AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(9): 21359–21370.
DOI: 10.3934/math.20231088
Received: 22 May 2023
Revised: 15 June 2023
Accepted: 15 June 2023
Published: 05 July 2023

Research article

Characterizing non-totally geodesic spheres in a unit sphere

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Abstract: A concircular vector field $\mathbf{u}$ on the unit sphere $\mathbf{S}^{n+1}$ induces a vector field $\mathbf{w}$ on an orientable hypersurface M of the unit sphere $\mathbf{S}^{n+1}$, simply called the induced vector field on the hypersurface M. Moreover, there are two smooth functions, f and σ, defined on the hypersurface M, where f is the restriction of the potential function $\overline{f}$ of the concircural vector field $\mathbf{u}$ on the unit sphere $\mathbf{S}^{n+1}$ to M and σ is defined as $g(\mathbf{u}, N)$, where N is the unit normal to the hypersurface. In this paper, we show that if function f on the compact hypersurface satisfies the Fischer–Marsden equation and the integral of the squared length of the vector field $\mathbf{w}$ has a certain lower bound, then a characterization of a small sphere in the unit sphere $\mathbf{S}^{n+1}$ is produced. Additionally, we find another characterization of a small sphere using a lower bound on the integral of the Ricci curvature of the compact hypersurface M in the direction of the vector field $\mathbf{w}$ with a non-zero function σ.

Keywords: small sphere; concircular vector field; the Fischer–Marsden equation; the Ricci curvature
Mathematics Subject Classification: 53C20, 53C99, 58J99

1. Introduction

Research into understanding the geometry of hypersurfaces in the unit sphere $\mathbf{S}^{n+1}$ is highly significant in differential geometry and has engaged the attention of several pioneering mathematicians [1, 5, 9, 14, 22, 24, 27, 32, 33, 36]. It is worth noting there are still fascinating open problems in the geometry of hypersurfaces in the unit sphere, such as the Chern's problem on isometric hypersurfaces ([40], Problem 105). Over the period, several celebrated results in this area have been obtained; for example, Okumura [25] gave a criterion for a hypersurface of a unit sphere of constant mean curvature to be totally umbilical and Chen [7] characterized minimal hypersurfaces. In [2], the rigidity of compact-oriented hypersurfaces with constant scalar curvature isometrically immersed into the unit Euclidean sphere was studied. The papers [6, 10] were devoted to the study of the Fisher–Marsden conjecture regarding the Kenmotsu manifold. In [3, 11], the authors considered Ricci solitons. The Clifford hypersurface in a unit sphere was considered in [23, 30]. A characterization of Euclidean spheres out of complete Riemannian manifolds was made by certain vector fields on complete Riemannian manifolds satisfying a partial differential equation on vector fields in [18]. Some characterizations of certain rank-one symmetric Riemannian manifolds by the existence of non-trivial solutions to certain partial differential equations on Riemannian manifolds are surveyed in [16].

There are two important hypersurfaces: the unit sphere $\mathbf{S}^{n+1}$, namely the totally geodesic hypersurfaces $\mathbf{S}^n$ known as great spheres, and $\mathbf{S}^n\left(\frac{1}{\alpha^2}\right)$, namely the small spheres. Some interesting results for the case of the unit sphere with constant curvature were received in [8, 20, 38, 39]. Hypersurfaces were studied in [12, 13, 19, 21, 28, 29, 31, 35, 37, 41]. In [4], authors have considered characterizing small spheres among compact hypersurfaces of the unit sphere $\mathbf{S}^{n+1}$ using the Fischer–Marsden equation satisfied by the support function σ of the hypersurface.

It is well known that there are several concircular vector fields on the unit sphere $\mathbf{S}^{n+1}$ obtained through tangential projections of constant vector fields on the ambient Euclidean space $\mathbf{E}^{n+2}$. Such a concircular vector field $\mathbf{u}$ on $\mathbf{S}^{n+1}$ satisfies $\overline{\nabla}_X\mathbf{u} = -\overline{f}X$, where X is a smooth vector field on $\mathbf{S}^{n+1}$ and $\overline{f}$ is a smooth function defined on $\mathbf{S}^{n+1}$ called the potential function of the concircular vector field $\mathbf{u}$. Given an orientable hypersurface M of the unit sphere $\mathbf{S}^{n+1}$ with unit normal N and shape operator A, one can express the restriction of the concircular vector field $\mathbf{u}$ to M as $\mathbf{u} = \mathbf{w} + \sigma N$, where $\mathbf{w}$ is tangent to the hypersurface M and $\sigma = g(\mathbf{u}, N)$ is a smooth function on M. We denote by f the restriction of the potential function $\overline{f}$ to the hypersurface M. In this paper, we call the vector field $\mathbf{w}$ as the induced vector field on the hypersurface M, the function f as the associated function, and the function σ as the support function of the hypersurface. We show that the associated function f for the special hypersurface the small sphere $\mathbf{S}^n(c)$ satisfies the Fischer–Marsden equation.

2. Small spheres and their properties

Consider the unit sphere $\mathbf{S}^{n+1}$ as the hypersurface of the Euclidean space $\mathbf{R}^{n+2}$ with unit normal ξ and shape operator $B = -I$, where I denotes the identity operator. For the constant vector field $Z = \frac{\partial}{\partial u^1}$ on the Euclidean space $\mathbf{R}^{n+2}$, where $u^1, ..., u^{n+2}$ are Euclidean coordinates on $\mathbf{R}^{n+2}$, we denote the tangential projection of Z by $\mathbf{u}$ to the unit sphere $\mathbf{S}^{n+1}$. Then, we have

$$Z = \mathbf{u} + \overline{f}\xi,$$

where $\overline{f} = \langle Z, \xi\rangle$, $\langle , \rangle$ is the Euclidean metric on $\mathbf{R}^{n+2}$. By differentiating the above equation with respect to a vector field X on the unit sphere $\mathbf{S}^{n+1}$ and using the Gauss–Weingarten formulae for hypersurface, we have

$$\overline{\nabla}_X\mathbf{u} = -\overline{f}X, \quad \mathrm{grad}\overline{f} = \mathbf{u},$$

where $\overline{\nabla}$ is the Riemannian connection on the unit sphere $\mathbf{S}^{n+1}$ with respect to the canonical metric g and $\mathrm{grad}\overline{f}$ is the gradient of the smooth function $\overline{f}$ on $\mathbf{S}^{n+1}$. The above equation shows that $\mathbf{u}$ is a concircular vector field on the unit sphere $\mathbf{S}^{n+1}$.

Now, consider the small sphere (non-totally geodesic sphere) $\mathbf{S}^n\left(\frac{1}{\alpha^2}\right)$ in the unit sphere $\mathbf{S}^{n+1}$ defined by

$$\mathbf{S}^n\left(\frac{1}{\alpha^2}\right)=\left\{(u^1,...,u^{n+2}):\sum_{i=1}^{n+1}\left(u^i\right)^2=\alpha^2,u^{n+2}=\sqrt{1-\alpha^2},0<\alpha<1\right\}.$$

Then, it follows that $\mathbf{S}^n\left(\frac{1}{\alpha^2}\right)$ is a hypersurface of the unit sphere $\mathbf{S}^{n+1}$ with unit normal vector field ζ given by

$$\zeta=\left(-\frac{\sqrt{1-\alpha^2}}{\alpha}u^1,...,-\frac{\sqrt{1-\alpha^2}}{\alpha}u^{n+1},\alpha\right).$$

We use the same letter g to denote the induced metric on the small sphere $\mathbf{S}^n\left(\frac{1}{\alpha^2}\right)$ and denote the Riemannian connection with respect to the induced metric g by ∇. Then, by a simple computation, we have

$$\overline{\nabla}_X\zeta=-\frac{\sqrt{1-\alpha^2}}{\alpha}X,\quad X\in\mathfrak{X}\left(\mathbf{S}^n\left(\frac{1}{\alpha^2}\right)\right).\tag{2.1}$$

That is, the shape operator A of the hypersurface $\mathbf{S}^n\left(\frac{1}{\alpha^2}\right)$ is given by

$$A=\frac{\sqrt{1-\alpha^2}}{\alpha}I=HI,\tag{2.2}$$

where H is the mean curvature of the hypersurface $\mathbf{S}^n\left(\frac{1}{\alpha^2}\right)$. It is clear that H is a non-zero constant, as $0<\alpha<1$. Now, we utilize $\mathbf{w}$ to denote the tangential projection of the vector field $\mathbf{u}$ to the small sphere $\mathbf{S}^n\left(\frac{1}{\alpha^2}\right)$ and define $\sigma=g\left(\mathbf{u},\zeta\right)$. Then, we have

$$\mathbf{u}=\mathbf{w}+\sigma\zeta.\tag{2.3}$$

However, using the definitions of $\mathbf{u}$ and ζ, we can easily see that

$$g\left(\mathbf{u},\zeta\right)=-\frac{\sqrt{1-\alpha^2}}{\alpha}f,$$

where f is the restriction of $\overline{f}$ to $\mathbf{S}^n\left(\frac{1}{\alpha^2}\right)$. Thus,

$$\sigma=-Hf.\tag{2.4}$$

Differentiating Eq (2.3) and using the Gauss–Weingarten formulae for hypersurface, we conclude on using Eqs (2.1) and (2.2) and on equating tangential components, that

$$\nabla_X\mathbf{w}=-(1+H^2)fX,\quad \mathrm{grad}\sigma=-H\mathbf{w},\tag{2.5}$$

for $X\in\mathfrak{X}\left(\mathbf{S}^n\left(\frac{1}{\alpha^2}\right)\right)$. Thus, in view of Eqs (2.4) and (2.5), the Laplace operator acting on the smooth function σ is given by

$$\Delta\sigma=-n(1+H^2)\sigma.$$

The Ricci tensor of the small sphere $\mathbf{S}^n\left(\frac{1}{\alpha^2}\right)$ is given by

$$Ric=(n-1)(1+H^2)g.$$

Additionally, using Eqs (2.4) and (2.5), we have

$$\operatorname{grad} f = \mathbf{w}$$

and consequently, we have

$$Hess(f)\,(X,Y) = -(1+H^2) f g\,(X,Y)\,, \quad \Delta f = -n(1+H^2) f.$$

Thus, we see that for the function f, we have

$$(\Delta f) g + f Ric = Hess(f). \tag{2.6}$$

Thus, the function f satisfies the Fischer–Marsden equation [15, 17, 26, 34].

3. Preliminaries

Let M be an orientable hypersurface of the unit sphere $\mathbf{S}^{n+1}$ with unit normal N and shape operator A. We denote the canonical metric on $\mathbf{S}^{n+1}$ by g and the induced metric on M by the same letter g. Additionally, utilize $\overline{\nabla}$ and ∇ to denote the Riemannian connections on the unit sphere $\mathbf{S}^{n+1}$ and the hypersurface M, respectively. Then, we have the following fundamental formulae for the hypersurface:

$$\overline{\nabla}_X Y = \nabla_X Y + g\,(AX,Y)\,N, \quad \overline{\nabla}_X N = -AX, \quad X,Y \in \mathfrak{X}(M), \tag{3.1}$$

where $\mathfrak{X}(M)$ is the Lie-algebra of smooth vector fields on the hypersurface M. The curvature tensor R, the Ricci tensor Ric, and the scalar curvature of the hypersurface are given by

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(AY,Z)AX - g(AX,Z)AY, \quad X,Y,Z \in \mathfrak{X}(M),$$

$$Ric(X,Y) = (n-1)g(X,Y) + nHg(AX,Y) - g(AX,AY), \quad X,Y \in \mathfrak{X}(M), \tag{3.2}$$

$$\tau = n(n-1) + n^2H^2 - \|A\|^2\,. \tag{3.3}$$

The Codazzi equation for the hypersurface is

$$(\nabla A)\,(X,Y) = (\nabla A)\,(Y,X), \quad X,Y \in \mathfrak{X}(M),$$

where $(\nabla A)\,(X,Y) = \nabla_X AY - A\nabla_X Y$. By using a local orthonormal frame $\{u_1,..,u_n\}$ on the hypersurface M and the mean curvature $H = \frac{1}{n} trA$ in Eq (3.3), the following expression for the gradient of the mean curvature function H is given:

$$n\operatorname{grad} H = \sum_{i=1}^{n} (\nabla A)\,(u_i,u_i)\,. \tag{3.4}$$

Recall that on the unit sphere $\mathbf{S}^{n+1}$, a concircular vector field $\mathbf{u}$ is defined using a constant vector field $Z = \frac{\partial}{\partial u^1}$ on the Euclidean space $\mathbf{R}^{n+2}$ as $Z = \mathbf{u} + \overline{f}\xi$, where the function $\overline{f} = \langle Z, \xi\rangle$, $\langle , \rangle$ is the Euclidean metric on $\mathbf{R}^{n+2}$ and we have

$$\overline{\nabla}_X \mathbf{u} = -\overline{f}X, \quad \operatorname{grad}\overline{f} = \mathbf{u}, \quad X \in \mathfrak{X}(\mathbf{S}^{n+1}).$$

We utilize f to denote the restriction of the function $\overline{f}$ to the hypersurface M. We define a vector field $\mathbf{w}$ on the hypersurface M by

$$\mathbf{u} = \mathbf{w} + \sigma N, \tag{3.5}$$

that is, $\mathbf{w}$ is the tangential component of the concircular vector field $\mathbf{u}$ to the hypersurface M and the function $\sigma = g(\mathbf{u}, N)$. We call the vector field $\mathbf{w}$ the induced vector field on the hypersurface, the function σ as the support function of the hypersurface, and the function f as the associated function of the hypersurface. Taking covariant derivative in Eq (3.5), and using formulae in (3.1), we get

$$\nabla_X \mathbf{w} = -fX + \sigma AX \quad \text{and} \quad \mathrm{grad}\sigma = -A\mathbf{w}, \quad X \in \mathfrak{X}(M). \tag{3.6}$$

Additionally, we have the tangential component $\left[\mathrm{grad}\overline{f}\right]^T = \mathrm{grad}f$ and that the normal component $\left[\mathrm{grad}\overline{f}\right]^\perp = \sigma N$.

4. Main results

Theorem 1. *Let M be an orientable, non-totally geodesic compact and connected hypersurface of the unit sphere $\mathbf{S}^{n+1}$, $n \geq 2$, with mean curvature H, induced vector field $\mathbf{w}$, and non-zero associated function f. Then, the potential function f is a non-trivial solution of the Fischer–Marsden Eq (2.6) and the inequality*

$$\int_M \|\mathbf{w}\|^2 \geq n \int_M \left(1 + H^2\right) f^2$$

holds if and only if H is a constant and M is isometric to the small sphere $\mathbf{S}^n(1 + H^2)$.

Proof. Suppose the associated function f of the hypersurface is a non-trivial solution of the Fischer–Marsden equation, that is,

$$(\Delta f)\, g + fRic = Hess(f).$$

Taking trace in above equation, we conclude

$$\Delta f = -\frac{\tau}{n-1} f. \tag{4.1}$$

Now, using (3.5), we have $\mathrm{grad}f = \mathbf{w}$ and Eq (3.6) implies $\mathrm{div}\mathbf{w} = n\,(-f + \sigma H)$. Thus, $\Delta f = n\,(-f + \sigma H)$ and combining it with (4.1), we have

$$-\frac{\tau}{n-1} f = n\,(-f + \sigma H)\,.$$

Using Eq (3.3) in above equation, we conclude

$$\frac{1}{n-1}\left(\|A\|^2 - nH^2\right) f^2 = n\sigma f H + nH^2 f^2. \tag{4.2}$$

Note by on using $\mathrm{grad}f = \mathbf{w}$ and $\mathrm{div}\mathbf{w} = n\,(-f + \sigma H)$, we have $\mathrm{div}\,(f\mathbf{w}) = \|\mathbf{w}\|^2 - nf^2 + nf\sigma H$. Thus, Eq (4.2) becomes

$$\frac{1}{n-1}\left(\|A\|^2 - nH^2\right) f^2 = nf^2 + nH^2 f^2 - \|\mathbf{w}\|^2 + \mathrm{div}\,(f\mathbf{w})$$

and integrating above equation, we have

$$\frac{1}{n-1}\int_M \left(\|A\|^2 - nH^2\right) f^2 = n\int_M \left(1+H^2\right) f^2 - \int_M \|\mathbf{w}\|^2 .$$

Note that owing to Schwartz's inequality $\|A\|^2 \geq nH^2$, the integral on the left hand side is non-negative, and consequently, using the condition in the statement, we conclude that

$$\frac{1}{n-1}\int_M \left(\|A\|^2 - nH^2\right) f^2 = 0.$$

Thus, the Schwartz's inequality is actually equality $\|A\|^2 = nH^2$, which holds if and only if $A = HI$. We compute $(\nabla A)(X,Y) = X(H)Y$ and summing the last equation over a local orthonormal frame $\{u_1, .., u_n\}$ on M, we conclude that

$$\sum_{i=1}^{n} (\nabla A)(u_i, u_i) = \text{grad}H$$

and combining this equation with Eq (3.4), we obtain $n\text{grad}H = \text{grad}H$. Since $n \geq 2$, we get $\text{grad}H = 0$, that is, H is a constant. Hence, we see that M is isometric to the small sphere $\mathbf{S}^n(1+H^2)$.

Conversely, suppose that the hypersurface M is isometric to the small sphere $\mathbf{S}^n(1+H^2)$. Then, from the introduction, it follows that the associated function f satisfies the Fischer–Marsden equation (cf. (2.6)) and that $\Delta f = -n(1+H^2)f$ implies that f has to be a non-trivial solution, for otherwise, we shall have $f = 0$ and $\mathbf{w} = 0$, which by equation (2.4) will imply $\sigma = 0$, and in turn Eq (2.3) will imply $\mathbf{u} = 0$. It will imply that $\overline{f} = 0$, and consequently, $Z = 0$, a contradiction. Moreover, we have

$$f\Delta f = -n(1+H^2)f^2,$$

which on integrating by parts, gives

$$\int_M \left\|\text{grad}f\right\|^2 = n(1+H^2)\int_M f^2.$$

Using $\mathbf{w} = \text{grad}f$, in above equation gives the equality

$$\int_M \|\mathbf{w}\|^2 = n(1+H^2)\int_M f^2.$$

Hence, the converse holds. □

In the following result, we shall use a lower bound on the integral of the Ricci curvature $Ric(\mathbf{w},\mathbf{w})$ of a compact non-totally geodesic hypersurface with non-zero potential function σ of the unit sphere $\mathbf{S}^{n+1}$, to find a characterization of a small sphere. Indeed we prove:

Theorem 2. *Let M be an orientable non-totally geodesic compact and connected hypersurface of the unit sphere $\mathbf{S}^{n+1}$, $n \geq 2$, with mean curvature H, induced vector field $\mathbf{w}$, non-zero support function σ. Then, the inequality*

$$\int_M Ric(\mathbf{w},\mathbf{w}) \geq (n-1)\int_M \left(n\left(\sigma^2H^2 - f^2\right) + 2\|\mathbf{w}\|^2\right)$$

holds if and only if H is a constant and M is isometric to the small sphere $\mathbf{S}^n(1+H^2)$.

Proof. Suppose M is an orientable non-totally geodesic compact and connected hypersurface of the unit sphere $\mathbf{S}^{n+1}$, $n \geq 2$, with a mean curvature H, induced vector field $\mathbf{w}$, and non-zero support function σ with the inequality

$$\int_M Ric(\mathbf{w},\mathbf{w}) \geq (n-1)\int_M \left(n\left(\sigma^2H^2 - f^2\right) + 2\|\mathbf{w}\|^2\right) \tag{4.3}$$

holds. Note that, by differentiating grad$\sigma = -A\mathbf{w}$, and using Eq (3.6), we have the following expression for the Hessian operator A_σ:

$$A_\sigma X = -\nabla_X Av = -\left[(\nabla A)(X,\mathbf{w}) + A(-fX + \sigma AX)\right], \quad X \in \mathfrak{X}(M),$$

that is,

$$A_\sigma X = -(\nabla A)(X,\mathbf{w}) + fAX - \sigma A^2X, \quad X \in \mathfrak{X}(M). \tag{4.4}$$

For a local orthonormal frame $\{u_1,..,u_n\}$ on M, using symmetry of the shape operator A and Eq (3.4), we have

$$\sum_{i=1}^n g\left((\nabla A)(u_i,\mathbf{w}), u_i\right) = \sum_{i=1}^n g\left(\mathbf{w}, (\nabla A)(u_i,u_i)\right) = n\mathbf{w}(H).$$

Taking trace in Eq (4.4), while using above equation, we get the following expression for the Laplacian $\Delta\sigma$

$$\Delta\sigma = -n\mathbf{w}(H) + nfH - \sigma\|A\|^2,$$

that is,

$$\sigma\Delta\sigma = -n\sigma\mathbf{w}(H) + n\sigma fH - \sigma^2\|A\|^2. \tag{4.5}$$

Note that Eq (3.6) gives, div$\mathbf{w} = n(-f + \sigma H)$, which implies

$$\mathrm{div}H(\sigma\mathbf{w}) = \sigma\mathbf{w}(H) + H\mathrm{div}(\sigma\mathbf{w}) = \sigma\mathbf{w}(H) + H(\mathbf{w}(\sigma) + n\sigma(-f + \sigma H)),$$

which on using grad$\sigma = -A\mathbf{w}$, gives

$$\mathrm{div}H(\sigma\mathbf{w}) = \sigma\mathbf{w}(H) - Hg(A\mathbf{w},\mathbf{w}) - nH\sigma f + n\sigma^2H^2.$$

Inserting the value of $\sigma\mathbf{w}(H)$ from above equation in Eq (4.5), we get

$$\sigma\Delta\sigma = -n\left(\mathrm{div}H(\sigma\mathbf{w}) + Hg(A\mathbf{w},\mathbf{w}) + nH\sigma f - n\sigma^2H^2\right) + n\sigma fH - \sigma^2\|A\|^2.$$

Integrating by parts the above equation, we get

$$-\int_M \left\|\mathrm{grad}\sigma\right\|^2 = \int_M \left(-nHg(A\mathbf{w},\mathbf{w}) - n(n-1)\sigma fH + n^2\sigma^2H^2 - \sigma^2\|A\|^2\right). \tag{4.6}$$

Now, using Eq (3.2) and grad$\sigma = -A\mathbf{w}$, that is,

$$-\int_M \left\|\mathrm{grad}\sigma\right\|^2 = \int_M \left(Ric(\mathbf{w},\mathbf{w}) - (n-1)\|\mathbf{w}\|^2 - nHg(A\mathbf{w},\mathbf{w})\right)$$

in Eq (4.6), we get

$$\int_M \left(Ric\,(\mathbf{w},\mathbf{w}) - (n-1)\,\|\mathbf{w}\|^2\right) = \int_M \left(-n(n-1)\sigma f H + n^2\sigma^2 H^2 - \sigma^2\,\|A\|^2\right),$$

that is,

$$\int_M \sigma^2\left(\|A\|^2 - nH^2\right) = \int_M \left(n(n-1)\sigma^2 H^2 - n(n-1)\sigma f H + (n-1)\,\|\mathbf{w}\|^2 - Ric\,(\mathbf{w},\mathbf{w})\right). \tag{4.7}$$

Also, using $\mathrm{grad} f = \mathbf{w}$, we get $\mathrm{div}\,(f\mathbf{w}) = \|\mathbf{w}\|^2 + f\mathrm{div}\,(\mathbf{w}) = \|\mathbf{w}\|^2 + nf\,(-f+\sigma H)$, that is,

$$nf\sigma H = \mathrm{div}\,(f\mathbf{w}) + nf^2 - \|\mathbf{w}\|^2\,.$$

Inserting above equation in the Eq (4.7), we get

$$\int_M \sigma^2\left(\|A\|^2 - nH^2\right) = \int_M \left(n(n-1)\left(\sigma^2H^2 - f^2\right) + 2(n-1)\,\|\mathbf{w}\|^2 - Ric\,(\mathbf{w},\mathbf{w})\right),$$

that is,

$$\int_M \sigma^2\left(\|A\|^2 - nH^2\right) = \int_M \left((n-1)\left[n\left(\sigma^2H^2 - f^2\right) + 2\,\|\mathbf{w}\|^2\right] - Ric\,(\mathbf{w},\mathbf{w})\right).$$

Using inequality (4.3), we conclude

$$\int_M \sigma^2\,\|A - HI\|^2 \leq 0,$$

that is, $\sigma^2\,\|A - HI\|^2 = 0$, which together with $\sigma \neq 0$ implies $A = HI$. Then, as $n \geq 2$, and the argument given in the Proof of above Theorem, we get H is constant and M is isometric to $\mathbf{S}^n(1+H^2)$.

Conversely, as M is non-totally geodesic hypersurface isometric to $\mathbf{S}^n(1+H^2)$, by Eq (2.4), we see $\sigma \neq 0$. Also, we have

$$Ric\,(\mathbf{w},\mathbf{w}) = (n-1)(1+H^2)\,\|\mathbf{w}\|^2 \tag{4.8}$$

and Eq (2.5) implies

$$\mathrm{div}\mathbf{w} = -n(1+H^2)f.$$

By using $\mathrm{div}\,(f\mathbf{w}) = \mathbf{w}\,(f) + f\mathrm{div}\mathbf{w} = \|\mathbf{w}\|^2 - n(1+H^2)f^2$, we get

$$\int_M \|\mathbf{w}\|^2 = n(1+H^2)\int_M f^2. \tag{4.9}$$

Using Eq (4.9) in the integral of Eq (4.8), we have

$$\int_M Ric\,(\mathbf{w},\mathbf{w}) = n(n-1)(1+H^2)^2\int_M f^2. \tag{4.10}$$

Now, using Eqs (2.4) and (4.9), we get

$$(n-1)\int\limits_{M}\left(n\left(\sigma^{2}H^{2}-f^{2}\right)+2\left\|\mathbf{w}\right\|^{2}\right)=(n-1)\int\limits_{M}\left(n\left(f^{2}H^{4}-f^{2}\right)+2n(1+H^{2})f^{2}\right),$$

that is,

$$(n-1)\int\limits_{M}\left(n\left(\sigma^{2}H^{2}-f^{2}\right)+2\left\|\mathbf{w}\right\|^{2}\right)=n(n-1)(1+H^{2})^{2}\int\limits_{M}f^{2}. \tag{4.11}$$

Equations (4.10) and (4.11) imply

$$\int\limits_{M}Ric\left(\mathbf{w},\mathbf{w}\right)=(n-1)\int\limits_{M}\left(n\left(\sigma^{2}H^{2}-f^{2}\right)+2\left\|\mathbf{w}\right\|^{2}\right).$$

Hence, all the requirements of the statement hold. □

5. Conclusions

In this paper, we asked whether the Fischer–Marsden equation is satisfied by the associated function f could be used to characterize small spheres in the unit sphere $\mathbf{S}^{n+1}$.

In the first result of this paper, we answered this question and obtained a characterization for a small sphere.

In yet other result, we obtained an interesting characterization of the small sphere using an appropriate lower bound on the integral of the Ricci curvature $Ric\left(\mathbf{w},\mathbf{w}\right)$.

It is known that for the small sphere $\mathbf{S}^{n}\left(1+H^{2}\right)$ in the unit sphere $\mathbf{S}^{n+1}$, its support function σ and the associated function f satisfies (see Eq (2.4))

$$\sigma=-Hf.$$

This initiates a natural question: Does a non-totally geodesic compact hypersurface M with support function σ, associated function f and mean curvature H of the unit sphere $\mathbf{S}^{n+1}$ satisfying the equation $\sigma=-Hf$ necessarily isometric to the small sphere $\mathbf{S}^{n}\left(1+H^{2}\right)$? Answering this question will be an interesting future study in the geometry of hypersurfaces of the unit sphere $\mathbf{S}^{n+1}$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) for funding and supporting this work through Research Partnership Program No. RP-21-09-10.

Conflict of interest

The authors declare no conflicts of interest.

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