsilon e^{-\eta s} + Q_j M_1\rho\Upsilon e^{-\eta s} e^{\eta \tau_{ji_0}^+}) \Big\} ds + a_{i_0}^+ \int_{t'-\alpha_{i_0}(t')}^{t'} M_1\rho\Upsilon e^{-\eta\tau} d\tau \\
&\quad + \sum_{j=1}^{m} s_{ji_0}^+ (P_j M_1\rho\Upsilon e^{-\eta t'} + Q_j M_1\rho\Upsilon e^{-\eta t'} e^{\eta \tau_{ji_0}^+}) \\
&\leq \{\frac{N_0}{M_1} a_{i_0}^+ e^{-a_{i_0}^- t'} + a_{i_0}^+ \frac{e^{-\eta t'} - e^{-a_{i_0}^- t'}}{a_{i_0}^- - \eta} h_{i_0}(\eta)\} M_1\rho\Upsilon + h_{i_0}(\eta) M_1\rho\Upsilon e^{-\eta t'} \\
&= \{\frac{N_0}{M_1} - \frac{h_{i_0}(\eta)}{a_{i_0}^- - \eta}\} a_{i_0}^+ e^{-a_{i_0}^- t'} M_1\rho\Upsilon + \{\frac{a_{i_0}^+}{a_{i_0}^- - \eta} h_{i_0}(\eta) + h_{i_0}(\eta)\} M_1\rho\Upsilon e^{-\eta t'} \\
&< M_1\rho\Upsilon e^{-\eta t'},
\end{aligned} \tag{4.11}
$$

where $h_{i_0}(\eta) = a_{i_0}^+ \frac{e^{\eta \alpha_{i_0}^+} - 1}{\eta} + \sum_{j=1}^{m} s_{ji_0}^+ (P_j + Q_j e^{\eta \tau_{ji_0}^+})$. Equations (4.10) and (4.11) are both in conflict with (4.9).

For $l_0 \in \{m+1, m+2, \cdots, 2m\}$, let $j_0 = l_0 - m$; similarly, from (H_5) and (4.4), we derive

$$\begin{aligned} |v_{j_0}(t')| &\leq \{\frac{N_0}{M_1} - \frac{1}{b_{j_0}^- - \eta}\overline{h}_{j_0}(\eta)\}e^{-b_{j_0}^- t'}M_1\rho\Upsilon + \frac{1}{b_{j_0}^- - \eta}\overline{h}_{j_0}(\eta)M_1\rho\Upsilon e^{-\eta t'} \\ &< M_1\rho\Upsilon e^{-\eta t'}, \end{aligned} \tag{4.12}$$

$$\begin{aligned} |v'_{j_0}(t')| &= \{\frac{N_0}{M_1} - \frac{\overline{h}_{j_0}(\eta)}{b_{j_0}^- - \eta}\}b_{j_0}^+ e^{-b_{j_0}^- t'}M_1\rho\Upsilon + \{\frac{b_{j_0}^+}{b_{j_0}^- - \eta}\overline{h}_{j_0}(\eta) + \overline{h}_{j_0}(\eta)\}M_1\rho\Upsilon e^{-\eta t'} \\ &< M_1\rho\Upsilon e^{-\eta t'}, \end{aligned} \tag{4.13}$$

where $\overline{h}_{j_0}(\eta) = b_{j_0}^+ \frac{e^{\eta\beta_{j_0}^+}-1}{\eta} + \sum_{i=1}^{m} t_{ij_0}^+(H_i e^{\eta\delta_{ij_0}^+} + K_i)$. Equations (4.12) and (4.13) are both in conflict with (4.9).

Therefore, (4.8) is not valid. For $t \in (0, t_1]$, we have that (4.7) holds. Let $\rho \to 1$; for $t \in (0, t_1]$, (4.6) holds.

For $t = t_1^+$, we have

$$\begin{aligned} |u_i(t_1^+)| &= |x_i(t_1^+) - \overline{x}_i(t_1^+)| \leq |x_i(t_1^-) - \overline{x}_i(t_1^-)| + |I_1(x_i(t_1)) - I_1(\overline{x}_i(t_1))| \\ &= |u_i(t_1^-)| + I|u_i(t_1^-)| \leq (1+I)M_1\Upsilon e^{-\eta t_1}, \\ |v_j(t_1^+)| &= |y_j(t_1^+) - \overline{y}_j(t_1^+)| \leq |y_j(t_1^-) - y_j^*(t_1^-)| + |J_1(y_j(t_1)) - J_1(y_j^*(t_1))| \\ &= |v_j(t_1^-)| + J|v_j(t_1^-)| \leq (1+J)M_1\Upsilon e^{-\eta t_1}. \end{aligned}$$

Therefore,

$$||w_l(t_1^+)||_\infty \leq N_1\Upsilon e^{-\eta t_1}. \tag{4.14}$$

For $t \in (t_1, t_2]$, $l = 1, 2, \cdots, 2m$, we certify that

$$||w_l(t)||_1 \leq M_2\Upsilon e^{-\eta t}. \tag{4.15}$$

In order to prove that (4.15) is true, we first prove that for any $\rho > 1$ and $t \in (t_1, t_2]$, the following inequality holds:

$$||w_l(t)||_1 < M_2\rho\Upsilon e^{-\eta t}. \tag{4.16}$$

If (4.16) is not established, there are $l_1 \in \{1, 2, \cdots, 2m\}$ and $t'' \in (t_1, t_2]$ such that

$$||w_{l_1}(t'')||_1 \geq M_2\rho\Upsilon e^{-\eta t'}, \quad ||w_l(t)||_1 < M_2\rho\Upsilon e^{-\eta t}, \quad t \in (t_1, t''), \tag{4.17}$$

where $t'' \in (t_1, t_2]$ is the smallest variable such that (4.17) holds. There is a constant $\sigma_1 \geq 1$ such that

$$||w_{l_1}(t')||_1 = \sigma_1 M_2\rho\Upsilon e^{-\eta t'} \tag{4.18}$$

and $||w_l(t)||_1 < M_2\rho\Upsilon e^{-\eta t}, t \in (t_1, t'')$.

For $l_1 \in \{1, 2, \cdots, m\}$, let $i_1 = l_1$; from (H_5) and (4.3), we derive

$$\begin{aligned}
|u_{i_1}(t'')| &\leq N_1\rho\Upsilon e^{-\eta t_1} e^{-a_{i_1}^-(t''-t_1^+)} + \int_{t_1^+}^{t''} e^{-\int_s^{t''} a_{i_1}(\tau)d\tau}\Big\{a_{i_1}^+ \int_{s-\alpha_{i_1}(s)}^{s} M_2\rho\Upsilon e^{-\eta\tau}d\tau \\
&\quad + \sum_{j=1}^{m} s_{ji_1}^+(P_j|u_j(s)| + Q_j|v_j(s-\tau_{ji_1}(s))|)\Big\}ds \\
&\leq N_1\rho\Upsilon e^{-\eta t_1} e^{-a_{i_1}^-(t''-t_1^+)} + e^{-a_{i_1}^- t''} M_2\rho\Upsilon \frac{e^{(a_{i_1}^- -\eta)t''} - e^{(a_{i_1}^- -\eta)t_1}}{a_{i_1}^- - \eta} h_{i_1}(\eta) \\
&= N_1\rho\Upsilon e^{-\eta t_1} e^{-a_{i_1}^-(t''-t_1^+)} + M_2\rho\Upsilon \frac{e^{-\eta t''} - e^{-a_{i_1}^-(t''-t_1)}e^{-\eta t_1}}{a_{i_1}^- - \eta} h_{i_1}(\eta) \\
&\leq \{\frac{N_1}{M_2} - \frac{h_{i_1}(\eta)}{a_{i_1}^- - \eta}\} e^{-a_{i_1}^-(t''-t_1)} M_2\rho\Upsilon e^{-\eta t_1} + \frac{1}{a_{i_1}^- - \eta} h_{i_1}(\eta) M_2\rho\Upsilon e^{-\eta t''} \\
&< M_2\rho\Upsilon e^{-\eta t''},
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
|u_{i_1}'(t'')| &\leq |u_{i_1}(t_1^+)| a_{i_1}^+ e^{-a_{i_1}^-(t''-t_1^+)} + a_{i_1}^+ e^{-a_{i_1}^- t''} \int_{t_1^+}^{t''} e^{a_{i_1}^- s}\Big\{a_{i_1}^+ \int_{s-\alpha_{i_1}(s)}^{s} |u_{i_1}'(\tau)|d\tau \\
&\quad + \sum_{j=1}^{m} s_{ji_1}^+(P_j|u_j(s)| + Q_j|v_j(s-\tau_{ji_1}(s))|)\Big\}ds + a_i^+ \int_{t''-\alpha_{i_1}(t'')}^{t''} |u_{i_1}'(\tau)|d\tau \\
&\quad + \sum_{j=1}^{m} s_{ji_1}^+(P_j|u_j(t'')| + Q_j|v_j(t''-\tau_{ji_1}(t''))|) \\
&\leq N_1\rho\Upsilon e^{-\eta t_1} a_{i_1}^+ e^{-a_{i_1}^-(t''-t_1)} + a_{i_1}^+ e^{-a_{i_1}^- t''} \int_{t_1^+}^{t''} e^{a_{i_1}^- s}\Big\{a_{i_1}^+ \int_{s-\alpha_{i_1}(s)}^{s} M_2\rho\Upsilon e^{-\eta\tau}d\tau \\
&\quad + \sum_{j=1}^{m} s_{ji_1}^+(P_j M_2\rho\Upsilon e^{-\eta s} + Q_j M_2\rho\Upsilon e^{-\eta s} e^{\eta\tau_{ji_1}^+})\Big\}ds + a_{i_1}^+ \int_{t''-\alpha_{i_1}(t'')}^{t''} M_2\rho\Upsilon e^{-\eta\tau}d\tau \\
&\quad + \sum_{j=1}^{m} s_{ji_1}^+(P_j M_2\rho\Upsilon e^{-\eta t''} + Q_j M_2\rho\Upsilon e^{-\eta t''} e^{\eta\tau_{ji_1}^+}) \\
&\leq \{\frac{N_1}{M_2} - \frac{h_{i_1}(\eta)}{a_{i_1}^- - \eta}\} a_{i_1}^+ e^{-a_{i_1}^-(t''-t_1)} M_2\rho\Upsilon e^{-\eta t_1} + \{\frac{a_{i_1}^+}{a_{i_1}^- - \eta} h_{i_1}(\eta) + h_{i_1}(\eta)\} M_2\rho\Upsilon e^{-\eta t''} \\
&< M_2\rho\Upsilon e^{-\eta t''},
\end{aligned} \tag{4.20}$$

where $h_{i_1}(\eta) = a_{i_1}^+ \frac{e^{\eta\alpha_{i_1}^+}-1}{\eta} + \sum_{j=1}^{m} s_{ji_1}^+(P_j + Q_j e^{\eta\tau_{ji_1}^+})$. Equations (4.19) and (4.20) are both in conflict with (4.18).

For $l_1 \in \{m+1, m+2, \cdots, 2m\}$, let $j_1 = l_1 - m$; similarly, from (H_5) and (4.4), we derive

$$\begin{aligned}
|v_{j_1}(t'')| &\leq \{\frac{N_1}{M_2} - \frac{\overline{h}_{j_1}(\eta)}{b_{j_1}^- - \eta}\} e^{-b_{j_1}^-(t''-t_1)} M_2\rho\Upsilon e^{-\eta t_1} + \frac{1}{b_{j_1}^- - \eta} \overline{h}_{j_1}(\eta) M_2\rho\Upsilon e^{-\eta t''} \\
&< M_2\rho\Upsilon e^{-\eta t''},
\end{aligned} \tag{4.21}$$

$$\begin{aligned} |v'_{j_1}(t'')| &\leq \{\frac{N_1}{M_2} - \frac{\overline{h}_{j_1}(\eta)}{b^-_{j_1} - \eta}\} b^+_{j_1} e^{-b^-_{j_1}(t''-t_1)} M_2 \rho \Upsilon e^{-\eta t_1} + \{\frac{b^+_{j_1}}{b^-_{j_1} - \eta} \overline{h}_{j_1}(\eta) + \overline{h}_{j_1}(\eta)\} M_2 \rho \Upsilon e^{-\eta t''} \\ &< M_2 \rho \Upsilon e^{-\eta t''}, \end{aligned} \tag{4.22}$$

where $\overline{h}_{j_1}(\eta) = b^+_{j_1} \frac{e^{\eta \beta^+_{j_1}} - 1}{\eta} + \sum_{i=1}^{m} t^+_{ij_1} (H_i e^{\eta \delta^+_{ij_1}} + K_i)$. Equations (4.21) and (4.22) are both in conflict with (4.18).

Therefore, (4.17) is not valid. For $t \in (t_1, t_2]$, we have that (4.16) holds. Let $\rho \to 1$; for $t \in (t_1, t_2]$, (4.15) holds.

Similar to the proof of (4.14) and (4.15), we deduce that

$$\begin{aligned} \|w_l(t^+_{k-1})\|_\infty \leq N_{k-1} \Upsilon e^{-\eta t_{k-1}}, \\ \|w_l(t)\|_1 \leq M_k \Upsilon e^{-\eta t}, t \in (t_{k-1}, t_k]. \end{aligned} \tag{4.23}$$

Let $M = \sup_{k \geq 1} \max\{M_k, N_{k-1}\}$; obviously, $M > 1$; we have

$$\|w_l(t)\|_1 \leq M \Upsilon e^{-\eta t}, t > 0. \tag{4.24}$$

Therefore, for $i, j = 1, 2, \cdots, m$,

$$\begin{aligned} \|x_i(t) - \overline{x}_i(t)\|_1 \leq M \Upsilon e^{-\eta t}, t > 0, \\ \|y_j(t) - \overline{y}_j(t)\|_1 \leq M \Upsilon e^{-\eta t}, t > 0. \end{aligned} \tag{4.25}$$

From Definition 2.8, the piecewise pseudo almost periodic solution

$$(\overline{x}_1(t), \overline{x}_2(t), \cdots, \overline{x}_m(t), \overline{y}_1(t), \overline{y}_2(t), \cdots, \overline{y}_m(t))^T$$

of system (1.1) with the initial value conditions

$$\overline{\phi} = (\overline{\varphi}_1(\theta), \overline{\varphi}_2(\theta), \cdots, \overline{\varphi}_m(\theta), \overline{\psi}_1(\theta), \overline{\psi}_2(\theta), \cdots, \overline{\psi}_m(\theta))^T$$

is said to be globally exponentially stable. □

5. Example

This section presents an example to illustrate the effectiveness of our results obtained in previous sections. Consider the following interval general BAM neural network with mixed time-varying delays and impulsive perturbations:

$$\begin{cases} x'_i(t) = -a_i(t) x_i(t - \alpha_i(t)) + \sum_{j=1}^{2} s_{ji}(t) f_j(x_j(t), y_j(t - \tau_{ji}(t))) + c_i(t), \\ t > 0, t \neq 2k, i = 1, 2, \\ \Delta x_i(2k) = I_k(x_i(2k)), k \in \mathbb{Z}^+, \\ y'_j(t) = -b_j(t) y_j(t - \beta_j(t)) + \sum_{i=1}^{2} t_{ij}(t) g_i(x_i(t - \delta_{ij}(t)), y_i(t)) + d_j(t), \\ t > 0, t \neq 2k, j = 1, 2, \\ \Delta y_j(2k) = J_k(y_j(2k)), k \in \mathbb{Z}^+, \end{cases} \tag{5.1}$$

where $f_j(x_j(t), y_j(t-\tau_{ji}(t))) = |x_j(t)|+|y_j(t-\tau_{ji}(t)|, g_i(x_i(t-\delta_{ij}(t)), y_i(t)) = |x_i(t-\delta_{ij}(t))|+|y_i(t)|, i, j = 1, 2,$ and

$$a(t) = \begin{pmatrix} 0.2 + 0.1\cos^2 t \\ 0.2 + 0.1\sin^4 t \end{pmatrix},\ b(t) = \begin{pmatrix} 0.2 + 0.1\cos^2 t \\ 0.2 + 0.1\sin^4 t \end{pmatrix},\ s(t) = \begin{pmatrix} \frac{1}{80} + \frac{1}{80}\sin t & \frac{1}{80} + \frac{1}{80}\cos t \\ \frac{1}{80} + \frac{1}{80}\sin t & \frac{1}{80} + \frac{1}{80}\cos t \end{pmatrix};$$

$$t(t) = \begin{pmatrix} \frac{1}{80} + \frac{1}{80}\sin t & \frac{1}{80} + \frac{1}{80}\cos t \\ \frac{1}{80} + \frac{1}{80}\sin t & \frac{1}{80} + \frac{1}{80}\cos t \end{pmatrix},\ c(t) = \begin{pmatrix} \frac{0.5+0.1\sin t}{10(1+t^2)} \\ \frac{0.2+0.1\cos t}{10(1+t^2)} \end{pmatrix},\ d(t) = \begin{pmatrix} \frac{0.3+0.1\sin t}{10(1+t^2)} \\ \frac{0.4+0.1\cos t}{10(1+t^2)} \end{pmatrix};$$

$$\alpha(t) = \begin{pmatrix} \frac{1}{8}|\sin t| \\ \frac{1}{8}|\cos t| \end{pmatrix},\ \beta(t) = \begin{pmatrix} \frac{1}{8}|\cos t| \\ \frac{1}{8}|\sin t| \end{pmatrix},\ \delta(t) = \begin{pmatrix} 0.4|\sin t| & 0.2|\cos t| \\ 0.3|\cos t| & 0.1|\sin t| \end{pmatrix};$$

$$\tau(t) = \begin{pmatrix} 0.5|\cos t| & 0.3|\sin t| \\ 0.2|\sin t| & 0.4|\cos t| \end{pmatrix},\ \begin{pmatrix} \Delta x_1(2k) \\ \Delta x_2(2k) \\ \Delta y_1(2k) \\ \Delta y_2(2k) \end{pmatrix} = \begin{pmatrix} -\frac{1}{80}x_1(2k) + \frac{1}{80}sin(x_1(2k)) + 4 \\ -\frac{1}{80}x_2(2k) + \frac{1}{80}cos(x_2(2k)) + 3 \\ -\frac{1}{80}y_1(2k) + \frac{1}{80}sin(y_1(2k)) + 2 \\ -\frac{1}{80}y_2(2k) + \frac{1}{80}cos(y_2(2k)) + 1.5 \end{pmatrix}.$$

Obviously,

$$a^+ = \begin{pmatrix} 0.3 \\ 0.3 \end{pmatrix},\ a^- = \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix},\ b^+ = \begin{pmatrix} 0.3 \\ 0.3 \end{pmatrix},\ b^- = \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix},\ \alpha^+ = \begin{pmatrix} \frac{1}{8} \\ \frac{1}{8} \end{pmatrix},\ s^+ = \begin{pmatrix} \frac{2}{80} & \frac{2}{80} \\ \frac{2}{80} & \frac{2}{80} \end{pmatrix};$$

$$\beta^+ = \begin{pmatrix} \frac{1}{8} \\ \frac{1}{8} \end{pmatrix},\ t^+ = \begin{pmatrix} \frac{2}{80} & \frac{2}{80} \\ \frac{2}{80} & \frac{2}{80} \end{pmatrix},\ \delta^+ = \begin{pmatrix} 0.4 & 0.2 \\ 0.3 & 0.1 \end{pmatrix},\ \tau^+ = \begin{pmatrix} 0.5 & 0.3 \\ 0.2 & 0.4 \end{pmatrix}.$$

By calculation, it is easy to acquire $\underline{\sigma} = 2$, $P_j = Q_j = H_i = K_i = 1$, $i, j = 1, 2, \cdots, m$, $I = J = \frac{1}{40}$ and

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \frac{11}{80} \\ \frac{11}{80} \\ \frac{11}{80} \\ \frac{11}{80} \end{pmatrix},\ \begin{pmatrix} \frac{I}{1-e^{-a_1^-\underline{\sigma}}} \\ \frac{I}{1-e^{-a_2^-\underline{\sigma}}} \\ \frac{J}{1-e^{-b_1^-\underline{\sigma}}} \\ \frac{J}{1-e^{-b_2^-\underline{\sigma}}} \end{pmatrix} = \begin{pmatrix} 0.0758 \\ 0.0758 \\ 0.0758 \\ 0.0758 \end{pmatrix}.$$

For $i, j = 1, 2, \cdots, m$, $p_{ij} = 0.7633 < 1$ and $0.2 > \frac{11}{80}$. That is, the conditions (H_1)–(H_5) are valid. From Theorems 3.4 and 4.1, system (5.1) has a piecewise pseudo almost periodic solution, which is globally exponentially stable. Computer simulations of the existence and stable behavior of the pseudo-almost periodic solutions are given in Figures 1 and 2, respectively.

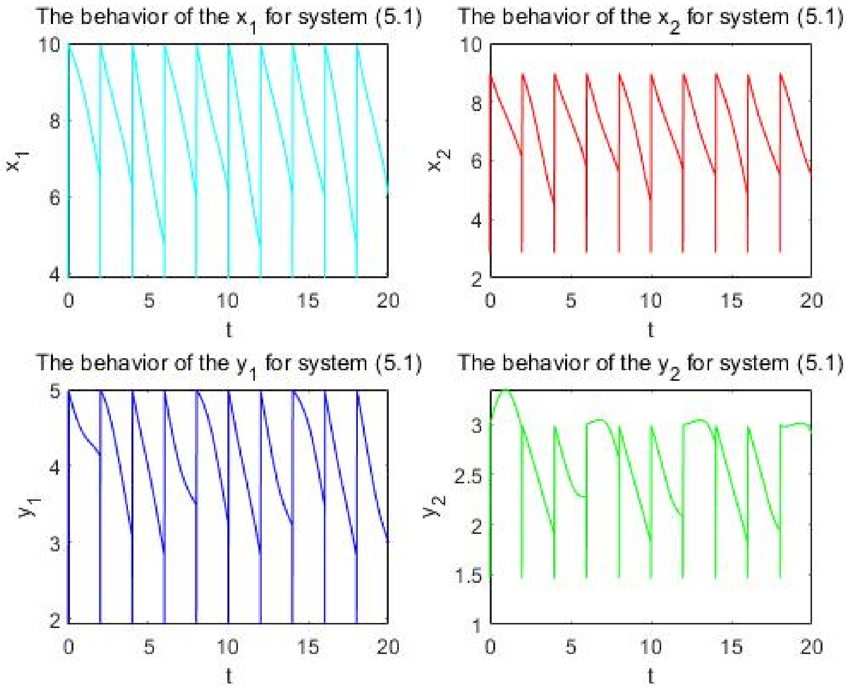

Figure 1. The behavior of x, y for system (5.1) satisfying an initial value condition.

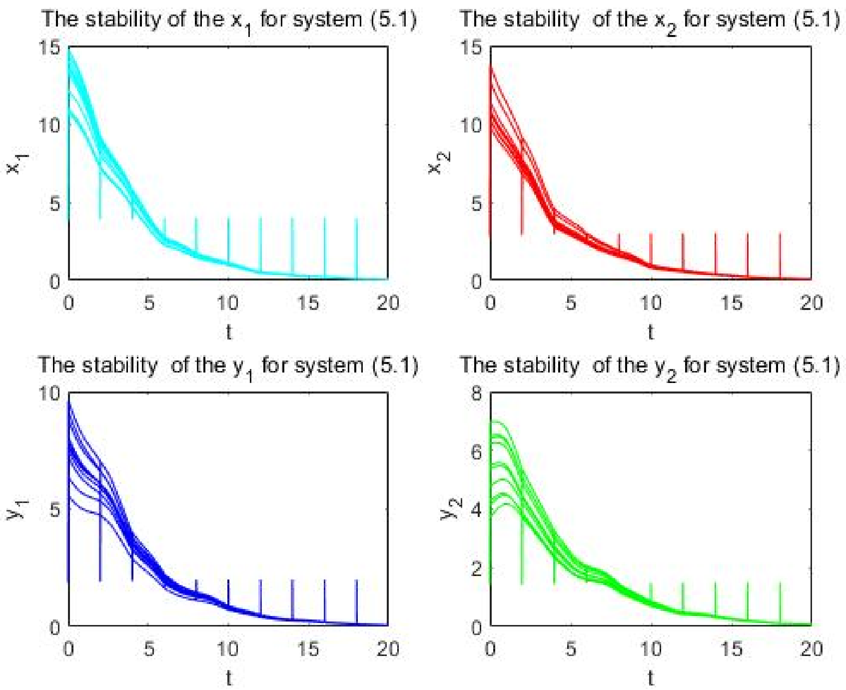

Figure 2. The behavior of x, y for system (5.1) satisfying the random 10 initial value conditions, respectively.

6. Conclusions

The pseudo almost periodic behaviors have been applied to the qualitative theory of differential equations. Piecewise pseudo almost periodic solutions of BAM neural networks are generalizations of almost periodic solutions, which have obvious scientific significance and application value in many fields such as signal processing, pattern recognition, associative memory, image processing and optimization problems.

This paper investigated the piecewise pseudo almost periodic solutions of the interval general BAM neural networks with mixed time-varying delays and impulsive perturbations. By employing the exponential dichotomy of linear differential equations, fixed-point theory for contraction mapping, differential inequality techniques and mathematical methods of induction, the effective conditions for the existence and global exponential stability of piecewise pseudo almost periodic solutions of system (1.1) have been established. From Theorems 3.4 and 4.1, the delays contained in the activation functions do not affect the existence and global exponential stability of piecewise pseudo almost periodic solutions of system (1.1). The existence and global exponential stability of the piecewise pseudo almost periodic solutions of the system (1.1) are determined by the negative feedback terms, leakage delays, connection weights, impulsive perturbations and the Lipschitz constants of the activation functions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no competing interests concerning the publication of this article.

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