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Research article

Cohomology of nonabelian embedding tensors on Hom-Lie algebras

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Abstract: In this paper, we generalize known results of nonabelian embedding tensor to the Hom setting. We introduce the concept of Hom-Leibniz-Lie algebra, which is the basic algebraic structure of nonabelian embedded tensors on Hom-Lie algebras and can also be regarded as a nonabelian generalization of Hom-Leibniz algebra. Moreover, we define a cohomology of nonabelian embedding tensors on Hom-Lie algebras with coefficients in a suitable representation. The first cohomology group is used to describe infinitesimal deformations as an application. In addition, Nijenhuis elements are used to describe trivial infinitesimal deformations.

Keywords: Hom-Lie algebra; Hom-Leibniz-Lie algebra; nonabelian embedding tensor; cohomology; deformation
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1. Introduction

This paper studies a special type of algebraic structure called Hom-type algebra structure. The definition identities of this algebraic structure are twisted by algebraic endomorphisms. These structures first appear in the q-deformation of Witt and Virasoro algebras in the form of Hom-Lie algebras [1]. Many scholars pay special attention to this algebraic structure because of its close relationship with discrete and deformed vector fields and differential calculus [1–3]. In particular, Hom-Lie algebras with symmetric invariant nondegenerate have been studied in [4]. Representations, cohomologies and deformations of Hom-algebras have been systematically studied in [5–10]. These conclusions provide a good starting point for further research.

Embedding tensors can be traced back to the study of gauge supergravity theory [11]. Bergshoeff et al. [12] studied the Bagger-Lambert theory of multiple $M2$-branes and the $N = 8$ supersymmetric gauge theories using the embedding tensor. In mathematics, the embedding tensor is called the average operator. Aguiar [13] explored the average operator on associative and Lie algebras. Later on, the deformation and cohomology theory of embedding tensors on associative algebras, Lie algebras and 3-Lie algebras are given in [14–16]. Nonabelian embedding tensors on Lie algebras are first introduced by Tang and Sheng [17], which is a nonabelian generalization of the embedding tensor.

Recently, $\mathcal{O}$-operators on Hom-Lie algebras have been studied by Mishra and Naolekar [18]. Nijenhuis operators on Hom-Lie algebras have been studied by Das and Sen [19]. Embedding tensors on Hom-Lie algebras have been studied by Das and Makhlouf [20]. Our main objective is to study nonabelian embedding tensors on Hom-Lie algebras. The method of this paper is based on the recent work in [17, 20]. More precisely, we introduce the concept of Hom-Leibniz-Lie algebras, which is the algebraic structure behind the nonabelian embedding tensor on Hom-Lie algebras. It can be regarded as a nonabelian generalization of the Hom-Leibniz algebra and as a twisted version of the Leibniz-Lie algebra [17]. The main content of this paper is to construct a suitable cohomology theory for a nonabelian embedding tensors on Hom-Lie algebras. We introduce the cohomology of nonabelian embedding tensor by the Loday-Pirashvili cohomology of Hom-Leibniz algebras. However, it is important to note that the cochain complex of Loday-Pirashvili cohomology starts only from 1-cochians, not from 0-cochains. The main difficulty is to choose 0-cochains appropriately and build a proper coboundary map from the set of 0-cochains to that of 1-cochains. Our strategy is to define the set of 0-cochains to be $C^0_\Gamma(L', L)$, then, construct the coboundary map explicitly (see Proposition 4.2). At this point, we need extra conditions, that is, the structural map of Hom-Lie algebras is reversible and the endomorphism in the representation of Hom-Lie algebras is an automorphism, so as to ensure that the composition $C^0_\Gamma(L', L) \xrightarrow{\delta_\Gamma} C^1_\Gamma(L', L) \xrightarrow{\delta_\Gamma} C^2_\Gamma(L', L)$ is the zero map. Finally, we classify the infinitesimal deformation of nonabelian embedding tensors on Hom-Lie algebras using the first cohomology group. All the results in this paper can be regarded as generalizations of embedding tensors on Hom-Lie algebras [20] and nonabelian embedding tensors on Lie algebras [17].

This paper is organized as follows. Section 2 first recalls some basic concepts of Hom-Lie algebras and Hom-Leibniz algebras. Then we introduce the coherent representation of Hom-Lie algebras and the notion of nonabelian embedding tensors on Hom-Lie algebras with respect to a coherent representation. In Section 3, the concept of Hom-Leibniz-Lie algebra is introduced as the basic algebraic structure of a nonabelian embedding tensor. Naturally, a Hom-Leibniz-Lie algebra induces a Hom-Leibniz algebra. In Section 4, the cohomology theory of nonabelian embedding tensors on Hom-Lie algebras is introduced. In particular, we obtain the 0-coboundary operator. Thus the first cohomology group is established. At last, as an application, we characterize the infinitesimal deformation using the first cohomology group. Moreover, Nijenhuis elements are used to describe trivial infinitesimal deformations.

All vector spaces, tensor products and algebras considered in this paper are on the field $\mathbb{K}$ with the characteristic of 0.

2. Nonabelian embedding tensors on Hom-Lie algebras

This section recalls some basic concepts of Hom-Lie algebras and Hom-Leibniz algebras. After that, we introduce the coherent representation of Hom-Lie algebras, and we introduce the concept of nonabelian embedding tensors on Hom-Lie algebras by its coherent representation as a twisted version of nonabelian embedding tensors on Lie algebras [17] and a nonabelian generalization of embedding tensors on Hom- Lie algebras [20].

Definition 2.1. *(see [8]) A Hom-Lie algebra (Hom-LieA) is a triple* $(L, [\cdot,\cdot], \alpha)$ *consisting of a vector space L, a bilinear skew-symmetric mapping* $[\cdot,\cdot] : L \otimes L \to L$*, and a linear transformation* $\alpha : L \to L$ *satisfying* $\alpha([a, b]) = [\alpha(a), \alpha(b)]$ *such that*

$$[\alpha(a), [b, c]] + [\alpha(c), [a, b]] + [\alpha(b), [c, a]] = 0, \tag{2.1}$$

for any $a, b, c \in L$*. Furthermore, if* $\alpha : L \to L$ *is a vector space automorphism of L, then L is called regular.*

A homomorphism between two Hom-LieAs $(L', [\cdot,\cdot]', \alpha')$ and $(L, [\cdot,\cdot], \alpha)$ is a linear map $\psi : L' \to L$ satisfying $\psi \circ \alpha' = \alpha \circ \psi$ and

$$\psi([u, v]') = [\psi(u), \psi(v)], \quad \forall\, u, v \in L'.$$

To introduce the concept of nonabelian embedding tensors on Lie algebras, Tang and Sheng [17] proposed the coherent action of a Lie algebra on another Lie algebra. Similarly, we propose the coherent representation of Hom-Lie algebras.

Definition 2.2. *(1) (see [10]) A representation of a Hom-LieA* $(L, [\cdot,\cdot], \alpha)$ *on a Hom-vector space* (L', α') *is a linear map* $\rho : L \to \mathrm{End}(L')$*, such that*

$$\rho(\alpha(\varsigma)) \circ \alpha' = \alpha' \circ \rho(\varsigma), \tag{2.2}$$

$$\rho([\varsigma, \tau]) \circ \alpha' = \rho(\alpha(\varsigma)) \circ \rho(\tau) - \rho(\alpha(\tau))) \circ \rho(\varsigma), \tag{2.3}$$

for all $\varsigma, \tau \in L$*.*
(2) A coherent representation of a Hom-LieA $(L, [\cdot,\cdot], \alpha)$ *on a Hom-LieA* $(L', [\cdot,\cdot]', \alpha')$ *is a linear map* $\rho : L \to \mathrm{End}(L')$ *satisfying Eqs* (2.2)*,* (2.3) *and*

$$\rho(\alpha(\varsigma))[\mu, \nu]' = [\rho(\varsigma)\mu, \alpha'(\nu)]' + [\alpha'(\mu), \rho(\varsigma)\nu]', \tag{2.4}$$

$$[\rho(\varsigma)\mu, \nu]' = 0, \tag{2.5}$$

for all $\varsigma \in L$ *and* $\mu, \nu \in L'$*. We denote a coherent representation by* $(L'; \rho^{\dagger}, \alpha')$*. Furthermore, if* $(L', [\cdot,\cdot]', \alpha')$ *is a regular Hom-LieA, then* $(L'; \rho^{\dagger}, \alpha')$ *is called a regular coherent representation of* $(L, [\cdot,\cdot], \alpha)$*.*

Example 2.3. *A 2-step nilpotent Hom-LieA is a Hom-LieA* $(L, [\cdot,\cdot], \alpha)$ *satisfying*

$$[[a, b], c] = 0, \forall a, b, c \in L. \tag{2.6}$$

For a 2-step nilpotent Hom-LieA, the adjoint representation $(L; \mathrm{ad}, \alpha)$ *of* $(L, [\cdot,\cdot], \alpha)$ *is a coherent adjoint representation.*

Definition 2.4. *(1) A nonabelian embedding tensor (nonabelian ET) on the Hom-LieA* $(L, [\cdot,\cdot], \alpha)$ *w.r.t the coherent representation* $(L'; \rho^{\dagger}, \alpha')$ *is a linear map* $\Gamma : L' \to L$ *satisfying the following equations:*

$$\Gamma \circ \alpha' = \alpha \circ \Gamma, \tag{2.7}$$

$$[\Gamma\mu, \Gamma\nu] = \Gamma(\rho(\Gamma\mu)\nu + [\mu, \nu]'), \tag{2.8}$$

for any $\mu, \nu \in L'$.

(2) A nonabelian embedding tensor Hom-Lie algebra (nonabelian ETHLA) is a triple (L, L', Γ) *consisting of a Hom-LieA* $(L, [\cdot,\cdot], \alpha)$, *a coherent representation* $(L'; \rho^{\dagger}, \alpha')$ *of* L *and a nonabelian ET* $\Gamma : L' \to L$. *We denote a nonabelian ETHLA* (L, L', Γ) *by the notation* $L' \xrightarrow{\Gamma} L$.

Remark 2.5. *If* $(L', [\cdot,\cdot]', \alpha')$ *is an abelian Hom-LieA, then we can get that* Γ *is a ET on Hom-LieA (see [20]). In addition, If* $\rho = 0$, *then* Γ *is a Hom-LieA homomorphism from* L' *to* L.

Example 2.6. *Let* L' *be a 3-dimensional linear space spanned by* $\{\zeta_1, \zeta_2, \zeta_3\}$. *We define a bilinear skew symmetric operation* $[\cdot,\cdot]' : L' \otimes L' \to L'$ *and a linear transformation* $\alpha' : L' \to L'$ *by*

$$[\zeta_1, \zeta_2]' = k\zeta_3, \alpha'(\zeta_1) = -\zeta_1, \alpha'(\zeta_2) = \zeta_2, \alpha'(\zeta_3) = -\zeta_3,$$

where $k \in \mathbb{K}$. *Then* $(L', [\cdot,\cdot]', \alpha')$ *is a 2-step nilpotent Hom-LieA. Moreover, for* $s, t \in \mathbb{K}$,

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s & 0 \\ t & 0 & 0 \end{pmatrix}$$

is a nonabelian ET on $(L', [\cdot,\cdot]', \alpha')$ *w.r.t the coherent adjoint representation* $(L'; \mathrm{ad}, \alpha')$.

Definition 2.7. *(see [9]) A Hom-Leibniz algebra (Hom-LeibA) is a vector space* $\mathcal{L}$ *together with a bilinear operation* $[\cdot,\cdot]_{\mathcal{L}} : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ *and a linear transformation* $\alpha_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}$ *satisfying* $\alpha_{\mathcal{L}}([x, y]) = [\alpha_{\mathcal{L}}(x), \alpha_{\mathcal{L}}(y)]$ *such that*

$$[\alpha_{\mathcal{L}}(x), [y, z]_{\mathcal{L}}]_{\mathcal{L}} = [[x, y]_{\mathcal{L}}, \alpha_{\mathcal{L}}(z)]_{\mathcal{L}} + [\alpha_{\mathcal{L}}(y), [x, y]_{\mathcal{L}}]_{\mathcal{L}}, \tag{2.9}$$

for any $x, y, z \in \mathcal{L}$.

A homomorphism between two Hom-LeibAs $(\mathcal{L}_1, [\cdot,\cdot]_{\mathcal{L}_1}, \alpha_{\mathcal{L}_1})$ and $(\mathcal{L}_2, [\cdot,\cdot]_{\mathcal{L}_2}, \alpha_{\mathcal{L}_2})$ is a linear map $\varphi : \mathcal{L}_1 \to \mathcal{L}_2$ satisfying $\varphi \circ \alpha_{\mathcal{L}_1} = \alpha_{\mathcal{L}_2} \circ \varphi$ and

$$\varphi([u, v]_{\mathcal{L}_1}) = [\varphi(u), \varphi(v)]_{\mathcal{L}_2}, \ \ \forall\, u, v \in \mathcal{L}_1.$$

We denote by HLeib the category of Hom-LeibAs and their homomorphisms.

Definition 2.8. *(see [6]) A representation of a Hom-LeibA* $(\mathcal{L}, [\cdot,\cdot]_{\mathcal{L}}, \alpha_{\mathcal{L}})$ *is a pair* $(\mathcal{V}, \beta_{\mathcal{V}})$ *of a vector space* $\mathcal{V}$ *and a linear transformation* $\beta_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ *equipped with two actions* $l : \mathcal{L} \otimes \mathcal{V} \to \mathcal{V}$ *and* $r : \mathcal{V} \otimes \mathcal{L} \to \mathcal{V}$ *satisfying the following conditions*

$$\begin{aligned}
\beta_{\mathcal{V}}(l(x, u)) &= l(\alpha_{\mathcal{L}}(x), \beta_{\mathcal{V}}(v)),\\
\beta_{\mathcal{V}}(r(u, x)) &= r(\beta_{\mathcal{V}}(v), \alpha_{\mathcal{L}}(x)),\\
r(\beta_{\mathcal{V}}(u), [x, y]_{\mathcal{L}}) &= r(r(u, x), \alpha_{\mathcal{L}}(y)) + l(\alpha_{\mathcal{L}}(x), r(u, y)),\\
l(\alpha_{\mathcal{L}}(x), r(u, y)) &= r(l(x, u), \alpha_{\mathcal{L}}(y)) + r(\beta_{\mathcal{V}}(u), [x, y]_{\mathcal{L}}),\\
l(\alpha_{\mathcal{L}}(x), l(y, u)) &= l([x, y]_{\mathcal{L}}, \beta_{\mathcal{V}}(u)) + l(\alpha_{\mathcal{L}}(y), l(x, u)),
\end{aligned}$$

for any $x, y \in \mathcal{L}, u \in \mathcal{V}$.

Proposition 2.9. *Let* $(L, [\cdot,\cdot], \alpha)$ *and* $(L', [\cdot,\cdot]', \alpha')$ *be two Hom-LieAs, and let* $(L'; \rho^{\dagger}, \alpha')$ *be a coherent representation of* L. *Then,* $L \oplus L'$ *is a Hom-LeibA under the following maps:*

$$\begin{aligned}
&(\alpha \oplus \alpha')(a + \mu) := \alpha(a) + \alpha'(\mu),\\
&[a + \mu, b + \nu]_{\rho} := [a, b] + \rho(a)\nu + [\mu, \nu]',
\end{aligned}$$

for any $a, b \in L$ and $\mu, \nu \in L'$. $(L \oplus L', [\cdot,\cdot]_\rho, \alpha \oplus \alpha')$ is called the nonabelian hemisemidirect product Hom-LeibA, and denoted by $L \ltimes_\rho L'$.

Proof. For all $a, b, c \in L, \mu, \nu, \omega \in L'$, by Eqs (2.1)–(2.5), we have

$$\begin{aligned}
&(\alpha \oplus \alpha')[a+\mu, b+\nu]_\rho \\
=&(\alpha \oplus \alpha')([a,b] + \rho(a)(\nu) + [\mu,\nu]') \\
=&\alpha([a,b]) + \alpha'(\rho(a)(\nu) + [\mu,\nu]') \\
=&[\alpha(a), \alpha(b)] + \rho(\alpha(a))\alpha'(\nu) + [\alpha'(\mu), \alpha'(\nu)]' \\
=&[\alpha(a) + \alpha'(\mu), \alpha(b) + \alpha'(\nu)]_\rho \\
=&[(\alpha \oplus \alpha')(a+\mu), (\alpha \oplus \alpha')(b+\nu)]_\rho
\end{aligned}$$

and

$$\begin{aligned}
&[[a+\mu, b+\nu]_\rho, (\alpha \oplus \alpha')(c+\omega)]_\rho + [(\alpha \oplus \alpha')(b+\nu), [a+\mu, c+\omega]_\rho]_\rho \\
&- [(\alpha \oplus \alpha')(a+\mu), [b+\nu, c+\omega]_\rho]_\rho \\
=&[[a,b] + \rho(a)\nu + [\mu,\nu]', \alpha(c) + \alpha'(\omega)]_\rho + [\alpha(b) + \alpha'(\nu), [a,c] + \rho(a)\omega + [\mu,\omega]']_\rho \\
&- [\alpha(a) + \alpha'(\mu), [b,c] + \rho(b)\omega + [\nu,\omega]']_\rho \\
=&[[a,b], \alpha(c)] + \rho([a,b])\alpha'(\omega) + [\rho(a)\nu + [\mu,\nu]', \alpha'(\omega)]' + [\alpha(b), [a,c]] \\
&+ \rho(\alpha(b))(\rho(a)\omega + [\mu,\omega]') + [\alpha'(\nu), \rho(a)\omega + [\mu,\omega]']' \\
&- [\alpha(a), [b,c]] - \rho(\alpha(a))(\rho(b)\omega + [\nu,\omega]') - [\alpha'(\mu), \rho(b)\omega + [\nu,\omega]']' \\
=&[[a,b], \alpha(c)] + \rho([a,b])\alpha'(\omega) + [\rho(a)\nu, \alpha'(\omega)]' + [[\mu,\nu]', \alpha'(\omega)]' + [\alpha(b), [a,c]] \\
&+ \rho(\alpha(b))\rho(a)\omega + \rho(\alpha(b))[\mu,\omega]' + [\alpha'(\nu), \rho(a)\omega]' + [\alpha'(\nu), [\mu,\omega]']' \\
&- [\alpha(a), [b,c]] - \rho(\alpha(a))\rho(b)\omega - \rho(\alpha(a))[\nu,\omega]' - [\alpha'(\mu), \rho(b)\omega]' - [\alpha'(\mu), [\nu,\omega]']' \\
=&0.
\end{aligned}$$

Thus, $(L \oplus L', [\cdot,\cdot]_\rho, \alpha \oplus \alpha')$ is a Hom-LeibA. □

In the following theorem, we use graphs to describe nonabelian ETs.

Theorem 2.10. *A linear map $\Gamma : L' \to L$ is a nonabelian ET on $(L, [\cdot,\cdot], \alpha)$ w.r.t the coherent representation $(L'; \rho^\dagger, \alpha')$ if and only if the graph $Gr(\Gamma) = \{\Gamma\nu + \nu \mid \nu \in L'\}$ is a subalgebra of the nonabelian hemisemidirect product Hom-LeibA $L \ltimes_\rho L'$.*

Proof. Let $\Gamma : L' \to L$ be a linear map. Then, for all $\mu, \nu \in L'$, we have

$$\begin{aligned}
(\alpha \oplus \alpha')(\Gamma\mu + \mu) &= \alpha(\Gamma\mu) + \alpha'(\mu), \\
[\Gamma\mu + \mu, \Gamma\nu + \nu]_\rho &= [\Gamma\mu, \Gamma\nu] + \rho(\Gamma\mu)\nu + [\mu,\nu]',
\end{aligned}$$

Thus, the graph $Gr(\Gamma) = \{\Gamma\mu + \mu \mid \mu \in L'\}$ is a subalgebra of the nonabelian hemisemidirect product Hom-LeibA $L \ltimes_\rho L'$ if and only if Γ meets Eqs (2.7) and (2.8), which implies that Γ is a nonabelian ET on L w.r.t the coherent representation $(L'; \rho^\dagger, \alpha')$. □

Because L' and $Gr(\Gamma)$ are isomorphic as linear spaces, thereby, we have the following corollary.

Corollary 2.11. *Let $L' \xrightarrow{\Gamma} L$ be a nonabelian ETHLA. If a linear map $[\cdot,\cdot]_\Gamma : L' \times L' \to L'$ is given by*

$$[\mu,\nu]_\Gamma = \rho(\Gamma\mu)\nu + [\mu,\nu]', \tag{2.10}$$

for any $\mu, \nu \in L'$. Then $(L', [\cdot,\cdot]_\Gamma, \alpha')$ is a Hom-LeibA. Moreover, Γ is a homomorphism from the Hom-LeibA $(L', [\cdot,\cdot]_\Gamma, \alpha')$ to the Hom-LieA $(L, [\cdot,\cdot], \alpha)$. The Hom-LeibA $(L', [\cdot,\cdot]_\Gamma, \alpha')$ is called the descendent Hom-LeibA.

The above corollary shows that a nonabelian ET on Hom-LieAs naturally induces a Hom-LeibA.

Definition 2.12. *Let $L' \xrightarrow{\Gamma} L$ and $L' \xrightarrow{\Gamma'} L$ be two nonabelian ETHLAs. Then a homomorphism from $L' \xrightarrow{\Gamma'} L$ to $L' \xrightarrow{\Gamma} L$ consists of two Hom-LieA homomorphisms $\psi_L : L \to L$ and $\psi_{L'} : L' \to L'$ meets the following equations*

$$\Gamma \circ \psi_{L'} = \psi_L \circ \Gamma', \tag{2.11}$$

$$\psi_{L'}(\rho(a)\mu) = \rho(\psi_L(a))\psi_{L'}(\mu), \tag{2.12}$$

for $a \in L$ and $\mu \in L'$. Furthermore, if ψ_L and $\psi_{L'}$ are nondegenerate, $(\psi_L, \psi_{L'})$ is called an isomorphism from $L' \xrightarrow{\Gamma'} L$ to $L' \xrightarrow{\Gamma} L$.

We denote by NETHLA the category of nonabelian ETHLAs and their homomorphisms.

It follows from the following proposition that there is a functor $\mathcal{F}$: NETHLA $\to$ HLeib.

Proposition 2.13. *Let $(\psi_L, \psi_{L'})$ be a homomorphism from $L' \xrightarrow{\Gamma'} L$ to $L' \xrightarrow{\Gamma} L$. Then, $\psi_{L'}$ is a homomorphism of descendent Hom-LeibA from $(L', [\cdot,\cdot,\cdot]_{\Gamma'}, \alpha')$ to $(L', [\cdot,\cdot,\cdot]_\Gamma, \alpha')$.*

Proof. For all $\mu, \nu \in L'$, by Eqs (2.10)–(2.12), we have

$$\begin{aligned}
\psi_{L'}([\mu,\nu]_{\Gamma'}) =& \psi_{L'}(\rho(\Gamma'\mu)\nu + [\mu,\nu]') \\
=& \rho(\psi_L(\Gamma'\mu))\psi_{L'}(\nu) + \psi_{L'}([\mu,\nu]') \\
=& \rho(\Gamma\psi_{L'}(\mu))\psi_{L'}(\nu) + [\psi_{L'}(\mu), \psi_{L'}(\nu)]' \\
=& [\psi_{L'}(\mu), \psi_{L'}(\nu)]_\Gamma.
\end{aligned}$$

Thus, we can get $\psi_{L'}$ is a homomorphism of descendent Hom-LeibAs from $(L', [\cdot,\cdot]_{\Gamma'}, \alpha')$ to $(L', [\cdot,\cdot]_\Gamma, \alpha')$. □

Proposition 2.14. *Let $\Gamma : L' \to L$ be a nonabelian ET on $(L, [\cdot,\cdot], \alpha)$ w.r.t the coherent representation $(L'; \rho^\dagger, \alpha')$. Let $\psi_L : L \to L$ and $\psi_{L'} : L' \to L'$ be two Hom-LieA isomorphisms such that Eqs* (2.11) *and* (2.12) *hold. Then, $\psi_L^{-1} \circ \Gamma \circ \psi_{L'}$ is a nonabelian ET on $(L, [\cdot,\cdot], \alpha)$ w.r.t the coherent representation $(L'; \rho^\dagger, \alpha')$.*

Proof. For all $\mu, \nu \in L'$, by Eqs (2.7), (2.8), (2.11) and (2.12), we have

$$\begin{aligned}
&(\psi_L^{-1} \circ \Gamma \circ \psi_{L'}) \circ \alpha' = (\psi_L^{-1} \circ \Gamma) \circ (\alpha' \circ \psi_{L'}) = \psi_L^{-1} \circ (\alpha \circ \Gamma) \circ \psi_{L'} = \alpha \circ (\psi_L^{-1} \circ \Gamma \circ \psi_{L'}), \\
&[(\psi_L^{-1} \circ \Gamma \circ \psi_{L'})\mu, (\psi_L^{-1} \circ \Gamma \circ \psi_{L'})\nu] = \psi_L^{-1}([\Gamma\psi_{L'}(\mu), \Gamma\psi_{L'}(\nu)]) \\
&= \psi_L^{-1}(\Gamma(\rho(\Gamma\psi_{L'}(\mu))\psi_{L'}(\nu) + [\psi_{L'}(\mu), \psi_{L'}(\nu)]')) \\
&= \psi_L^{-1}(\Gamma(\psi_{L'}(\rho(\psi_L^{-1}(\Gamma\psi_{L'}\mu))\nu) + \psi_{L'}([\mu,\nu]'))) \\
&= (\psi_L^{-1} \circ \Gamma \circ \psi_{L'})(\rho((\psi_L^{-1} \circ \Gamma \circ \psi_{L'})(\mu))\nu + [\mu,\nu]').
\end{aligned}$$

Thus, $\psi_L^{-1} \circ \Gamma \circ \psi_{L'}$ is a nonabelian ET. □

3. Hom-Leibniz-Lie algebras

In [17], Tang and Sheng construct Leibniz- Lie algebra as the algebraic structure of nonabelian embedding tensors on Lie algebras. In this section, developing some techniques in [17], we introduce the concept of Hom-Leibniz-Lie algebra as the basic algebraic structure of nonabelian ETHLA, and as a twisted version of Leibniz-Lie algebra [17]. It is also proved that a Hom-Leibniz-Lie algebra induces a Hom-LeibA.

Definition 3.1. *A Hom-Leibniz-Lie algebra (Hom-LeibLieA)* $(L', [\cdot,\cdot]', \lhd, \alpha')$ *consists of a Hom-LieA* $(L', [\cdot,\cdot]', \alpha')$ *and a bilinear product* $\lhd : L' \otimes L' \to L'$ *satisfying* $\alpha'(\mu \lhd \nu) = \alpha'(\mu) \lhd \alpha'(\nu)$ *such that*

$$\alpha'(\mu) \lhd (\nu \lhd \omega) = (\mu \lhd \nu) \lhd \alpha'(\omega) + \alpha'(\nu) \lhd (\mu \lhd \omega) + [\mu, \nu]' \lhd \alpha'(\omega), \tag{3.1}$$

$$\alpha'(\mu) \lhd [\nu, \omega]' = [\mu \lhd \nu, \alpha'(\omega)]' = 0, \tag{3.2}$$

for any $\mu, \nu, \omega \in L'$.

A homomorphism between two Hom-LeibLieAs $(L'_1, [\cdot,\cdot]'_1, \lhd_1, \alpha'_1)$ and $(L'_2, [\cdot,\cdot]'_2, \lhd_2, \alpha'_2)$ is a Hom-LieA homomorphism $\psi : L'_1 \to L'_2$ such that

$$\psi(\mu \lhd_1 \nu) = \psi(\mu) \lhd_2 \psi(\nu), \quad \forall\, \mu, \nu \in L'_1.$$

We denote by HLLAlg the category of Hom-LeibLieAs and their homomorphisms.

Remark 3.2. *A Hom-LeibA* $(\mathcal{L}, [\cdot,\cdot]_{\mathcal{L}}, \alpha_{\mathcal{L}})$ *is naturally a Hom-LeibLieA if the Hom-LieA is abelian.*

Example 3.3. *Let* $(L', [\cdot,\cdot]', \alpha')$ *be a 3-dimensional Hom-LieA given in* Example 2.6. *We define a bilinear product* $\lhd : L' \otimes L' \to L'$ *by*

$$\zeta_1 \lhd \zeta_1 = \zeta_3, \zeta_1 \lhd \zeta_2 = -\zeta_3, \zeta_2 \lhd \zeta_1 = \zeta_3, \zeta_2 \lhd \zeta_2 = -\zeta_3.$$

Then, $(L', [\cdot,\cdot]', \lhd, \alpha')$ *is a Hom-LeibLieA.*

The following theorem shows that a Hom-LeibLieA naturally induces a Hom-LeibA and a representation of Hom-LeibA.

Theorem 3.4. *Let* $(L', [\cdot,\cdot]', \lhd, \alpha')$ *be a Hom-LeibLieA. Then, the binary product* $[\cdot,\cdot]_{\lhd} : L' \otimes L' \to L'$ *given by*

$$[\mu, \nu]_{\lhd} := \mu \lhd \nu + [\mu, \nu]', \tag{3.3}$$

for any $\mu, \nu \in L'$*, defines a Hom-LeibA structure on* L'*, which is denoted by* $(L', [\cdot,\cdot]_{\lhd}, \alpha')$ *and called the subadjacent Hom-LeibA.*

Furthermore, we define $l_{\lhd} : L' \otimes L' \to L'$ *by*

$$l_{\lhd}(\mu, \nu) = \mu \lhd \nu.$$

Then, $(L', l_{\lhd}, 0, \alpha')$ *is a representation of* $(L', [\cdot,\cdot]_{\lhd}, \alpha')$.

Proof. For any $\mu, \nu, \omega \in L'$, by Eqs (2.1), (3.1), (3.2) and (3.3), we have

$$\alpha'([\mu, \nu]_{\lhd}) = \alpha'(\mu \lhd \nu + [\mu, \nu]') = \alpha'(\mu) \lhd \alpha'(\nu) + [\alpha'(\mu), \alpha'(\nu)]' = [\alpha'(\mu), \alpha'(\nu)]_{\lhd},$$

$$\begin{aligned}
&[[\mu, \nu]_{\lhd}, \alpha'(\omega)]_{\lhd} + [\alpha'(\nu), [\mu, \omega]_{\lhd}]_{\lhd} - [\alpha'(\mu), [\nu, \omega]_{\lhd}]_{\lhd} \\
=&[\mu \lhd \nu + [\mu, \nu]', \alpha'(\omega)]_{\lhd} + [\alpha'(\nu), \mu \lhd \omega + [\mu, \omega]']_{\lhd} - [\alpha'(\mu), \nu \lhd \omega + [\nu, \omega]']_{\lhd} \\
=&(\mu \lhd \nu) \lhd \alpha'(\omega) + [\mu, \nu]' \lhd \alpha'(\omega) + [\mu \lhd \nu, \alpha'(\omega)]' + [[\mu, \nu]', \alpha'(\omega)]' + \alpha'(\nu) \lhd (\mu \lhd \omega) \\
&+ \alpha'(\nu) \lhd [\mu, \omega]' + [\alpha'(\nu), \mu \lhd \omega]' + [\alpha'(\nu), [\mu, \omega]']' - \alpha'(\mu) \lhd (\nu \lhd \omega) - \alpha'(\mu) \lhd [\nu, \omega]' \\
&- [\alpha'(\mu), \nu \lhd \omega]' - [\alpha'(\mu), [\nu, \omega]']' \\
=&0.
\end{aligned}$$

Therefore, we deduce that $(L', [\cdot,\cdot]_{\lhd}, \alpha')$ is a Hom-LeibA.

Furthermore, by Eqs (3.1) and (3.3), we have $l_{\lhd}(\alpha'(\mu), l_{\lhd}(\nu,\omega)) = l_{\lhd}([\mu,\nu]_{\lhd}, \alpha'(\omega)) + l_{\lhd}(\alpha'(\nu), l_{\lhd}(\mu,\omega))$ which implies that $(L', l_{\lhd}, 0, \alpha')$ is a representation of the Hom-LeibA $(L', [\cdot,\cdot]_{\lhd}, \alpha')$. □

The following theorem shows that a nonabelian ETHLA induces a Hom-LeibLieA.

Theorem 3.5. *Let $L' \stackrel{\Gamma}{\longrightarrow} L$ be a nonabelian ETHLA. Then, $(L', [\cdot,\cdot]', \lhd_\Gamma, \alpha')$ is a Hom-LeibLieA, where*

$$\mu \lhd_\Gamma \nu := \rho(\Gamma\mu)\nu, \tag{3.4}$$

for any $\mu,\nu \in L'$.

Proof. For any $\mu,\nu,\omega \in L'$, by Eqs (2.2), (2.3), (2.7) and (2.8), we have

$$\begin{aligned}
&\alpha'(\mu \lhd_\Gamma \nu) = \alpha'(\rho(\Gamma\mu)\nu) = \rho(\alpha(\Gamma\mu))\alpha'(\nu) = \rho(\Gamma\alpha'(\mu))\alpha'(\nu) = \alpha'(\mu) \lhd_\Gamma \alpha'(\mu),\\
&(\mu \lhd_\Gamma \nu) \lhd_\Gamma \alpha'(\omega) + \alpha'(\nu) \lhd_\Gamma (\mu \lhd_\Gamma w) + [\mu,\nu]' \lhd_\Gamma \alpha'(\omega) - \alpha'(\mu) \lhd_\Gamma (\nu \lhd_\Gamma \omega)\\
=&\rho(\Gamma\rho(\Gamma\mu)\nu)\alpha'(\omega) + \rho(\Gamma\alpha'(\nu))(\rho(\Gamma\mu)\omega) + \rho(\Gamma[\mu,\nu]')\alpha'(\omega) - \rho(\Gamma\alpha'(\mu))\rho(\Gamma\nu)\omega\\
=&\rho(\Gamma\rho(\Gamma\mu)\nu)\alpha'(\omega) + \rho(\alpha(\Gamma\nu))(\rho(\Gamma\mu)\omega) + \rho([\Gamma\mu,\Gamma\nu] - T\rho(\Gamma\mu)\nu)\alpha'(\omega) - \rho(\alpha(\Gamma\mu))\rho(\Gamma\nu)\omega\\
=&\rho(\alpha(\Gamma\nu))(\rho(\Gamma\mu)\omega) + \rho([\Gamma\mu,\Gamma\nu])\alpha'(\omega) - \rho(\alpha(\Gamma\mu))\rho(\Gamma\nu)\omega\\
=&0.
\end{aligned}$$

Moreover, by Eqs (2.4) and (2.5), we have

$$\begin{aligned}
\alpha'(\mu) \lhd_\Gamma [\nu,\omega]' =&\rho(\Gamma\alpha'(\mu))[\nu,\omega]' = \rho(\alpha(\Gamma\mu))[\nu,\omega]' = [\rho(\Gamma\mu)\nu, \alpha'(\omega)]' + [\alpha'(\nu), \rho(\Gamma\mu)\omega]'\\
=&[\rho(\Gamma\mu)\nu, \alpha'(\omega)]'\\
=&[\mu \lhd_\Gamma \nu, \alpha'(\omega)]' = 0.
\end{aligned}$$

Therefore, $(L', [\cdot,\cdot]', \lhd_\Gamma, \alpha')$ is a Hom-LeibLieA. □

It follows from the following proposition that there is a functor $\mathcal{J} : NETHLA \to HLLAlg$.

Proposition 3.6. *Let $(\psi_L, \psi_{L'})$ be a homomorphism from $L' \stackrel{\Gamma'}{\longrightarrow} L$ to $L' \stackrel{\Gamma}{\longrightarrow} L$. Then, $\psi_{L'}$ is a homomorphism of Hom-LeibLieAs from $(L', [\cdot,\cdot]', \lhd_{\Gamma'}, \alpha')$ to $(L', [\cdot,\cdot]', \lhd_\Gamma, \alpha')$.*

Proof. For any $\mu,\nu \in L'$, by Eqs (2.11), (2.12) and (3.4), we have

$$\begin{aligned}
\psi_{L'}(\mu \lhd_{\Gamma'} \nu) =&\psi_{L'}(\rho(\Gamma'\mu)\nu)\\
=&\rho(\psi_L(\Gamma'\mu))\psi_{L'}(\nu)\\
=&\rho(\Gamma\psi_{L'}(\mu))\psi_{L'}(\nu)\\
=&\psi_{L'}(\mu) \lhd_\Gamma \psi_{L'}(\nu).
\end{aligned}$$

Therefore, $\psi_{L'}$ is a homomorphism of Hom-LeibLieAs from $(L', [\cdot,\cdot]', \lhd_{\Gamma'}, \alpha')$ to $(L', [\cdot,\cdot]', \lhd_\Gamma, \alpha')$. □

It follows from Proposition 2.13, Theorem 3.4 and Proposition 3.6 that there is a functor $\mathcal{G} : HLLAlg \to HLeib$.

4. Cohomology theory and infinitesimal deformations of nonabelian embedding tensors on Hom-Lie algebras

In [17], Tang and Sheng established the cohomology theory and linear deformations of nonabelian embedding tensors on Lie algebras via the Loday-Pirashvili cohomology of Leibniz algebra.

Developing some techniques in [17], this section introduces the cohomology theory of nonabelian ET on Hom-LieA via the Loday-Pirashvili cohomology of Hom-LeibA (see [6]). At last, as an application, we characterize the infinitesimal deformation by using the first cohomology.

For $n \geq 1$, an n-cochain on a Hom-LeibA $(\mathcal{L}, [\cdot,\cdot]_{\mathcal{L}}, \alpha_{\mathcal{L}})$ with coefficients in a representation $(V; l, r, \beta)$ is a linear map $f : \otimes^n \mathcal{L} \to V$ satisfying $\beta \circ f = f \circ \alpha_{\mathcal{L}}^{\otimes n}$. The space generated by n-cochains is denoted as $C^n_{\mathrm{HLei}}(\mathcal{L}, V)$. The coboundary map $\delta : C^n_{\mathrm{HLei}}(\mathcal{L}, V) \to C^{n+1}_{\mathrm{HLei}}(\mathcal{L}, V)$, for $x_1, \cdots, x_{n+1} \in \mathcal{L}$, as

$$\begin{aligned}&(\delta f)(x_1, x_2, \cdots, x_{n+1})\\ =&\sum_{1\leq j<k\leq n+1}(-1)^j f(\alpha_{\mathcal{L}}(x_1), \cdots, \widehat{x_j}, \cdots, \alpha_{\mathcal{L}}(x_{k-1}), [x_j, x_k]_{\mathcal{L}}, \alpha_{\mathcal{L}}(x_{k+1}), \cdots, \alpha_{\mathcal{L}}(x_{n+1}))\\ &+\sum_{j=1}^{n}(-1)^{j+1} l(\alpha_{\mathcal{L}}^{n-1}(x_j), f(x_1, \cdots, \widehat{x_j}, \cdots, x_{n+1})) + (-1)^{n+1} r(f(x_1, \cdots, x_n), \alpha_{\mathcal{L}}^{n-1}(x_{n+1})).\end{aligned}$$

We denote by $\mathcal{H}^{\bullet}_{\mathrm{HLei}}(\mathcal{L}, V)$ the corresponding Loday-Pirashvili cohomology groups.

One can refer to [6, 9] for more information on Hom-LeibAs and (co)homology theory.

Lemma 4.1. *Let Γ be a nonabelian ET on the Hom-LieA $(L, [\cdot,\cdot], \alpha)$ w.r.t the coherent representation $(L'; \rho^{\dagger}, \alpha')$. Define two actions*

$$l_\Gamma : L' \otimes L \to L, r_\Gamma : L \otimes L' \to L,$$

by

$$\begin{aligned}l_\Gamma(v, a) &= [\Gamma v, a],\\ r_\Gamma(a, v) &= [a, \Gamma v] - \Gamma\rho(a)v,\end{aligned}$$

for any $v \in L', a \in L$. Then, $(L; l_\Gamma, r_\Gamma, \alpha)$ is a representation of the descendent Hom-LeibA $(L', [\cdot,\cdot]_\Gamma, \alpha')$.

Proof. For all $\mu, v \in L'$ and $a \in L$, by Eqs (2.2) and (2.7), we have

$$\begin{aligned}l_\Gamma(\alpha'(\mu), \alpha(a)) &= [\Gamma\alpha'(\mu), \alpha(a)] = [\alpha(\Gamma\mu), \alpha(a)] = \alpha([\Gamma\mu, a])\\ &= \alpha(l_\Gamma(\mu, a)),\\ r_\Gamma(\alpha(a), \alpha'(\mu)) &= [\alpha(a), \Gamma\alpha'(\mu)] - \Gamma\rho(\alpha(a))\alpha'(\mu)\\ &= [\alpha(a), \alpha(\Gamma\mu)] - \Gamma\alpha'(\rho(a)\mu)\\ &= \alpha([a, \Gamma\mu]) - \alpha(\Gamma(\rho(a)\mu)\\ &= \alpha(r_\Gamma(a, \mu)).\end{aligned}$$

By Eqs (2.1), (2.3), (2.4), (2.5), (2.7) and (2.8), we have

$$\begin{aligned}&r_\Gamma(\alpha(a), [\mu, v]_\Gamma) - r_\Gamma(r_\Gamma(a, \mu), \alpha'(v)) - l_\Gamma(\alpha'(\mu), r_\Gamma(a, v))\\ =&[\alpha(a), \Gamma(\rho(\Gamma\mu)v + [\mu, v]')] - \Gamma\rho(\alpha(a))(\rho(\Gamma\mu)v + [\mu, v]') - [[a, \Gamma\mu] - \Gamma\rho(a)\mu, \Gamma\alpha'(v)]\\ &+ \Gamma\rho([a, \Gamma\mu] - \Gamma\rho(a)\mu)\alpha'(v) - [\Gamma\alpha'(\mu), [a, \Gamma v] - \Gamma\rho(a)v]\\ =&[\alpha(a), [\Gamma\mu, \Gamma v]] - \Gamma\rho(\alpha(a))\rho(\Gamma\mu)v - \Gamma\rho(\alpha(a))[\mu, v]' - [[a, \Gamma\mu], \alpha(\Gamma v)] + [\Gamma\rho(a)\mu, \Gamma\alpha'(v)]\\ &+ \Gamma\rho([a, \Gamma\mu])\alpha'(v) - \Gamma\rho(\Gamma\rho(a)\mu)\alpha'(v) - [\alpha(\Gamma\mu), [a, \Gamma v]] + [\Gamma\alpha'(\mu), \Gamma\rho(a)v]\end{aligned}$$

$$
\begin{aligned}
&= - \Gamma\rho(\alpha(a))\rho(\Gamma\mu)\nu - \Gamma\rho(\alpha(a))[\mu,\nu]' + \Gamma\rho(\Gamma\rho(a)\mu)\alpha'(\nu) + \Gamma[\rho(a)\mu,\alpha'(\nu)]' \\
&\quad + \Gamma\rho([a,\Gamma\mu])\alpha'(\nu) - \Gamma\rho(\Gamma\rho(a)\mu)\alpha'(\nu) + \Gamma\rho(\Gamma\alpha'(\mu))\rho(a)\nu + \Gamma[\alpha'(\mu),\rho(a)\nu]' \\
&=0, \\
&\quad l_\Gamma(\alpha'(\mu), r_\Gamma(a,\nu)) - r_\Gamma(l_\Gamma(\mu,a),\alpha'(\nu)) - r_\Gamma(\alpha(a),[\mu,\nu]_\Gamma) \\
&=[\Gamma\alpha'(\mu),[a,\Gamma\nu] - \Gamma\rho(a)\nu] - [[\Gamma\mu,a],\Gamma\alpha'(\nu)] + \Gamma\rho([\Gamma\mu,a])\alpha'(\nu) - [\alpha(a),\Gamma(\rho(\Gamma\mu)\nu + [\mu,\nu]')] \\
&\quad + \Gamma\rho(\alpha(a))(\rho(\Gamma\mu)\nu + [\mu,\nu]') \\
&=[\alpha(\Gamma\mu),[a,\Gamma\nu]] - [\Gamma\alpha'(\mu),\Gamma\rho(a)\nu] - [[\Gamma\mu,a],\alpha(\Gamma\nu)] + \Gamma\rho([\Gamma\mu,a])\alpha'(\nu) - [\alpha(a),[\Gamma\mu,\Gamma\nu]] \\
&\quad + \Gamma\rho(\alpha(a))\rho(\Gamma\mu)\nu + \Gamma\rho(\alpha(a))[\mu,\nu]' \\
&= - \Gamma\rho(\alpha(\Gamma\mu))\rho(a)\nu - \Gamma[\alpha'(\mu),\rho(a)\nu]' + \Gamma\rho([\Gamma\mu,a])\alpha'(\nu) + \Gamma\rho(\alpha(a))\rho(\Gamma\mu)\nu + \Gamma\rho(\alpha(a))[\mu,\nu]' \\
&=0, \\
&\quad l_\Gamma(\alpha'(\mu), l_\Gamma(\nu,a)) - l_\Gamma([\mu,\nu]_\Gamma,\alpha(a)) - l_\Gamma(\alpha'(\nu), l_\Gamma(\mu,a)) \\
&=[\Gamma\alpha'(\mu),[\Gamma\nu,a]] - [\Gamma(\rho(\Gamma\mu)\nu + [\mu,\nu]'),\alpha(a)] - [\Gamma\alpha'(\nu),[\Gamma\mu,a]] \\
&=[\alpha(\Gamma\mu),[\Gamma\nu,a]] - [[\Gamma\mu,\Gamma\nu],\alpha(a)] - [\alpha(\Gamma\nu),[\Gamma\mu,a]] \\
&=0.
\end{aligned}
$$

Thus, $(L; l_\Gamma, r_\Gamma, \alpha)$ is a representation of $(L', [\cdot,\cdot]_\Gamma, \alpha')$. □

Let Γ be a nonabelian ET on $(L, [\cdot,\cdot], \alpha)$ w.r.t the coherent representation $(L'; \rho^\dagger, \alpha')$. When $n \geq 1$, let $\delta_\Gamma : C^n_\Gamma(L', L) \to C^{n+1}_\Gamma(L', L)$, be the Loday-Pirashvili coboundary operator of the Hom-LeibA $(L', [\cdot,\cdot]_\Gamma, \alpha')$ with coefficients in the representation $(L; l_\Gamma, r_\Gamma, \alpha)$. More precisely, for all $f \in C^n_\Gamma(L', L), \nu_1, \cdots, \nu_{n+1} \in L'$, we have

$$
\begin{aligned}
&(\delta_\Gamma f)(\nu_1, \nu_2, \cdots, \nu_{n+1}) \\
&= \sum_{1\leq j<k\leq n+1} (-1)^j f(\alpha'(\nu_1), \cdots, \widehat{\nu_j}, \cdots, \alpha'(\nu_{k-1}), \rho(\Gamma\nu_j)\nu_k + [\nu_j,\nu_k]', \alpha'(\nu_{k+1}), \cdots, \alpha'(\nu_{n+1})) \\
&\quad + \sum_{j=1}^{n} (-1)^{j+1} [\Gamma\alpha'^{n-1}(\nu_j), f(\nu_1, \cdots, \widehat{\nu_j}, \cdots, \nu_{n+1})] \\
&\quad + (-1)^{n+1}[f(\nu_1, \cdots, \nu_n), \Gamma\alpha'^{n-1}(\nu_{n+1})] - (-1)^{n+1}\Gamma\rho(f(\nu_1, \cdots, \nu_n))\alpha'^{n-1}(\nu_{n+1}).
\end{aligned}
$$

In particular, for $f \in C^1_\Gamma(L', L) := \{\theta \in \mathrm{Hom}(L', L) \mid \alpha \circ \theta = \theta \circ \alpha'\}$ and $\mu, \nu \in L'$, we have

$$
(\delta_\Gamma f)(\mu,\nu) = - f(\rho(\Gamma\mu)\nu) - f([\mu,\nu]') + [\Gamma\mu, f(\nu)] + [f(\mu), \Gamma\nu] - \Gamma\rho(f(\mu))\nu.
$$

Next, when $n = 0$, in order to get the first cohomology group, we need additional conditions, that is, Hom-LieA is regular and the endomorphism in the representation of Hom-LieA is an automorphism.

For any $\xi \in C^0_\Gamma(L', L) := \{\varsigma \in L \mid \alpha(\varsigma) = \varsigma\}$, we define $\delta_\Gamma : C^0_\Gamma(L', L) \to C^1_\Gamma(L', L), \xi \mapsto \hbar(\xi)$ by

$$
\hbar(\xi)\nu = \Gamma\rho(\xi)\alpha'^{-1}(\nu) - [\xi, \Gamma\alpha'^{-1}(\nu)], \forall \nu \in L',
$$

where $\alpha' : L' \to L'$ is a Hom-LieA isomorphism.

Proposition 4.2. *Let Γ be a nonabelian ET on the regular Hom-LieA $(L, [\cdot,\cdot], \alpha)$ w.r.t the regular coherent representation $(L'; \rho^\dagger, \alpha')$, Then, $\delta_\Gamma(\hbar(a)) = 0$, that is the composition*

$C^0_\Gamma(L',L) \xrightarrow{\delta_\Gamma} C^1_\Gamma(L',L) \xrightarrow{\delta_\Gamma} C^2_\Gamma(L',L)$ *is the zero map.*

Proof. For any $\mu,\nu \in L'$, by Eqs (2.1), (2.2), (2.3), (2.4), (2.7) and (2.8), we have

$$\begin{aligned}
&(\delta_\Gamma \hbar(\xi))(\mu,\nu)\\
=&-\hbar(\xi)(\rho(\Gamma\mu)\nu)-\hbar(\xi)([\mu,\nu]')+[\Gamma\mu,\hbar(\xi)(\nu)]+[\hbar(\xi)(\mu),\Gamma\nu]-\Gamma\rho(\hbar(\xi)(\mu))\nu\\
=&-\Gamma\rho(\xi)\alpha'^{-1}(\rho(\Gamma\mu)\nu)+[\xi,\Gamma\alpha'^{-1}(\rho(\Gamma\mu)\nu)]-\Gamma\rho(\xi)\alpha'^{-1}([\mu,\nu]')+[\xi,\Gamma\alpha'^{-1}([\mu,\nu]')]\\
&+[\Gamma\mu,\Gamma\rho(\xi)\alpha'^{-1}(\nu)-[\xi,\Gamma\alpha'^{-1}(\nu)]]+[\Gamma\rho(\xi)\alpha'^{-1}(\mu)-[\xi,\Gamma\alpha'^{-1}(\mu)],\Gamma\nu]\\
&-\Gamma\rho(\Gamma\rho(\xi)\alpha'^{-1}(\mu)-[\xi,\Gamma\alpha'^{-1}(\mu)])\nu\\
=&-\Gamma\rho(\xi)\rho(\Gamma\alpha'^{-1}(\mu))\alpha'^{-1}(\nu)+[\xi,\Gamma\rho(\Gamma\alpha'^{-1}(\mu))\alpha'^{-1}(\nu)]-\Gamma\rho(\xi)[\alpha'^{-1}(\mu),\alpha'^{-1}(\nu)]'\\
&+[\xi,\Gamma[\alpha'^{-1}(\mu),\alpha'^{-1}(\nu)]']+[\Gamma\mu,\Gamma\rho(\xi)\alpha'^{-1}(\nu)]-[\Gamma\mu,[\xi,\Gamma\alpha'^{-1}(\nu)]]+[\Gamma\rho(\xi)\alpha'^{-1}(\mu),\Gamma\nu]\\
&-[[\xi,\Gamma\alpha'^{-1}(\mu)],\Gamma\nu]-\Gamma\rho(\Gamma\rho(\xi)\alpha'^{-1}(\mu))\nu+\Gamma\rho([\xi,\Gamma\alpha'^{-1}(\mu)])\nu\\
=&-\Gamma\rho(\xi)\rho(\Gamma\alpha'^{-1}(\mu))\alpha'^{-1}(\nu)-\Gamma\rho(\xi)[\alpha'^{-1}(\mu),\alpha'^{-1}(\nu)]'+[\xi,[\Gamma\alpha'^{-1}(\mu),\Gamma\alpha'^{-1}(\nu)]]\\
&+\Gamma\rho(\Gamma\mu)\rho(\xi)\alpha'^{-1}(\nu)+\Gamma[\mu,\rho(\xi)\alpha'^{-1}(\nu)]'-[\alpha(\Gamma\alpha'^{-1}(\mu)),[\xi,\Gamma\alpha'^{-1}(\nu)]]+\Gamma[\rho(\xi)\alpha'^{-1}(\mu),\nu]'\\
&-[[\xi,\Gamma\alpha'^{-1}(\mu)],\alpha(\Gamma\alpha'^{-1}(\nu))]+\Gamma\rho([\xi,\Gamma\alpha'^{-1}(\mu)])\nu\\
=&-\Gamma\rho(\xi)\rho(\Gamma\alpha'^{-1}(\mu))\alpha'^{-1}(\nu)+\Gamma\rho(\alpha(\Gamma\alpha'^{-1}(\mu)))\rho(\xi)\alpha'^{-1}(\nu)+\Gamma\rho([\xi,\Gamma\alpha'^{-1}(\mu)])\nu\\
&-\Gamma\rho(\xi)[\alpha'^{-1}(\mu),\alpha'^{-1}(\nu)]'+\Gamma[\mu,\rho(\xi)\alpha'^{-1}(\nu)]'+\Gamma[\rho(\xi)\alpha'^{-1}(\mu),\nu]'\\
=&0.
\end{aligned}$$

Therefore, $\delta_\Gamma(\hbar(\xi))=0$. □

Definition 4.3. *Let Γ be a nonabelian ET on $(L,[\cdot,\cdot],\alpha)$ w.r.t the coherent representation $(L';\rho^\dagger,\alpha')$. The cohomology of the nonabelian ET on Hom-LieA is defined as the cohomology of the cochain complex $(C^\bullet_\Gamma(L',L):=\oplus_{n=0}^{+\infty}C^n_\Gamma(L',L),\delta_\Gamma)$.*

For $n\geq 1$, we denote the set of n-cocycles by $\mathcal{Z}^n_\Gamma(L',L)=\{f\in C^n_\Gamma(L',L)\mid \delta_\Gamma f=0\}$, the set of n-coboundaries by $\mathcal{B}^n_\Gamma(L',L)=\{\delta_\Gamma f\mid f\in C^{n-1}_\Gamma(L',L)\}$ and the n-th cohomology group of the nonabelian ET Γ by $\mathcal{H}^n_\Gamma(L',L)=\mathcal{Z}^n_\Gamma(L',L)/\mathcal{B}^n_\Gamma(L',L)$.

Finally, we study infinitesimal deformations of nonabelian ET on Hom-LieA by using the first cohomology group.

Let $(L,[\cdot,\cdot],\alpha)$ be a Hom-LieA over $\mathbb{K}$ and $\mathbb{K}[t]$ be the polynomial ring in one variable t. Then, $\mathbb{K}[t]/(t^2)\otimes L$ is an $\mathbb{K}[t]/(t^2)$-module. Moreover, $\mathbb{K}[t]/(t^2)\otimes L$ is a Hom-LieA over $\mathbb{K}[t]/(t^2)$, where the Hom-LieA structure is defined by

$$\begin{aligned}
&[f_1(t)\otimes a_1,f_2(t)\otimes a_2]=f_1(t)f_2(t)\otimes[a_1,a_2],\\
&\alpha(f_1(t)\otimes a_1)=f_1(t)\otimes\alpha(a_1),
\end{aligned}$$

for any $f_1(t),f_2(t)\in\mathbb{K}[t]/(t^2)$ and $a_1,a_2\in L$. In the sequel, we denote $f(t)\otimes a$ by $f(t)a$, where $f(t)\in\mathbb{K}[t]/(t^2)$.

Definition 4.4. *Let $\Gamma: L'\to L$ be a nonabelian ET on $(L,[\cdot,\cdot],\alpha)$ w.r.t the coherent representation $(L';\rho^\dagger,\alpha')$, and let $\Im: L'\to L$ be a linear map. If $\Gamma_t=\Gamma+t\Im \pmod{t^2}$ satisfies*

$$\Gamma_t\circ\alpha'=\alpha\circ\Gamma_t, \tag{4.1}$$

$$[\Gamma_t\mu,\Gamma_t\nu]=\Gamma_t(\rho(\Gamma_t\mu)\nu+[\mu,\nu]'), \tag{4.2}$$

for all $\mu, \nu \in L'$. Then, we say that $\Im$ generates an infinitesimal deformation of the nonabelian ET Γ.

Clearly Eqs (4.1) and (4.2) are equivalent to the following equations

$$\Im \circ \alpha' = \alpha \circ \Im, \tag{4.3}$$
$$[\Gamma\mu, \Im\nu] + [\Im\mu, \Gamma\nu] = \Im\rho(\Gamma\mu)\nu + \Gamma\rho(\Im\mu)\nu + \Im[\mu, \nu]', \tag{4.4}$$
$$[\Im\mu, \Im\nu] = \Im\rho(\Im\mu)\nu, \tag{4.5}$$

for all $\mu, \nu \in L'$. Thus, Γ_t is an infinitesimal deformation of Γ if and only if Eqs (4.3)–(4.5) hold. From Eqs (4.3) and (4.5) it follows that the map $\Im$ is a ET on the Hom-LieA $(L, [\cdot, \cdot, \cdot], \alpha)$ w.r.t the representation $(L'; \rho, \alpha')$ (see [20]).

Proposition 4.5. *Let $\Gamma_t = \Gamma + t\Im$ is an infinitesimal deformation of a nonabelian ET Γ on $(L, [\cdot, \cdot], \alpha)$ w.r.t the coherent representation $(L'; \rho^\dagger, \alpha')$. Then $\Im$ is a 1-cocycle of the nonabelian ET Γ. Moreover, the 1-cocycle $\Im$ is called the infinitesimal of the infinitesimal deformation Γ_t of Γ.*

Proof. We observe that Eq (4.4) implies $\delta_\Gamma \Im = 0$. □

Next, we discuss equivalent infinitesimal deformations.

Definition 4.6. *Let $\Gamma : L' \to L$ be a nonabelian ET on the regular Hom-LieA $(L, [\cdot, \cdot], \alpha)$ w.r.t the regular coherent representation $(L'; \rho^\dagger, \alpha')$. Two infinitesimal deformations $\Gamma_t^1 = \Gamma + t\Im_1$ and $\Gamma_t^2 = \Gamma + t\Im_2$ are said to be equivalent if there exists an element $\xi \in L$ such that $\alpha(\xi) = \xi$ and the pair $(\mathrm{Id}_L + t\alpha^{-1}(\mathrm{ad}(\xi)), \mathrm{Id}_{L'} + t\alpha'^{-1}(\rho(\xi)))$ is a homomorphism from $L' \xrightarrow{\Gamma_t^2} L$ to $L' \xrightarrow{\Gamma_t^1} L$.*

Let's recall Definition 2.12 that the the pair $(\mathrm{Id}_L + t\alpha^{-1}(\mathrm{ad}(\xi)), \mathrm{Id}_{L'} + t\alpha'^{-1}(\rho(\xi)))$ is a homomorphism from $L' \xrightarrow{\Gamma_t^2} L$ to $L' \xrightarrow{\Gamma_t^1} L$ if the following conditions are true:

(1) $\mathrm{Id}_L + t\alpha^{-1}(\mathrm{ad}(\xi)) : L \to L, x \mapsto x + t[\xi, \alpha^{-1}(x)]$ and $\mathrm{Id}_{L'} + t\alpha'^{-1}(\rho(\xi)) : L' \to L', \nu \mapsto \nu + t\rho(\xi)\alpha'^{-1}(\nu)$ are two Hom-LieA homomorphisms,

(2) $\rho(x)\nu + t\alpha'^{-1}(\rho(\xi)\rho(x)\nu) = \rho(x + t\alpha^{-1}(\mathrm{ad}(\xi)x))(\nu + t\alpha'^{-1}(\rho(\xi)\nu))$,

(3) $(\Gamma + t\Im_1)(\nu + t\alpha'^{-1}(\rho(\xi)\nu)) = (\mathrm{Id}_L + t\alpha^{-1}(\mathrm{ad}(\xi)))(\Gamma\nu + t\Im_2\nu)$, $\forall x \in L, \nu \in L'$.

By the condition (1), we have

$$[[\xi, \alpha^{-1}(x)], [\xi, \alpha^{-1}(y)]] = 0, \forall x, y \in L. \tag{4.6}$$

From the condition (2), we get

$$\rho([\xi, \alpha^{-1}(x)])\rho(\xi)\alpha'^{-1}(\nu) = 0, \forall x \in L, \nu \in L'. \tag{4.7}$$

From the condition (3), we have the following proposition.

Proposition 4.7. *Let Γ be a nonabelian ET on the regular Hom-LieA $(L, [\cdot, \cdot], \alpha)$ w.r.t the regular coherent representation $(L'; \rho^\dagger, \alpha')$. If two infinitesimal deformations $\Gamma_t^1 = \Gamma + t\Im_1$ and $\Gamma_t^2 = \Gamma + t\Im_2$ of Γ are equivalent, then $\Im_1$ and $\Im_2$ belong to the same cohomology class in $\mathcal{H}_\Gamma^1(L', L)$.*

Proof. When we compare the t^1 coefficients on both sides of the above condition (3), we can get

$$\begin{aligned} \Im_2\nu - \Im_1\nu =& \Gamma\alpha'^{-1}(\rho(\xi)\nu) - \alpha^{-1}([\xi, \Gamma\nu]) \\ =& \Gamma\rho(\xi)\alpha'^{-1}(\nu) - [a, \Gamma\alpha'^{-1}(\nu)] \\ =& \hbar(\xi)\nu \in \mathcal{B}_\Gamma^1(L', L), \end{aligned} \tag{4.8}$$

this means that $\Im_2$ and $\Im_1$ belong to the same cohomology class in $\mathcal{H}_\Gamma^1(L', L)$. □

Conversely, any 1-cocycle $\Im_1$ gives rise to the infinitesimal deformation $\Gamma_t^1 = \Gamma + t\Im_1$. In addition, cohomologous 1-cocycles correspond to equivalent infinitesimal deformations. To sum up, we get the following result.

Theorem 4.8. *Let $\Gamma : L' \to L$ be a nonabelian ET on the regular Hom-LieA $(L, [\cdot,\cdot], \alpha)$ w.r.t the regular coherent representation $(L'; \rho^\dagger, \alpha')$. Then, there is a bijection between the set of all equivalence classes of infinitesimal deformations of Γ and the first cohomology group $\mathcal{H}^1_\Gamma(L', L)$.*

Definition 4.9. *Let Γ be a nonabelian ET on the regular Hom-LieA $(L, [\cdot,\cdot], \alpha)$ w.r.t the regular coherent representation $(L'; \rho^\dagger, \alpha')$. An infinitesimal deformation $\Gamma_t = \Gamma + t\Im$ is said to be trivial if it is equivalent to the deformation Γ.*

Definition 4.10. *Let Γ be a nonabelian ET on the regular Hom-LieA $(L, [\cdot,\cdot], \alpha)$ w.r.t the regular coherent representation $(L'; \rho^\dagger, \alpha')$. An element $\xi \in L$ is called a Nijenhuis element associated to the nonabelian ET Γ if ξ satisfies $\alpha(\xi) = \xi$, Eqs (4.6), (4.7) and the equation*

$$[\xi, \Gamma\rho(\xi)\alpha'^{-1}(\nu) - [\xi, \Gamma\alpha'^{-1}(\nu)]] = 0, \forall \nu \in L'. \tag{4.9}$$

The set of all Nijenhuis elements is denoted as $\mathrm{Nij}(\Gamma)$.

According to Eqs (4.6)–(4.8), a trivial infinitesimal deformation will produce a Nijenhuis element. On the contrary, a Nijenhuis element can also produce a trivial infinitesimal deformation, as shown in the following theorem.

Theorem 4.11. *Let $\Gamma : L' \to L$ be a nonabelian ET on the regular Hom-LieA $(L, [\cdot,\cdot], \alpha)$ w.r.t the regular coherent representation $(L'; \rho^\dagger, \alpha')$. For any $\xi \in \mathrm{Nij}(\Gamma)$, the infinitesimal deformation $\Gamma_t = \Gamma + t\Im$ generated by $\Im := \hbar(\xi)$ is a trivial deformation of Γ.*

Proof. First, we need to show that $\Im$ satisfies Eqs (4.3)–(4.5). Obviously, for all $\mu, \nu \in L'$, we have

$$\begin{aligned}
&\alpha \circ \Im = \alpha \circ \hbar(\xi) = \hbar(\xi) \circ \alpha' = \Im \circ \alpha',\\
&[\Gamma\mu, \Im\nu] + [\Im\mu, \Gamma\nu] - \Im\rho(\Gamma\mu)\nu - \Gamma\rho(\Im\mu)\nu - \Im[\mu,\nu]'\\
=&[\Gamma\mu, \hbar(\xi)\nu] + [\hbar(\xi)\mu, \Gamma\nu] - \hbar(\xi)\rho(\Gamma\mu)\nu - \Gamma\rho(\hbar(\xi)\mu)\nu - \hbar(\xi)[\mu,\nu]'\\
=&0, \text{(by Proposition 4.2)}\\
&[\Im\mu, \Im\nu] - \Im\rho(\Im\mu)\nu\\
=&[\hbar(\xi)\mu, \hbar(\xi)\nu] - \hbar(\xi)\rho(\hbar(\xi)\mu)\nu\\
=&[\Gamma\rho(\xi)\alpha'^{-1}(\mu) - [\xi, \Gamma\alpha'^{-1}(\mu)], \Gamma\rho(\xi)\alpha'^{-1}(\nu) - [\xi, \Gamma\alpha'^{-1}(\nu)]]\\
&- \Gamma\rho(\xi)\alpha'^{-1}(\rho(\Gamma\rho(\xi)\alpha'^{-1}(\mu) - [\xi, \Gamma\alpha'^{-1}(\mu)])\nu) + [\xi, \Gamma\alpha'^{-1}(\rho(\Gamma\rho(\xi)\alpha'^{-1}(\mu) - [\xi, \Gamma\alpha'^{-1}(\mu)])\nu)]\\
=&[\Gamma\rho(\xi)\alpha'^{-1}(\mu), \Gamma\rho(\xi)\alpha'^{-1}(\nu)] + [[\xi, \Gamma\alpha'^{-1}(\mu)], [\xi, \Gamma\alpha'^{-1}(\nu)]] - [\Gamma\rho(\xi)\alpha'^{-1}(\mu), [\xi, \Gamma\alpha'^{-1}(\nu)]]\\
&- [[\xi, \Gamma\alpha'^{-1}(\mu)], \Gamma\rho(\xi)\alpha'^{-1}(\nu)] - \Gamma\rho(\xi)\alpha'^{-1}(\rho(\Gamma\rho(\xi)\alpha'^{-1}(\mu))\nu)\\
&+ \Gamma\rho(\xi)\alpha'^{-1}(\rho([\xi, \Gamma\alpha'^{-1}(\mu)])\nu) + [\xi, \alpha^{-1}\Gamma(\rho(\Gamma\rho(\xi)\alpha'^{-1}(\mu) - [\xi, \Gamma\alpha'^{-1}(\mu)])\nu)]\\
=&0. \text{ (by Eqs (4.6), (4.7) and (4.9))}
\end{aligned}$$

Thus, $\Gamma_t = \Gamma + t\Im$ is an infinitesimal deformation of Γ.

Next, we need to show that the infinitesimal deformation Γ_t is trivial. By $\xi \in \mathrm{Nij}(\Gamma)$, it immediately follows that the pair $(\mathrm{Id}_L + t\alpha^{-1}(\mathrm{ad}(\xi)), \mathrm{Id}_{L'} + t\alpha'^{-1}(\rho(\xi)))$ is a homomorphism from $L' \xrightarrow{\Gamma_t} L$ to $L' \xrightarrow{\Gamma} L$.
□

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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