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Research article

Three solutions for a three-point boundary value problem with instantaneous and non-instantaneous impulses

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Abstract: In this paper, we consider the multiplicity of solutions for the following three-point boundary value problem of second-order p-Laplacian differential equations with instantaneous and non-instantaneous impulses:

$$\begin{cases} -(\rho(t)\Phi_p(u'(t)))' + g(t)\Phi_p(u(t)) = \lambda f_j(t,u(t)), \quad t \in (s_j, t_{j+1}],\ j = 0,1,...,m, \\ \Delta(\rho(t_j)\Phi_p(u'(t_j))) = \mu I_j(u(t_j)), \quad j = 1,2,...,m, \\ \rho(t)\Phi_p(u'(t)) = \rho(t_j^+)\Phi_p(u'(t_j^+)), \quad t \in (t_j, s_j],\ j = 1,2,...,m, \\ \rho(s_j^+)\Phi_p(u'(s_j^+)) = \rho(s_j^-)\Phi_p(u'(s_j^-)), \quad j = 1,2,...,m, \\ u(0) = 0, \quad u(1) = \zeta u(\eta), \end{cases}$$

where $\Phi_p(u) := |u|^{p-2}u,\ p > 1,\ 0 = s_0 < t_1 < s_1 < t_2 < ... < s_{m_1} < t_{m_1+1} = \eta < ... < s_m < t_{m+1} = 1,\ \zeta > 0,\ 0 < \eta < 1$, $\Delta(\rho(t_j)\Phi_p(u'(t_j))) = \rho(t_j^+)\Phi_p(u'(t_j^+)) - \rho(t_j^-)\Phi_p(u'(t_j^-))$ for $u'(t_j^\pm) = \lim_{t\to t_j^\pm} u'(t)$, $j = 1,2,...,m$, and $f_j \in C((s_j,t_{j+1}]\times\mathbb{R},\mathbb{R})$, $I_j \in C(\mathbb{R},\mathbb{R})$. $\lambda \in (0,+\infty)$, $\mu \in \mathbb{R}$ are two parameters. $\rho(t) \geq 1$, $1 \leq g(t) \leq c$ for $t \in (s_j,t_{j+1}]$, $\rho(t),\ g(t) \in L^p[0,1]$, and c is a positive constant. By using variational methods and the critical points theorems of Bonanno-Marano and Ricceri, the existence of at least three classical solutions is obtained. In addition, several examples are presented to illustrate our main results.

Keywords: three-point boundary value problem; variational method; critical points theorem; instantaneous impulse; non-instantaneous impulse
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1. Introduction

In this paper, we consider the following problem:

$$\begin{cases} -(\rho(t)\Phi_p(u'(t)))' + g(t)\Phi_p(u(t)) = \lambda f_j(t,u(t)), \quad t \in (s_j, t_{j+1}],\ j = 0,1,...,m, \\ \Delta(\rho(t_j)\Phi_p(u'(t_j))) = \mu I_j(u(t_j)), \quad j = 1,2,...,m, \\ \rho(t)\Phi_p(u'(t)) = \rho(t_j^+)\Phi_p(u'(t_j^+)), \quad t \in (t_j, s_j],\ j = 1,2,...,m, \\ \rho(s_j^+)\Phi_p(u'(s_j^+)) = \rho(s_j^-)\Phi_p(u'(s_j^-)), \quad j = 1,2,...,m, \\ u(0) = 0, \quad u(1) = \zeta u(\eta), \end{cases} \tag{1.1}$$

where $\Phi_p(u) := |u|^{p-2}u,\ p > 1,\ 0 = s_0 < t_1 < s_1 < t_2 < ... < s_{m_1} < t_{m_1+1} = \eta < ... < s_m < t_{m+1} = 1,\ \zeta > 0,\ 0 < \eta < 1$, $\Delta(\rho(t_j)\Phi_p(u'(t_j))) = \rho(t_j^+)\Phi_p(u'(t_j^+)) - \rho(t_j^-)\Phi_p(u'(t_j^-))$ for $u'(t_j^{\pm}) = \lim\limits_{t\to t_j^{\pm}} u'(t)$, $j = 1,2,...,m$, and $f_j \in C((s_j, t_{j+1}]\times\mathbb{R},\mathbb{R})$, $I_j \in C(\mathbb{R},\mathbb{R})$. $\lambda \in (0,+\infty)$, $\mu \in \mathbb{R}$ are two parameters. $\rho(t) \ge 1$, $1 \le g(t) \le c$ for $t \in (s_j, t_{j+1}]$, $\rho(t),\ g(t) \in L^p[0,1]$, and c is a positive constant. The instantaneous impulses occur at the points t_j and the non-instantaneous impulses continue on the intervals $(t_j, s_j]$.

In recent years, the study of the differential equations has received extensive attention owing to their wide applications in many different areas of science and technology [1–4], especially the differential equations with impulses. It is worth noting that there are two popular types of impulses in the literature, that is, instantaneous impulses and non-instantaneous impulses. As far as we know, the instantaneous impulse was first presented by Milman-Myshkis [5] and the non-instantaneous impulse was first introduced by Hernández-O'Regan [6]. More details of these two types are given in [7]. Up to now, there are many methods that has been applied to study the differential equations with impulsive effects, such as fixed point theorem, topological degree theory, upper and lower solutions method, and theory of analytic semigroup, see for instance [6, 8–12].

Since the pioneering works of Tian-Ge [13] and Nieto-O'Regan [14], variational approach has become one of the important methods in the study of impulsive differential equations [15–19]. Recently, the study of existence and multiplicity of solutions for the differential equations with both instantaneous and non-instantaneous impulses by using variational methods and critical point theory has gained much attention. In [20], Tian-Zhang first considered the following second-order differential equations with instantaneous and non-instantaneous impulses:

$$\begin{cases} -u''(t) = f_j(t,u(t)), \quad t \in (s_j, t_{j+1}],\ j = 0,1,2,...,m, \\ \Delta u'(t_j) = I_j(u(t_j)), \quad j = 1,2,...,m, \\ u'(t) = u'(t_j^+), \quad t \in (t_j, s_j],\ j = 1,2,...,m, \\ u'(s_j^+) = u'(s_j^-), \quad j = 1,2,...,m, \\ u(0) = u(T) = 0, \end{cases} \tag{1.2}$$

where $0 = s_0 < t_1 < s_1 < t_2 < s_2 < ... < s_m < t_{m+1} = T$, $f_j \in C((s_j, t_{j+1}] \times \mathbb{R}, \mathbb{R})$, $I_j \in C(\mathbb{R},\mathbb{R})$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$, the instantaneous impulses occur at the points t_j and the non-instantaneous impulses continue on the intervals $(t_j, s_j]$. The authors obtained the problem (1.2) has at least one classical solution by applying Ekeland's variational principle. From then on, the different types of the differential equations with instantaneous and non-instantaneous impulses were investigated by means of variational methods and some excellent and interesting results were obtained, see for instance [21–25].

On the other hand, there has been increasing interest in studying three-point boundary value problems of differential equations due to their extensive applications in physics and engineering in recent years. The existence results for three-point boundary value problems have been studied by many different methods [26–30], such as fixed-point theory, upper and lower solutions method and variational approach. Especially, in [30], Lian-Bai-Du used variational method to consider the following three-point boundary value problem:

$$\begin{cases} (P(t)u'(t))' + f(t,u(t)) = 0, \quad a.e.\ 0 < t < 1, \\ u(0) = 0, \quad u(1) = \zeta u(\eta), \end{cases} \tag{1.3}$$

where $P : [0,1] \to \mathbb{R}^{n\times n}$ is a continuously symmetric matrix. $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function and locally Lipschitz continuous. The interesting point of the paper is the boundary value conditions are imposed on an appropriate space rather than the functionals. Thus, the authors proposed a different idea to deal with the non-local boundary value problem (1.3) and gave the variational structure. Finally, they proved that the problem (1.3) possess a nontrivial solution, a positive and a negative solution by using mountain pass lemma. Inspired by the study of [30], Wei-Shang-Bai [31] first considered the following second-order p-Laplacian differential equations involving instantaneous and non-instantaneous impulses with three-point boundary conditions:

$$\begin{cases} -(\rho(t)\Phi_p(u'(t)))' + g(t)\Phi_p(u(t)) = f_j(t,u(t)), \quad t \in (s_j, t_{j+1}],\ j = 0,1,...,m, \\ \Delta(\rho(t_j)\Phi_p(u'(t_j))) = I_j(u(t_j)), \quad j = 1,2,...,m, \\ \rho(t)\Phi_p(u'(t)) = \rho(t_j^+)\Phi_p(u'(t_j^+)), \quad t \in (t_j, s_j],\ j = 1,2,...,m, \\ \rho(s_j^+)\Phi_p(u'(s_j^+)) = \rho(s_j^-)\Phi_p(u'(s_j^-)), \quad j = 1,2,...,m, \\ u(0) = 0, \quad u(1) = \zeta u(\eta), \end{cases} \tag{1.4}$$

where $\Phi_p(u) := |u|^{p-2}u,\ p > 1,\ \rho(t),\ g(t) \in L^p[0,1],\ 0 = s_0 < t_1 < s_1 < ... < s_{m_1} = \eta < t_{m_1+1} < ... < s_m < t_{m+1} = 1,\ \zeta > 0,\ 0 < \eta < 1$, and $\Delta(\rho(t_j)\Phi_p(u'(t_j))) = \rho(t_j^+)\Phi_p(u'(t_j^+)) - \rho(t_j^-)\Phi_p(u'(t_j^-))$ for $u'(t_j^\pm) = \lim_{t\to t_j^\pm} u'(t),\ j = 1,2,...,m$, and $f_j \in C((s_j,t_{j+1}] \times \mathbb{R}, \mathbb{R}),\ I_j \in C(\mathbb{R},\mathbb{R})$. The authors obtained that the problem (1.4) has at least two classical solutions and infinitely many classical solutions by the virtue of variational methods and critical point theory. On the basis of [31], Yao [32] revisited the problem (1.4) and obtained the existence of at least one classical solution and infinitely many classical solutions by applying the minimization methods, mountain pass theorem and symmetric mountain pass theorem.

To the best of our knowledge, the study of solutions for a three-point boundary value problem with instantaneous and non-instantaneous impulses using variational methods has received considerably less attention. Motivated by the above mentioned works, in this paper, our aim is to study the existence of at least three classical solutions of the problem (1.1) via three critical points theorems obtained by Bonanno-Marano [33] and Ricceri [34]. Our main results are obtained depending on two parameters μ and λ. In addition, the problem (1.1) is reduced to the problem (1.4) when $\mu = \lambda = 1$. Consequently, our work will generalize the existing results in [31, 32].

The rest of this paper is arranged as follows. In Section 2, we give some preliminary results. In Section 3, we will present and prove our main results. Finally, in Section 4, two examples are given to verify our results.

2. Preliminaries

In this section, we first introduce some necessary definitions, lemmas and theorems.

Theorem 2.1. *[33, 35] Let X be a reflexive real Banach space, $\varphi : X \to \mathbb{R}$ be a sequentially weakly lower semi-continuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$\inf_{x\in X} \varphi(x) = \varphi(0) = \psi(0) = 0.$$

Assume that there exist $r > 0$ and $\tilde{x} \in X$, with $r < \varphi(\tilde{x})$ such that

(i) $\sup\{\psi(x) : \varphi(x) \le r\} < r\frac{\psi(\tilde{x})}{\varphi(\tilde{x})}$,

(ii) for each $\lambda \in \Lambda_r = \left(\frac{\varphi(\tilde{x})}{\psi(\tilde{x})}, \frac{r}{\sup\{\psi(x):\varphi(x)\le r\}}\right)$, the functional $\varphi - \lambda\psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$, the functional $\varphi - \lambda\psi$ has at least three distinct critical points in X.

Remark 2.1. *[24, 35] In Theorem 2.1, if* $\sup\{\psi(x) : \varphi(x) \le r\} = 0$*, then it is possible to consider the interval of parameters $\left(\frac{\varphi(\tilde{x})}{\psi(\tilde{x})}, +\infty\right)$.*

Definition 2.1. *[34] If X is a real Banach space, we denote by Γ_X the class of all functionals $\varphi : X \to \mathbb{R}$ possessing the following property: if $\{x_n\}$ is a sequence in X converging weakly to $x \in X$ and* $\liminf_{n\to\infty} \varphi(x_n) \le \varphi(x)$*, then $\{x_n\}$ has a subsequence converging strongly to x.*

Theorem 2.2. *[34] Let X be a separable and reflexive real Banach space; let $\varphi : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semi-continuous C^1 functional, bounded on each bounded subset of X, appertaining to Γ_X and whose derivative has a continuous inverse on X^*; $w : X \to \mathbb{R}$ is a C^1 functional with compact derivative. Suppose that there exists a strict local minimum x_0 of φ such that $\varphi(x_0) = w(x_0) = 0$. Finally, setting*

$$\rho_1 = \max\left\{0, \limsup_{\|x\|\to+\infty} \frac{w(x)}{\varphi(x)}, \limsup_{\|x\|\to x_0} \frac{w(x)}{\varphi(x)}\right\}, \quad \rho_2 = \sup_{x\in\varphi^{-1}(0,+\infty)} \frac{w(x)}{\varphi(x)},$$

assume that $\rho_1 < \rho_2$. Then, for each compact interval $[\theta_1, \theta_2] \subset \left(\frac{1}{\rho_2}, \frac{1}{\rho_1}\right)$ (with the conventions $\frac{1}{0} = +\infty$, $\frac{1}{+\infty} = 0$), there exists $R > 0$ satisfying the property: for each $\lambda \in [\theta_1, \theta_2]$ and any C^1 functional $\phi : X \to \mathbb{R}$ with compact derivative, there exists $\xi > 0$ such that, for each $\mu \in [0, \xi]$, the equation $\varphi'(x) - \mu\phi'(x) - \lambda w'(x) = 0$ has at least three solutions in X whose norms are less than R.

Let $X = \{u \in W^{1,p}([0, 1], \mathbb{R}) : u(0) = 0, u(1) = \zeta u(\eta)\}$ with the norm

$$\|u\|_X = \left(\int_0^1 (\rho(t)|u'(t)|^p + g(t)|u(t)|^p)dt\right)^{\frac{1}{p}}.$$

As shown in [36], X is a separable and reflexive real Banach space. According to [31], we can obtain that

$$\|u\| = \left(\int_0^1 \rho(t)|u'(t)|^p dt\right)^{\frac{1}{p}}, \quad \forall u \in X$$

is equivalent to the norm $\|u\|_X$, i.e., there exist $c_0 \ge \left(\frac{c}{\rho(t)} + 1\right)^{\frac{1}{p}}$, such that

$$\|u\| \le \|u\|_X \le c_0\|u\|.$$

The functionals $\varphi : X \to \mathbb{R}$ and $\psi : X \to \mathbb{R}$ are defined as follows:

$$\varphi(u) = \frac{1}{p}\|u\|^p + \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} g(t)|u(t)|^p dt, \tag{2.1}$$

$$\psi(u) = \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} F_j(t, u(t))dt - \frac{\mu}{\lambda} \sum_{j=1}^{m} J_j(u(t_j)), \tag{2.2}$$

where $F_j(t, u) = \int_0^u f_j(t, s)ds$ and $J_j(u) = \int_0^u I_j(s)ds$. It is clear that $\inf_{u\in X} \varphi(u) = \varphi(0) = 0$ and $\psi(0) = \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} F_j(t, 0)dt - \frac{\mu}{\lambda} \sum_{j=1}^{m} J_j(0) = 0$. By using the continuity of f_j, $j = 0, 1, ..., m$ and I_j, $j = 1, 2, ..., m$, we can obtain φ and ψ are continuous Gâteaux differentiable. For any $v \in X$, we have

$$\langle \varphi'(u), v\rangle = \int_0^1 \rho(t)|u'(t)|^{p-2}u'(t)v'(t)dt + \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} g(t)|u(t)|^{p-2}u(t)v(t)dt, \tag{2.3}$$

$$\langle \psi'(u), v\rangle = \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} f_j(t, u(t))v(t)dt - \frac{\mu}{\lambda} \sum_{j=1}^{m} I_j(u(t_j))v(t_j). \tag{2.4}$$

Lemma 2.1. *[30, Lemma 2.5] and [31, Lemma 1] The space X is compactly embedded in* $C([0, 1], \mathbb{R})$.

Lemma 2.2. *[31, Lemma 2] For each $u \in X$, there is $\|u\|_\infty \le \|u\|$.*

Lemma 2.3. *A function $u \in X$ is a weak solution of the problem* (1.1)*, then the following identity*

$$\int_0^1 \rho(t)|u'(t)|^{p-2}u'(t)v'(t)dt + \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} g(t)|u(t)|^{p-2}u(t)v(t)dt + \mu \sum_{j=1}^{m} I_j(u(t_j))v(t_j)$$
$$= \lambda \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} f_j(t, u(t))v(t)dt$$

holds for any $v \in X$.

Lemma 2.4. *If $u \in X$ is a weak solution of the problem* (1.1)*, then u is a classical solution of the problem* (1.1).

Remark 2.2. *The proofs of Lemmas 2.3 and 2.4 are similar to that of Lemma 6 in [31], so we omit them. In addition, from Lemma 2.3, the critical points of $\varphi - \lambda\psi$ are weak solutions of the problem* (1.1)*. According to Lemma 2.4, the weak solution of the problem* (1.1) *is also a classical one.*

3. Main results

In this section, our main results are proved by using two kinds of three critical points theorems.

Theorem 3.1. *Assume that the following conditions hold:*

($H1$) *There exist positive constants $K_0,\ K_1, ..., K_m,\ L_1,\ L_2, ..., L_m,\ k,\ l_1,\ l_2, ..., l_m$ with $k < p$ and $l_j < p$, $j = 1, 2, ..., m$ such that for all $t \in [0, 1]$, $u \in \mathbb{R}$,*

$$F_j(t, u) \le K_j\left(1 + |u|^k\right), \quad -J_j(u) \le L_j\left(1 + |u|^{l_j}\right).$$

$(H2)$ *There exist* $r > 0$ *and* $\tilde{u} \in X$ *such that* $\|\tilde{u}\|^p + p\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} g(t)|\tilde{u}(t)|^p dt > pr$,

$$\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j(t,\tilde{u}(t))dt > 0, \quad \sum_{j=1}^{m} J_j(\tilde{u}(t_j)) > 0$$

and the following inequality holds:

$$A_l := \frac{\frac{1}{p}\|\tilde{u}\|^p + \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} g(t)|\tilde{u}(t)|^p dt}{\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j(t,\tilde{u}(t))dt} < A_r := \frac{r}{\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} \max\limits_{|u|\le(pr)^{\frac{1}{p}}} F_j(t,u(t))dt}. \tag{3.1}$$

Then, for every $\lambda \in \Lambda_r = (A_l, A_r)$, *there exists*

$$\gamma := \min\left\{\frac{r - \lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} \max\limits_{|u|\le(pr)^{\frac{1}{p}}} F_j(t,u(t))dt}{\max\limits_{|u|\le(pr)^{\frac{1}{p}}} \sum_{j=1}^{m}(-J_j(u))}, \frac{\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j(t,\tilde{u}(t))dt - \frac{1}{p}\|\tilde{u}\|^p - \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} g(t)|\tilde{u}(t)|^p dt}{\sum_{j=1}^{m} J_j(\tilde{u}(t_j))}\right\}$$

such that, for each $\mu \in [0,\gamma)$, *the problem* (1.1) *has at least three classical solutions.*

Proof. we need three steps to complete the proof.

Step 1. The functional φ is sequentially weakly lower semi-continuous, coercive and its derivative admits a continuous inverse on X^*.

Suppose that $\{u_n\} \in X$, $u_n \rightharpoonup u$ as $n \to \infty$. The continuity and convexity of $\|u\|^p$ imply $\|u\|^p$ is sequentially weakly lower semi-continuous. Moreover, by Lemma 2.1, $\{u_n\}$ is convergent uniformly to u in $C([0,1])$. So

$$\liminf_{n\to\infty}\varphi(u_n) = \frac{1}{p}\|u_n\|^p + \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} g(t)|u_n(t)|^p dt \ge \frac{1}{p}\|u\|^p + \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} g(t)|u(t)|^p dt = \varphi(u).$$

Thus, φ is a sequentially weakly lower semi-continuous functional. From (2.1), we have $\varphi(u) \ge \frac{1}{p}\|u\|^p$, which shows that $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$. Thus, φ is coercive.

Next, we prove that φ' admits a continuous inverse on X^*. In fact, for any $u \in X\backslash\{0\}$, by (2.3), we have

$$\begin{aligned}\lim_{\|u\|\to+\infty}\frac{\langle\varphi'(u),u\rangle}{\|u\|} &= \lim_{\|u\|\to+\infty}\frac{\|u\|^p + \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} g(t)|u(t)|^p dt}{\|u\|}\\ &= \lim_{\|u\|\to+\infty}\|u\|^{p-1} + \frac{\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} g(t)|u(t)|^p dt}{\|u\|}\\ &= +\infty,\end{aligned}$$

which implies that φ' is coercive. For any $u,\ v \in X$,

$$\begin{aligned}\langle \varphi'(u) - \varphi'(v), u - v\rangle = &\int_0^1 \rho(t)(|u'(t)|^{p-2}u'(t) - |v'(t)|^{p-2}v'(t))(u'(t) - v'(t))dt \\ &+ \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} g(t)(|u(t)|^{p-2}u(t) - |v(t)|^{p-2}v(t))(u(t) - v(t))dt.\end{aligned}$$

By [37, Eq (2.2)], there exist constants $c_p, d_p > 0$, such that

$$\begin{aligned}\langle \varphi'(u) - \varphi'(v), u - v\rangle \geq &\begin{cases} c_p\left(\int_0^1 \rho(t)|u'(t) - v'(t)|^p dt + \sum_{j=0}^m \int_{s_j}^{t_{j+1}} g(t)|u(t) - v(t)|^p dt\right), & p \geq 2,\\ d_p\left(\int_0^1 \frac{\rho(t)|u'(t)-v'(t)|^2}{(|u'(t)|+|v'(t)|)^{2-p}}dt + \sum_{j=0}^m \int_{s_j}^{t_{j+1}} \frac{g(t)|u(t)-v(t)|^2}{(|u(t)|+|v(t)|)^{2-p}}dt\right), & 1 < p < 2.\end{cases}\\ \geq &\begin{cases} c_p \int_0^1 \rho(t)|u'(t) - v'(t)|^p dt, & p \geq 2,\\ d_p \int_0^1 \frac{\rho(t)|u'(t)-v'(t)|^2}{(|u'(t)|+|v'(t)|)^{2-p}}dt, & 1 < p < 2.\end{cases}\end{aligned}$$

If $p \geq 2$, we have

$$\langle \varphi'(u) - \varphi'(v), u - v\rangle \geq c_p\|u - v\|^p. \tag{3.2}$$

If $1 < p < 2$, by the Hölder's inequality, we find that

$$\begin{aligned}\int_0^1 \rho(t)|u'(t) - v'(t)|^p dt \leq &\left(\int_0^1 \frac{\rho(t)|u'(t) - v'(t)|^2}{(|u'(t)| + |v'(t)|)^{2-p}}dt\right)^{\frac{p}{2}} \left(\int_0^1 \rho(t)(|u'(t)| + |v'(t)|)^p dt\right)^{\frac{2-p}{2}} \\ \leq &2^{\frac{(p-1)(2-p)}{2}} \left(\int_0^1 \frac{\rho(t)|u'(t) - v'(t)|^2}{(|u'(t)| + |v'(t)|)^{2-p}}dt\right)^{\frac{p}{2}} (\|u\| + \|v\|)^{\frac{(2-p)p}{2}}.\end{aligned} \tag{3.3}$$

It follows from $1 < p < 2$ and (3.3) that

$$\langle \varphi'(u) - \varphi'(v), u - v\rangle \geq \frac{2^{\frac{(p-1)(p-2)}{p}} d_p}{(\|u\| + \|v\|)^{2-p}} \left(\int_0^1 \rho(t)|u'(t) - v'(t)|^p dt\right)^{\frac{2}{p}} = \frac{2^{\frac{(p-1)(p-2)}{p}} d_p\|u - v\|^2}{(\|u\| + \|v\|)^{2-p}}. \tag{3.4}$$

In view of (3.2) and (3.4), we know φ' is uniformly monotone. By [38, Theorem 26.A(d)], we see that $(\varphi')^{-1}$ exists and is continuous on X^*.

Step 2. $\psi' : X \to X^*$ is a continuous and compact functional.

Obviously, ψ' is continuous. Next, we mainly prove that $\psi' : X \to X^*$ is a compact functional. Assume that $u_n \rightharpoonup u$ in X as $n \to \infty$, then $\{u_n\} \subset X$ converges uniformly to u in $C[0, 1]$. Owing to the functions $f_j \in C((s_j, t_{j+1}] \times \mathbb{R}, \mathbb{R})$ and $I_j \in (\mathbb{R}, \mathbb{R})$, we have $f_j(t, u_n) \to f(t, u)$ and $I_j(u_n(t_j)) \to I_j(u(t_j))$ as $n \to \infty$. Therefrom, we obtain $\psi'(u_n) \to \psi'(u)$ as $n \to \infty$. Thus, ψ' is strongly continuous on X. Furthermore, by [38, Proposition 26.2], we can conclude that ψ' is a compact operator.

Step 3. The conditions (i) and (ii) of Theorem 2.1 are satisfied.

Let $u \in X$ with $\varphi(u) \leq r$, then by (2.1) and Lemma 2.2, we have $\varphi(u) \geq \frac{1}{p}\|u\|^p > \frac{1}{p}\|u\|_\infty^p$. It follows that

$$\{u \in X : \varphi(u) \leq r\} \subseteq \left\{u : \frac{1}{p}\|u\|_\infty^p \leq r\right\} = \left\{u : \|u\|_\infty \leq (pr)^{\frac{1}{p}}\right\}.$$

In view of $\lambda > 0, \mu \geq 0$, we have

$$\sup\{\psi(u) : \varphi(u) \leq r\} = \sup\left\{\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j(t, u(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^{m} J_j(u(t_j)) : \varphi(u) \leq r\right\}$$
$$\leq \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} \max_{|u|\leq(pr)^{\frac{1}{p}}} F_j(t, u(t))dt + \frac{\mu}{\lambda}\max_{|u|\leq(pr)^{\frac{1}{p}}}\sum_{j=1}^{m}(-J_j(u)).$$

If $\max_{|u|\leq(pr)^{\frac{1}{p}}} \sum_{j=1}^{m}(-J_j(u)) = 0$, then using $\lambda < A_r$, one has

$$\sup\{\psi(u) : \varphi(u) \leq r\} < \frac{r}{\lambda}. \tag{3.5}$$

If $\max_{|u|\leq(pr)^{\frac{1}{p}}} \sum_{j=1}^{m}(-J_j(u)) > 0$, then from $\mu \in [0, \gamma)$, the inequality (3.5) also holds.
On the other hand, by $\mu < \gamma$, we have

$$\psi(\tilde{u}) = \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j(t, \tilde{u}(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^{m} J_j(\tilde{u}(t_j)) > \frac{\varphi(\tilde{u})}{\lambda}. \tag{3.6}$$

By combining (3.5) and (3.6), we get

$$\frac{\psi(\tilde{u})}{\varphi(\tilde{u})} > \frac{1}{\lambda} > \frac{\sup\{\psi(u) : \varphi(u) \leq r\}}{r}.$$

Hence, the condition (i) in Theorem 2.1 holds. Finally, we will show that, for each $\lambda \in \Lambda_r = (A_l, A_r)$, the functional $\varphi - \lambda\psi$ is coercive. For any $u \in X$, by (H1), we obtain

$$\begin{aligned}\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j(t, u(t))dt &\leq \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} K_j\left(1 + |u(t)|^k\right)dt \\ \leq \left(1 + \|u\|_\infty^k\right)\sum_{j=0}^{m} K_j(t_{j+1} - s_j) &\leq \left(1 + \|u\|^k\right)\sum_{j=0}^{m} K_j(t_{j+1} - s_j)\end{aligned} \tag{3.7}$$

and

$$\sum_{j=1}^{m}(-J_j(u(t_j))) \leq \sum_{j=1}^{m} L_j\left(1 + |u(t_j)|^{l_j}\right) \leq \sum_{j=1}^{m} L_j\left(1 + \|u\|_\infty^{l_j}\right) \leq \sum_{j=1}^{m} L_j\left(1 + \|u\|^{l_j}\right). \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\psi(u) \leq \left(1 + \|u\|^k\right)\sum_{j=0}^{m} K_j(t_{j+1} - s_j) + \frac{\mu}{\lambda}\sum_{j=1}^{m} L_j\left(1 + \|u\|^{l_j}\right)$$
$$= \sum_{j=0}^{m} K_j(t_{j+1} - s_j) + \frac{\mu}{\lambda}\sum_{j=1}^{m} L_j + \|u\|^k\sum_{j=0}^{m} K_j(t_{j+1} - s_j) + \frac{\mu}{\lambda}\sum_{j=1}^{m} L_j\|u\|^{l_j}.$$

Hence,

$$\varphi(u) - \lambda\psi(u) \geq \frac{1}{p}\|u\|^p + \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} g(t)|u(t)|^p dt - \left(\lambda\sum_{j=0}^{m} K_j(t_{j+1} - s_j) + \mu\sum_{j=1}^{m} L_j\right.$$
$$\left. + \lambda\|u\|^k\sum_{j=0}^{m} K_j(t_{j+1} - s_j) + \mu\sum_{j=1}^{m} L_j\|u\|^{l_j}\right)$$
$$\geq \frac{1}{p}\|u\|^p - \lambda\sum_{j=0}^{m} K_j(t_{j+1} - s_j) - \mu\sum_{j=1}^{m} L_j - \lambda\|u\|^k\sum_{j=0}^{m} K_j(t_{j+1} - s_j) - \mu\sum_{j=1}^{m} L_j\|u\|^{l_j}.$$

Notice that $k < p$ and $l_j < p$, $j = 1, 2, ..., m$, we get $\varphi - \lambda\psi$ is coercive on X. Therefore, applying Theorem 2.1, we obtain that the functional $\varphi - \lambda\psi$ has at least three different critical points, i.e., the problem (1.1) has at least three distinct classical solutions. □

We note that the parameter μ is positive values in Theorem 3.1. In fact, we can consider negative values for the parameter μ and have the following result.

Theorem 3.2. *Assume that the following conditions hold:*

($H1^*$) *There exist positive constants $K_0, K_1, ..., K_m, L_1, L_2, ..., L_m, k, l_1, l_2, ..., l_m$ with $k < p$ and $l_j < p$, $j = 1, 2, ..., m$, such that for all $t \in [0, 1]$, $u \in \mathbb{R}$,*

$$F_j(t, u) \leq K_j\left(1 + |u|^k\right), \quad J_j(u) \leq L_j\left(1 + |u|^{l_j}\right).$$

($H2^*$) *There exist $r > 0$ and $\tilde{u} \in X$ such that $\|\tilde{u}\|^p + p\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} g(t)|\tilde{u}(t)|^p dt > pr$,*

$$\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j(t, \tilde{u}(t))dt > 0, \quad \sum_{j=1}^{m} J_j(\tilde{u}(t_j)) < 0$$

and (3.1) *holds.*

Then, for every $\lambda \in \Lambda_r = (A_l, A_r)$, there exists

$$\gamma^* := \min\left\{\frac{\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} \max_{|u|\leq(pr)^{\frac{1}{p}}} F_j(t, u(t))dt - r}{\max_{|u|\leq(pr)^{\frac{1}{p}}} \sum_{j=1}^{m} J_j(u)},\right.$$
$$\left.\frac{\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j(t, \tilde{u}(t))dt - \frac{1}{p}\|\tilde{u}\|^p - \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} g(t)|\tilde{u}(t)|^p dt}{\sum_{j=1}^{m} J_j(\tilde{u}(t_j))}\right\}$$

such that, for each $\mu \in (\gamma^*, 0]$, the problem (1.1) *has at least three classical solutions.*

Proof. Similarly to the proof of Theorem 3.1, we can prove that Theorem 3.2 holds. Next, we mainly demonstrate that the conditions (i) and (ii) of Theorem 2.1 are fulfilled. Since $\lambda > 0$ and $\mu \in (\gamma^*, 0]$, we have

$$\sup\{\psi(u) : \varphi(u) \leq r\} = \sup\left\{\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j(t, u(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^{m} J_j(u(t_j)) : \varphi(u) \leq r\right\}$$

$$\leq \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} \max_{|u|\leq (pr)^{\frac{1}{p}}} F_j(t,u(t))dt - \frac{\mu}{\lambda}\max_{|u|\leq (pr)^{\frac{1}{p}}}\sum_{j=1}^{m} J_j(u) < \frac{r}{\lambda},$$

which holds if $\max_{|u|\leq (pr)^{\frac{1}{p}}} \sum_{j=1}^{m} J_j(u) = 0$ since $\lambda < A_r$ and it is also true for $\mu \in (\gamma^*, 0]$ if $\max_{|u|\leq (pr)^{\frac{1}{p}}} \sum_{j=1}^{m} J_j(u) > 0$.

On the other hand, for $\mu \in (\gamma^*, 0]$, we can obtain

$$\psi(\tilde{u}) = \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j(t,\tilde{u}(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^{m} J_j(\tilde{u}(t_j)) > \frac{\varphi(\tilde{u})}{\lambda}.$$

A simple computation yields

$$\frac{\psi(\tilde{u})}{\varphi(\tilde{u})} > \frac{1}{\lambda} > \frac{\sup\{\psi(u) : \varphi(u) \leq r\}}{r}.$$

Now, we show that the functional $\varphi - \lambda\psi$ is coercive for each $\lambda \in \Lambda_r = (A_l, A_r)$. By (H1*), we can obtain

$$\sum_{j=1}^{m} J_j(u(t_j)) \leq \sum_{j=1}^{m} L_j\left(1 + \|u\|^{l_j}\right). \tag{3.9}$$

For all $u \in X$, combining with (3.7) and (3.9), we have

$$\begin{aligned}\varphi(u) - \lambda\psi(u) \geq &\frac{1}{p}\|u\|^p + \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} g(t)|u(t)|^p dt - \lambda\sum_{j=0}^{m} K_j(t_{j+1} - s_j) + \mu\sum_{j=1}^{m} L_j\\ &- \lambda\|u\|^k\sum_{j=0}^{m} K_j(t_{j+1} - s_j) + \mu\sum_{j=1}^{m} L_j\|u\|^{l_j}\\ \geq &\frac{1}{p}\|u\|^p - \lambda\sum_{j=0}^{m} K_j(t_{j+1} - s_j) + \mu\sum_{j=1}^{m} L_j - \lambda\|u\|^k\sum_{j=0}^{m} K_j(t_{j+1} - s_j) + \mu\sum_{j=1}^{m} L_j\|u\|^{l_j}.\end{aligned}$$

□

Since $k < p$ and $l_j < p$, $j = 1, 2, ..., m$, the coercivity of the functional $\varphi - \lambda\psi$ is obtained.

Theorem 3.3. *Suppose that there exist non-negative constants k_j, $j = 0, 1, ..., m$, and a function $\vartheta(t) \in X\backslash\{0\}$, such that*

$(H3)$ $\max\left\{\limsup_{|u|\to 0} \frac{F_j(t,u)}{|u|^p}, \limsup_{|u|\to\infty} \frac{F_j(t,u)}{|u|^p}\right\} \leq k_j, \quad j = 0, 1, ..., m.$

$(H4)$ $pc_0^p \max_{0\leq j\leq m}\{k_j\} < \frac{\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j(t,\vartheta(t))dt}{\|\vartheta\|^p}.$

Then, for each compact interval $[\theta_1, \theta_2] \subset \left(\frac{1}{\rho_2}, \frac{1}{\rho_1}\right)$ (ρ_1 and ρ_2 are as defined in Theorem 2.2), there exists $R > 0$ satisfying the property: for every $\lambda \in [\theta_1, \theta_2]$, there exists $\xi > 0$ such that, for each $\mu \in [0, \xi]$, the problem (1.1) *has at least three classical solutions u_i in X with $\|u_i\| < R$, $i = 1, 2, 3$.*

Proof. Define the functionals $\phi, w : X \to \mathbb{R}$ as follows:

$$\phi(u) = -\sum_{j=1}^{m} \int_0^{u(t_j)} I_j(s)ds, \quad w(u) = \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} F_j(t, u(t))dt. \tag{3.10}$$

Clearly, ϕ and w are continuous Gâteaux differentiable and their Gâteaux derivatives are

$$\langle \phi'(u), v\rangle = -\sum_{j=1}^{m} I_j(u(t_j))v(t_j), \quad \langle w'(u), v\rangle = \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} f_j(t, u(t))v(t)dt,$$

respectively, for any $v \in X$. Obviously, the weak solutions to the problem (1.1) are the corresponding critical points of functional $\varphi - \mu\phi - \lambda w$.

From the proof of Theorem 3.1, φ is a coercive, sequentially weakly lower semi-continuous C^1 functional whose Gâteaux derivative admits a continuous inverse on X^*. Analogous to [39], we can obtain φ belongs to Γ_X. Additionally, we can get that φ is bounded on each bounded subset of X. Assume that M is the bound of a subset of X, i.e, $\|u\| \le M$. Then, by (2.1), we have

$$\varphi(u) = \frac{1}{p}\|u\|^p + \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} g(t)|u(t)|^p dt \le \left(\frac{1}{p} + c\right)\|u\|^p \le \left(\frac{1}{p} + c\right) M^p.$$

Similarly to the proof of Theorem 3.1, we can obtain the derivatives of ϕ and w are compact. Moreover, φ has a strict local minimum 0 with $\varphi(0) = w(0) = 0$.

On the other hand, according to (H3), there exist $\varepsilon_1, \varepsilon_2 > 0$, such that

$$F_j(t, u(t)) \le k_j|u(t)|^p, \quad t \in [0, 1],\ |u| \in (0, \varepsilon_1) \cup (\varepsilon_2, +\infty). \tag{3.11}$$

By the continuity of F_j, $j = 0, 1, ..., m$, we know that $F_j(t, u(t))$ is bounded for any $|u| \in [\varepsilon_1, \varepsilon_2]$. Thus, we can choose $r > 0$ and $\sigma > p$, such that

$$F_j(t, u(t)) \le k_j|u(t)|^p + r|u(t)|^\sigma, \quad \text{for } t \in [0, 1],\ u \in \mathbb{R}. \tag{3.12}$$

Denote $k^* = \max_{0\le j\le m}\{k_j\}$, by (3.12) and Lemma 2.2, we have

$$w(u) = \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} F_j(t, u(t))dt \le k^*\|u\|^p + r\|u\|^\sigma.$$

Hence,

$$\limsup_{u\to 0} \frac{w(u)}{\varphi(u)} \le \limsup_{u\to 0} \frac{k^*\|u\|^p + r\|u\|^\sigma}{\frac{1}{p}\|u\|^p + \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} g(t)|u(t)|^p dt} \le \limsup_{u\to 0} \frac{k^*\|u\|^p + r\|u\|^\sigma}{\frac{1}{p}\|u\|^p} = pk^*. \tag{3.13}$$

In addition, if $|u| \le \varepsilon_2$, then $\int_{s_j}^{t_{j+1}} F_j(t, u(t))dt \le h_j$, where $j = 0, 1, ..., m$. Then it follows from (3.11) that

$$
\begin{aligned}
\limsup_{u\to\infty}\frac{w(u)}{\varphi(u)} &\le \limsup_{u\to\infty}\frac{\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}F_j(t,u(t))dt}{\frac{1}{p}\|u\|^p}\\
&= \limsup_{u\to\infty}\frac{\sum_{j=0}^{m}\int_{|u|\le\varepsilon_2}F_j(t,u(t))dt}{\frac{1}{p}\|u\|^p} + \limsup_{u\to\infty}\frac{\sum_{j=0}^{m}\int_{|u|>\varepsilon_2}F_j(t,u(t))dt}{\frac{1}{p}\|u\|^p} \qquad (3.14)\\
&\le \limsup_{u\to\infty}\frac{\sum_{j=0}^{m}h_j}{\frac{1}{p}\|u\|^p} + \limsup_{u\to\infty}\frac{k^*\|u\|^p}{\frac{1}{p}\|u\|^p} \le pk^*.
\end{aligned}
$$

Combining (3.13) with (3.14), one has

$$
\rho_1 = \max\left\{0, \limsup_{\|u\|\to+\infty}\frac{w(u)}{\varphi(u)}, \limsup_{\|u\|\to 0}\frac{w(u)}{\varphi(u)}\right\} \le pk^*.
$$

Furthermore, from (H4), we can obtain

$$
\begin{aligned}
\rho_2 = \sup_{u\in\varphi^{-1}(0,+\infty)}\frac{w(u)}{\varphi(u)} = \sup_{u\in X\setminus\{0\}}\frac{w(u)}{\varphi(u)} &\ge \frac{\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}F_j(t,\vartheta(t))dt}{\frac{1}{p}\|\vartheta(t)\|^p + \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}g(t)|\vartheta(t)|^p dt}\\
\ge \frac{\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}F_j(t,\vartheta(t))dt}{\|\vartheta(t)\|_X^p} &\ge \frac{\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}F_j(t,\vartheta(t))dt}{c_0^p\|\vartheta(t)\|^p} > pk^* \ge \rho_1.
\end{aligned}
$$

Therefore, by Theorem 2.2 and Lemma 2.4, for each compact interval $[\theta_1, \theta_2] \subset (\frac{1}{\rho_2}, \frac{1}{\rho_1})$, there exists $R > 0$ satisfying the property: for every $\lambda \in [\theta_1, \theta_2]$, there exists $\xi > 0$ such that, for each $\mu \in [0, \xi]$, the problem (1.1) has at least three classical solutions u_i in X with $\|u_i\|$ <R, $i = 1, 2, 3$. □

4. Examples

In this section, we illustrate the applications of our main results with two examples.

Example 4.1. Let $\rho(t) = g(t) = 1, m = p = \zeta = 2, \eta = \frac{1}{3}, s_0 = 0,\ t_1 = \frac{1}{3},\ s_1 = \frac{1}{2},\ t_2 = \frac{7}{12},\ s_2 = \frac{2}{3},\ t_3 = 1.$ Let us choose the functions

$$
F_j(u) = \begin{cases} 0, & u < 0,\\ \kappa(u), & 0 \le u \le \ell,\\ 15\left(1 + u^{\frac{3}{2}}\right), & \ell < u, \end{cases}
$$

where $\kappa(u)$ is a function which is differentiable and increasing on $[0, \ell]$ and satisfies $\kappa(0) = 0, \kappa'(0) = 0, \kappa(1) = 1, \kappa(\ell) = 15\left(1 + \ell^{\frac{3}{2}}\right)$, and $\kappa'(\ell) = \frac{45}{2}\sqrt{\ell}$, $\ell = \sqrt{\frac{10}{143/18}} \approx 1.1219$.

Let us take the function $\tilde{u}(t)$ as

$$
\tilde{u}(t) = \begin{cases} 3\ell t, & t \in [0, \frac{1}{3}),\\ \ell, & t \in [\frac{1}{3}, \frac{2}{3}],\\ (3t-1)\ell, & t \in (\frac{2}{3}, 1]. \end{cases}
$$

Through direct calculation, we know that $\tilde{u} \in X$ and $\varphi(\tilde{u}) = \frac{143}{36}\ell^2$. Let $r = \frac{1}{2}$. The inequality (3.1) of (H2) takes the form

$$\frac{15}{2}F_j(1) = \frac{15}{2}\max_{|u|\le 1} F_j(u(t)) < \sum_{j=0}^{2}\int_{s_j}^{t_{j+1}} F_j(\tilde{u}(t))dt.$$

Obviously, we have

$$\sum_{j=0}^{2}\int_{s_j}^{t_{j+1}} F_j(\tilde{u}(t))dt \ge \int_{\frac{1}{3}}^{\frac{2}{3}} F_j(\tilde{u}(t))dt = \frac{1}{3}F_j(\ell).$$

Note that,

$$\frac{45}{2} = \frac{45}{2}F_j(1) < F_j(\ell) = 15\left(1+\ell^{\frac{2}{3}}\right) \approx 32.8247,$$

which implies that the inequality (3.1) is satisfied. In addition, we can take $I_j(s) = \frac{1}{71}s^{\frac{1}{71}}$, then $J_j(u) = \int_0^u I_j(s)ds = \frac{1}{72}u^{\frac{72}{71}}$ and $\sum_{j=1}^{2} J_j(\tilde{u}(t_j)) = \frac{1}{36}\ell^{\frac{72}{71}} > 0$, $j = 1, 2$. Hence, for every

$$\lambda \in \left(\frac{143\ell^2}{36\sum_{j=0}^{2}\int_{s_j}^{t_{j+1}} F_j(\tilde{u}(t))dt}, \frac{2}{3F_j(1)}\right),$$

there exists

$$\gamma = \min\left\{\frac{18-27\lambda F_j(1)}{\max\limits_{|u|\le 1}(-u^{\frac{72}{71}})}, \frac{36\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j(t,\tilde{u}(t))dt - 143\ell^2}{\ell^{\frac{72}{71}}}\right\},$$

such that, for each $\mu \in [0, \gamma)$, the problem (1.1) has at least three classical solutions by Theorem 3.1.

Example 4.2. Consider the following problem:

$$\begin{cases} -(|u'(t)|u'(t))' + |u(t)|u(t) = \lambda f_j(t, u(t)), \quad t \in (s_j, t_{j+1}],\ j = 0, 1, \\ \Delta(|u'(t_1)|u'(t_1)) = \mu I_1(u(t_1)), \\ |u'(t)|u'(t) = |u'(t_1^+)|u'(t_1^+), \quad t \in (t_1, s_1], \\ |u'(s_1^+)|u'(s_1^+) = |u'(s_1^-)|u'(s_1^-), \\ u(0) = 0, \quad u(1) = 2u(\frac{1}{3}), \end{cases} \tag{4.1}$$

where $p = 3$, $m = 1$, $\eta = \frac{1}{3}$, $0 = s_0 < t_1 = \eta = \frac{1}{3} < s_1 = \frac{1+\eta}{2} = \frac{2}{3} < t_2 = 1$, $\zeta = 2$, $\rho(t) = 1$, $g(t) = 1$. Choose functions $F_j(t, u) = e^{-|u|}u^4$ and

$$\vartheta(t) = \begin{cases} 3t, & t \in [0, \frac{1}{3}), \\ 1, & t \in [\frac{1}{3}, \frac{2}{3}], \\ 3t-1, & t \in (\frac{2}{3}, 1]. \end{cases}$$

Obviously, $F_j(t, 0) = 0$ and $F_j(t, u)$ are all C^1 functionals in u. By direct calculation, we obtain that $\limsup\limits_{|u|\to\infty}\frac{F(t,u)}{|u|^3} = \limsup\limits_{|u|\to 0}\frac{F(t,u)}{|u|^3} = 0$ and $\|\vartheta\| \approx 2.6207$. Let $k^* = 0.003$ and $c_0 = 2^{\frac{1}{3}} \approx 1.2599$. We can verify that $\frac{\int_0^{\frac{1}{3}} F_j(t,\vartheta)dt + \int_{\frac{2}{3}}^{1} F_j(t,\vartheta)dt}{c_0^3\|\vartheta\|^3} \approx 0.0117 > 3k^* = 0.009$. Therefore, all conditions in Theorem 3.3 are satisfied. Applying Theorem 3.3, for each compact interval $[\theta_1, \theta_2] \subset (85.4652, 111.1111)$, there exists $R > 0$ with the following property: for every $\lambda \in [\theta_1, \theta_2]$, there exists $\xi > 0$ such that, for each $\mu \in [0, \xi]$, the problem (4.1) has at least three classical solutions whose norms are less than R.

5. Conclusions

In this paper, we investigate a class of three-point boundary value problem of second-order p-Laplacian differential equations with instantaneous and non-instantaneous impulses. By using variational methods and two kinds of three critical points theorems, we obtain the non-local boundary value problem (1.1) has at least three classical solutions. Furthermore, we provide two examples to illustrate the main results. On the other hand, the study of fractional differential equations has attracted much attention recently. So, we will study a class of fractional differential equations with three-point boundary conditions and instantaneous and non-instantaneous impulses in the follow-up work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

References

1. H. Khan, K. Alam, H. Gulzar, S. Etemadc, S. Rezapourc, A case study of fractal-fractional tuberculosis model in China: Existence and stability theories along with numerical simulations, *Math. Comput. Simul.*, **198** (2022), 455–473. https://doi.org/10.1016/j.matcom.2022.03.009
2. H. Mohammadi, S. Kumar, S. Rezapour, S. Etemad, A theoretical study of the Caputo-Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control, *Chaos Solitons Fractals*, **144** (2021), 110668. https://doi.org/10.1016/j.chaos.2021.110668
3. M. Ahmad, A. Zada, M. Ghaderi, R. George, S. Rezapour, On the existence and stability of a neutral stochastic fractional differential system, *Fractal Fract.*, **6** (2022), 203. https://doi.org/10.3390/fractalfract6040203
4. S. Hussain, E. N. Madi, H. Khan, H. Gulzar, S. Etemad, S. Rezapour, et al., On the stochastic modeling of COVID-19 under the environmental white noise, *J. Funct. Space*, **2022** (2022), 4320865. https://doi.org/10.1155/2022/4320865
5. V. D. Milman, A. D. Myshkis, On the stability of motion in the presence of impulses, *Sibirsk. Mat. Zh.*, **1** (1960), 233–237.

6. E. Hernández, D. O'Regan, On a new class of abstract impulsive differential equations, *Proc. Amer. Math. Soc.*, **141** (2013), 1641–1649. https://doi.org/10.1090/S0002-9939-2012-11613-2
7. R. Agarwal, D. O'Regan, S. Hristova, Stability by Lyapunov like functions of nonlinear differential equations with non-instantaneous impulses, *J. Appl. Math. Comput.*, **53** (2017), 147–168. https://doi.org/10.1007/s12190-015-0961-z
8. J. L. Li, J. J. Nieto, Existence of positive solutions for multipoint boundary value problem on the half-line with impulses, *Bound. Value Probl.*, **2009** (2009), 834158. https://doi.org/10.1155/2009/834158
9. D. B. Qian, X. Y. Li, Periodic solutions for ordinary differential equations with sublinear impulsive effects, *J. Math. Anal. Appl.*, **303** (2005), 288–303. https://doi.org/10.1016/j.jmaa.2004.08.034
10. L. J. Chen, J. T. Sun, Nonlinear boundary value problem of first order impulsive functional differential equations, *J. Math. Anal. Appl.*, **318** (2006), 726–741. https://doi.org/10.1016/j.jmaa.2005.08.012
11. M. Pierri, D. O'Regan, V. Rolnik, Existence of solutions for semilinear abstract differential equations with not instantaneous impulses, *Appl. Math. Comput.*, **219** (2013), 6743–6749. https://doi.org/10.1016/j.amc.2012.12.084
12. V. Colao, L. Muglia, H. K. Xu, Existence of solutions for a second-order differential equation with non-instantaneous impulses and delay, *Ann. Mat. Pura Appl.*, **195** (2016), 697–716. https://doi.org/10.1007/s10231-015-0484-0
13. Y. Tian, W. G. Ge, Applications of variational methods to boundary-value problem for impulsive differential equations, *Proc. Edinburngh Math. Soc.*, **51** (2008), 509–527. https://doi.org/10.1017/S0013091506001532
14. J. J. Nieto, D. O'Regan, Variational approach to impulsive differential equations, *Nonlinear Anal.-Real World Appl.*, **10** (2009), 680–690. https://doi.org/10.1016/j.nonrwa.2007.10.022
15. Y. L. Zhao, C. L. Luo, H. B. Chen, Existence results for non-instantaneous impulsive nonlinear fractional differential equation via variational methods, *Bull. Malays. Math. Sci. Soc.*, **43** (2020), 2151–2169. https://doi.org/10.1007/s40840-019-00797-7
16. W. J. Yao, Variational approach to non-instantaneous impulsive differential equations with p-Laplacian operator, *AIMS Mathematics*, **7** (2022), 17269–17285. https://doi.org/10.3934/math.2022951
17. H. H. Wang, D. Lu, H. Q. Lu, Multiplicity results for second order impulsive differential equations via variational methods, *Engineering*, **13** (2021), 82–93. https://doi.org/10.4236/eng.2021.132007
18. X. L. Fan, T. T. Xue, Y. S. Jiang, Dirichlet problems of fractional p-Laplacian equation with impulsive effects, *Math. Biosci. Eng.*, **20** (2023), 5094–5116. https://doi.org/10.3934/mbe.2023236
19. S. H. Wang, Y. Tian, Variational methods to the fourth-order linear and nonlinear differential equations with non-instantaneous impulses, *J. Appl. Anal. Comput.*, **10** (2020), 2521–2536. https://doi.org/10.11948/20190413
20. Y. Tian, M. Zhang, Variational method to differential equations with instantaneous and non-instantaneous impulses, *Appl. Math. Lett.*, **94** (2019), 160–165. https://doi-org-ssl.3436.top/10.1016/j.aml.2019.02.034

21. D. P. Li, F. Q. Chen, Y. H. Wu, Y. K. An, Multiple solutions for a class of p-Laplacian type fractional boundary value problems with instantaneous and non-instantaneous impulses, *Appl. Math. Lett.*, **106** (2020), 106352. https://doi.org/10.1016/j.aml.2020.106352

22. J. W. Zhou, Y. M. Deng, Y. N. Wang, Variational approach to p-Laplacian fractional differential equations with instantaneous and non-instantaneous impulses, *Appl. Math. Lett.*, **104** (2020), 106251. https://doi.org/10.1016/j.aml.2020.106251

23. W. J. Yao, Variational approach to instantaneous and noninstantaneous impulsive system of differential equations, *Bound. Value Probl.*, **2022** (2022), 71. https://doi.org/10.1186/s13661-022-01653-7

24. W. Zhang, J. B. Ni, Study on a new p-Laplacian fractional differential model generated by instantaneous and non-instantaneous impulsive effects, *Chaos Solitons Fractals*, **168** (2023), 113143. https://doi.org/10.1016/j.chaos.2023.113143

25. Y. Qiao, F. Q. Chen, Y. K. An, Variational method for p-Laplacian fractional differential equations with instantaneous and non-instantaneous impulses, *Math. Meth. Appl. Sci.*, **44** (2021), 8543–8553. https://doi.org/10.1002/mma.7276

26. X. P. Liu, M. Jia, W. G. Ge, The method of lower and upper solutions for mixed fractional four-point boundary value problem with p-Laplacian operator, *Appl. Math. Lett.*, **65** (2017), 56–62. https://doi.org/10.1016/j.aml.2016.10.001

27. X. P. Liu, M. Jia, Solvability and numerical simulations for BVPs of fractional coupled systems involving left and right fractional derivatives, *Appl. Math. Comput.*, **353** (2019), 230–242. https://doi.org/10.1016/j.amc.2019.02.011

28. M. ur Rehman, R. A. Khan, Existence and uniqueness of solutions for multi-point boundary value problems for fractional differential equations, *Appl. Math. Lett.*, **23** (2010), 1038–1044. https://doi.org/10.1016/j.aml.2010.04.033

29. Y. F. Wei, Z. B. Bai, Multiple solutions for some nonlinear impulsive differential equations with three-point boundary conditions via variational approach, *J. Appl. Anal. Comput.*, **11** (2021), 3031–3043. https://doi.org/10.11948/20210113

30. W. Lian, Z. B. Bai, Z. J. Du, Existence of solution of a three-point boundary value problem via variational approach, *Appl. Math. Lett.*, **104** (2020), 106283. https://doi.org/10.1016/j.aml.2020.106283

31. Y. F. Wei, S. M. Shang, Z. B. Bai, Applications of variational methods to some three-point boundary value problems with instantaneous and noninstantaneous impulses, *Nonlinear Anal. Model. Control*, **27** (2022), 466–478. https://doi.org/10.15388/namc.2022.27.26253

32. W. J. Yao, Existence and multiplicity of solutions for three-point boundary value problems with instantaneous and noninstantaneous impulses, *Bound. Value Probl.*, **2023** (2023), 15. https://doi.org/10.1186/s13661-023-01702-9

33. G. Bonanno, S. A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, *Appl. Anal.*, **89** (2010), 1–10. https://doi.org/10.1080/00036810903397438

34. B. Ricceri, A further three critical points theorem, *Nonlinear Anal.*, **71** (2009), 4151–4157. https://doi.org/10.1016/j.na.2009.02.074

35. R. Rodríguez-López, S. Tersian, Multiple solutions to boundary value problem for impulsive fractional differential equations, *Fract. Calc. Appl. Anal.*, **17** (2014), 1016–1038. https://doi.org/10.2478/s13540-014-0212-2

36. S. Abaspour, S. Khademloo, S. H. Rasouli, On the existence of multiple solutions for a three-point nonlinear boundary value problem of p-Laplacian type, *Afr. Mat.*, **31** (2020), 305–313. https://doi.org/10.1007/s13370-019-00725-9

37. J. Simon, Régularité de la solution d'une équation non linéaire dans R^N, In: *Lecture notes in mathematics*, Heidelberg: Springer, Berlin, 2006, 205–227. https://doi.org/10.1007/BFb0061807

38. E. Zeidler, *Nonlinear functional analysis and its applications: II/B: nonlinear monotone operators*, New York: Springer, 1990. https://doi.org/10.1007/978-1-4612-0981-2

39. H. W. Chen, Z. M. He, New results for perturbed Hamiltonian systems with impulses, *Appl. Math. Comput.*, **218** (2012), 9489-9497. https://doi.org/10.1016/j.amc.2012.03.038