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Research article

ϕ-pluriharmonicity in quasi bi-slant conformal $\xi^{\perp}$-submersions: a comprehensive study

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Abstract: This paper delves into quasi bi-slant conformal $\xi^{\perp}$-submersions from Sasakian manifolds onto Riemannian manifolds, which is a generalization of quasi hemi-slant conformal submersions. Our research involves studying the integrability conditions for distributions, taking into account the geometry of their leaves. We also provide decomposition theorems for quasi bi-slant conformal $\xi^{\perp}$-submersions, and showcase non-trivial examples to illustrate our findings. Furthermore, we analyze the φ-pluriharmonicity of such submersions.

Keywords: Sasakian manifold; Riemannian submersions; bi-slant submersions; quasi bi-slant submersions; Riemannian submersions
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Abbreviations: RS: Riemannian submersion; RM: Riemannian Manifold; ACM manifold: Almost contact metric manifold; $\mathcal{QBSC}$ $\xi^{\perp}$-submersion: Quasi bi-slant conformal $\xi^{\perp}$-submersion; G: gradient

1. Introduction

Immersions and submersions play crucial roles in differential geometry, with slant submersions being a particularly intriguing subject in the fields of differential, complex and contact geometry. The study of Riemannian submersions between Riemannian manifolds was first explored by O'Neill [23] and Gray [14], independently, and subsequently led to investigations of Riemannian submersions between almost Hermitian manifolds, known as almost Hermitian submersions, by

Watson in 1976 [34]. Riemannian submersions have many applications in mathematics and physics, especially in Yang-Mills theory [8, 35] and in Kaluza-Klein theory [18, 21].

Semi-invariant submersions, a generalization of holomorphic submersions and anti-invariant submersions, were introduced by Sahin in 2013 [30]. In 2016, Tatsan, Sahin, and Yanan studied hemi-slant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds, and presented several decomposition theorems for them [33]. R. Prasad et al. further examined quasi bi-slant submersions from almost contact metric manifolds onto Riemannian manifolds [25], as well as from Kenmotsu manifolds [26], which represents a step forward in the study of Riemannian submersions.

Since then, many authors have explored different types of Riemannian submersions, including anti-invariant submersions [4, 29], slant submersions [11, 31], semi-slant submersions [17, 24] and hemi-slant submersions [1, 20], from both almost Hermitian manifolds and almost contact metric manifolds. These studies have greatly expanded our understanding of the geometrical structures of Riemannian manifolds.

The concept of almost contact Riemannian submersions from almost contact manifold was introduced by Chinea in [9]. Chinea also examined the fibre space, base space and total space using a differential geometric perspective. To generalize Riemannian submersions, Gundmundsson and Wood [15, 16] presented horizontally conformal submersion, defined as: Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds of dimension m_1 and m_2, respectively. A smooth map $\Psi : (M_1, g_1) \to (M_2, g_2)$ is called a horizontally conformal submersion, if there is a positive function λ such that

$$\lambda^2 g_1(X_1, X_2) = g_2(\Psi_* X_1, \Psi_* X_2), \tag{1.1}$$

for all $X_1, X_2 \in \Gamma(ker\Psi_*)^\perp$. Thus, Riemannian submersion is a particular horizontally conformal submersion with $\lambda = 1$. Later on, Fuglede [13] and Ishihara [19] separately studied horizontally conformal submersions. Additionally, various other kind of submersions, such as conformal slant submersions [3], conformal anti-invariant submersions [6], conformal semi-slant submersions [2], conformal semi-invariant submersions [5] and conformal anti-invariant submersions [27] have been studied by Akyol and Sahin and R. Prasad et al. [27]. Furthermore, Shuaib and Fatima recently explored conformal hemi-slant Riemannian submersions from almost product manifolds onto Riemannian manifolds [32].

In this paper, we study quasi bi-slant conformal $\xi^\perp$-submersions from Sasakian manifold onto a Riemannian manifold considering the Reeb vector field ξ horizontal. This paper is divided into six sections. Section 2 contains definitions of almost contact metric manifolds and, in particular, Sasakian manifolds. In section 3, fundamental results for quasi bi-slant conformal submersion are investigated, which are necessary for our main results. The conditions of integrability and total geodesicness of distributions are explored in Section 4. Section 5 provides some condition under which a Riemannian submersion becomes totally geodesic as well as some decomposition theorems for quasi bi-slant conformal submersion are obtained. The last section discusses ϕ-pluriharmonicity of quasi bi-slant conformal $\xi^\perp$-submersions.

2. Preliminaries

Let M be a $(2n+1)$-dimensional almost contact manifold with almost contact structures (ϕ, ξ, η), where a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \ \ \phi\xi = 0, \ \ \eta \circ \phi = 0, \ \ \eta(\xi) = 1, \tag{2.1}$$

where I is the identity tensor. The almost contact structure is said to be normal if $N + d\eta \otimes \xi = 0$, where N is the Nijenhuis tensor of ϕ. Suppose that a Riemannian metric tensor g is given in M and satisfies the condition

$$g(\phi\widehat{U}, \phi\widehat{V}) = g(\widehat{U}, \widehat{V}) - \eta(\widehat{U})\eta(\widehat{V}), \ \ \eta(\widehat{U}) = g(\widehat{U}, \xi). \tag{2.2}$$

Then (ϕ, ξ, η, g)-structure is called an almost contact metric structure. Define a tensor field Φ of type $(0,2)$ by $\Phi(\widehat{X}, \widehat{Y}) = g(\phi\widehat{X}, \widehat{Y})$. If $d\eta = \Phi$, then an almost contact metric structure is said to be normal contact metric structure. Let Φ be the fundamental 2-form on M, i.e, $\Phi(\widehat{U}, \widehat{V}) = g(\widehat{U}, \phi\widehat{V})$. If $\Phi = d\eta$, M is said to be a contact manifold. A normal contact metric structure is called a Sasakian structure, which satisfies

$$(\nabla_{\widehat{U}}\phi)\widehat{V} = g(\widehat{U}, \widehat{V})\xi - \eta(\widehat{V})\widehat{U}, \tag{2.3}$$

where ∇ is the Levi-Civita connection of g. From above formula, we have for Sasakian manifold

$$\nabla_{\widehat{U}}\xi = -\phi\widehat{U}. \tag{2.4}$$

The covariant derivative of ϕ is defined by

$$(\nabla_{\widehat{U}_1}\phi)\widehat{V}_1 = \nabla_{\widehat{U}_1}\phi\widehat{V}_1 - \phi\nabla_{\widehat{U}_1}\widehat{V}_1, \tag{2.5}$$

for any vector fields $\widehat{U}_1, \widehat{V}_1 \in \Gamma(TM)$. Now, we provide a definition for conformal submersion and discuss some useful results that help us to achieve our main results.

Definition 2.1. *Let Ψ be a Riemannian submersion (RS) from an ACM manifold $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (RM) $(\bar{Q}_2, g_2)$. Then Ψ is called a horizontally conformal submersion, if there is a positive function λ such that*

$$g_1(\widehat{U}_1, \widehat{V}_1) = \frac{1}{\lambda^2} g_2(\Psi_*\widehat{U}_1, \Psi_*\widehat{V}_1), \tag{2.6}$$

for any $\widehat{U}_1, \widehat{V}_1 \in \Gamma(ker\Psi_*)^\perp$. It is obvious that every RS is a particularly horizontally conformal submersion with $\lambda = 1$.

Let $\Psi : (\bar{Q}_1, \phi, \xi, \eta, g_1) \to (\bar{Q}_2, g_2)$ be a RS. A vector field $\widehat{X}$ on $\bar{Q}_1$ is called a basic vector field if $\widehat{X} \in \Gamma(\ker \Psi_*)^\perp$ and Ψ-related with a vector field $\widehat{X}$ on $\bar{Q}_2$ i.e $\Psi_*(\widehat{X}(q)) = \widehat{X}\Psi(q)$ for $q \in \bar{Q}_1$.

The formulas provide the two $(1,2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ by O'Neill are

$$\mathcal{A}_{E_1}F_1 = \mathcal{H}\nabla_{\mathcal{H}E_1}\mathcal{V}F_1 + \mathcal{V}\nabla_{\mathcal{H}E_1}\mathcal{H}F_1, \tag{2.7}$$

$$\mathcal{T}_{E_1}F_1 = \mathcal{H}\nabla_{\mathcal{V}E_1}\mathcal{V}F_1 + \mathcal{V}\nabla_{\mathcal{V}E_1}\mathcal{H}F_1, \tag{2.8}$$

for any $E_1, F_1 \in \Gamma(T\bar{Q}_1)$ and ∇ is Levi-Civita connection of g_1. Note that a RS $\Psi : (\bar{Q}_1, \phi, \xi, \eta, g_1) \to (\bar{Q}_2, g_2)$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. From Eqs (2.7) and (2.8), we can deduce

$$\nabla_{\widehat{U}_1}\widehat{V}_1 = \mathcal{T}_{\widehat{U}_1}\widehat{V}_1 + \mathcal{V}\nabla_{\widehat{U}_1}\widehat{V}_1, \tag{2.9}$$

$$\nabla_{\widehat{U}_1}\widehat{X}_1 = \mathcal{T}_{\widehat{U}_1}\widehat{X}_1 + \mathcal{H}\nabla_{\widehat{U}_1}\widehat{X}_1, \tag{2.10}$$

$$\nabla_{\widehat{X}_1}\widehat{U}_1 = \mathcal{A}_{\widehat{X}_1}\widehat{U}_1 + \mathcal{V}_1\nabla_{\widehat{X}_1}\widehat{U}_1, \tag{2.11}$$

$$\nabla_{\widehat{X}_1}\widehat{Y}_1 = \mathcal{H}\nabla_{\widehat{X}_1}\widehat{Y}_1 + \mathcal{A}_{\widehat{X}_1}\widehat{Y}_1, \tag{2.12}$$

for any vector fields $\widehat{U}_1, \widehat{V}_1 \in \Gamma(ker\Psi_*)$ and $\widehat{X}_1, \widehat{Y}_1 \in \Gamma(ker\Psi_*)^\perp$ [12].

It is easily seen that $\mathcal{T}$ and $\mathcal{A}$ are skew-symmetric, that is

$$g(\mathcal{A}_{\widehat{X}}E_1, F_1) = -g(E_1, \mathcal{A}_{\widehat{X}}F_1), g(\mathcal{T}_{\widehat{V}}E_1, F_1) = -g(E_1, \mathcal{T}_{\widehat{V}}F_1), \tag{2.13}$$

for any vector fields $E_1, F_1 \in \Gamma(T_p\bar{Q}_1)$.

Definition 2.2. *A horizontally conformally submersion $\Psi : \bar{Q}_1 \to \bar{Q}_2$ is called horizontally homothetic if the gradient (G) of its dilation λ is vertical, i.e.,*

$$H(G\lambda) = 0, \tag{2.14}$$

at $p \in TM_1$, where H is the complement orthogonal distribution to $\nu = ker\ \Psi_*$ in $\Gamma(T_pM)$.

The second fundamental form of smooth map Ψ is given by the formula

$$(\nabla\Psi_*)(\widehat{U}_1, \widehat{V}_1) = \nabla^{\Psi}_{\widehat{U}_1}\Psi_*\widehat{V}_1 - \Psi_*\nabla_{\widehat{U}_1}\widehat{V}_1, \tag{2.15}$$

and the map be totally geodesic if $(\nabla\Psi_*)(\widehat{U}_1, \widehat{V}_1) = 0$ for all $\widehat{U}_1, \widehat{V}_1 \in \Gamma(T_pM)$ where ∇ and ∇^{Ψ} are Levi-Civita and pullback connections.

Lemma 2.1. *Let $\Psi : \bar{Q}_1 \to \bar{Q}_2$ be a horizontal conformal submersion. Then, we have*

(i) $(\nabla\Psi_*)(\widehat{X}_1, \widehat{Y}_1) = \widehat{X}_1(ln\lambda)\Psi_*(\widehat{Y}_1) + \widehat{Y}_1(ln\lambda)\Psi_*(\widehat{X}_1) - g_1(\widehat{X}_1, \widehat{Y}_1)\Psi_*(grad\ ln\lambda)$,

(ii) $(\nabla\Psi_*)(\widehat{U}_1, \widehat{V}_1) = -\Psi_*(\mathcal{T}_{\widehat{U}_1}\widehat{V}_1)$,

(iii) $(\nabla\Psi_*)(\widehat{X}_1, \widehat{U}_1) = -\Psi_*(\nabla_{\widehat{X}_1}\widehat{U}_1) = -\Psi_*(\mathcal{A}_{\widehat{X}_1}\widehat{U}_1)$,

for any horizontal vector fields $\widehat{X}_1, \widehat{Y}_1$ and vertical vector fields $\widehat{U}_1, \widehat{V}_1$ [7].

Definition 2.3. *Suppose $\mathfrak{D}$ is a k-dimensional smooth distribution on M. Then An immersed submanifold $i : N \hookrightarrow M$ is called an integral manifold for $\mathfrak{D}$ if for every $x \in N$, the image of $d_iN : T_pN \to T_pM$ is $\mathfrak{D}_p$. We say the distribution $\mathfrak{D}_p$ is integrable if through each point of M there exists an integral manifold of $\mathfrak{D}$.*

Further, A distribution $\mathfrak{D}$ is involutive if it satisfies the Frobenius condition such that if $X, Y \in \Gamma(TM)$ belongs to $\mathfrak{D}$, so $[X, Y] \in \mathfrak{D}$. Frobenius theorem state that an involutive distribution is integrable.

Definition 2.4. *Let M be n-dimensional smooth manifold. A foliation $\mathfrak{F}$ of M is a decomposition of M into a union of disjoint connected submanifolds $M = \cup_{L\in\mathfrak{F}}L$, called the leaves of the foliation, such that for each $m \in M$, there is a neighborhood U of M and a smooth submersion $f_U : U \to R^k$ with $f_U^{-1}(x)$ a leaf of $\mathfrak{F}|_U$ the restriction of the foliation to U, for each $x \in R^k$.*

Definition 2.5. *Let M be a Riemannian manifold, and let $\mathfrak{F}$ be a foliation on M. $\mathfrak{F}$ is totally geodesic if each leaf L is a totally geodesic submanifold of M; that is, any geodesic tangent to L at some point must lie within L.*

3. Quasi bi-slant conformal $\xi^{\perp}$-submersions

Definition 3.1. *Let $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ be a ACM manifold and $(\bar{Q}_2, g_2)$ a Riemannian manifold. A RS $\Psi : \bar{Q}_1 \to \bar{Q}_2$ where $\xi \in \Gamma(ker\Psi_*)^{\perp}$ is called quasi bi-slant conformal $\xi^{\perp}$-submersion ($\mathcal{QBSC}$ $\xi^{\perp}$-submersion) if there exists three mutually orthogonal distributions $\mathfrak{D}$, $\mathfrak{D}_{\theta_g}$ and $\mathfrak{D}_{\theta_y}$ such that*

(i) $ker\Psi_* = \mathfrak{D} \oplus \mathfrak{D}_{\theta_g} \oplus \mathfrak{D}_{\theta_y}$,
(ii) $\mathfrak{D}$ is invariant. i.e., $\phi\mathfrak{D} = \mathfrak{D}$,
(iii) $\phi\mathfrak{D}_{\theta_g} \perp \mathfrak{D}_{\theta_y}$ and $\phi\mathfrak{D}_{\theta_y} \perp \mathfrak{D}_{\theta_g}$,
(iv) for any non-zero vector field $\widehat{V}_1 \in (\mathfrak{D}_{\theta_g})_p$, $p \in \bar{Q}_1$ the angle θ_1 between $(\mathfrak{D}_{\theta_g})_p$ and $\phi\widehat{V}_1$ is constant and independent of the choice of the point p and $\widehat{V}_1 \in (\mathfrak{D}_{\theta_g})_p$,
(v) for any non-zero vector field $\widehat{V}_1 \in (\mathfrak{D}_{\theta_y})_q$, $q \in \bar{Q}_1$ the angle θ_2 between $(\mathfrak{D}_{\theta_y})_q$ and $\phi\widehat{V}_1$ is constant and independent of the choice of the point q and $\widehat{V}_1 \in (\mathfrak{D}_{\theta_y})_q$,

where θ_1 and θ_2 are called the slant angles of submersion.

If we suppose m_1, m_2 and m_3 are the dimensions of $\mathfrak{D}$, $\mathfrak{D}_{\theta_g}$ and $\mathfrak{D}_{\theta_y}$ respectively, then we have the following:

(i) If $m_1 \neq 0$, $m_2 = 0$ and $m_3 = 0$, then Ψ is an invariant submersion.
(ii) If $m_1 \neq 0$, $m_2 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $m_3 = 0$, then Ψ is a proper semi-slant submersion.
(iii) If $m_1 = 0$, $m_2 = 0$ and $m_3 \neq 0, 0 < \frac{\pi}{2}$, then Ψ is a slant submersion with slant angle θ_2.
(iv) If $m_1 = 0, m_2 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $m_3 \neq 0, \theta_2 = \frac{\pi}{2}$, then Ψ proper hemi-slant submersion.
(v) If $m_1 = 0, m_2 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $m_3 \neq 0, 0 <, \theta_2 < \frac{\pi}{2}$, then Ψ is proper bi-slant submersion with slant angles θ_1 and θ_2.
(vi) If $m_1 \neq 0, m_2 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $m_3 \neq 0, 0 < \theta_2 < \frac{\pi}{2}$, then Ψ is proper quasi bi-slant submersion with slant angles θ_1 and θ_2.

We construct an example of $\mathcal{QBSC}$ $\xi^{\perp}$-submersions from Sasakian manifold. Let $\left(R^{2k+1}, g_{2k+1}, \phi, \xi, \eta\right)$ denote the manifold with its usual Sasakian structure given by

$$\phi\left(\sum_{i=1}^{k}\left(X_i\frac{\partial}{\partial x_i} + Y_i\frac{\partial}{\partial y_i}\right) + Z\frac{\partial}{\partial z}\right) = \sum_{i=1}^{k}\left(Y_i\frac{\partial}{\partial x_i} - X_i\frac{\partial}{\partial y_i} + Y_i y_i\frac{\partial}{\partial z}\right),$$

where $x_1, \ldots, x_k, y_1 \ldots, y_k, z$ are the cartesian coordinates. Its Riemannian metric g is defined as $g = \eta\otimes \eta + \frac{1}{4}\left(\sum_{i=1}^{k}\left[dx_i \otimes dx_i + dy_i \otimes dy_i\right]\right)$, where η is its usual contact form and given as $\eta = \frac{1}{2}\left(dz - \sum_{i=1}^{k} Y_i dx_i\right)$. The characteristic vector field ξ is given by $2\frac{\partial}{\partial z}$. The vector fields $E_i = 2\frac{\partial}{\partial y_i}, E_{k+i} = 2\left(\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z}\right)$ and ξ form a ϕ-basis for the Sasakian structure and $\left(R^{2k+1}, \phi, \xi, \eta, g\right)$ is a Sasakian manifold. Throughout this section we will use the notation.

Example. Let $\left(R^{15}, \phi, \xi, \eta, g\right)$ be Sasakian manifold. Let $\Psi : R^{15} \to R^7$ be map defined by $\Psi(x_1, \ldots, x_7, y_1, \ldots, y_7, z) = \sqrt{\pi}(\cos\theta_1 x_3 + \sin\theta_1 x_4, x_5, x_7, y_3, \cos\theta_2 y_5 + \sin\theta_2 y_6, y_7, z)$, which is a quasi bi-slant conformal submersion with dilation $\lambda = \sqrt{\pi}$ such that

$$\begin{aligned} V_1 &= 2\left(\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial z}\right), V_2 = 2\left(\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial z}\right), \\ V_3 &= 2\left[\sin\theta_1\left(\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial z}\right) - \cos\theta_1\left(\frac{\partial}{\partial x_4} + y_4\frac{\partial}{\partial z}\right)\right], \\ V_4 &= 2\left(\frac{\partial}{\partial x_6} + y_6\frac{\partial}{\partial z}\right), V_5 = 2\frac{\partial}{\partial y_1}, V_6 = 2\frac{\partial}{\partial y_2}, \\ V_7 &= 2\frac{\partial}{\partial y_4}, V_8 = 2\left(\sin\theta_2\frac{\partial}{\partial y_5} - \cos\theta_2\frac{\partial}{\partial y_6}\right). \end{aligned}$$

$\ker\Psi_* = \mathfrak{D} \oplus \mathfrak{D}_{\theta_g} \oplus \mathfrak{D}_{\theta_y}$, where

$$\begin{aligned} \mathfrak{D} &= \left\langle V_1 = 2\left(\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial z}\right), V_2 = 2\left(\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial z}\right), V_5 = 2\frac{\partial}{\partial y_1}, V_6 = 2\frac{\partial}{\partial y_2}\right\rangle, \\ \mathfrak{D}_{\theta_g} &= \left\langle V_3 = 2\left[\sin\theta_1\left(\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial z}\right) - \cos\theta_1\left(\frac{\partial}{\partial x_4} + y_4\frac{\partial}{\partial z}\right)\right], V_7 = 2\frac{\partial}{\partial y_4}\right\rangle, \\ \mathfrak{D}_{\theta_y} &= \left\langle V_4 = 2\left(\frac{\partial}{\partial x_6} + y_6\frac{\partial}{\partial z}\right), V_8 = 2\left(\sin\theta_2\frac{\partial}{\partial y_5} - \cos\theta_2\frac{\partial}{\partial y_6}\right)\right\rangle, \end{aligned}$$

and

$$(\ker\Psi_*)^\perp = \left\langle 2\left(\cos\theta_1\frac{\partial}{\partial x_3} + \sin\theta_1\frac{\partial}{\partial x_4}\right), 2\frac{\partial}{\partial x_5}, 2\frac{\partial}{\partial x_7}, 2\frac{\partial}{\partial y_3}, 2\left(\cos\theta_2\frac{\partial}{\partial y_5} + \sin\theta_2\frac{\partial}{\partial y_6}\right), 2\frac{\partial}{\partial y_7}, 2\frac{\partial}{\partial z}.\right\rangle,$$

where θ_g and θ_y are the slant angles of the submersion for the distribution $\mathfrak{D}_{\theta_g}$ and $\mathfrak{D}_{\theta_y}$, respectively.

Let Ψ be a $\mathcal{QBSC}$ $\xi^\perp$-submersion from an ACM manifold $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{Q}_2, g_2)$. Then, for any $U \in (ker\Psi_*)$, we have

$$\widehat{U} = \mathfrak{P}_g\widehat{U} + \mathfrak{P}_y\widehat{U} + \mathfrak{P}_v\widehat{U}, \tag{3.1}$$

where $\mathfrak{P}_g, \mathfrak{P}_y$ and $\mathfrak{P}_v$ are the projections morphism onto $\mathfrak{D}, \mathfrak{D}_{\theta_g}$, and $\mathfrak{D}_{\theta_y}$. Now, for any $\widehat{U} \in (ker\Psi_*)$, we have

$$\phi\widehat{U} = \alpha\widehat{U} + \beta\widehat{U}, \tag{3.2}$$

where $\alpha\widehat{U} \in \Gamma(ker\Psi_*)$ and $\beta\widehat{U} \in \Gamma(ker\Psi_*)^\perp$. From Eqs (3.1) and (3.2), we have

$$\phi\widehat{U} = \phi(\mathfrak{P}_g\widehat{U}) + \phi(\mathfrak{P}_y\widehat{U}) + \phi(\mathfrak{P}_v\widehat{U}) = \alpha(\mathfrak{P}_g\widehat{U}) + \beta(\mathfrak{P}_g\widehat{U}) + \alpha(\mathfrak{P}_y\widehat{U}) + \beta(\mathfrak{P}_y\widehat{U}) + \alpha(\mathfrak{P}_v\widehat{U}) + \beta(\mathfrak{P}_v\widehat{U}).$$

Since $\phi\mathfrak{D} = \mathfrak{D}$ and $\beta(\mathfrak{P}_g\widehat{U}) = 0$, we have

$$\phi\widehat{U} = \alpha(\mathfrak{P}_g\widehat{U}) + \alpha(\mathfrak{P}_y\widehat{U}) + \beta(\mathfrak{P}_y\widehat{U}) + \alpha(\mathfrak{P}_v\widehat{U}) + \beta(\mathfrak{P}_v\widehat{U}).$$

Hence we have the decomposition as

$$\phi(ker\Psi_*) = \alpha\mathfrak{D} \oplus \alpha\mathfrak{D}_{\theta_g} \oplus \alpha\mathfrak{D}_{\theta_y} \oplus \beta\mathfrak{D}_{\theta_g} \oplus \beta\mathfrak{D}_{\theta_y}. \tag{3.3}$$

From Eq (3.3), we have the following decomposition:

$$(ker\Psi_*)^\perp = \beta\mathfrak{D}_{\theta_g} \oplus \beta\mathfrak{D}_{\theta_y} \oplus \mu, \tag{3.4}$$

where μ is the orthogonal complement to $\beta\mathfrak{D}_{\theta_g} \oplus \beta\mathfrak{D}_{\theta_y}$ in $(ker\Psi_*)^\perp$ such that $\mu = (\phi\mu) \oplus <\xi>$ and μ is invariant with respect to ϕ. Now, for any $\widehat{X} \in \Gamma(ker\Psi_*)^\perp$, we have

$$\phi\widehat{X} = \mathbb{C}\widehat{X} + \mathbb{B}\widehat{X}, \tag{3.5}$$

where $\mathbb{C}\widehat{X} \in \Gamma(ker\Psi_*)$ and $\mathbb{B}\widehat{X} \in \Gamma(ker\Psi_*)^\perp$.

Lemma 3.1. *Let $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ be an ACM manifold and $(\bar{Q}_2, g_2)$ be a RM. If $\Psi : \bar{Q}_1 \to \bar{Q}_2$ is a $\mathcal{QBSC}$ $\xi^\perp$-submersion, then we have*

$$-\widehat{U} = \alpha^2\widehat{U} + \mathbb{C}\beta\widehat{U},\ \beta\alpha\widehat{U} + \mathbb{B}\beta\widehat{U} = 0,$$

$$-\widehat{X} + \eta(\widehat{X})\xi = \beta\mathbb{C}\widehat{X} + \mathbb{B}^2\widehat{X},\ \alpha\mathbb{C}\widehat{X} + \mathbb{C}\mathbb{B}\widehat{X} = 0,$$

for $\widehat{U} \in \Gamma(ker\Psi_*)$ and $\widehat{X} \in \Gamma(ker\Psi_*)^\perp$.

Proof. On using Eqs (2.1), (3.2) and (3.5), we get the desired results. □

Since $\Psi : \bar{Q}_1 \to \bar{Q}_2$ is a $\mathcal{QBSC}$ $\xi^\perp$-submersion, Then let us provide some helpful findings that will be utilise throughout the paper.

Lemma 3.2. *Let Ψ be a $\mathcal{QBSC}$ $\xi^\perp$-submersion from an ACM manifold $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{Q}_2, g_2)$, then we have*

(i) $\alpha^2\widehat{U} = -\cos^2\theta_1\,\widehat{U}$,
(ii) $g_1(\alpha\widehat{U}, \alpha\widehat{V}) = \cos^2\theta_1\, g_1(\widehat{U}, \widehat{V})$,
(iii) $g(\beta\widehat{U}, \beta\widehat{V}) = \sin^2\theta_1\, g_1(\widehat{U}, \widehat{V})$,

for any vector fields $\widehat{U}, \widehat{V} \in \Gamma(\mathfrak{D}_{\theta_g})$.

Lemma 3.3. *Let Ψ be a $\mathcal{QBSC}$ $\xi^\perp$-submersion from an ACM manifold $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{Q}_2, g_2)$, then we have*

(i) $\alpha^2\widehat{Z} = -\cos^2\theta_2\,\widehat{Z}$,
(ii) $g_1(\alpha\widehat{Z}, \alpha\widehat{W}) = \cos^2\theta_2\, g_1(\widehat{Z}, \widehat{W})$,
(iii) $g_1(\beta\widehat{Z}, \beta\widehat{W}) = \sin^2\theta_2\, g_1(\widehat{Z}, \widehat{W})$,

for any vector fields $\widehat{Z}, \widehat{W} \in \Gamma(\mathfrak{D}_{\theta_y})$.

Proof. The proof of above Lemmas is similar to the proof of the Theorem 2.2 of [10]. □

Let $(\bar{Q}_2, g_2)$ be a Riemannian manifold and $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ be a Sasakian manifold. We now consider how the Sasakian structure on $\bar{Q}_1$ affects the tensor fields $\mathcal{T}$ and $\mathcal{A}$ of a $\mathcal{QBSC}\xi^\perp$-submersion $\Psi : (\bar{Q}_1, \phi, \xi, \eta, g_1) \to (\bar{Q}_2, g_2)$.

Lemma 3.4. *Let Ψ be a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion from Sasakian manifold $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{Q}_2, g_2)$, then we have*

$$\mathcal{A}_{\widehat{X}}\mathbb{C}\widehat{Y} + \mathcal{H}\nabla_{\widehat{X}}\mathbb{B}\widehat{Y} = \mathbb{B}\mathcal{H}\nabla_{\widehat{X}}\widehat{Y} + \beta\mathcal{A}_{\widehat{X}}\widehat{Y} + g_1(\widehat{X}, \widehat{Y})\xi - \eta(\widehat{Y})\widehat{X}, \tag{3.6}$$

$$\mathcal{V}\nabla_{\widehat{X}}\mathbb{C}\widehat{Y} + \mathcal{A}_{\widehat{X}}\mathbb{B}\widehat{Y} = \mathbb{C}\mathcal{H}\nabla_{\widehat{X}}\widehat{Y} + \alpha\mathcal{A}_{\widehat{X}}\widehat{Y}, \tag{3.7}$$

$$\mathcal{V}\nabla_{\widehat{X}}\alpha\widehat{V} + \mathcal{A}_{\widehat{X}}\beta\widehat{V} = \mathbb{C}\mathcal{A}_{\widehat{X}}\widehat{V} + \alpha\mathcal{V}\nabla_{\widehat{X}}\widehat{V}, \tag{3.8}$$

$$\mathcal{A}_{\widehat{X}}\alpha\widehat{V} + \mathcal{H}\nabla_{\widehat{X}}\beta\widehat{V} = \mathbb{B}\mathcal{A}_{\widehat{X}}\widehat{V} + \beta\mathcal{V}\nabla_{\widehat{X}}\widehat{V}, \tag{3.9}$$

$$\mathcal{V}\nabla_{\widehat{V}}\mathbb{C}\widehat{X} + \mathcal{T}_{\widehat{V}}\mathbb{B}\widehat{X} = \alpha\mathcal{T}_{\widehat{V}}\mathbb{B}\widehat{X} + \mathbb{C}\mathcal{H}\nabla_{\widehat{V}}\widehat{X} - \eta(\widehat{X})\widehat{V}, \tag{3.10}$$

$$\mathcal{T}_{\widehat{V}}\mathbb{C}\widehat{X} + \mathcal{H}\nabla_{\widehat{V}}\mathbb{B}\widehat{X} = \beta\mathcal{T}_{\widehat{V}}\widehat{X} + \mathbb{B}\mathcal{H}\nabla_{\widehat{V}}\widehat{X}, \tag{3.11}$$

$$\mathcal{V}\nabla_{\widehat{U}}\alpha\widehat{V} + \mathcal{T}_{\widehat{U}}\beta\widehat{V} = \mathbb{C}\mathcal{T}_{\widehat{U}}\widehat{V} + \alpha\mathcal{V}\nabla_{\widehat{U}}\widehat{V}, \tag{3.12}$$

$$\mathcal{T}_{\widehat{U}}\alpha\widehat{V} + \mathcal{H}\nabla_{\widehat{U}}\beta\widehat{V} = \mathbb{B}\mathcal{T}_{\widehat{U}}\widehat{V} + \beta\mathcal{V}\nabla_{\widehat{U}}\widehat{V} + g_1(\widehat{U}, \widehat{V})\xi, \tag{3.13}$$

for any vector fields $\widehat{U}, \widehat{V} \in \Gamma(\ker\Psi_*)$ and $\widehat{X}, \widehat{Y} \in \Gamma(\ker\Psi_*)^{\perp}$.

Proof. From (2.5), (2.12) and (3.5), we obtained the conditions (3.6) and (3.7). Again using Eqs (3.2), (3.5), (2.9)–(2.12) and (2.5), finish he result. □

Now we will go through some fundamental results that can be used to investigate the geometry of $\mathcal{QBSC}$ $\xi^{\perp}$-submersion $\Psi : \bar{Q}_1 \to \bar{Q}_2$. For this, define the following:

$$(\nabla_{\widehat{U}}\alpha)\widehat{V} = \mathcal{V}\nabla_{\widehat{U}}\alpha\widehat{V} - \alpha\mathcal{V}\nabla_{\widehat{U}}\widehat{V}, \tag{3.14}$$

$$(\nabla_{\widehat{U}}\beta)\widehat{V} = \mathcal{H}\nabla_{\widehat{U}}\beta\widehat{V} - \beta\mathcal{V}\nabla_{\widehat{U}}\widehat{V}, \tag{3.15}$$

$$(\nabla_{\widehat{X}}\mathbb{C})\widehat{Y} = \mathcal{V}\nabla_{\widehat{X}}\mathbb{C}\widehat{Y} - \mathbb{C}\mathcal{H}\nabla_{\widehat{X}}\widehat{Y}, \tag{3.16}$$

$$(\nabla_{\widehat{X}}\mathbb{B})\widehat{Y} = \mathcal{H}\nabla_{\widehat{X}}\mathbb{B}\widehat{Y} - \mathbb{B}\mathcal{H}\nabla_{\widehat{X}}\widehat{Y}, \tag{3.17}$$

for any vector fields $\widehat{U}, \widehat{V} \in \Gamma(\ker\Psi_*)$ and $\widehat{X}, \widehat{Y} \in \Gamma(\ker\Psi_*)^{\perp}$.

Lemma 3.5. *Let $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ be a Sasakian manifold and $(\bar{Q}_2, g_2)$ be a RM. If $\Psi : \bar{Q}_1 \to \bar{Q}_2$ is a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion, then we have*

$$(\nabla_{\widehat{U}}\alpha)\widehat{V} = \mathbb{C}\mathcal{T}_{\widehat{U}}\widehat{V} - \mathcal{T}_{\widehat{U}}\beta\widehat{V},$$

$$(\nabla_{\widehat{U}}\beta)\widehat{V} = \mathbb{B}\mathcal{T}_{\widehat{U}}\widehat{V} - \mathcal{T}_{\widehat{U}}\alpha\widehat{V},$$

$$(\nabla_{\widehat{X}}\mathbb{C})\widehat{Y} = \alpha\mathcal{A}_{\widehat{X}}\widehat{Y} - \mathcal{A}_{\widehat{X}}\mathbb{B}\widehat{Y},$$

$$(\nabla_{\widehat{X}}\mathbb{B})\widehat{Y} = \beta\mathcal{A}_{\widehat{X}}\widehat{Y} - \mathcal{A}_{\widehat{X}}\mathbb{C}\widehat{Y},$$

for all vector fields $\widehat{U}, \widehat{V} \in \Gamma(ker\Psi_*)$ and $\widehat{X}, \widehat{Y} \in \Gamma(ker\Psi_*)^{\perp}$.

Proof. On using Eqs (2.5), (2.9)–(2.12) and (3.14)–(3.17), we get the desired result. □

If α and β, the tensor fields, are parallel with respect to the connection ∇ of $\bar{Q}_1$ then, we have

$$\mathbb{C}\mathcal{T}_{\widehat{U}}\widehat{V} = \mathcal{T}_{\widehat{U}}\beta\widehat{V},\ \mathbb{B}\mathcal{T}_{\widehat{U}}\widehat{V} = \mathcal{T}_{\widehat{U}}\alpha\widehat{V},$$

for any vector fields $\widehat{U}, \widehat{V} \in \Gamma(T\bar{Q}_1)$.

4. Integrability and totally geodesicness of distributions

Since $\Psi : \bar{Q}_1 \to \bar{Q}_2$ is a $\mathcal{QBSC}$ $\xi^\perp$-submersion, where $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ represents a Kenmotsu manifold and $(\bar{Q}_2, g_2)$ a Riemannian manifold. The definition of $\mathcal{QBSC}$ $\xi^\perp$-submersion ensures the existence of three mutually orthogonal distributions, which include an invariant distribution $\mathfrak{D}$, a pair of slant distributions $\mathfrak{D}^{\theta_g}$ and $\mathfrak{D}^{\theta_y}$. We start the discussion on the integrability of distributions by determining the integrability of the slant distribution in the manner described below:

Theorem 4.1. *Let Ψ be a $\mathcal{QBSC}$ $\xi^\perp$-submersion from Sasakian manifold $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{Q}_2, g_2)$. Then slant distribution $\mathfrak{D}_{\theta_g}$ is integrable if and only if*

$$\begin{aligned}
&\frac{1}{\lambda^2}\{g_2(\nabla^{\Psi}_{\widehat{U}_1}\Psi_*\beta\widehat{V}_1 + \nabla^{\Psi}_{\widehat{V}_1}\Psi_*\beta\widehat{U}_1, \Psi_*\beta\mathfrak{P}_v\widehat{Z})\} \\
&= \frac{1}{\lambda^2}\{g_2((\nabla\Psi_*)(\widehat{U}_1, \beta\widehat{V}_1) + (\nabla\Psi_*)(\widehat{V}_1, \beta\widehat{U}_1), \Psi_*\beta\mathfrak{P}_v\widehat{Z})\} \\
&\quad - g_1(\nabla_{\widehat{V}_1}\beta\alpha\widehat{U}_1 - \nabla_{\widehat{U}_1}\beta\alpha\widehat{V}_1, \widehat{Z}) - g_1(\mathcal{T}_{\widehat{U}_1}\beta\widehat{V}_1 - \mathcal{T}_{\widehat{V}_1}\beta\widehat{U}_1, \phi\mathfrak{P}_g\widehat{Z} + \alpha\mathfrak{P}_v\widehat{Z}),
\end{aligned} \tag{4.1}$$

for any $\widehat{U}_1, \widehat{V}_1 \in \Gamma(\mathfrak{D}_{\theta_g})$ and $\widehat{Z} \in \Gamma(\mathfrak{D} \oplus \mathfrak{D}_{\theta_y})$.

Proof. For all $\widehat{U}_1, \widehat{V}_1 \in \Gamma(\mathfrak{D}_{\theta_g})$ and $\widehat{Z} \in \Gamma(\mathfrak{D} \oplus \mathfrak{D}_{\theta_y})$ with using Eqs (2.2), (2.5), (2.14) and (3.2), we get

$$g_1([\widehat{U}_1, \widehat{V}_1], \widehat{Z}) = g_1(\nabla_{\widehat{U}_1}\alpha\widehat{V}_1, \phi\widehat{Z}) + g_1(\nabla_{\widehat{U}_1}\beta\widehat{V}_1, \phi\widehat{Z}) - g_1(\nabla_{\widehat{V}_1}\alpha\widehat{U}_1, \phi\widehat{Z}) - g_1(\nabla_{\widehat{V}_1}\beta\widehat{U}_1, \phi\widehat{Z}).$$

By using Eqs (2.5), (2.14) and (3.2), we have

$$\begin{aligned}
g_1([\widehat{U}_1, \widehat{V}_1], \widehat{Z}) &= -g_1(\nabla_{\widehat{U}_1}\alpha^2\widehat{V}_1, \widehat{Z}) - g_1(\nabla_{\widehat{U}_1}\beta\alpha\widehat{V}_1, \widehat{Z}) + g_1(\nabla_{\widehat{V}_1}\alpha^2\widehat{U}_1, \widehat{Z}) \\
&\quad + g_1(\nabla_{\widehat{V}_1}\beta\alpha\widehat{U}_1, \widehat{Z}) + g_1(\nabla_{\widehat{U}_1}\beta\widehat{V}_1, \phi\mathfrak{P}_g\widehat{Z} + \alpha\mathfrak{P}_v\widehat{Z} + \beta\mathfrak{P}_v\widehat{Z}) \\
&\quad - g_1(\nabla_{\widehat{V}_1}\beta\widehat{U}_1, \phi\mathfrak{P}_g\widehat{Z} + \alpha\mathfrak{P}_v\widehat{Z} + \beta\mathfrak{P}_v\widehat{Z}).
\end{aligned}$$

Taking account the fact of Lemma 3.2 with Eq (2.10), we get

$$\begin{aligned}
g_1([\widehat{U}_1, \widehat{V}_1], \widehat{Z}) &= cos^2\theta_1 g_1([\widehat{U}_1, \widehat{V}_1], \widehat{Z}) + g_1(\nabla_{\widehat{V}_1}\beta\alpha\widehat{U}_1 - \nabla_{\widehat{U}_1}\beta\alpha\widehat{V}_1, \widehat{Z}) \\
&\quad + g_1(\mathcal{T}_{\widehat{U}_1}\beta\widehat{V}_1 - \mathcal{T}_{\widehat{V}_1}\beta\widehat{U}_1, \phi\mathfrak{P}_g\widehat{Z} + \alpha\mathfrak{P}_v\widehat{Z}) \\
&\quad + g_1(\mathcal{H}\nabla_{\widehat{U}_1}\beta\widehat{V}_1 - \mathcal{H}\nabla_{\widehat{V}_1}\beta\widehat{U}_1, \beta\mathfrak{P}_v\widehat{Z}).
\end{aligned}$$

On using Eq (2.6), formula (2.15) with Lemma 2.1, we finally get

$$\begin{aligned}
&sin^2\theta_1 g_1([\widehat{U}_1, \widehat{V}_1], \widehat{Z}) \\
&= \frac{1}{\lambda^2}\{g_2(\nabla^{\Psi}_{\widehat{U}_1}\Psi_*\beta\widehat{V}_1 - \nabla^{\Psi}_{\widehat{V}_1}\Psi_*\beta\widehat{U}_1, \Psi_*\beta\mathfrak{P}_v\widehat{Z})\} \\
&\quad + \frac{1}{\lambda^2}\{g_2((\nabla\Psi_*)(\widehat{U}_1, \beta\widehat{V}_1), \Psi_*\beta\mathfrak{P}_v\widehat{Z}) + g_2((\nabla\Psi_*)(\widehat{V}_1, \beta\widehat{U}_1), \Psi_*\beta\mathfrak{P}_v\widehat{Z})\} \\
&\quad + g_1(\mathcal{T}_{\widehat{U}_1}\beta\widehat{V}_1 - \mathcal{T}_{\widehat{V}_1}\beta\widehat{U}_1, \phi\mathfrak{P}_g\widehat{Z} + \alpha\mathfrak{P}_v\widehat{Z}) + g_1(\nabla_{\widehat{V}_1}\beta\alpha\widehat{U}_1 - \nabla_{\widehat{U}_1}\beta\alpha\widehat{V}_1, \widehat{Z}).
\end{aligned}$$

□

In a same manner, we can obained the condition of integrability for $\mathfrak{D}_{\theta_y}$ as follows:

Theorem 4.2. *Let* $\Psi : (\bar{Q}_1, \phi, \xi, \eta, g_1) \rightarrow (\bar{Q}_2, g_2)$ *be a* $\mathcal{QBSC}$ $\xi^\perp$*-submersion, where* $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ *a Sasakian manifold and* $(\bar{Q}_2, g_2)$ *a RM. Then slant distribution* $\mathfrak{D}_{\theta_y}$ *is integrable if and only if*

$$\begin{aligned}
&-\frac{1}{\lambda^2}\{g_2((\nabla\Psi_*)(\widehat{U}_2, \beta\widehat{V}_2) - (\nabla\Psi_*)(\widehat{V}_2, \beta\widehat{U}_2), \Psi_*\beta\mathfrak{P}_y\widehat{Z})\} \\
&= g_1(\mathcal{T}_{\widehat{V}_2}\beta\alpha\widehat{U}_2 - \mathcal{T}_{\widehat{U}_2}\beta\alpha\widehat{V}_2, \widehat{Z}) + g_1(\mathcal{T}_{\widehat{U}_2}\beta\widehat{V}_2 - \mathcal{T}_{\widehat{V}_2}\beta\widehat{U}_2, \phi\mathfrak{P}_g\widehat{Z} + \alpha\mathfrak{P}_y\widehat{Z}) \\
&\quad + \frac{1}{\lambda^2}\{g_2(\nabla^\Psi_{\widehat{U}_2}\Psi_*\beta\widehat{V}_2 - \nabla^\Psi_{\widehat{V}_2}\Psi_*\beta\widehat{U}_2, \Psi_*\beta\mathfrak{P}_y\widehat{Z})\},
\end{aligned}$$

for any $\widehat{U}_2, \widehat{V}_2 \in \Gamma(\mathfrak{D}_{\theta_y})$ *and* $\widehat{Z} \in \Gamma(\mathfrak{D} \oplus \mathfrak{D}_{\theta_g})$.

Proof. On using Eqs (2.2), (2.5), (2.14) and (3.2), we have

$$\begin{aligned}
g_1([\widehat{U}_2, \widehat{V}_2], \widehat{Z}) &= g_1(\nabla_{\widehat{V}_2}\alpha^2\widehat{U}_2, \widehat{Z}) + g_1(\nabla_{\widehat{V}_2}\beta\alpha\widehat{U}_2, \widehat{Z}) - g_1(\nabla_{\widehat{U}_2}\alpha^2\widehat{V}_2, \widehat{Z}) \\
&\quad - g_1(\nabla_{\widehat{U}_2}\beta\alpha\widehat{V}_2, \widehat{Z}) + g_1(\nabla_{\widehat{U}_2}\beta\widehat{V}_2 - \nabla_{\widehat{V}_2}\beta\widehat{U}_2, \phi\widehat{Z}),
\end{aligned}$$

for any $\widehat{U}_2, \widehat{V}_2 \in \Gamma(\mathfrak{D}_{\theta_y})$ and $\widehat{Z} \in \Gamma(\mathfrak{D} \oplus \mathfrak{D}_{\theta_g})$. From Eq (2.10) and Lemma 3.3, we get

$$\begin{aligned}
sin^2\theta_2 g_1([\widehat{U}_2, \widehat{V}_2], \widehat{Z}) &= g_1(\mathcal{T}_{\widehat{V}_2}\beta\alpha\widehat{U}_2 - \mathcal{T}_{\widehat{U}_2}\beta\alpha\widehat{V}_2, \widehat{Z}) + g_1(\mathcal{T}_{\widehat{U}_2}\beta\widehat{V}_2 - \mathcal{T}_{\widehat{V}_2}\beta\widehat{U}_2, \phi\mathfrak{P}_g\widehat{Z} + \alpha\mathfrak{P}_y\widehat{Z}) \\
&\quad + g_1(\mathcal{H}\nabla_{\widehat{U}_2}\beta\widehat{V}_2 - \mathcal{H}\nabla_{\widehat{V}_2}\beta\widehat{U}_2, \beta\mathfrak{P}_y\widehat{Z}).
\end{aligned}$$

Since Ψ is $\mathcal{QBSC}$ $\xi^\perp$-submersion, using conformality condition with Eqs (2.6) and (2.15), we finally get

$$\begin{aligned}
sin^2\theta_2 g_1([\widehat{U}_2, \widehat{V}_2], \widehat{Z}) &= g_1(\mathcal{T}_{\widehat{V}_2}\beta\alpha\widehat{U}_2 - \mathcal{T}_{\widehat{U}_2}\beta\alpha\widehat{V}_2, \widehat{Z}) + g_1(\mathcal{T}_{\widehat{U}_2}\beta\widehat{V}_2 - \mathcal{T}_{\widehat{V}_2}\beta\widehat{U}_2, \phi\mathfrak{P}_g\widehat{Z} + \alpha\mathfrak{P}_y\widehat{Z}) \\
&\quad + \frac{1}{\lambda^2}\{g_2((\nabla\Psi_*)(\widehat{U}_2, \beta\widehat{V}_2) - (\nabla\Psi_*)(\widehat{V}_2, \beta\widehat{U}_2), \Psi_*\beta\mathfrak{P}_y\widehat{Z})\} \\
&\quad + \frac{1}{\lambda^2}\{g_2(\nabla^\Psi_{\widehat{U}_2}\Psi_*\beta\widehat{V}_2 - \nabla^\Psi_{\widehat{V}_2}\Psi_*\beta\widehat{U}_2, \Psi_*\beta\mathfrak{P}_y\widehat{Z})\}.
\end{aligned}$$

This completes the proof of the theorem. □

Since, the invariant distribution is mutually orthogonal to the slant distributions in accordance with the concept of $\mathcal{QBSC}$ $\xi^\perp$-submersion, this led us to investigate the necessary and sufficient condition for the invariant distribution to be integrable.

Theorem 4.3. *Let* $\Psi : (\bar{Q}_1, \phi, \xi, \eta, g_1) \rightarrow (\bar{Q}_2, g_2)$ *be a* $\mathcal{QBSC}$ $\xi^\perp$*-submersion, where* $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ *a Sasakian manifold and* $(\bar{Q}_2, g_2)$ *a RM. Then the invariant distribution* $\mathfrak{D}$ *is integrable if and only if*

$$g_1(\mathcal{T}_{\widehat{U}}\alpha\mathfrak{P}_g\widehat{V} - \mathcal{T}_{\widehat{V}}\alpha\mathfrak{P}_g\widehat{U}, \beta\mathfrak{P}_y\widehat{Z} + \beta\mathfrak{P}_v\widehat{W}) - g_1(\mathcal{V}\nabla_{\widehat{U}}\alpha\mathfrak{P}_g\widehat{V} - \mathcal{V}\nabla_{\widehat{V}}\alpha\mathfrak{P}_g\widehat{U}, \alpha\mathfrak{P}_y\widehat{Z} + \alpha\mathfrak{P}_v\widehat{Z}) = 0, \quad (4.2)$$

for any $\widehat{U}, \widehat{V} \in \Gamma(\mathfrak{D})$ *and* $\widehat{Z} \in \Gamma(\mathfrak{D}_{\theta_g} \oplus \mathfrak{D}_{\theta_y})$.

Proof. For all $\widehat{U}, \widehat{V} \in \Gamma(\mathfrak{D})$ and $\widehat{Z} \in \Gamma(\mathfrak{D}_{\theta_g} \oplus \mathfrak{D}_{\theta_y})$ with using Eqs (2.2), (2.9), (2.14) and decomposition (3.1), we have

$$g_1([\widehat{U}, \widehat{V}], \widehat{Z}) = g_1(\nabla_{\widehat{U}} \alpha \mathfrak{P}_g \widehat{V}, \phi \mathfrak{P}_y \widehat{Z} + \phi \mathfrak{P}_v \widehat{Z}) - g_1(\nabla_{\widehat{V}} \alpha \mathfrak{P}_g \widehat{U}, \phi \mathfrak{P}_y \widehat{Z} + \phi \mathfrak{P}_v \widehat{Z}).$$

On using Eq (3.2), we finally have

$$g_1([\widehat{U}, \widehat{V}], \widehat{Z}) = g_1(\mathcal{T}_{\widehat{U}} \alpha \mathfrak{P}_g \widehat{V} - \mathcal{T}_{\widehat{V}} \alpha \mathfrak{P}_g \widehat{U}, \beta \mathfrak{P}_y \widehat{Z} + \beta \mathfrak{P}_v \widehat{Z}) + g_1(\mathcal{V} \nabla_{\widehat{U}} \alpha \mathfrak{P}_g \widehat{V} - \mathcal{V} \mathcal{A}_{\widehat{V}} \alpha \mathfrak{P}_g \widehat{U}, \alpha \mathfrak{P}_y \widehat{Z} + \alpha \mathfrak{P}_v \widehat{Z}).$$

This completes the proof of theorem. □

The necessary and sufficient prerequisites that must also exist in order for distributions to be totally geodesic will now be discussed after the necessary conditions for distributions integrability. We start with investigating the necessary and sufficient conditions for distributions to be totally geodesic.

Theorem 4.4. *Let $\Psi : (\bar{Q}_1, \phi, \xi, \eta, g_1) \to (\bar{Q}_2, g_2)$ be a $\mathcal{QBSC}$ $\xi^\perp$-submersion, where $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ a Sasakian manifold and $(\bar{Q}_2, g_2)$ a RM. Then invariant distribution $\mathfrak{D}$ defines totally geodesic foliation on $\bar{Q}_1$ if and only if*

(i) $\lambda^{-2} g_2\{((\nabla \Psi_*)(\widehat{U}, \phi \widehat{V}), \Psi_* \beta \widehat{Z})\} = g_1(\mathcal{V} \nabla_{\widehat{U}} \phi \widehat{V}, \alpha \widehat{Z})$,
(ii) $\lambda^{-2} \{g_2((\nabla \Psi_*)(\widehat{U}, \phi \widehat{V}), \Psi_* \mathbb{B} \widehat{X})\} = g_1(\mathcal{V} \nabla_{\widehat{U}} \phi \widehat{V}, \mathbb{C} \widehat{X}) + g_1(\phi \widehat{U}, \widehat{V}) \eta(\widehat{X})$,

for any $\widehat{U}, \widehat{V} \in \Gamma(\mathfrak{D})$ and $\widehat{Z} \in \Gamma(\mathfrak{D}_{\theta_g} \oplus \mathfrak{D}_{\theta_y})$.

Proof. For any $\widehat{U}, \widehat{V} \in \Gamma(\mathfrak{D})$ and $\widehat{Z} \in \Gamma(\mathfrak{D}_{\theta_g} \oplus \mathfrak{D}_{\theta_y})$ with using Eqs (2.2), (2.5), (2.14) and (3.2), we may write

$$g_1(\nabla_{\widehat{U}} \widehat{V}, \widehat{Z}) = g_1(\mathcal{V} \nabla_{\widehat{U}} \phi \widehat{V}, \alpha \widehat{Z}) + g_1(\mathcal{T}_{\widehat{U}} \phi \widehat{V}, \beta \widehat{Z}).$$

On using the conformality of Ψ with Eqs (2.6) and (2.15), we get

$$g_1(\nabla_{\widehat{U}} \widehat{V}, \widehat{Z}) = g_1(\mathcal{V} \nabla_{\widehat{U}} \phi \widehat{V}, \alpha \widehat{Z}) - \lambda^{-2} g_2((\nabla \Psi_*)(\widehat{U}, \phi \widehat{V}), \Psi_* \beta \widehat{Z}).$$

On the other hand, using Eqs (2.2), (2.5) and (2.14) with conformality of Ψ, we finally have

$$g_1(\nabla_{\widehat{U}} \widehat{V}, \widehat{X}) = g_1(\mathcal{V} \nabla_{\widehat{U}} \phi \widehat{V}, \mathbb{C} \widehat{X}) - \lambda^{-2} g_2((\nabla \Psi_*)(\widehat{U}, \phi \widehat{V}), \Psi_* \mathbb{B} \widehat{X}) + g_1(\phi \widehat{U}, \widehat{V}) \eta(\widehat{X}),$$

from which we get the desired result. □

In same manner, we can examine the geometry of leaves of $\mathfrak{D}_{\theta_g}$ as follows:

Theorem 4.5. *Let Ψ be a $\mathcal{QBSC}$ $\xi^\perp$-submersion from Sasakian manifold $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{Q}_2, g_2)$. Then slant distribution $\mathfrak{D}_{\theta_g}$ defines totally geodesic foliation on $\bar{Q}_1$ if and only if*

$$\begin{aligned}
&\frac{1}{\lambda^2} g_2(\nabla^{\Psi}_{\widehat{Z}} \Psi_* \beta \mathfrak{P}_y \widehat{W}, \Psi_* \beta \mathfrak{P}_v \widehat{W}) \\
=&cos^2\theta_1 g_1(\nabla_{\widehat{Z}} \mathfrak{P}_y \widehat{W}, \widehat{U}) - g_1(\mathcal{T}_{\widehat{Z}} \beta \alpha \mathfrak{P}_y \widehat{W}, \widehat{U}) + g_1(\mathcal{T}_{\widehat{Z}} \beta \alpha \mathfrak{P}_y \widehat{W}, \phi \mathfrak{P}_g \widehat{U}) \\
&+ g_1(\mathcal{T}_{\widehat{Z}} \beta \mathfrak{P}_y \widehat{W}, \alpha \mathfrak{P}_v \widehat{U} - \frac{1}{\lambda^2} g_2((\nabla \Psi_*)(\beta \mathfrak{P}_y \widehat{W}, \widehat{Z}), \Psi_* \mathfrak{P}_v \widehat{U})
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}&\lambda^{-2}\{g_2(\nabla^{\Psi}_{\widehat{Z}}\Psi_*\beta\alpha\mathfrak{P}_y\widehat{W},\Psi_*\widehat{X})\}-g_1(\alpha\mathfrak{P}_y\widehat{W},\widehat{Z})\eta(\widehat{X})\\ =&\frac{1}{\lambda^2}g_2((\nabla\Psi_*)(\widehat{Z},\beta\alpha\mathfrak{P}_y\widehat{W}),\Psi_*\widehat{X}))-\frac{1}{\lambda^2}g_2((\nabla\Psi_*)(\widehat{Z},\beta\alpha\mathfrak{P}_y\widehat{W}),\Psi_*\mathbb{B}\widehat{X}))\\ &+cos^2\theta_1 g_1(\nabla_{\widehat{Z}}\mathfrak{P}_y\widehat{W},\widehat{X})+g_1(\mathcal{T}_{\widehat{Z}}\beta\alpha\mathfrak{P}_y\widehat{W},\mathbb{C}\widehat{X})-g_2(\nabla^{\Psi}_{\widehat{Z}}\Psi_*\beta\alpha\mathfrak{P}_y\widehat{W},\Psi_*\mathbb{B}\widehat{X}),\end{aligned}\tag{4.4}$$

for any $\widehat{Z},\widehat{W}\in\Gamma(\mathfrak{D}_{\theta_g})$, $\widehat{U}\in\Gamma(\mathfrak{D}\oplus\mathfrak{D}_{\theta_y})$ *and* $\widehat{X}\in\Gamma(ker\Psi_*)^{\perp}$.

Proof. By using Eqs (2.2), (2.5), (2.14) and (3.2), we get

$$g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{U})=g_1(\nabla_{\widehat{Z}}\beta\mathfrak{P}_y\widehat{W},\phi(\mathfrak{P}_g\widehat{U}+\mathfrak{P}_v\widehat{U}))-g_1(\phi\nabla_{\widehat{Z}}\alpha\mathfrak{P}_y\widehat{W},\widehat{U}),$$

for $\widehat{Z},\widehat{W}\in\Gamma(\mathfrak{D}_{\theta_g})$ and $\widehat{U}\in\Gamma(\mathfrak{D}\oplus\mathfrak{D}_{\theta_y})$. Again using Eqs (2.2), (2.5), (2.10), (2.14) and (3.2) with Lemma 3.2, we may write

$$\begin{aligned}g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{U})=&cos^2\theta_1 g_1(\nabla_{\widehat{Z}}\mathfrak{P}_y\widehat{W},\widehat{U})-g_1(\mathcal{T}_{\widehat{Z}}\beta\alpha\mathfrak{P}_y\widehat{W},\widehat{U})+g_1(\mathcal{T}_{\widehat{Z}}\beta\alpha\mathfrak{P}_y\widehat{W},\phi\mathfrak{P}_g\widehat{U})\\ &+g_1(\mathcal{T}_{\widehat{Z}}\beta\mathfrak{P}_y\widehat{W},\alpha\mathfrak{P}_v\widehat{U})+g_1(\mathcal{H}\nabla_{\widehat{Z}}\beta\mathfrak{P}_y\widehat{W},\beta\mathfrak{P}_v\widehat{U}).\end{aligned}$$

Since, Ψ is conformal, using Lemma 2.1 with Eqs (2.6) and (2.15), we have

$$\begin{aligned}g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{U})=&cos^2\theta_1 g_1(\nabla_{\widehat{Z}}\mathfrak{P}_y\widehat{W},\widehat{U})-g_1(\mathcal{T}_{\widehat{Z}}\beta\alpha\mathfrak{P}_y\widehat{W},\widehat{U})+g_1(\mathcal{T}_{\widehat{Z}}\beta\alpha\mathfrak{P}_y\widehat{W},\phi\mathfrak{P}_g\widehat{U})\\ &+g_1(\mathcal{T}_{\widehat{Z}}\beta\mathfrak{P}_y\widehat{W},\alpha\mathfrak{P}_v\widehat{U})-\frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\widehat{Z}}\Psi_*\beta\mathfrak{P}_y\widehat{W},\Psi_*\beta\mathfrak{P}_v\widehat{W})\\ &-\frac{1}{\lambda^2}g_2((\nabla\Psi_*)(\beta\mathfrak{P}_y\widehat{W},\widehat{Z}),\Psi_*\mathfrak{P}_v\widehat{U}).\end{aligned}\tag{4.5}$$

On the other hand, for $\widehat{Z},\widehat{W}\in\Gamma(\mathfrak{D}_{\theta_g})$ and $\widehat{X}\in\Gamma(ker\Psi_*)^{\perp}$, with using Eqs (2.2), (2.5), (2.14) and (3.2), we get

$$g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{X})=g_1(\nabla_{\widehat{Z}}\alpha\mathfrak{P}_y\widehat{W},\phi\widehat{X})+g_1(\nabla_{\widehat{Z}}\beta\mathfrak{P}_y\widehat{W},\phi\widehat{X}).$$

From Lemma 3.2 with Eqs (2.10) and (3.5), the above equation takes the form

$$\begin{aligned}g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{X})=&cos^2\theta_1 g_1(\nabla_{\widehat{Z}}\mathfrak{P}_y\widehat{W},\widehat{X})-g_1(\mathcal{H}\nabla_{\widehat{Z}}\beta\alpha\mathfrak{P}_y\widehat{W},\widehat{X})+g_1(\alpha\mathfrak{P}_y\widehat{W},\widehat{Z})\eta(\widehat{X})\\ &+g_1(\mathcal{T}_{\widehat{Z}}\beta\alpha\mathfrak{P}_y\widehat{W},\mathbb{C}\widehat{X})+g_1(\mathcal{H}\nabla_{\widehat{Z}}\beta\alpha\mathfrak{P}_y\widehat{W},\mathbb{B}\widehat{X}).\end{aligned}$$

Since Ψ is conformal and from Eqs (2.6) and (2.15), we have

$$\begin{aligned}g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{X})=&cos^2\theta_1 g_1(\nabla_{\widehat{Z}}\mathfrak{P}_y\widehat{W},\widehat{X})+g_1(\mathcal{T}_{\widehat{Z}}\beta\alpha\mathfrak{P}_y\widehat{W},\mathbb{C}\widehat{X})+g_1(\alpha\mathfrak{P}_y\widehat{W},\widehat{Z})\eta(\widehat{X})\\ &+\frac{1}{\lambda^2}g_2((\nabla\Psi_*)(\beta\alpha\mathfrak{P}_y\widehat{W},\widehat{Z}),\Psi_*\widehat{X})-\frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\widehat{Z}}\Psi_*\beta\alpha\mathfrak{P}_y\widehat{W},\Psi_*\widehat{X})\\ &-\frac{1}{\lambda^2}g_2((\nabla\Psi_*)(\beta\alpha\mathfrak{P}_y\widehat{W},\widehat{Z}),\Psi_*\mathbb{B}\widehat{X})+\frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\widehat{Z}}\Psi_*\beta\alpha\mathfrak{P}_y\widehat{W},\Psi_*\mathbb{B}\widehat{X}),\end{aligned}$$

from which we get the result. □

In the following theorem, we study the necessary and sufficient conditions for slant distribution $\mathfrak{D}_{\theta_y}$ to be totally geodesic.

Theorem 4.6. *Let* $\Psi : (\bar{Q}_1, \phi, \xi, \eta, g_1) \to (\bar{Q}_2, g_2)$ *be a* $\mathcal{QBSC}$ $\xi^{\perp}$*-submersion, where* $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ *a Sasakian manifold and* $(\bar{Q}_2, g_2)$ *a RM. Then slant distribution* $\mathfrak{D}_{\theta_y}$ *defines totally geodesic foliation on* $\bar{Q}_1$ *if and only if*

$$\begin{aligned}
&\frac{1}{\lambda^2} g_2(\nabla^{\Psi}_{\widehat{Z}} \Psi_* \beta \mathfrak{P}_y \widehat{W}, \Psi_* \beta \mathfrak{P}_v \widehat{W}) \\
=& cos^2\theta_1 g_1(\nabla_{\widehat{Z}} \mathfrak{P}_y \widehat{W}, \widehat{V}) - g_1(\mathcal{T}_{\widehat{Z}} \beta\alpha \mathfrak{P}_y \widehat{W}, \widehat{V}) + g_1(\mathcal{T}_{\widehat{Z}} \beta\alpha \mathfrak{P}_y \widehat{W}, \phi \mathfrak{P}_g \widehat{V}) \\
&+ g_1(\mathcal{T}_{\widehat{Z}} \beta \mathfrak{P}_y \widehat{W}, \alpha \mathfrak{P}_v \widehat{V} - \frac{1}{\lambda^2} g_2((\nabla \Psi_*)(\beta \mathfrak{P}_y \widehat{W}, \widehat{Z}), \Psi_* \mathfrak{P}_v \widehat{V})
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
&\lambda^{-2}\{g_2(\nabla^{\Psi}_{\widehat{Z}} \Psi_* \beta\alpha \mathfrak{P}_v \widehat{W}, \Psi_* \widehat{Y}) - g_2(\nabla^{\Psi}_{\widehat{Z}} \Psi_* \beta\alpha \mathfrak{P}_v \widehat{W}, \Psi_* \mathbb{B} \widehat{Y})\} \\
=& \frac{1}{\lambda^2} g_2((\nabla \Psi_*)(\widehat{Z}, \beta\alpha \mathfrak{P}_v \widehat{W}), \Psi_* \widehat{Y}) - \frac{1}{\lambda^2} g_2((\nabla \Psi_*)(\widehat{Z}, \beta\alpha \mathfrak{P}_v \widehat{W}), \Psi_* \mathbb{B} \widehat{Y}) \\
&+ cos^2\theta_2 g_1(\nabla_{\widehat{Z}} \mathfrak{P}_v \widehat{W}, \widehat{Y}) + g_1(\mathcal{T}_{\widehat{Z}} \beta\alpha \mathfrak{P}_v \widehat{W}, \mathbb{C} \widehat{Y}),
\end{aligned} \tag{4.7}$$

for any $\widehat{Z}, \widehat{W} \in \Gamma(\mathfrak{D}_{\theta_y})$, $\widehat{V} \in \Gamma(\mathfrak{D} \oplus \mathfrak{D}_{\theta_g})$ *and* $\widehat{Y} \in \Gamma(ker\Psi_*)^{\perp}$.

Proof. The proof of above theorem is similar to the proof of Theorem 4.5. □

Since, Ψ is $\mathcal{QBSC}$ $\xi^{\perp}$-submersion, its vertical and horizontal distribution are $(ker\Psi_*)$ and $(ker\Psi_*)^{\perp}$, respectively. Now, we examine the conditions under which distributions defines totally geodesic foliation on $\bar{Q}_1$. With regards to the totally geodesicness of vertical distribution, we have

Theorem 4.7. *Let* $\Psi : (\bar{Q}_1, \phi, \xi, \eta, g_1) \to (\bar{Q}_2, g_2)$ *be a* $\mathcal{QBSC}$ $\xi^{\perp}$*-submersion, where* $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ *a Sasakian manifold and* $(\bar{Q}_2, g_2)$ *a RM. Then* $ker\Psi_*$ *defines totally geodesic foliation on* $\bar{Q}_1$ *if and only if*

$$\begin{aligned}
&\frac{1}{\lambda^2}\{g_2(\nabla^{\Psi}_{\widehat{U}} \Psi_* \beta\alpha \mathfrak{P}_y \widehat{V} + \nabla^{\Psi}_{\widehat{U}} \Psi_* \beta\alpha \mathfrak{P}_v \widehat{V}, \Psi_* \widehat{X})\} \\
=& g_1(\mathcal{T}_{\widehat{U}} \mathfrak{P}_g \widehat{V} + cos^2\theta_1 \mathcal{T}_{\widehat{U}} \mathfrak{P}_y \widehat{V} + cos^2\theta_2 \mathcal{T}_{\widehat{U}} \mathfrak{P}_v \widehat{V}, \widehat{X}) + g_1(\mathcal{T}_{\widehat{U}} \beta \widehat{V}, \mathbb{C} \widehat{X}) \\
&+ \frac{1}{\lambda^2}\{g_2((\nabla \Psi_*)(\widehat{U}, \beta\alpha \mathfrak{P}_y \widehat{V}) - (\nabla \Psi_*)(\widehat{U}, \beta\alpha \mathfrak{P}_v \widehat{V}), \Psi_* \widehat{X})\} \\
&+ \frac{1}{\lambda^2}\{g_2(\nabla^{\Psi}_{\widehat{U}} \Psi_* \beta \widehat{V} - (\nabla \Psi_*)(\widehat{U}, \beta \widehat{V}), \Psi_* \mathbb{B} \widehat{X})\} \\
&- \eta(\widehat{X}) g_1(\phi \widehat{U} + \alpha \widehat{U}, \widehat{V} - \mathfrak{P}_g \widehat{V}),
\end{aligned} \tag{4.8}$$

for any $\widehat{U}, \widehat{V} \in \Gamma(ker\Psi_*)$ *and* $\widehat{X} \in \Gamma(ker\Psi_*)^{\perp}$.

Proof. For any $\widehat{U}, \widehat{V} \in \Gamma(ker\Psi_*)$ and $\widehat{X} \in \Gamma(ker\Psi_*)^{\perp}$ with using Eqs (2.2), (2.5), (2.14) with decomposition (3.1), we get

$$g_1(\nabla_{\widehat{U}} \widehat{V}, \widehat{X}) = g_1(\nabla_{\widehat{U}} \phi \mathfrak{P}_g \widehat{V}, \phi \widehat{X}) + g_1(\nabla_{\widehat{U}} \phi \mathfrak{P}_y \widehat{V}, \phi \widehat{X}) + g_1(\nabla_{\widehat{U}} \phi \mathfrak{P}_v \widehat{V}, \phi \widehat{X}) + g_1(\alpha \widehat{U}, \widehat{V} - \mathfrak{P}_g \widehat{V}) \eta(\widehat{X}).$$

On using Eq (3.2) with Lemmas 3.2 and 3.3, we have

$$\begin{aligned}
g_1(\nabla_{\widehat{U}} \widehat{V}, \widehat{X}) =& g_1(\nabla_{\widehat{U}} \mathfrak{P}_g \widehat{V}, \widehat{X}) + cos^2\theta_1 g_1(\nabla_{\widehat{U}} \mathfrak{P}_y \widehat{V}, \widehat{X}) + cos^2\theta_2 g_1(\nabla_{\widehat{U}} \mathfrak{P}_v \widehat{V}, \widehat{X}) \\
&+ g_1(\nabla_{\widehat{U}} \beta \mathfrak{P}_y \widehat{V}, \phi \widehat{X}) - g_1(\nabla_{\widehat{U}} \beta\alpha \mathfrak{P}_y \widehat{V}, \widehat{X}) - g_1(\nabla_{\widehat{U}} \beta\alpha \mathfrak{P}_v \widehat{V}, \widehat{X}) \\
&+ g_1(\nabla_{\widehat{U}} \beta \mathfrak{P}_v \widehat{V}, \phi \widehat{X}) - \eta(\widehat{X}) g_1(\phi \widehat{U} + \alpha \widehat{U}, \widehat{V} - \mathfrak{P}_g \widehat{V}).
\end{aligned}$$

From Eqs (2.9), (2.10) and (3.5), we may yields

$$g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X}) = g_1(\mathcal{T}_{\widehat{U}}\mathfrak{P}_g\widehat{V} + cos^2\theta_1\mathcal{T}_{\widehat{U}}\mathfrak{P}_y\widehat{V} + cos^2\theta_2\mathcal{T}_{\widehat{U}}\mathfrak{P}_v\widehat{V},\widehat{X})$$
$$- g_1(\mathcal{H}\nabla_{\widehat{U}}\beta\alpha\mathfrak{P}_y\widehat{V} + \mathcal{H}\nabla_{\widehat{U}}\beta\alpha\mathfrak{P}_v\widehat{V},\widehat{X}) + g_1(\mathcal{T}_{\widehat{U}}\beta\mathfrak{P}_y\widehat{V} + \mathcal{T}_{\widehat{U}}\beta\mathfrak{P}_v\widehat{V},\mathbb{C}\widehat{X})$$
$$+ g_1(\mathcal{H}\nabla_{\widehat{U}}\beta\mathfrak{P}_y\widehat{V} + \mathcal{H}\nabla_{\widehat{U}}\beta\mathfrak{P}_v\widehat{V},\mathbb{B}\widehat{X}) - \eta(\widehat{X})g_1(\phi\widehat{U} + \alpha\widehat{U},\widehat{V} - \mathfrak{P}_g\widehat{V}).$$

From decomposition (3.1), the above equation takes the form

$$g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X}) = g_1(\mathcal{T}_{\widehat{U}}\mathfrak{P}_g\widehat{V} + cos^2\theta_1\mathcal{T}_{\widehat{U}}\mathfrak{P}_y\widehat{V} + cos^2\theta_2\mathcal{T}_{\widehat{U}}\mathfrak{P}_v\widehat{V},\widehat{X}) + g_1(\mathcal{T}_{\widehat{U}}\beta\widehat{V},\mathbb{C}\widehat{X})$$
$$- g_1(\mathcal{H}\nabla_{\widehat{U}}\beta\alpha\mathfrak{P}_y\widehat{V} + \mathcal{H}\nabla_{\widehat{U}}\beta\alpha\mathfrak{P}_v\widehat{V},\widehat{X}) + g_1(\mathcal{H}\nabla_{\widehat{U}}\beta\widehat{V},\mathbb{B}\widehat{X})$$
$$- \eta(\widehat{X})g_1(\phi\widehat{U} + \alpha\widehat{U},\widehat{V} - \mathfrak{P}_g\widehat{V}).$$

Using the conformality of Ψ with Eqs (2.6) and (2.15), we have

$$g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X}) = g_1(\mathcal{T}_{\widehat{U}}\mathfrak{P}_g\widehat{V} + cos^2\theta_1\mathcal{T}_{\widehat{U}}\mathfrak{P}_y\widehat{V} + cos^2\theta_2\mathcal{T}_{\widehat{U}}\mathfrak{P}_v\widehat{V},\widehat{X}) + g_1(\mathcal{T}_{\widehat{U}}\beta\widehat{V},\mathbb{C}\widehat{X})$$
$$+ \frac{1}{\lambda^2}\{g_2((\nabla\Psi_*)(\widehat{U},\beta\alpha\mathfrak{P}_y\widehat{V}) - (\nabla\Psi_*)(\widehat{U},\beta\alpha\mathfrak{P}_v\widehat{V}),\Psi_*\widehat{X})\}$$
$$- \frac{1}{\lambda^2}\{g_2(\nabla^{\Psi}_{\widehat{U}}\Psi_*\beta\alpha\mathfrak{P}_y\widehat{V} + \nabla^{\Psi}_{\widehat{U}}\Psi_*\beta\alpha\mathfrak{P}_v\widehat{V},\Psi_*\widehat{X})\}$$
$$+ \frac{1}{\lambda^2}\{g_2(\nabla^{\Psi}_{\widehat{U}}\Psi_*\beta\widehat{V} - (\nabla\Psi_*)(\widehat{U},\beta\widehat{V}),\Psi_*\mathbb{B}\widehat{X})\}$$
$$- \eta(\widehat{X})g_1(\phi\widehat{U} + \alpha\widehat{U},\widehat{V} - \mathfrak{P}_g\widehat{V}).$$

This completes the proof of the theorem. □

We can now talk about the geometry of leaves of horizontal distribution. The following theorem presents the necessary and sufficient condition under which horizontal distribution defines totally geodesic foliation on $\bar{Q}_1$.

Theorem 4.8. *Let Ψ be a $\mathcal{QBSC}$ $\xi^\perp$-submersion from Sasakian manifold $(\bar{Q}_1,\phi,\xi,\eta,g_1)$ onto a RM $(\bar{Q}_2,g_2)$,. Then $(ker\Psi_*)^\perp$ defines totally geodesic foliation on $\bar{Q}_1$ if and only if*

$$\begin{aligned}
&- \frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\widehat{X}}\Psi_*\mathbb{B}\widehat{Y},\Psi_*\beta Z) + \frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\widehat{X}}\Psi_*\widehat{Y},\Psi_*\beta\alpha\mathfrak{P}_y Z)\\
&= \cos^2\theta_1\{\eta(\widehat{Y})g_1(\mathbb{C}\widehat{X},\mathfrak{P}_y Z) + g_1(\mathcal{A}_{\widehat{X}}\widehat{Y},\mathfrak{P}_y Z)\} + \cos^2\theta_2\{\eta(\widehat{Y})g_1(\mathbb{C}\widehat{X},\mathfrak{P}_v Z) + \mathcal{A}_{\widehat{X}}\widehat{Y},\mathfrak{P}_v Z\}\\
&+ g_1(\mathcal{V}\nabla_{\widehat{X}}\mathbb{C}\widehat{Y},\alpha\mathfrak{P}_g Z) + g_1(\mathcal{A}_{\widehat{X}}\mathbb{B}\widehat{Y},\alpha\mathfrak{P}_g Z) + g_1(\mathcal{A}_{\widehat{X}}\mathbb{C}\widehat{Y},\beta Z) + \eta(\widehat{Y})g_1(\phi Z,\widehat{X})\\
&+ \frac{1}{\lambda^2}g_2(\widehat{X}(\ln\lambda)\Psi_*\mathbb{B}\widehat{Y} + \mathbb{B}\widehat{Y}(\ln\lambda)\Psi_*\widehat{X} - g_1(\widehat{X},\mathbb{B}\widehat{Y})\Psi_*(G\ln\lambda),\Psi_*\beta Z)\\
&+ \frac{1}{\lambda^2}g_2(\widehat{X}(\ln\lambda)\Psi_*\widehat{Y} + \widehat{Y}(\ln\lambda)\Psi_*\widehat{X} - g_1(\widehat{X},\widehat{Y})\Psi_*(G\ln\lambda),\Psi_*\beta\alpha\mathfrak{P}_y Z)\\
&+ \frac{1}{\lambda^2}g_2(\widehat{X}(\ln\lambda)\Psi_*\widehat{Y} + \widehat{Y}(\ln\lambda)\Psi_*\widehat{X} - g_1(\widehat{X},\widehat{Y})\Psi_*(G\ln\lambda),\Psi_*\beta\alpha\mathfrak{P}_v Z)\\
&+ \frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\widehat{X}}\Psi_*\widehat{Y},\Psi_*\beta\alpha\mathfrak{P}_v Z),
\end{aligned} \tag{4.9}$$

for any $\widehat{X},\widehat{Y}\in\Gamma(ker\Psi_*)^\perp$ and $\widehat{Z}\in\Gamma(ker\Psi_*)$.

Proof. For any $\widehat{X}, \widehat{Y} \in \Gamma(ker\Psi_*)^\perp$ and $\widehat{Z} \in \Gamma(ker\Psi_*)$ with using Eqs (2.2), (2.5) and (2.14) with decomposition (3.1), we get

$$g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{Z}) = g_1(\nabla_{\widehat{X}}\phi\widehat{Y},\phi\mathfrak{P}_g\widehat{Z}) + g_1(\nabla_{\widehat{X}}\phi\widehat{Y},\phi\mathfrak{P}_y\widehat{Z}) + g_1(\nabla_{\widehat{X}}\phi\widehat{Y},\phi\mathfrak{P}_v\widehat{Z}).$$

From Eqs (2.11) and (3.2) with Lemma 3.2, we have

$$\begin{aligned}
&g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{Z})\\
=&g_1(\mathcal{V}\nabla_{\widehat{X}}\mathbb{C}\widehat{Y},\alpha\mathfrak{P}_gZ) + g_1(\mathcal{A}_{\widehat{X}}\mathbb{B}\widehat{Y},\alpha\mathfrak{P}_gZ) + g_1(\phi\nabla_{\widehat{X}}\phi\widehat{Y},\phi\alpha\mathfrak{P}_yZ)\\
&+ g_1(\nabla_{\widehat{X}}\mathbb{C}\widehat{Y},\beta\mathfrak{P}_yZ) + g_1(\nabla_{\widehat{X}}\mathbb{B}\widehat{Y},\beta\mathfrak{P}_yZ) + g_1(\phi\nabla_{\widehat{X}}\phi\widehat{Y},\phi\alpha\mathfrak{P}_vZ)\\
&+ g_1(\nabla_{\widehat{X}}\mathbb{C}\widehat{Y},\beta\mathfrak{P}_vZ) + g_1(\nabla_{\widehat{X}}\mathbb{B}\widehat{Y},\beta\mathfrak{P}_vZ) + \eta(\widehat{Y})g_1(\widehat{X},\phi Z).
\end{aligned}$$

Since $\beta\mathfrak{P}_yZ + \beta\mathfrak{P}_vZ = \beta Z$ and with using the Eqs (2.12) and (3.2), we get

$$\begin{aligned}
&g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{Z})\\
=&g_1(\mathcal{V}\nabla_{\widehat{X}}\mathbb{C}\widehat{Y},\alpha\mathfrak{P}_gZ) + g_1(\mathcal{A}_{\widehat{X}}\mathbb{B}\widehat{Y},\alpha\mathfrak{P}_gZ) + g_1(\mathcal{A}_{\widehat{X}}\mathbb{C}\widehat{Y},\beta Z) + \eta(\widehat{Y})g_1(\phi Z,\widehat{X})\\
&+ g_1(\mathcal{H}\nabla_{\widehat{X}}\mathbb{B}\widehat{Y},\beta Z) - g_1(\mathcal{H}\nabla_{\widehat{X}}\widehat{Y},\beta\alpha\mathfrak{P}_yZ) - g_1(\mathcal{H}\nabla_{\widehat{X}}\widehat{Y},\beta\alpha\mathfrak{P}_vZ)\\
&+ \cos^2\theta_1\{\eta(\widehat{Y})g_1(\mathbb{C}\widehat{X},\mathfrak{P}_yZ) + g_1(\mathcal{A}_{\widehat{X}}\widehat{Y},\mathfrak{P}_yZ)\}\\
&+ \cos^2\theta_2\{\eta(\widehat{Y})g_1(\mathbb{C}\widehat{X},\mathfrak{P}_vZ) + g_1(\mathcal{A}_{\widehat{X}}\widehat{Y},\mathfrak{P}_vZ)\}.
\end{aligned}$$

From formula (2.6) and (2.15), we yields that

$$\begin{aligned}
&g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{Z})\\
=&g_1(\mathcal{V}\nabla_{\widehat{X}}\mathbb{C}\widehat{Y},\alpha\mathfrak{P}_gZ) + g_1(\mathcal{A}_{\widehat{X}}\mathbb{B}\widehat{Y},\alpha\mathfrak{P}_gZ) + g_1(\mathcal{A}_{\widehat{X}}\mathbb{C}\widehat{Y},\beta Z) + \eta(\widehat{Y})g_1(\phi Z,\widehat{X})\\
&+ \frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\widehat{X}}\Psi_*\mathbb{B}\widehat{Y},\Psi_*\beta Z) - \frac{1}{\lambda^2}g_2((\nabla\Psi_*)(\widehat{X},\mathbb{B}\widehat{Y}),\Psi_*\beta Z)\\
&- \frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\widehat{X}}\Psi_*\widehat{Y},\Psi_*\beta\alpha\mathfrak{P}_yZ) + \frac{1}{\lambda^2}g_2((\nabla\Psi_*)(\widehat{X},\widehat{Y}),\Psi_*\beta\alpha\mathfrak{P}_yZ)\\
&- \frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\widehat{X}}\Psi_*\widehat{Y},\Psi_*\beta\alpha\mathfrak{P}_vZ) + \frac{1}{\lambda^2}g_2((\nabla\Psi_*)(\widehat{X},\widehat{Y}),\Psi_*\beta\alpha\mathfrak{P}_vZ)\\
&+ \cos^2\theta_1\{\eta(\widehat{Y})g_1(\mathbb{C}\widehat{X},\mathfrak{P}_yZ) + g_1(\mathcal{A}_{\widehat{X}}\widehat{Y},\mathfrak{P}_yZ)\}\\
&+ \cos^2\theta_2\{\eta(\widehat{Y})g_1(\mathbb{C}\widehat{X},\mathfrak{P}_vZ) + g_1(\mathcal{A}_{\widehat{X}}\widehat{Y},\mathfrak{P}_vZ)\}.
\end{aligned}$$

Since Ψ is conformal submersion, then we finally get

$$
\begin{aligned}
& g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{Z}) \\
=& \cos^2\theta_1\{\eta(\widehat{Y})g_1(\mathbb{C}\widehat{X},\mathfrak{P}_yZ)+g_1(\mathcal{A}_{\widehat{X}}\widehat{Y},\mathfrak{P}_yZ)\}+\cos^2\theta_2\{\eta(\widehat{Y})g_1(\mathbb{C}\widehat{X},\mathfrak{P}_vZ)+\mathcal{A}_{\widehat{X}}\widehat{Y},\mathfrak{P}_vZ\} \\
&+g_1(\mathcal{V}\nabla_{\widehat{X}}\mathbb{C}\widehat{Y},\alpha\mathfrak{P}_gZ)+g_1(\mathcal{A}_{\widehat{X}}\mathbb{B}\widehat{Y},\alpha\mathfrak{P}_gZ)+g_1(\mathcal{A}_{\widehat{X}}\mathbb{C}\widehat{Y},\beta Z)+\eta(\widehat{Y})g_1(\phi Z,\widehat{X}) \\
&+\frac{1}{\lambda^2}g_2(\widehat{X}(\ln\lambda)\Psi_*\mathbb{B}\widehat{Y}+\mathbb{B}\widehat{Y}(\ln\lambda)\Psi_*\widehat{X}-g_1(\widehat{X},\mathbb{B}\widehat{Y})\Psi_*(G\ln\lambda),\Psi_*\beta Z) \\
&+\frac{1}{\lambda^2}g_2(\widehat{X}(\ln\lambda)\Psi_*\widehat{Y}+\widehat{Y}(\ln\lambda)\Psi_*\widehat{X}-g_1(\widehat{X},\widehat{Y})\Psi_*(G\ln\lambda),\Psi_*\beta\alpha\mathfrak{P}_yZ) \\
&+\frac{1}{\lambda^2}g_2(\widehat{X}(\ln\lambda)\Psi_*\widehat{Y}+\widehat{Y}(\ln\lambda)\Psi_*\widehat{X}-g_1(\widehat{X},\widehat{Y})\Psi_*(G\ln\lambda),\Psi_*\beta\alpha\mathfrak{P}_vZ) \\
&+\frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\widehat{X}}\Psi_*\mathbb{B}\widehat{Y},\Psi_*\beta Z)-\frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\widehat{X}}\Psi_*\widehat{Y},\Psi_*\beta\alpha\mathfrak{P}_yZ) \\
&+\frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\widehat{X}}\Psi_*\widehat{Y},\Psi_*\beta\alpha\mathfrak{P}_vZ).
\end{aligned}
$$

This completes the proof of theorem. □

We currently have a few prerequisites that must be met in order for $\mathcal{QBSC}$ $\xi^\perp$-submersion $\Psi:\bar{Q}_1\to\bar{Q}_2$ to be a totally geodesic map. In this regard, we offer the subsequent finding.

Theorem 4.9. *Let Ψ be a $\mathcal{QBSC}$ $\xi^\perp$-submersion from Sasakian manifold $(\bar{Q}_1,\phi,\xi,\eta,g_1)$ onto a RM $(\bar{Q}_2,g_2)$. Then $\Psi:(\bar{Q}_1,\phi,\xi,\eta,g_1)\to(\bar{Q}_1,g_2)$ is totally geodesic map if and only if*

$$
\begin{aligned}
&\Psi_*\{cos^2\theta_1\nabla_{\widehat{U}}\mathfrak{P}_y\widehat{V}+cos^2\theta_2\nabla_{\widehat{U}}\mathfrak{P}_v\widehat{V}-\nabla_{\widehat{U}}\beta\alpha\mathfrak{P}_y\widehat{V}-\nabla_{\widehat{U}}\beta\alpha\mathfrak{P}_v\widehat{V}-g_1(\phi\widehat{U},\mathfrak{P}_g\widehat{V})\xi\} \\
=&\Psi_*\{\mathbb{B}(\mathcal{H}\nabla_{\widehat{U}}\beta\mathfrak{P}_y\widehat{V}+\mathcal{H}\nabla_{\widehat{U}}\beta\mathfrak{P}_v\widehat{V}+\mathcal{T}_{\widehat{U}}\alpha\mathfrak{P}_g\widehat{V})\}+\Psi_*\{\beta(\mathcal{T}_{\widehat{U}}\beta\mathfrak{P}_y\widehat{V}+\mathcal{T}_{\widehat{U}}\beta\mathfrak{P}_v\widehat{V}+\mathcal{V}\nabla_{\widehat{U}}\alpha\mathfrak{P}_g\widehat{V})\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\Psi_*\{cos^2\theta_1\nabla_{\widehat{X}}\mathfrak{P}_y\widehat{U}+cos^2\theta_2\nabla_{\widehat{X}}\mathfrak{P}_v\widehat{U}-\nabla_{\widehat{X}}\beta\alpha\mathfrak{P}_y\widehat{U}-\nabla_{\widehat{X}}\beta\alpha\mathfrak{P}_v\widehat{U}\} \\
=&\Psi_*\{\mathbb{B}(\mathcal{A}_{\widehat{X}}\alpha\mathfrak{P}_g\widehat{U}+\mathcal{H}\nabla_{\widehat{X}}\beta\mathfrak{P}_y\widehat{U}+\mathcal{H}\nabla_{\widehat{X}}\beta\mathfrak{P}_v\widehat{U})\}-g_1(P\widehat{X},\widehat{U})\Psi_*\xi \\
&+\Psi_*\{\beta(\mathcal{V}\nabla_{\widehat{X}}\alpha\mathfrak{P}_g\widehat{U}+\mathcal{A}_{\widehat{X}}\beta\mathfrak{P}_y\widehat{U}+\mathcal{A}_{\widehat{X}}\beta\mathfrak{P}_v\widehat{U})\},
\end{aligned}
$$

for any $\widehat{U},\widehat{V}\in\Gamma(ker\Psi_*)$ and $\widehat{X},\widehat{Y}\in\Gamma(ker\Psi_*)^\perp$.

Proof. Now, using Eqs (2.1), (2.5), (2.14) and (2.15). we can write

$$(\nabla\Psi_*)(\widehat{U},\widehat{V})=\Psi_*\{-\eta(\nabla_{\widehat{U}}\widehat{V})\xi+\phi\nabla_{\widehat{U}}\phi\widehat{V}\},$$

for any $\widehat{U},\widehat{V}\in\Gamma(ker\Psi_*)$. From decomposition (3.1) and Eq (3.2), we have

$$(\nabla\Psi_*)(\widehat{U},\widehat{V})=\Psi_*\{\phi\nabla_{\widehat{U}}\alpha\mathfrak{P}_g\widehat{V}+\phi\nabla_{\widehat{U}}\alpha\mathfrak{P}_y\widehat{V}+\phi\nabla_{\widehat{U}}\beta\mathfrak{P}_y\widehat{V}+\phi\nabla_{\widehat{U}}\alpha\mathfrak{P}_v\widehat{V}+\phi\nabla_{\widehat{U}}\beta\mathfrak{P}_v\widehat{V}-g_1(\phi\widehat{U},\widehat{V})\xi\}.$$

By using Eqs (2.9) and (2.10), the above equation takes the form

$$
\begin{aligned}
(\nabla\Psi_*)(\widehat{U},\widehat{V})=&\Psi_*\{\phi\mathcal{T}_{\widehat{U}}\alpha\mathfrak{P}_g\widehat{V}+\phi\mathcal{V}\nabla_{\widehat{U}}\alpha\mathfrak{P}_g\widehat{V}\}+\Psi_*(\nabla_{\widehat{U}}\phi\alpha\mathfrak{P}_y\widehat{V}) \\
&+\Psi_*(\phi\mathcal{T}_{\widehat{U}}\beta\mathfrak{P}_y\widehat{V}+\phi\mathcal{H}\nabla_{\widehat{U}}\beta\mathfrak{P}_y\widehat{V})+\Psi_*(\nabla_{\widehat{U}}\phi\alpha\mathfrak{P}_y\widehat{V}) \\
&+\Psi_*\{\phi\mathcal{T}_{\widehat{U}}\beta\mathfrak{P}_v\widehat{V}+\phi\mathcal{H}\nabla_{\widehat{U}}\beta\mathfrak{P}_v\widehat{V}+g_1(\phi\widehat{U},\mathfrak{P}_g\widehat{V})\xi\}.
\end{aligned}
$$

Since Ψ is conformal submersion, by using Lemmas 3.2 and 3.3 with Eq (3.2), we finally get

$$\begin{aligned}(\nabla\Psi_*)(\widehat{U},\widehat{V}) = \Psi * \{&\mathbb{B}(\mathcal{H}\nabla_{\widehat{U}}\beta\mathfrak{P}_y\widehat{V} + \mathcal{H}\nabla_{\widehat{U}}\beta\mathfrak{P}_v\widehat{V} + \mathcal{T}_{\widehat{U}}\alpha\mathfrak{P}_g\widehat{V})\\ &+ \beta(\mathcal{V}\nabla_{\widehat{U}}\alpha\mathfrak{P}_g\widehat{V} + \mathcal{T}_{\widehat{U}}\alpha\mathfrak{P}_y\widehat{V} + \mathcal{T}_{\widehat{U}}\alpha\mathfrak{P}_v\widehat{V})\}\\ &- \Psi_*\{cos^2\theta_1\nabla_{\widehat{U}}\mathfrak{P}_y\widehat{V} + cos^2\theta_2\nabla_{\widehat{U}}\mathfrak{P}_v\widehat{V} - \nabla_{\widehat{U}}\beta\alpha\mathfrak{P}_y\widehat{V}\\ &- \nabla_{\widehat{U}}\beta\alpha\mathfrak{P}_v\widehat{V} - g_1(\phi\widehat{U},\mathfrak{P}_g\widehat{V})\xi\}.\end{aligned}$$

From this, the (i) part of theorem proved. On the other hand, for any $\widehat{U} \in \Gamma(ker\Psi_*)$ and $\widehat{X} \in \Gamma(ker\Psi_*)^\perp$ with using Eqs (2.1), (2.5), (2.14) and (2.15), we can write

$$(\nabla\Psi_*)(\widehat{X},\widehat{U}) = \Psi_*(\phi\nabla_{\widehat{X}}\phi\widehat{U} - \eta(\nabla_{\widehat{X}}\widehat{U})\xi).$$

On using decomposition (3.1) with Eq (3.2), we have

$$(\nabla\Psi_*)(\widehat{X},\widehat{U}) = \Psi_*\{\phi(\nabla_{\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \nabla_{\widehat{X}}\alpha\mathfrak{P}_y\widehat{U} + \nabla_{\widehat{X}}\beta\mathfrak{P}_y\widehat{U} + \nabla_{\widehat{X}}\alpha\mathfrak{P}_v\widehat{U} + \nabla_{\widehat{X}}\beta\mathfrak{P}_v\widehat{U})\} - g_1(P\widehat{X},\widehat{U})\Psi_*\xi.$$

By taking account the fact from Eqs (2.11) and (2.12), we get

$$\begin{aligned}(\nabla\Psi_*)(\widehat{X},\widehat{U}) = \Psi_*\{&\phi(\mathcal{A}_{\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \mathcal{V}\nabla_{\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \nabla_{\widehat{X}}\phi\alpha\mathfrak{P}_y\widehat{U})\\ &+ \phi(\mathcal{H}\nabla_{\widehat{X}}\beta\mathfrak{P}_y\widehat{U} + \mathcal{A}_{\widehat{X}}\beta\mathfrak{P}_y\widehat{U}) + \nabla_{\widehat{X}}\phi\alpha\mathfrak{P}_v\widehat{U}\\ &+ \phi(\mathcal{H}\nabla_{\widehat{X}}\beta\mathfrak{P}_v\widehat{U} + \mathcal{A}_{\widehat{X}}\beta\mathfrak{P}_v\widehat{U})\} - g_1(P\widehat{X},\widehat{U})\Psi_*\xi.\end{aligned}$$

Finally, from conformality of RS Ψ and Lemmas 3.2 and 3.3, we can write

$$\begin{aligned}(\nabla\Psi_*)(\widehat{X},\widehat{U}) = &\Psi_*\{\beta(\mathcal{V}\nabla_{\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \mathcal{A}_{\widehat{X}}\beta\mathfrak{P}_y\widehat{U} + \mathfrak{P}_{g\widehat{X}}\beta\mathfrak{P}_v\widehat{U})\} - g_1(P\widehat{X},\widehat{U})\Psi_*\xi\\ &- \Psi_*(cos^2\theta_1\nabla_{\widehat{X}}\mathfrak{P}_y\widehat{U} + cos^2\theta_2\nabla_{\widehat{X}}\mathfrak{P}_v\widehat{U} - \nabla_{\widehat{X}}\beta\alpha\mathfrak{P}_y\widehat{U} - \nabla_{\widehat{X}}\beta\alpha\mathfrak{P}_v\widehat{U})\\ &+ \Psi_*\{\mathbb{B}(\mathcal{A}_{\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \mathcal{H}\nabla_{\widehat{X}}\beta\mathfrak{P}_y\widehat{U} + \mathcal{H}\nabla_{\widehat{X}}\beta\mathfrak{P}_v\widehat{U})\}.\end{aligned}$$

From which we obtain (ii) part of theorem. This completes the proof of theorem. □

5. Decomposition theorems

In this section, we recall the following result from [28] and discuss some decomposition theorems by using prior theorems. Let us suppose that g be a Riemannian metric on the manifold $M = \bar{Q}_1 \times \bar{Q}_2$, then

(i) $M = \bar{Q}_1 \times_\lambda \bar{Q}_2$ is a locally product if and only if $\bar{Q}_1$ and $\bar{Q}_2$ are totally geodesic foliations,
(ii) a warped product $\bar{Q}_1 \times_\lambda \bar{Q}_2$ if and only if $\bar{Q}_1$ is a totally geodesic foliation and $\bar{Q}_2$ is a spherics foliation, i.e., it is umbilic and its mean curvature vector field is parallel,
(ii) $M = \bar{Q}_1 \times_\lambda \bar{Q}_2$ is a twisted product if and only if $\bar{Q}_1$ is a totally geodesic foliation and $\bar{Q}_2$ is a totally umbilic foliation.

The presence of three orthogonal complementary distributions $\mathfrak{D}$, $\mathfrak{D}^{\theta_g}$, and $\mathfrak{D}^{\theta_y}$, which satisfy some conditions of integrable and totally geodesic that we have stated previously, is ensured by the fact that $\Psi : (\bar{Q}_1, \phi, \xi, \eta, g_1) \to (\bar{Q}_2, g_2)$ is $\mathcal{QBSC}$ $\xi^\perp$-submersion. It makes sense to now look for the conditions in which the total space $\bar{Q}_1$ converts into locally twisted product manifolds. Now, we are giving the following result.

Theorem 5.1. *Let Ψ be a $\mathcal{QBSC}$ $\xi^{\perp}$-submersion from Sasakian manifold (M,ϕ,ξ,η,g_1) onto a RM (M_2,g_2). Then $\bar{Q}_1$ is locally twisted product of the form $\bar{Q}_{1(ker\Psi_*)} \times \bar{Q}_{1(ker\Psi_*)^{\perp}}$ if and only if*

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla f_*)(\widehat{U},\beta\widehat{V}),f_*\mathbb{B}\widehat{X}) &= g_1(\mathcal{T}_{\widehat{U}}\alpha\widehat{V},\mathbb{B}\widehat{X}) + g_1(\widehat{\mathcal{V}}\nabla_{\widehat{U}}\alpha\widehat{V} + \mathcal{T}_{\widehat{U}}\beta\widehat{V},\mathbb{C}\widehat{X}) \\ &+ \frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{U}\beta\widehat{V},\Psi_*\mathbb{B}\widehat{X}) - g_1(\widehat{U},\widehat{V})\eta(\widehat{X})\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}g(\widehat{X},\widehat{Y})H &= -P\mathcal{A}_{\widehat{X}}P\widehat{Y} - \alpha\nabla_{\widehat{X}}\mathbb{C}\widehat{Y} - \alpha\mathcal{A}_{\widehat{X}}\mathbb{B}\widehat{Y} - \phi\Psi_*(\nabla^{\Psi}_{\widehat{X}}\Psi_*\mathbb{B}\widehat{Y}) + \widehat{X}(\ln\lambda)P\mathbb{B}\widehat{Y} \\ &+ \mathbb{B}\widehat{Y}(\ln\lambda)\mathbb{C}\widehat{X} - P(G\ln\lambda)g(\widehat{X},\mathbb{B}\widehat{Y}),\end{aligned} \tag{5.2}$$

where H is a mean curvature vector and for any $\widehat{U},\widehat{V} \in \Gamma(ker\Psi_*)$ and $\widehat{X}_1,\widehat{X}_2 \in \Gamma(ker\Psi_*)^{\perp}$.

Proof. For any $\widehat{X}_1,\widehat{X}_2 \in \Gamma(ker\Psi_*)^{\perp}$ and $\widehat{U} \in \Gamma(ker\Psi_*)$ and using Eqs (2.2), (2.5), (2.11), (2.12) and (2.14), we have

$$g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X}) = g_1(\mathcal{T}_{\widehat{U}}\alpha V,\mathbb{B}X) + g_1(\mathcal{V}\nabla_{\widehat{U}}\alpha V + \mathcal{T}_{\widehat{U}}\beta V,\mathbb{C}X) + g_1(\phi\widehat{U},V)\eta(X) - g_1(\mathcal{H}\nabla_{\widehat{U}}\beta V,\mathbb{B}X).$$

From using formula (2.6), (2.15) and with conformality of RS Ψ, the above equation finally takes the form

$$\begin{aligned}g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X}) &= g_1(\mathcal{T}_{\widehat{U}}\alpha\widehat{V},\mathbb{B}\widehat{X}) + g_1(\widehat{\mathcal{V}}\nabla_{\widehat{U}}\alpha\widehat{V} + \mathcal{T}_{\widehat{U}}\beta\widehat{V},\mathbb{C}\widehat{X}) + g_1(\phi\widehat{U},V)\eta(X) \\ &- \frac{1}{\lambda^2}g_2((\nabla f_*)(\widehat{U},\beta\widehat{V}),f_*\mathbb{B}\widehat{X}) + \frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{U}\beta\widehat{V},\Psi_*\mathbb{B}\widehat{X}).\end{aligned}$$

It follows that the Eq (5.1) satisfies if and only if $\bar{Q}_{1(ker\Psi_*)}$ is totally geodesic. On the other hand, for $\widehat{U} \in \Gamma(ker\Psi_*)$ and $\widetilde{X},\widehat{Y} \in \Gamma(ker\Psi_*)^{\perp}$ with using Eqs (2.2), (2.5), (2.14) and (3.5), we get

$$g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{U}) = g_1(\nabla_{\widehat{X}}P\widehat{Y},\phi\widehat{U}) + g_1(\mathcal{A}_{\widehat{X}}\mathbb{B}\widehat{Y},\alpha\widehat{U}) + g_1(\mathcal{H}\nabla_{\widehat{X}}\mathbb{B}\widehat{Y},\beta\widehat{U}).$$

By using the Eq (2.15) with definition of conformality of Ψ, we deduce that

$$\begin{aligned}g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{U}) &= -\frac{1}{\lambda^2}g_2((\nabla\Psi_*)(\widehat{X},\mathbb{B}\widehat{Y}),\Psi_*\beta\widehat{U}) + \frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\widehat{X}}\Psi_*\mathbb{B}\widehat{Y},\Psi_*\beta\widehat{U}) \\ &+ g_1(\nabla_{\widehat{X}}P\widehat{Y},\phi\widehat{U}) + g_1(\mathcal{A}_{\widehat{X}}\mathbb{B}\widehat{Y},\alpha\widehat{U}).\end{aligned}$$

Considering the (i) part of Lemma 2.1, above equation turns in to

$$\begin{aligned}g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{U}) &= \frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\widehat{X}}\Psi_*\mathbb{B}\widehat{Y},\Psi_*\beta\widehat{U}) + g_1(\nabla_{\widehat{X}}\mathbb{C}\widehat{Y},\phi\widehat{U}) + g_1(\mathcal{A}_{\widehat{X}}\mathbb{B}\widehat{Y},\alpha\widehat{U}) \\ &- g_1(G\ln\lambda,\widehat{X})g_1(\mathbb{B}\widehat{Y},\beta\widehat{U}) - g_1(G\ln\lambda,\mathbb{B}\widehat{Y})g_1(\widehat{X},\beta\widehat{U}) \\ &+ g_1(G\ln\lambda,\beta\widehat{U})g_1(\widehat{X},\mathbb{B}\widehat{Y}).\end{aligned}$$

By direct calculation, finally we get

$$\begin{aligned}g_1(\widehat{X},\widehat{Y})H &= -P\mathcal{A}_{\widehat{X}}P\widehat{Y} - \alpha\nabla_{\widehat{X}}\mathbb{C}\widehat{Y} - \alpha\mathcal{A}_{\widehat{X}}\mathbb{B}\widehat{Y} - \phi\Psi_*(\nabla^{\Psi}_{\widehat{X}}\Psi_*\mathbb{B}\widehat{Y}) + \widehat{X}(\ln\lambda)P\mathbb{B}\widehat{Y} \\ &+ \mathbb{B}\widehat{Y}(\ln\lambda)\mathbb{C}\widehat{X} - P(G\ln\lambda)g_1(\widehat{X},\mathbb{B}\widehat{Y}).\end{aligned}$$

From the above equation we conclude that $\bar{Q}_{1(ker\Psi_*)^{\perp}}$ is totally umbilical if and only if Eq (5.2) satisfied. □

6. ϕ-pluriharmonicity of quasi bi-slant conformal $\xi^\perp$-submersion

Y. Ohnita established J-pluriharminicity from a almost hermitian manifold in [22]. In this section, we extend the concept of ϕ-pluriharmonicity to almost contact metric manifolds.

Let Ψ be a $\mathcal{QBSC}$ $\xi^\perp$-submersion from Sasakian manifold $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{Q}_2, g_2)$ with slant angles θ_1 and θ_2. Then $\mathcal{QBSC}$ submersion is ϕ-pluriharmonic, $\mathfrak{D}$-ϕ-pluriharmonic, $\mathfrak{D}^{\theta_i}$-ϕ-pluriharmonic, $(\mathfrak{D} - \mathfrak{D}^{\theta_i})$-$\phi$ pluriharmonic (where $i = 1, 2$), $ker\Psi_*$-ϕ-pluriharmonic, $(ker\Psi_*)^\perp$-ϕ-pluriharmonic and $((ker\Psi_*)^\perp - ker\Psi_*)$-$\phi$-pluriharmonic if

$$(\nabla\Psi_*)(\widehat{U}, \widehat{V}) + (\nabla\Psi_*)(\phi\widehat{U}, \phi\widehat{V}) = 0, \tag{6.1}$$

for any $\widehat{U}, \widehat{V} \in \Gamma(\mathfrak{D})$, for any $\widehat{U}, \widehat{V} \in \Gamma(\mathfrak{D}^{\theta_i)}$, for any $\widehat{U} \in \Gamma(\mathfrak{D}), \widehat{V} \in \Gamma(\mathfrak{D}^{\theta_i})$ (where $i = 1, 2$), for any $\widehat{U}, \widehat{V} \in \Gamma(ker\Psi_*)$, for any $\widehat{U}, \widehat{V} \in \Gamma(ker\Psi_*)^\perp$ and for any $\widehat{U} \in \Gamma(ker\Psi_*)^\perp, \widehat{V} \in \Gamma(ker\Psi_*)$.

Theorem 6.1. *Let Ψ be a $\mathcal{QBSC}$ $\xi^\perp$-submersion from Sasakian manifold $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{Q}_2, g_2)$ with slant angles θ_1 and θ_2. Suppose that Ψ is $\mathfrak{D}_{\theta_g}$-ϕ-pluriharmonic. Then $\mathfrak{D}_{\theta_g}$ defines totally geodesic foliation $\bar{Q}_1$ if and only if*

$$\begin{aligned}
&\Psi_*(\beta\mathcal{T}_{\alpha\widehat{U}}\beta\alpha\widehat{V} + \mathbb{B}\mathcal{H}\nabla_{\alpha\widehat{U}}\beta\alpha\widehat{V}) - \Psi_*(\mathcal{A}_{\beta\widehat{U}}\alpha\widehat{V} + \mathcal{H}\nabla_{\alpha\widehat{U}}\beta\widehat{V}) \\
&= \cos^2\theta_1\Psi_*(\mathbb{B}\mathcal{T}_{\alpha\widehat{U}}\widehat{V} + \beta\mathcal{V}\nabla_{\alpha\widehat{U}}\widehat{V}) + \nabla^\Psi_{\alpha\widehat{U}}\Psi_*\phi\widehat{V} + g_1(\alpha\widehat{U}, \alpha\widehat{V})\Psi_*\xi \\
&\quad - \beta\widehat{U}(\ln\lambda)\Psi_*\beta\widehat{V} - \beta\widehat{V}(\ln\lambda)\Psi_*\beta\widehat{U} + g_1(\beta\widehat{U}, \beta\widehat{V})\Psi_*(G\ln\lambda),
\end{aligned}$$

for any $\widehat{U}, \widehat{V} \in \Gamma(\mathfrak{D}_{\theta_g})$.

Proof. For any $\widehat{U}, \widehat{V} \in \Gamma(\mathfrak{D}_{\theta_g})$ and since, Ψ is $\mathfrak{D}_{\theta_g}$-ϕ-pluriharmonic, then by using Eqs (2.9) and (2.15), we have

$$\begin{aligned}
0 &= (\nabla\Psi_*)(\widehat{U}, \widehat{V}) + (\nabla\Psi_*)(\phi\widehat{U}, \phi\widehat{V}), \\
\Psi_*(\nabla_{\widehat{U}}\widehat{V}) &= -\Psi_*(\nabla_{\phi\widehat{U}}\phi\widehat{V}) + \nabla^\Psi_{\phi\widehat{U}}\Psi_*(\phi\widehat{V}) \\
&= -\Psi_*(\mathcal{A}_{\beta\widehat{U}}\alpha\widehat{V} + \mathcal{V}\nabla_{\beta\widehat{U}}\alpha\widehat{V} + \mathcal{T}_{\alpha\widehat{U}}\beta\widehat{V} + \mathcal{H}\nabla_{\alpha\widehat{U}}\beta\widehat{V}) \\
&\quad + (\nabla\Psi_*)(\beta\widehat{U}, \beta\widehat{V}) - \nabla^\Psi_{\beta\widehat{U}}\Psi_*\beta\widehat{V} + \nabla^\Psi_{\phi\widehat{U}}\Psi_*\phi\widehat{V} \\
&\quad + \Psi_*(\phi\nabla_{\alpha\widehat{U}}\phi\alpha\widehat{V} - \eta(\nabla_{\alpha\widehat{U}}\alpha\widehat{V})\xi).
\end{aligned}$$

On using Eqs (3.2) and (3.5) with Lemmas 2.1 and 3.2, the above equation finally takes the form

$$\begin{aligned}
\Psi_*(\nabla_{\widehat{U}}V) = &-\cos^2\theta_1\Psi_*(P\mathcal{T}_{\alpha\widehat{U}}\widehat{V} + \mathbb{B}\mathcal{T}_{\alpha\widehat{U}}\widehat{V} + \alpha\mathcal{V}\nabla_{\alpha\widehat{U}}\widehat{V} + \beta\mathcal{V}\nabla_{\alpha\widehat{U}}\widehat{V}) \\
&+ \Psi_*(\alpha\mathcal{T}_{\alpha\widehat{U}}\beta\alpha\widehat{V} + \beta\mathcal{T}_{\alpha\widehat{U}}\beta\alpha\widehat{V} + P\mathcal{H}\nabla_{\alpha\widehat{U}}\beta\alpha\widehat{V} + \mathbb{B}\mathcal{H}\nabla_{\alpha\widehat{U}}\beta\alpha\widehat{V}) \\
&- \Psi_*(\mathcal{A}_{\beta\widehat{U}}\alpha\widehat{V} + \mathcal{V}\nabla_{\beta\widehat{U}}\alpha\widehat{V} + \mathcal{T}_{\alpha\widehat{U}}\beta\widehat{V} + \mathcal{H}\nabla_{\alpha\widehat{U}}\beta\widehat{V}) \\
&+ \beta\widehat{U}(\ln\lambda)\Psi_*\beta\widehat{V} + \beta\widehat{V}(\ln\lambda)\Psi_*\beta\widehat{U} - g_M(\beta\widehat{U}, \beta\widehat{V})\Psi_*(grad\ln\lambda) \\
&+ g_1(\alpha\widehat{U}, \alpha\widehat{V})\Psi_*\xi - \nabla^\Psi_{\beta\widehat{U}}\Psi_*\beta\widehat{V} + \nabla^\Psi_{\phi\widehat{U}}\Psi_*\phi\widehat{V}
\end{aligned}$$

from which we get the desired result. □

Theorem 6.2. *Let $\vec{f}$ be a QBSC $\xi^{\perp}$-submersion from Sasakian manifold $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{Q}_2, g_2)$ with slant angles θ_1 and θ_2. Suppose that Ψ is $\mathfrak{D}_{\theta_y}$-ϕ-pluriharmonic. Then $\mathfrak{D}_{\theta_y}$ defines totally geodesic foliation $\bar{Q}_1$ if and only if*

$$\begin{aligned}&\Psi_*(\beta\mathcal{T}_{\alpha\widehat{Z}}\beta\alpha\widehat{W} + \mathbb{B}\mathcal{H}\nabla_{\alpha\widehat{Z}}\beta\alpha\widehat{W}) - \Psi_*(\mathcal{A}_{\beta\widehat{Z}}\alpha\widehat{W} + \mathcal{H}\nabla_{\alpha\widehat{Z}}\beta\widehat{W})\\ &= \cos^2\theta_2\Psi_*(\mathbb{B}\mathcal{T}_{\alpha\widehat{Z}}\widehat{W} + \beta\widehat{\mathcal{W}}\nabla_{\alpha\widehat{Z}}\widehat{W}) + \nabla^{\Psi}_{\alpha\widehat{Z}}\Psi_*\phi\widehat{W} + g_1(\alpha\widehat{Z}, \alpha\widehat{W})\Psi_*\xi\\ &\quad - \beta\widehat{Z}(\ln\lambda)\Psi_*\beta\widehat{W} - \beta\widehat{W}(\ln\lambda)\Psi_*\beta\widehat{Z} + g_M(\beta\widehat{Z}, \beta\widehat{W})\Psi_*(grad\ln\lambda),\end{aligned}$$

for any $\widehat{Z}, \widehat{W} \in \Gamma(\mathfrak{D}_{\theta_y})$.

Proof. The proof of the theorem is similar to the proof of Theorem 6.1. □

Theorem 6.3. *Let $\vec{f}$ be a QBSC $\xi^{\perp}$-submersion from Sasakian manifold $(\bar{Q}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{Q}_2, g_2)$ with slant angles θ_1 and θ_2. Suppose that Ψ is $((ker\Psi_*)^{\perp} - ker\Psi_*)$-$\phi$-pluriharmonic. Then the following assertion are equivalent.*

(i) The horizontal distribution $(ker\Psi_*)^{\perp}$ defines totally geodesic foliation on $\bar{Q}_1$.

(ii) $$\begin{aligned}&(cos^2\theta_1 + cos^2\theta_2)\Psi_*\{\mathbb{B}\mathcal{T}_{\mathbb{C}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \beta\mathcal{V}\nabla_{\mathbb{C}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \mathbb{B}\mathcal{A}_{\mathbb{B}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \beta\mathcal{V}\nabla_{\mathbb{B}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U}\}\\ &+ \Psi_*\{\beta\mathcal{A}_{\mathbb{B}\widehat{X}}\beta\alpha\mathfrak{P}_y\widehat{U} + \beta\mathcal{A}_{\mathbb{B}\widehat{X}}\beta\alpha\mathfrak{P}_v\widehat{U} - \mathcal{H}\nabla_{\mathbb{C}\widehat{X}}\beta\widehat{U}\} + \nabla^{\Psi}_{\mathbb{B}\widehat{X}}\Psi_*\beta\alpha\mathfrak{P}_y\widehat{U} + \nabla^{\Psi}_{\mathbb{B}\widehat{X}}\Psi_*\beta\alpha\mathfrak{P}_v\widehat{U}\\ &= \Psi_*\{\mathbb{B}\mathcal{T}_{\mathbb{C}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \beta\mathcal{V}\nabla_{\mathbb{C}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \mathbb{B}\mathcal{A}_{\mathbb{B}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \beta\mathcal{H}\nabla_{\mathbb{B}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U}\}\\ &- \Psi_*\{\beta\mathcal{T}_{\mathbb{C}\widehat{X}}\beta\alpha\mathfrak{P}_y\widehat{U} + \mathbb{B}\mathcal{H}\nabla_{\mathbb{C}\widehat{X}}\beta\alpha\mathfrak{P}_y\widehat{U} + \beta\mathcal{T}_{\mathbb{C}\widehat{X}}\beta\alpha\mathfrak{P}_v\widehat{U} + \mathbb{B}\mathcal{H}\nabla_{\mathbb{C}\widehat{X}}\beta\alpha\mathfrak{P}_v\widehat{U}\}\\ &+ \mathbb{B}\widehat{X}(\ln\lambda)\Psi_*\beta\alpha\mathfrak{P}_y\widehat{U} + \beta\alpha\mathfrak{P}_y\widehat{U}(\ln\lambda)\Psi_*\mathbb{B}\widehat{X} - g_1(\mathbb{B}\widehat{X}, \beta\alpha\mathfrak{P}_y\widehat{U})\Psi_*(grad\ln\lambda)\\ &+ \mathbb{B}\widehat{X}(\ln\lambda)\Psi_*\beta\alpha\mathfrak{P}_v\widehat{U} + \beta\alpha\mathfrak{P}_v\widehat{U}(\ln\lambda)\Psi_*\mathbb{B}\widehat{X} - g_1(\mathbb{B}\widehat{X}, \beta\alpha\mathfrak{P}_v\widehat{U})\Psi_*(grad\ln\lambda)\\ &+ \Psi_*(\nabla_{\widehat{X}}\widehat{U}) + \nabla^{\Psi}_{\phi\widehat{X}}\Psi_*\beta\widehat{U} + g_1(P\widehat{X}, \alpha\widehat{U})\Psi_*\xi,\end{aligned}$$

for any $\widehat{X} \in \Gamma(ker\Psi_*)^{\perp}$ and $\widehat{U} \in \Gamma(ker\Psi_*)$.

Proof. For any $\widehat{X} \in \Gamma(ker\Psi_*)^{\perp}$ and $\widehat{U} \in \Gamma(ker\Psi_*)$, since Ψ is $((ker\Psi_*)^{\perp} - ker\Psi_*)$-$\phi$-pluriharmonic, then by using (2.15), (3.2) and (3.5), we get

$$\Psi_*(\nabla_{\mathbb{B}\widehat{X}}\beta\widehat{U}) = -\Psi_*(\nabla_{\mathbb{C}\widehat{X}}\alpha\widehat{U} + \nabla_{\mathbb{C}\widehat{X}}\beta\widehat{U} + \nabla_{\mathbb{B}\widehat{X}}\alpha\widehat{U}) + \Psi_*(\nabla_{\widehat{X}}\widehat{U}) + \nabla^{\Psi}_{\phi\widehat{X}}\Psi_*\beta\widehat{U}.$$

Taking account the fact from (2.1) and (2.10), we have

$$\begin{aligned}\Psi_*(\nabla_{\mathbb{B}\widehat{X}}\beta\widehat{U}) = &-\Psi_*(\mathcal{T}_{\mathbb{C}\widehat{X}}\beta\widehat{U} + \mathcal{H}\nabla_{\mathbb{C}\widehat{X}}\beta\widehat{U}) + \Psi_*(\nabla_{\widehat{X}}\widehat{U}) + \nabla^{\Psi}_{\phi\widehat{X}}\Psi_*\beta\widehat{U}\\ &+ \Psi_*\{\phi\nabla_{\mathbb{C}\widehat{X}}\phi\alpha\widehat{U} - \eta(\nabla_{P\widehat{X}}\alpha\widehat{U})\xi\}\\ &+ \Psi_*\{\phi\nabla_{\mathbb{B}\widehat{X}}\phi\alpha\widehat{U} - \eta(\nabla_{\mathbb{B}\widehat{X}}\alpha\widehat{U})\xi\}.\end{aligned}$$

Now on using decomposition (3.1), Lemmas 3.2 and 3.3 with Eq (3.2), we may yields

$$\begin{aligned}\Psi_*(\nabla_{\mathbb{B}\widehat{X}}\beta\widehat{U}) = &\Psi_*\{\phi\nabla_{\mathbb{C}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} - \cos^2\theta_1\phi\nabla_{\mathbb{C}\widehat{X}}\alpha\widehat{U} - \cos^2\theta_2\phi\nabla_{\mathbb{C}\widehat{X}}\alpha\widehat{U} + g_1(P\widehat{X}, \alpha\widetilde{U})\xi\\ &+ \Psi_*\{\phi\nabla_{\mathbb{B}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} - \cos^2\theta_1\phi\nabla_{\mathbb{B}\widehat{X}}\alpha\widehat{U} - \cos^2\theta_2\phi\nabla_{\mathbb{B}\widehat{X}}\alpha\widehat{U} + g_1(\mathbb{B}\widehat{X}, \alpha\widetilde{U})\xi\\ &+ \Psi_*\{\phi\nabla_{\mathbb{C}\widehat{X}}\beta\alpha\mathfrak{P}_y\widehat{U} + \phi\nabla_{\mathbb{C}\widehat{X}}\beta\alpha\mathfrak{P}_v\widehat{U} + \phi\nabla_{\mathbb{B}\widehat{X}}\beta\alpha\mathfrak{P}_y\widehat{U} + \phi\nabla_{\mathbb{B}\widehat{X}}\beta\alpha\mathfrak{P}_v\widehat{U}\}\\ &- \Psi_*(\mathcal{H}\nabla_{\mathbb{C}\widehat{X}}\beta\widehat{U}) + \Psi_*(\nabla_{\widehat{X}}\widehat{U}) + \nabla^{\Psi}_{\phi\widehat{X}}\Psi_*\beta\widehat{U}.\end{aligned}$$

From Eqs (2.9)–(2.12) and after simple calculation, we may write

$$\begin{aligned}\Psi_*(\nabla_{\mathbb{B}\widehat{X}}\beta\widehat{U}) = &-(cos^2\theta_1 + cos^2\theta_2)\Psi_*\{\mathbb{B}\mathcal{T}_{\mathbb{C}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \beta\mathcal{V}\nabla_{\mathbb{C}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \mathbb{B}\mathcal{A}_{\mathbb{B}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} \\ &+ \beta\mathcal{V}\nabla_{\mathbb{B}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U}\} - \Psi_*\{\beta\mathcal{A}_{\mathbb{B}\widehat{X}}\beta\alpha\mathfrak{P}_y\widehat{U} + \beta\mathcal{A}_{\mathbb{B}\widehat{X}}\beta\alpha\mathfrak{P}_v\widehat{U} - \mathcal{H}\nabla_{\mathbb{C}\widehat{X}}\beta\widehat{U}\} \\ &+ \Psi_*\{\mathbb{B}\mathcal{T}_{\mathbb{C}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \beta\mathcal{V}\nabla_{\mathbb{C}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \mathbb{B}\mathcal{A}_{\mathbb{B}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \beta\mathcal{H}\nabla_{\mathbb{B}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U}\} \\ &- \Psi_*\{\beta\mathcal{T}_{\mathbb{C}\widehat{X}}\beta\alpha\mathfrak{P}_y\widehat{U} + \mathbb{B}\mathcal{H}\nabla_{\mathbb{C}\widehat{X}}\beta\alpha\mathfrak{P}_y\widehat{U} + \beta\mathcal{T}_{\mathbb{C}\widehat{X}}\beta\alpha\mathfrak{P}_v\widehat{U} + \mathbb{B}\mathcal{H}\nabla_{\mathbb{C}\widehat{X}}\beta\alpha\mathfrak{P}_v\widehat{U}\} \\ &- \Psi_*(\mathbb{B}\mathcal{H}\nabla_{\mathbb{B}\widehat{X}}\beta\alpha\mathfrak{P}_y\widehat{U} + \mathbb{B}\mathcal{H}\nabla_{\mathbb{B}\widehat{X}}\beta\alpha\mathfrak{P}_y\widehat{U}) + \Psi_*(\nabla_{\widehat{X}}\widehat{U}) \\ &+ \nabla^{\Psi}_{\phi\widehat{X}}\Psi_*\beta\widehat{U} + g_1(P\widehat{X}, \alpha\widehat{U})\Psi_*\xi.\end{aligned}$$

Since Ψ is conformal Riemannian submersion, the by using Eq (2.15) and from Lemma 2.1, we finally have

$$\begin{aligned}&\Psi_*(\nabla_{\mathbb{B}\widehat{X}}\beta\widehat{U}) \\ =& -(cos^2\theta_1 + cos^2\theta_2)\Psi_*\{\mathbb{B}\mathcal{T}_{\mathbb{C}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \beta\mathcal{V}\nabla_{\mathbb{C}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \mathbb{B}\mathcal{A}_{\mathbb{B}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \beta\mathcal{V}\nabla_{\mathbb{B}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U}\} \\ &+ \Psi_*\{\mathbb{B}\mathcal{T}_{\mathbb{C}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \beta\mathcal{V}\nabla_{\mathbb{C}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \mathbb{B}\mathcal{A}_{\mathbb{B}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U} + \beta\mathcal{H}\nabla_{\mathbb{B}\widehat{X}}\alpha\mathfrak{P}_g\widehat{U}\} \\ &- \Psi_*\{\beta\mathcal{T}_{\mathbb{C}\widehat{X}}\beta\alpha\mathfrak{P}_y\widehat{U} + \mathbb{B}\mathcal{H}\nabla_{\mathbb{C}\widehat{X}}\beta\alpha\mathfrak{P}_y\widehat{U} + \beta\mathcal{T}_{\mathbb{C}\widehat{X}}\beta\alpha\mathfrak{P}_v\widehat{U} + \mathbb{B}\mathcal{H}\nabla_{\mathbb{C}\widehat{X}}\beta\alpha\mathfrak{P}_v\widehat{U}\} \\ &+ \mathbb{B}\widehat{X}(\ln\lambda)\Psi_*\beta\alpha\mathfrak{P}_y\widehat{U} + \beta\alpha\mathfrak{P}_y\widehat{U}(\ln\lambda)\Psi_*\mathbb{B}\widehat{X} - g_1(\mathbb{B}\widehat{X}, \beta\alpha\mathfrak{P}_y\widehat{U})\Psi_*(grad\ln\lambda) \\ &+ \mathbb{B}\widehat{X}(\ln\lambda)\Psi_*\beta\alpha\mathfrak{P}_v\widehat{U} + \beta\alpha\mathfrak{P}_v\widehat{U}(\ln\lambda)\Psi_*\mathbb{B}\widehat{X} - g_1(\mathbb{B}\widehat{X}, \beta\alpha\mathfrak{P}_v\widehat{U})\Psi_*(grad\ln\lambda) \\ &- \Psi_*\{\beta\mathcal{A}_{\mathbb{B}\widehat{X}}\beta\alpha\mathfrak{P}_y\widehat{U} + \beta\mathcal{A}_{\mathbb{B}\widehat{X}}\beta\alpha\mathfrak{P}_v\widehat{U} - \mathcal{H}\nabla_{\mathbb{C}\widehat{X}}\beta\widehat{U}\} + g_1(P\widehat{X}, \alpha\widehat{U})\Psi_*\xi. \\ &+ \Psi_*(\nabla_{\widehat{X}}\widehat{U}) + \nabla^{\Psi}_{\phi\widehat{X}}\Psi_*\beta\widehat{U} - \nabla^{\Psi}_{\mathbb{B}\widehat{X}}\Psi_*\beta\alpha\mathfrak{P}_y\widehat{U} - \nabla^{\Psi}_{\mathbb{B}\widehat{X}}\Psi_*\beta\alpha\mathfrak{P}_v\widehat{U},\end{aligned}$$

which completes the proof of theorem. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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