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Robust stability and boundedness of uncertain conformable fractional-order delay systems under input saturation

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Abstract: In this article, a class of uncertain conformable fractional-order delay systems under input saturation is considered. By establishing the Lyapunov boundedness theorem for conformable fractional-order delay systems, some sufficient conditions for robust stability and boundedness of the systems are obtained. Examples are given to illustrate the obtained theory.

Keywords: conformable fractional-order delay systems; robust boundedness; robust stability; saturation
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1. Introduction

It is generally known that the fractional-order derivative generalizes the integer-order derivative, which attracts extensive attention for its tremendous application potentials in the domains of earthquake dynamics, electrical circuits, fluid dynamics, control theory and so forth. Compared with classical integer-order derivatives, its fractional-order counterpart can better simulate natural physical phenomena and dynamical system processes. In 2014, Khalil et al. [1] proposed a novel definition of the fractional-order derivative named conformable fractional-order derivative. It shares some advantages that neither the Caputo derivative or Riemann-Liouville derivative have. For instance, conformable fractional-order derivatives satisfy the chain rule $T^{\alpha}(\xi \circ \eta)(s) = T^{\alpha}\xi(\eta(s))T^{\alpha}\eta(s)$ and the Leibniz rule $T^{\alpha}(\xi(s)\eta(s)) = \xi(s)T^{\alpha}\eta(s) + \eta(s)T^{\alpha}\xi(s)$, but both the Caputo derivative and the Riemann-Liouville derivative fail to provide such admirable properties.

In the view of control, stability of fractional-order differential systems is currently a hot topic. Up to now, various meaningful and brilliant results related to stability or boundedness of fractional-order differential systems have been derived by Riemann-Liouville derivative or Caputo derivative [2–7]. Recently, Shahri et al. [8] proposed the Lyapunov method for the stability of uncertain fractional-order systems under input saturation. Advanced and interesting as their result is, the addressed systems fail to take delay effects into account. However, it is worth noting that time delays are also ubiquitous phenomenon due to some factors like limitation of transmission speed. It is reported that time delays cannot be ignored readily because the existence of delay could severely exert undesired influence on systems, which inevitably leads to instability, unboundedness, divergence, chaos, oscillation, divergence or other performance deterioration of systems [9]. On the other hand, in most practical systems, there exist many unavoidable constraints, one of which is input saturation. As a matter of fact, input saturation effects commonly exist owing to physical limitations like finite actuation power of systems. Hence, it is imperative to introduce both input saturation and delay effects into the dynamical behaviors of fractional-order differential systems. In the past decades, many differential systems with input saturation, delay effects or both have been widely investigated, and various intriguing results have been obtained [10–16], all of which limit the scope of the stability problem of fractional-order differential systems. However, stability may not be achieved sometimes because of some inevitable factors like external perturbations, which motivates us to further study the bounds of systems and try to confine it within a small range to realize boundedness of systems. So far, the problem with respect to the boundedness for integer-order systems has been studied widely [17–19]. Recently, many scholars have tried to study the boundedness of fractional-order systems, and some meaningful results have been reported [20–23]. However, most of these boundedness results are limited to the Caputo fractional-order systems. Therefore, it is necessary and meaningful to further study the boundedness problem of conformable fractional-order delay differential systems under input saturation.

In the existing works, two techniques are widely utilized for the investigation of asymptotic behavior of fractional-order differential systems, one of which is to establish fractional-order differential inequalities. Though estimating the solution of the fractional-order differential inequalities is an effective technique to investigate the stability of fractional-order differential systems, such methods share some limitations. The other technique to study the stability of fractional-order differential systems is Lyapunov's first method and Lyapunov's second method. As we all know, Lyapunov's first method is a powerful tool for studying the asymptotic behavior of fractional-order differential systems. Lyapunov's second method is sometimes challenging to apply to a fractional-order differential system since it is by no means an easy task to compute or estimate the fractional-order derivative of the Lyapunov function in the sense of the Riemann-Liouville derivative or Caputo derivative. However, conformable fractional-order derivative enjoys some well-behaved properties, which is analogous to integer-order derivatives such that Lyapunov's second method can be applied to fractional-order differential systems more easily. Various excellent results concerning the theory and application of the fractional-order Lyapunov function are proposed in [24–28]. Despite this progress, boundedness analysis of conformable fractional-order delay systems under input saturation based on the Lyapunov method is still in infancy, and limited research is available on the boundedness problem of conformable fractional-order delay systems under input saturation.

Originating from the above-mentioned discussions, this article mainly focuses on a class of uncertain conformable fractional-order delay systems under input saturation. By building the Lyapunov boundedness theorem for conformable fractional-order delayed systems, some sufficient conditions for robust stability and boundedness of the systems are obtained. Finally, numerical examples are presented to illustrate the feasibility of obtained theory. The main contributions of this article are listed as follows:

(i) Concerned with the problem of robust stability and boundedness of conformable fractional systems and take fully into account the effects of time delays and input saturation.

(ii) Lyapunov boundedness theorem for conformable fractional-order delay systems is proposed.

(iii) Based on the Lyapunov boundedness theorem for conformable fractional-order delay systems, some sufficient conditions for robust stability and boundedness of the systems are obtained.

The remainder of this article is organized as follows. In Section 2, some preliminaries are introduced. By establishing the Lyapunov boundedness theorem for conformable fractional-order delay systems, some sufficient conditions for robust stability and boundedness of the systems are proposed in Section 3. In Section 4, two examples are provided to illustrate the effectiveness of the main results. Finally, the conclusion is stated in Section 5.

2. Preliminaries and problem formulation

2.1. Preliminaries

Let $||\mathbf{x}|| = \sqrt{x^T x}$ denote the Euclidean norm of a vector $\mathbf{x}$, $||\mathbf{U}|| = \sqrt{eig_{\max}(\mathbf{U}^T\mathbf{U})}$ denote the trace norm of a matrix $\mathbf{U}$ and $\lambda_{\max}(\cdot)(\lambda_{\min}(\cdot))$ be the maximal(minimum) eigenvalue of a real symmetric square matrix.

Definition 2.1. [1,26] *For $h : [s_0, \infty) \to \mathbb{R}$, the conformable fractional derivative starting from s_0 of order $\alpha \in (0, 1]$ for h is defined by*

$$\mathcal{T}^{\alpha}_{s_0} h(s) = \lim_{\varkappa \to 0} \frac{h\left(s + \varkappa(s - s_0)^{1-\alpha}\right) - h(s)}{\varkappa}, \ s > s_0. \tag{2.1}$$

The conformable fractional derivative at s_0 is defined as $\mathcal{T}^{\alpha}_{s_0} h(s_0) = \lim_{s \to s_0^+} \mathcal{T}^{\alpha}_{s_0} h(s)$.

Lemma 2.2. [26] *Let $h : [t_0, \infty) \to \mathbb{R}$ be a continuous function such that $\mathcal{T}^{\alpha}_{t_0} h(t)$ exists on (t_0, ∞), if $\mathcal{T}^{\alpha}_{t_0} \geq 0$, for all $t \in (t_0, \infty)$, then h is an increasing function.*

Lemma 2.3. [26] *Let $\mathbf{x} : [t_0, \infty) \to \mathbb{R}^n$ such that $\mathcal{T}^{\alpha}_{t_0}$ exists on (t_0, ∞) and $\mathbf{Q}$ is a symmetric positive definite matrix. Then $\mathcal{T}^{\alpha}_{t_0}(\mathbf{x}^T\mathbf{Q}\mathbf{x})$ exists on (t_0, ∞) and the following relation is satisfied:*

$$\mathcal{T}^{\alpha}_{t_0}(\mathbf{x}^T\mathbf{Q}\mathbf{x}) = 2\mathbf{x}^T\mathbf{Q}\mathcal{T}^{\alpha}_{t_0}\mathbf{x}, \ t > t_0. \tag{2.2}$$

Definition 2.4. [26] *The conformable fractional exponential function is defined for every $a \geq 0$ by*

$$E_{\alpha}(b, a) = e^{b\frac{a^{\alpha}}{\alpha}},$$

where $\alpha \in (0, 1]$ and $b \in \mathbb{R}$.

Lemma 2.5. [29] *For any given matrices $\mathbf{U}$ and $\mathbf{V}$ with appropriate dimensions, there exists a positive scalar ϵ such that the following relationship holds:*

$$\mathbf{U}^T\mathbf{V} \leq \epsilon^{-1}\mathbf{U}^T\mathbf{V} + \epsilon\mathbf{V}^T\mathbf{V}. \tag{2.3}$$

2.2. Problem formulation

In this article, we will study the following uncertain fractional-order delay system:

$$\mathcal{T}^{\alpha}_{t_0}\mathbf{x}(t) = (\mathbf{U} + \Delta\mathbf{U}(t))\mathbf{x}(t) + (\mathbf{V} + \Delta\mathbf{V}(t))\mathbf{x}(t - \varsigma) + (\mathbf{C} + \Delta\mathbf{C}(t))\mathbf{sat}(\mathbf{u}(t)) + \mathbf{d}(\mathbf{x}, \mathbf{x}(t - \varsigma)), \tag{2.4}$$

where $\mathbf{x} \in \mathbb{R}^n$ denotes the state vector, $\mathbf{U}$, $\mathbf{V} \in \mathbb{R}^{n\times n}$ and $\mathbf{C} \in \mathbb{R}^{n\times m}$ represent constant system matrices corresponding to the linear part of the system dynamics and input vector, respectively, $\mathbf{u}(t) \in \mathbb{R}^m$ is the control input, $\mathbf{sat}(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is the saturation function (its definition will be given later), and $\mathbf{d}(\mathbf{x}, \mathbf{x}(t-\varsigma)) \in \mathbb{R}^n$ is the disturbance signal satisfying the following assumption.

Assumption 2.6. *There are three positive constants l_1, l_2 and l_3 such that*

$$\|\mathbf{d}(\mathbf{x}(t), \mathbf{x}(t-\varsigma))\|^2 \leq l_1\|\mathbf{x}\|^2 + l_2\|\mathbf{x}(\mathbf{t}-\varsigma)\|_\varsigma^2 + l_3. \tag{2.5}$$

Moreover, $\Delta\mathbf{U}(t) \in \mathbb{R}^n$ and $\Delta\mathbf{V}(t) \in \mathbb{R}^n$ stand for time-varying uncertain terms regarding to the mismatch model of the linear term, and $\Delta\mathbf{C}(t) \in \mathbb{R}^{n\times m}$ is the input matrix uncertainty satisfying the following assumption.

Assumption 2.7. *There are three positive constants α, β and γ such that*

$$\|\Delta\mathbf{U}\| \leq \alpha,\ \|\Delta\mathbf{V}\| \leq \beta,\ \|\Delta\mathbf{C}\| \leq \gamma. \tag{2.6}$$

Remark 2.8. [30] *A nonlinear function $h(\cdot)$ meets the Lipchitz condition if and only if*

$$\|h(\mathbf{y_1}) - h(\mathbf{y_2})\| \leq L_h\|\mathbf{y_1} - \mathbf{y_2}\|, \tag{2.7}$$

where $L_h > 0$ is the Lipschitz constant and $\mathbf{y_1}, \mathbf{y_2} \in \mathbb{R}^n$.

Remark 2.9. [31] *The saturation function denoted by $\mathbf{sat}(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$, $\mathbf{sat}(\mathbf{u}) = (sat(u_1), sat(u_2), \cdots, sat(u_m))^T$, $sat(u_i) = \min(\|u_i\|, 1)sign(u_i)$ satisfies the Lipschitz condition.*

Remark 2.10. [32] *Let $\mathbf{K} \in \mathbb{R}^{m\times n}$ be a constant matrix, $\varphi(x) = \mathbf{sat}(\mathbf{Kx}) - \mathbf{Kx}$, then there is a constant $l_\varphi \geq 0$ such that*

$$\|\varphi(\mathbf{x_1}) - \varphi(\mathbf{x_2})\| \leq l_\varphi\|\mathbf{x_1} - \mathbf{x_2}\|. \tag{2.8}$$

3. Main results

In this section, we will study the robust stability and boundedness of the system (2.4) via Lyapunov methods. To begin with, let us introduce the following Lyapunov boundedness theorem for conformable fractional-order delay systems.

Theorem 3.1. *Suppose that $\mathbf{x}$ is a solution of the conformable fractional-order delay system*

$$\begin{cases} \mathcal{T}_{t_0}^{\alpha}\mathbf{x} = f(t, \mathbf{x}, \mathbf{x}(t-\varsigma)), \\ \mathbf{x}(t_0+\nu) = \varphi(\nu),\ \nu \in [-\varsigma, 0],\ \varphi \in C[[-\varsigma, 0], \mathbb{R}^n]. \end{cases} \tag{3.1}$$

If there exist a Lyapunov function $G(t, \mathbf{x}(t))$ and positive numbers ϱ_i ($i = 1, 2, ..., 5$) with $\varrho_1\varrho_3 > \varrho_2\varrho_4$ such that

$$\varrho_1\|\mathbf{x}\|^2 \leq G(t, \mathbf{x}) \leq \varrho_2\|\mathbf{x}\|^2,\ (t, \mathbf{x}) \in [t_0 - \varsigma, \infty) \times \mathbb{R}^n, \tag{3.2}$$

$$\mathcal{T}_{t_0}^{\alpha}G(t, \mathbf{x}) \leq -\varrho_3\|\mathbf{x}\|^2 + \varrho_4\|\mathbf{x}(t)\|_\varsigma^2 + \varrho_5,\ (t, \mathbf{x}) \in [t_0, \infty) \times \mathbb{R}^n, \tag{3.3}$$

where $[G(t,\mathbf{x}(t))]_\varsigma = \sup_{-\varsigma\le\nu\le 0} G(t+\nu,\mathbf{x}(t+\nu))$. *Then, system (3.1) is exponentially ultimately bounded and the solution of system (3.1) obeys*

$$\|\mathbf{x}\| \le \sqrt{\frac{\rho_2}{\varrho_1}}\|\phi\|E_\alpha(-\frac{\vartheta}{2}, t-t_0)) + \sqrt{\frac{\varrho_2\varrho_5}{\varrho_1\varrho_3-\varrho_2\varrho_4}},\ t\ge t_0, \tag{3.4}$$

where $\vartheta > 0$ *is a solution of the following inequality:*

$$\frac{\varrho_4}{\varrho_1}E_\alpha(\vartheta,\varsigma) - \frac{\varrho_3}{\varrho_2} + \vartheta < 0. \tag{3.5}$$

Proof. Using (3.2) and (3.3), we have

$$\mathcal{T}^\alpha_{t_0} G(t,\mathbf{x}) \le -\frac{\varrho_3}{\varrho_2}G(t,\mathbf{x}) + \frac{\varrho_4}{\varrho_1}[G(t,\mathbf{x}(t))]_\varsigma + \varrho_5,\ (t,\mathbf{x})\in[t_0,\infty)\times\mathbb{R}^n. \tag{3.6}$$

Next, we claim that

$$G(t,\mathbf{x}) \le [G(t_0,\mathbf{x}(t_0))]_\varsigma E_\alpha(-\vartheta,(t-t_0)\vee 0) + \frac{\varrho_1\varrho_2\varrho_5}{\varrho_1\varrho_3-\varrho_2\varrho_4},\ t\in[t_0-\varsigma,\infty). \tag{3.7}$$

Let

$$\zeta(t) = [G(t_0,\mathbf{x}(t_0))]_\varsigma E_\alpha(-\vartheta,(t-t_0)\vee 0) + \frac{\varrho_1\varrho_2\varrho_5}{\varrho_1\varrho_3-\varrho_2\varrho_4},\ t\in[t_0-\varsigma,\infty). \tag{3.8}$$

If (3.7) is false, then, by Lemma 2.2, there is a $t^\star > t_0$ such that

$$G(t^\star,\mathbf{x}) = \zeta(t^\star), \tag{3.9}$$
$$\mathcal{T}^\alpha_{t_0} G(t^\star,\mathbf{x}) \ge \mathcal{T}^\alpha_{t_0}\zeta(t^\star), \tag{3.10}$$
$$G(t,\mathbf{x}) \le \zeta(t), t\in[t_0,t^\star). \tag{3.11}$$

According to Definition 2.1, we have

$$\begin{aligned}
\mathcal{T}^\alpha_{t_0}\zeta(t) =& \lim_{\eta\to 0}\{\frac{1}{\eta}\cdot[[G(t_0,\mathbf{x}(t_0))]_\varsigma e^{-\vartheta\frac{(t+\eta(t-t_0)^{1-\alpha}-t_0)^\alpha}{\alpha}} - [G(t_0,\mathbf{x}(t_0))]_\varsigma e^{-\vartheta\frac{(t-t_0)^\alpha}{\alpha}}]\}\\
=& \lim_{\eta\to 0}[-\frac{\vartheta}{\alpha}[G(t_0,\mathbf{x}(t_0))]_\varsigma e^{-\vartheta\frac{(t+\eta(t-t_0)^{1-\alpha}-t_0)^\alpha}{\alpha}}\alpha(t+\eta(t-t_0)^{1-\alpha}-t_0)^{\alpha-1}(t-t_0)^{1-\alpha}]\\
=& -\vartheta[G(t_0,\mathbf{x}(t_0))]_\varsigma e^{-\vartheta\frac{(t-t_0)^\alpha}{\alpha}}(t-t_0)^{\alpha-1}(t-t_0)^{1-\alpha}\\
=& -\vartheta[G(t_0,\mathbf{x}(t_0))]_\varsigma e^{-\vartheta\frac{(t-t_0)^\alpha}{\alpha}},\ t\ge t_0.
\end{aligned} \tag{3.12}$$

Therefore, by (3.12) one has

$$\mathcal{T}^\alpha_{t_0}\zeta(t^\star) = -\vartheta[G(t_0,\mathbf{x}(t_0))]_\varsigma e^{-\vartheta\frac{(t^\star-t_0)^\alpha}{\alpha}}. \tag{3.13}$$

It follows from (3.6), (3.8), (3.9) and (3.13) that

$$
\begin{aligned}
\mathcal{T}_{t_0}^{\alpha}G(t^\star,\mathbf{x}) \le & -\frac{\varrho_3}{\varrho_2}G(t^\star,\mathbf{x})+\frac{\varrho_4}{\varrho_1}[G(t^\star,\mathbf{x}(t))]_{\varsigma}+\varrho_5\\
\le & [G(t_0,\mathbf{x}(t_0))]_{\varsigma}[-\frac{\varrho_3}{\varrho_2}E_\alpha(-\vartheta,t^\star-t_0)+\frac{\varrho_4}{\varrho_1}E_\alpha(-\vartheta,(t^\star-\varsigma-t_0)\vee 0]\\
& -\frac{\varrho_3}{\varrho_2}\cdot\frac{\varrho_1\varrho_2\varrho_5}{\varrho_1\varrho_3-\varrho_2\varrho_4}+\frac{\varrho_4}{\varrho_1}\cdot\frac{\varrho_1\varrho_2\varrho_5}{\varrho_1\varrho_3-\varrho_2\varrho_4}+\varrho_5\\
\le & [G(t_0,\mathbf{x}(t_0))]_{\varsigma}[-\frac{\varrho_3}{\varrho_2}E_\alpha(-\vartheta,t^\star-t_0)+\frac{\varrho_4}{\varrho_1}E_\alpha(-\vartheta,t^\star-t_0)E_\alpha(\vartheta,\varsigma)]\\
\le & [G(t_0,\mathbf{x}(t_0))]_{\varsigma}[-\frac{\varrho_3}{\varrho_2}+\frac{\varrho_4}{\varrho_1}E_\alpha(\vartheta,\varsigma)]E_\alpha(-\vartheta,t^\star-t_0)\\
< & -\vartheta[G(t_0,\mathbf{x}(t_0))]_{\varsigma}E_\alpha(-\vartheta,t^\star-t_0)\le\mathcal{T}_{t_0}^{\alpha}\zeta(t^\star).
\end{aligned}
\tag{3.14}
$$

This contradicts (3.10). Thus, one has

$$
G(t,\mathbf{x})\le[G(t_0,\mathbf{x}(t_0))]_{\varsigma}E_\alpha(-\vartheta,t-t_0)+\frac{\varrho_1\varrho_2\varrho_5}{\varrho_1\varrho_3-\varrho_2\varrho_4},\ t\in[t_0,+\infty).
\tag{3.15}
$$

From this together with the condition (3.2), we know that (3.4) holds. The proof is completed.

Corollary 3.2. *Under the assumptions of Theorem 3.1, if $\varrho_5=0$, then system (3.1) is exponentially stable and the solution of system (3.1) obeys*

$$
\|\mathbf{x}\|\le\sqrt{\frac{\varrho_2}{\varrho_1}}\|\phi\|E_\alpha(-\frac{\vartheta}{2},t-t_0)),\ t\ge t_0,
\tag{3.16}
$$

where $\vartheta>0$ is a solution of the inequality

$$
\frac{\varrho_4}{\varrho_1}E_\alpha(\vartheta,\varsigma)-\frac{\varrho_3}{\varrho_2}+\vartheta<0.
\tag{3.17}
$$

Proof. By Theorem 3.1, the corollary follows.

Remark 3.3. *Theorem 1 in [26] is a special case of our Corollary 3.2.*

If we consider a state feedback $\mathbf{u}=\mathbf{K}\mathbf{x},\ \mathbf{K}\in\mathbb{R}^{m\times n}$, satisfying $-u_0\le u\le u_0$, then the closed-loop system can be written as

$$
\mathcal{T}_t^{\alpha}\mathbf{x}=(\mathbf{U}_{cl}+\Delta\mathbf{U})\mathbf{x}+(\mathbf{V}+\Delta\mathbf{V})\mathbf{x}(\mathbf{t}-\varsigma)+\mathbf{C}\varphi(\mathbf{x},t)+\Delta\mathbf{C}\mathbf{sat}(\mathbf{K}\mathbf{x})+\mathbf{d}(\mathbf{x},\mathbf{x}(t-\varsigma)),
\tag{3.18}
$$

where $\mathbf{U}_{cl}=\mathbf{U}+\mathbf{C}\mathbf{K}$ and $\varphi(\mathbf{x},t)=\mathbf{sat}(\mathbf{K}\mathbf{x})-\mathbf{K}\mathbf{x}$.

Remark 3.4. *If $0<u_i\le 1$, then the saturation function works in linear domain, $sat(u)=u$ and the entire closed-loop system is*

$$
\begin{aligned}
&\mathcal{T}_t^{\alpha}\mathbf{x}=(\mathbf{U}+\Delta\mathbf{U})\mathbf{x}+(\mathbf{V}+\Delta\mathbf{V})\mathbf{x}(\mathbf{t}-\varsigma)+(\mathbf{C}+\Delta\mathbf{C})\mathbf{K}\mathbf{x}+\mathbf{d}(\mathbf{x},\mathbf{x}(\mathbf{t}-\varsigma)),\\
&\mathcal{T}_t^{\alpha}\mathbf{x}=(\mathbf{U}_{cl}+\Delta\mathbf{U}+\Delta\mathbf{C}\mathbf{K})\mathbf{x}+(\mathbf{V}+\Delta\mathbf{V})\mathbf{x}(\mathbf{t}-\varsigma)+\mathbf{d}(\mathbf{x},\mathbf{x}(\mathbf{t}-\varsigma)).
\end{aligned}
\tag{3.19}
$$

Theorem 3.5. *Consider the closed-loop system (3.19) with $\mathbf{d}(\mathbf{x},\mathbf{x}(\mathbf{t}-\varsigma))=0$ under the Assumption 2.7. If there exist positive constants ϵ_1, ϵ_2, $\hat{\varrho}_3$ with $\hat{\varrho}_3>\epsilon_1+\epsilon_2$ and the controller matrix $\mathbf{K}$ such that the following relationship is satisfied:*

$$
\mathbf{U}_{cl}+\alpha\mathbf{I}+\gamma\|\mathbf{K}\|\mathbf{I}+\epsilon_2^{-1}\beta^2\mathbf{I}+\epsilon_1^{-1}\mathbf{V}\mathbf{V}^T+\hat{\varrho}_3\mathbf{I}\le 0.
\tag{3.20}
$$

Then, the closed-loop system (3.19) is robustly exponentially stable and the solution obeys

$$\|\mathbf{x}\| \leq \|\phi\| E_\alpha(-\frac{\vartheta}{2}, t - t_0)), \ t \geq t_0, \tag{3.21}$$

where $\vartheta > 0$ *is a solution of the following inequality:*

$$2(\epsilon_1 + \epsilon_2) E_\alpha(\vartheta, \varsigma) - 2\hat{\varrho}_3 + \vartheta < 0. \tag{3.22}$$

Proof. Let us choose the following Lyapunov function:

$$G = \frac{1}{2}\mathbf{x}^T\mathbf{x}. \tag{3.23}$$

Using Lemmas 2.3 and 2.5, we can derive that

$$\begin{aligned}\mathcal{T}_t^\alpha G =& \mathbf{x}^T(\mathbf{U}_{cl} + \Delta\mathbf{U} + \Delta\mathbf{CK})\mathbf{x} + \mathbf{x}^T(\mathbf{V} + \Delta\mathbf{V})\mathbf{x(t - \varsigma)} \\ \leq& \mathbf{x}^T(\mathbf{U}_{cl} + \|\Delta\mathbf{U}\| + \|\Delta\mathbf{CK}\|)\mathbf{x} + \mathbf{x}^T\mathbf{V}\mathbf{x(t - \varsigma)} + \mathbf{x}^T\Delta\mathbf{V}\mathbf{x(t - \varsigma)} \\ \leq& \mathbf{x}^T(\mathbf{U}_{cl} + \|\Delta\mathbf{U}\| + \|\Delta\mathbf{CK}\|)\mathbf{x} + \epsilon_1^{-1}\mathbf{x}^T\mathbf{V}\mathbf{V}^T\mathbf{x} + \epsilon_2^{-1}\mathbf{x}^T\Delta\mathbf{V}\Delta\mathbf{V}^T\mathbf{x} \\ &+ \epsilon_1\mathbf{x(t - \varsigma)}^T\mathbf{x(t - \varsigma)} + \epsilon_2\mathbf{x(t - \varsigma)}^T\mathbf{x(t - \varsigma)}.\end{aligned} \tag{3.24}$$

From Assumption 2.7 and (3.24), we have

$$\begin{aligned}\mathcal{T}_t^\alpha G \leq& \mathbf{x}^T(\mathbf{U}_{cl} + \alpha\mathbf{I} + \gamma\|\mathbf{K}\|\mathbf{I} + \epsilon_2^{-1}\beta^2\mathbf{I})\mathbf{x} + \epsilon_1^{-1}\mathbf{x}^T\mathbf{V}\mathbf{V}^T\mathbf{x} + (\epsilon_1 + \epsilon_2)\mathbf{x(t - \varsigma)}^T\mathbf{x(t - \varsigma)} \\ \leq& -\hat{\varrho}_3\|x\|^2 + (\epsilon_1 + \epsilon_2)\|\mathbf{x}(t)\|_\varsigma^2.\end{aligned} \tag{3.25}$$

Based on Corollary 3.2, the closed-loop system (3.19) with $\mathbf{d}(\mathbf{x}, \mathbf{x(t - \varsigma)}) = 0$ is robustly exponentially stable and the solution obeys (3.21). The proof is completed.

Theorem 3.6. *Consider the closed-loop system (3.19) under Assumptions 2.6 and 2.7. If there exist a positive symmetric definite matrix* $\mathbf{Q}$*, the controller matrix* $\mathbf{K}$ *and positive scalars* $\hat{\varrho}_3$*,* $\hat{\varrho}_4$ *and* ϵ_i*,* $i = 1, 2, ...5$*, such that the following relationships hold:*

$$\mathbf{Q}(\mathbf{U}_{cl} + \alpha\mathbf{I} + \gamma\|\mathbf{K}\|\mathbf{I}) + (\epsilon_3^{-1} + \epsilon_4^{-1} + \epsilon_5^{-1})\mathbf{Q}\mathbf{Q}^T + \epsilon_5 l_1\mathbf{I} + \hat{\varrho}_3\mathbf{I} \leq 0, \tag{3.26}$$

$$(\epsilon_3\mathbf{V}^T\mathbf{V} + \epsilon_4\beta^2\mathbf{I}) + \epsilon_5 l_2\mathbf{I} - \hat{\varrho}_4\mathbf{I} \leq 0, \tag{3.27}$$

$$\lambda_{\min}(\mathbf{Q})\hat{\varrho}_3 - \lambda_{\max}(\mathbf{Q})\hat{\varrho}_4 > 0. \tag{3.28}$$

Then, the closed-loop system (3.19) is robustly exponentially ultimately bounded and and the solution obeys

$$\|\mathbf{x}\| \leq \sqrt{\frac{\lambda_{\max}(\mathbf{Q})}{\lambda_{\min}(\mathbf{Q})}}\|\phi\| E_\alpha(-\frac{\vartheta}{2}, t - t_0)) + \sqrt{\frac{\lambda_{\max}(\mathbf{Q})\epsilon_5 l_3}{\lambda_{\min}(\mathbf{Q})\hat{\varrho}_3 - \lambda_{\max}(\mathbf{Q})\hat{\varrho}_4}}, \ t \geq t_0, \tag{3.29}$$

where $\vartheta > 0$ *is determined by the following inequality:*

$$\frac{2\hat{\varrho}_4}{\lambda_{\min}(\mathbf{Q})} E_\alpha(\vartheta, \varsigma) - \frac{2\hat{\varrho}_3}{\lambda_{\max}(\mathbf{Q})} + \vartheta < 0. \tag{3.30}$$

Proof. Let us choose the following Lyapunov function:

$$G = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x}. \tag{3.31}$$

Clearly,

$$\frac{1}{2}\lambda_{\min}(\mathbf{Q})||\mathbf{x}||^2 \le G(t, \mathbf{x}(t)) \le \frac{1}{2}\lambda_{\max}(\mathbf{Q})||\mathbf{x}||^2. \tag{3.32}$$

Using Lemmas 2.3 and 2.5, we can derive that

$$\begin{aligned}\mathcal{T}_t^{\alpha}G =&\mathbf{x}^T\mathbf{Q}((\mathbf{U}_{cl} + \Delta\mathbf{U} + \Delta\mathbf{C}\mathbf{K})\mathbf{x} + (\mathbf{V} + \Delta\mathbf{V})\mathbf{x}(\mathbf{t}-\varsigma) + \mathbf{d}(\mathbf{x}, \mathbf{x}(\mathbf{t}-\varsigma)))\\ =&\mathbf{x}^T\mathbf{Q}((\mathbf{U}_{cl} + \Delta\mathbf{U} + \Delta\mathbf{C}\mathbf{K})\mathbf{x} + (\mathbf{V} + \Delta\mathbf{V})\mathbf{x}(\mathbf{t}-\varsigma)) + \mathbf{x}^T\mathbf{Q}\mathbf{d}(\mathbf{x}, \mathbf{x}(\mathbf{t}-\varsigma))\\ \le&\mathbf{x}^T\mathbf{Q}(\mathbf{U}_{cl} + ||\Delta\mathbf{U}|| + ||\Delta\mathbf{C}\mathbf{K}||)\mathbf{x} + \mathbf{x}^T\mathbf{Q}\mathbf{V}\mathbf{x}(\mathbf{t}-\varsigma) + \mathbf{x}^T\mathbf{Q}\Delta\mathbf{V}\mathbf{x}(\mathbf{t}-\varsigma) + \mathbf{x}^T\mathbf{Q}\mathbf{d}(\mathbf{x}, \mathbf{x}(\mathbf{t}-\varsigma))\\ \le&\mathbf{x}^T\mathbf{Q}(\mathbf{U}_{cl} + ||\Delta\mathbf{U}|| + ||\Delta\mathbf{C}\mathbf{K}||)\mathbf{x} + \epsilon_3^{-1}\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x} + \epsilon_4^{-1}\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x} + \epsilon_5^{-1}\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x}\\ &+ \epsilon_3\mathbf{x}(\mathbf{t}-\varsigma)^T\mathbf{V}^T\mathbf{V}\mathbf{x}(\mathbf{t}-\varsigma) + \epsilon_4\mathbf{x}(\mathbf{t}-\varsigma)^T\Delta\mathbf{V}^T\Delta\mathbf{V}\mathbf{x}(\mathbf{t}-\varsigma) + \epsilon_5\mathbf{d}(\mathbf{x}, \mathbf{x}(\mathbf{t}-\varsigma))^T\mathbf{d}(\mathbf{x}, \mathbf{x}(\mathbf{t}-\varsigma)).\end{aligned} \tag{3.33}$$

Then, using Assumptions 2.6 and 2.7, we have

$$\begin{aligned}\mathcal{T}_t^{\alpha}G \le&\mathbf{x}^T\mathbf{Q}(\mathbf{U}_{cl} + \alpha\mathbf{I} + \gamma||\mathbf{K}||\mathbf{I})\mathbf{x} + (\epsilon_3^{-1} + \epsilon_4^{-1} + \epsilon_5^{-1})\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x}\\ &+ \epsilon_3\mathbf{x}(\mathbf{t}-\varsigma)^T\mathbf{V}^T\mathbf{V}\mathbf{x}(\mathbf{t}-\varsigma) + \epsilon_4\mathbf{x}(\mathbf{t}-\varsigma)^T\Delta\mathbf{V}^T\Delta\mathbf{V}\mathbf{x}(\mathbf{t}-\varsigma)\\ &+ \epsilon_5[l_1||\mathbf{x}|| + l_2||\mathbf{x}(\mathbf{t}-\varsigma)||_{\varsigma} + l_3]\\ \le&\mathbf{x}^T\mathbf{Q}(\mathbf{U}_{cl} + \alpha\mathbf{I} + \gamma||\mathbf{K}||\mathbf{I})\mathbf{x} + (\epsilon_3^{-1} + \epsilon_4^{-1} + \epsilon_5^{-1})\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x}\\ &+ \mathbf{x}(\mathbf{t}-\varsigma)^T(\epsilon_3\mathbf{V}^T\mathbf{V} + \epsilon_4\beta^2\mathbf{I})\mathbf{x}(\mathbf{t}-\varsigma) + \epsilon_5(l_1||\mathbf{x}||^2 + l_2||\mathbf{x}(\mathbf{t}-\varsigma)||_{\varsigma}^2 + l_3).\end{aligned}$$

This together with (3.26) and (3.27), we have

$$\mathcal{T}_t^{\alpha}G \le -\hat{\varrho}_3||\mathbf{x}||^2 + \hat{\varrho}_4||\mathbf{x}(\mathbf{t}-\varsigma)||_{\varsigma}^2 + \epsilon_5 l_3. \tag{3.34}$$

Then, with the help of (3.28), (3.32) and (3.34), one can apply Theorem 3.1 to conclude that the closed-loop system (3.19) is robustly exponentially ultimately bounded and the solution obeys (3.29). The proof is completed.

Theorem 3.7. *Consider the closed-loop system (3.18) with* $\mathbf{d}(\mathbf{x}, \mathbf{x}(\mathbf{t}-\varsigma)) = 0$*. Suppose that Assumption 2.7 holds. If there exist a positive symmetric definite matrix* $\mathbf{Q}$*, the controller matrix* $\mathbf{K}$ *and positive scalars* $\hat{\varrho}_3$*,* $\hat{\varrho}_4$ *and* ϵ_i*,* $i = 1, 2, ...5$*, such that the following relationships hold:*

$$\begin{aligned}&\mathbf{Q}\mathbf{U}_{cl} + (\epsilon_1^{-1} + \epsilon_2^{-1} + \epsilon_3^{-1} + \epsilon_5^{-1}\gamma^2)\mathbf{Q}\mathbf{Q}^T + \epsilon_4^{-1}\mathbf{Q}CC^T\mathbf{Q}^T\\ &+ (\epsilon_1\alpha^2 + \epsilon_4 l_{\varphi}^2 + \epsilon_5(l_{\varphi} + ||K||)^2)\mathbf{I} + \hat{\varrho}_3\mathbf{I} \le 0,\end{aligned} \tag{3.35}$$

$$\epsilon_2\mathbf{V}^T\mathbf{V} + \epsilon_3\beta^2\mathbf{I} - \hat{\varrho}_4\mathbf{I} \le 0, \tag{3.36}$$

$$\lambda_{\min}(\mathbf{Q})\hat{\varrho}_3 - \lambda_{\max}(\mathbf{Q})\hat{\varrho}_4 > 0. \tag{3.37}$$

Then, the closed-loop system (3.18) with $\mathbf{d}(\mathbf{x}, \mathbf{x}(\mathbf{t}-\varsigma)) = 0$ *is robustly exponentially stable and the solution obeys*

$$||\mathbf{x}|| \le ||\phi||E_{\alpha}(-\frac{\vartheta}{2}, t - t_0)),\ t \ge t_0, \tag{3.38}$$

where $\vartheta > 0$ is a solution of the inequality

$$\frac{2\hat{\varrho}_4}{\lambda_{\min}(\mathbf{Q})} E_\alpha(\vartheta, \varsigma) - \frac{2\hat{\varrho}_3}{\lambda_{\max}(\mathbf{Q})} + \vartheta < 0. \tag{3.39}$$

Proof. Let us choose the following Lyapunov function:

$$G(t, x(t)) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x}. \tag{3.40}$$

Using Lemmas 2.3 and 2.5, we can derive that

$$\begin{aligned}
\mathcal{T}_t^\alpha G =& \mathbf{x}^T\mathbf{Q}((\mathbf{U}_{cl} + \Delta\mathbf{U})\mathbf{x} + (\mathbf{V} + \Delta\mathbf{V})\mathbf{x(t-\varsigma)} + \mathbf{C}\varphi(\mathbf{x}, t) + \Delta\mathbf{C}\mathbf{sat}(\mathbf{Kx})) \\
=& \mathbf{x}^T\mathbf{Q}\mathbf{U}_{cl}\mathbf{x} + \mathbf{x}^T\mathbf{Q}\Delta\mathbf{U}\mathbf{x} + \mathbf{x}^T\mathbf{Q}\mathbf{V}\mathbf{x(t-\varsigma)} + \mathbf{x}^T\mathbf{Q}\Delta\mathbf{V}\mathbf{x(t-\varsigma)} + \mathbf{x}^T\mathbf{Q}\mathbf{C}\varphi(\mathbf{x}, t) + \mathbf{x}^T\mathbf{Q}\Delta\mathbf{C}\mathbf{sat}(\mathbf{Kx})) \\
\leq& \mathbf{x}^T\mathbf{Q}\mathbf{U}_{cl}\mathbf{x} + \epsilon_1^{-1}\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x} + \epsilon_1\mathbf{x}^T\Delta\mathbf{U}^T\Delta\mathbf{U}\mathbf{x} + \epsilon_2^{-1}\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x} + \epsilon_2\mathbf{x(t-\varsigma)}^T\mathbf{V}^T\mathbf{V}\mathbf{x(t-\varsigma)} \\
&+ \epsilon_3^{-1}\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x} + \epsilon_3\mathbf{x(t-\varsigma)}^T\Delta\mathbf{V}^T\Delta\mathbf{V}\mathbf{x(t-\varsigma)} + \epsilon_4^{-1}\mathbf{x}^T\mathbf{Q}CC^T\mathbf{Q}^T\mathbf{x} + \epsilon_4\varphi(\mathbf{x}, t)^T\varphi(\mathbf{x}, t) \\
&+ \epsilon_5^{-1}\mathbf{x}^T\mathbf{Q}\Delta\mathbf{C}\Delta\mathbf{C}^T\mathbf{Q}^T\mathbf{x} + \epsilon_5\mathbf{sat}(\mathbf{Kx})^T\mathbf{sat}(\mathbf{Kx}).
\end{aligned} \tag{3.41}$$

Then, using Assumption 2.7 and Remark 2.10, we have

$$\begin{aligned}
\mathcal{T}_t^\alpha G \leq& \mathbf{x}^T\mathbf{Q}\mathbf{U}_{cl}\mathbf{x} + \epsilon_1^{-1}\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x} + \epsilon_1\mathbf{x}^T\alpha^2\mathbf{x} + \epsilon_2^{-1}\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x} + \epsilon_2\mathbf{x(t-\varsigma)}^T\mathbf{V}^T\mathbf{V}\mathbf{x(t-\varsigma)} \\
&+ \epsilon_3^{-1}\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x} + \epsilon_3\mathbf{x(t-\varsigma)}^T\beta^2\mathbf{x(t-\varsigma)} + \epsilon_4^{-1}\mathbf{x}^T\mathbf{Q}CC^T\mathbf{Q}^T\mathbf{x} + \epsilon_4\mathbf{x}^T l_\varphi^2\mathbf{x} \\
&+ \epsilon_5^{-1}\mathbf{x}^T\mathbf{Q}\gamma^2\mathbf{Q}^T\mathbf{x} + \epsilon_5\mathbf{x}^T(l_\varphi + \|K\|)^2\mathbf{x} \\
=& \mathbf{x}^T(\mathbf{Q}\mathbf{U}_{cl} + (\epsilon_1^{-1} + \epsilon_2^{-1} + \epsilon_3^{-1} + \epsilon_5^{-1}\gamma^2)\mathbf{Q}\mathbf{Q}^T + \epsilon_4^{-1}\mathbf{Q}CC^T\mathbf{Q}^T \\
&+ (\epsilon_1\alpha^2 + \epsilon_4 l_\varphi^2 + \epsilon_5(l_\varphi + \|K\|)^2)\mathbf{I})\mathbf{x} + \mathbf{x(t-\varsigma)}^T(\epsilon_2\mathbf{V}^T\mathbf{V} + \epsilon_3\beta^2\mathbf{I})\mathbf{x(t-\varsigma)}.
\end{aligned} \tag{3.42}$$

This together with (3.35) and (3.36), we have

$$\mathcal{T}_t^\alpha G \leq -\hat{\varrho}_3\|\mathbf{x}\|^2 + \hat{\varrho}_4\|\mathbf{x(t-\varsigma)}\|_\varsigma^2. \tag{3.43}$$

Based on the Corollary 3.2, the closed-loop system (3.18) with $\mathbf{d}(\mathbf{x}, \mathbf{x(t-\varsigma)}) = 0$ is robustly exponentially stable and the solution obeys (3.38). The proof is completed.

Theorem 3.8. *Consider the closed-loop system (3.18) under Assumptions 2.6 and 2.7. If there exist a positive symmetric definite matrix $\mathbf{Q}$, the controller matrix $\mathbf{K}$ and positive scalars $\hat{\varrho}_3$, $\hat{\varrho}_4$ and ϵ_i, $i = 1, 2, ...6$, such that the following relationships hold:*

$$\begin{aligned}
&\mathbf{Q}\mathbf{U}_{cl} + (\epsilon_1^{-1} + \epsilon_2^{-1} + \epsilon_3^{-1} + \epsilon_6^{-1} + \epsilon_5^{-1}\gamma^2)\mathbf{Q}\mathbf{Q}^T + \epsilon_4^{-1}\mathbf{Q}CC^T\mathbf{Q}^T \\
&+ (\epsilon_1\alpha^2 + \epsilon_4 l_\varphi^2 + \epsilon_6 l_1 + \epsilon_5(l_\varphi + \|K\|)^2)\mathbf{I} + \hat{\varrho}_3\mathbf{I} \leq 0,
\end{aligned} \tag{3.44}$$

$$\epsilon_2\mathbf{V}^T\mathbf{V} + \epsilon_3\beta^2\mathbf{I} + \epsilon_6 l_2\mathbf{I} - \hat{\varrho}_4\mathbf{I} \leq 0, \tag{3.45}$$

$$\lambda_{\min}(\mathbf{Q})\hat{\varrho}_3 - \lambda_{\max}(\mathbf{Q})\hat{\varrho}_4 > 0. \tag{3.46}$$

Then, the closed-loop system (3.18) is robustly exponentially ultimately bounded and the solution obeys

$$\|\mathbf{x}\| \leq \sqrt{\frac{\lambda_{\max}(\mathbf{Q})}{\lambda_{\min}(\mathbf{Q})}}\|\phi\|E_\alpha(-\frac{\vartheta}{2}, t - t_0)) + \sqrt{\frac{\lambda_{\max}(\mathbf{Q})\epsilon_6 l_3}{\lambda_{\min}(\mathbf{Q})\hat{\varrho}_3 - \lambda_{\max}(\mathbf{Q})\hat{\varrho}_4}},\ t \geq t_0, \tag{3.47}$$

where $\vartheta > 0$ is a solution of the following inequality:

$$\frac{2\hat{\varrho}_4}{\lambda_{\min}(\mathbf{Q})}E_\alpha(\vartheta, \varsigma) - \frac{2\hat{\varrho}_3}{\lambda_{\max}(\mathbf{Q})} + \vartheta < 0. \tag{3.48}$$

Proof. Let us choose the following Lyapunov function:

$$G = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x}. \tag{3.49}$$

Using Lemmas 2.3 and 2.5, we can derive that

$$\begin{aligned}
\mathcal{T}_t^\alpha G =& \mathbf{x}^T\mathbf{Q}((\mathbf{U}_{cl} + \Delta\mathbf{U})\mathbf{x} + (\mathbf{V} + \Delta\mathbf{V})\mathbf{x}(\mathbf{t} - \varsigma) + \mathbf{C}\varphi(\mathbf{x}, t) + \Delta\mathbf{C}\mathbf{sat}(\mathbf{K}\mathbf{x}) + \mathbf{d}(\mathbf{x}, \mathbf{x}(\mathbf{t} - \varsigma)))\\
=& \mathbf{x}^T\mathbf{Q}\mathbf{U}_{cl}\mathbf{x} + \mathbf{x}^T\mathbf{Q}\Delta\mathbf{U}\mathbf{x} + \mathbf{x}^T\mathbf{Q}\mathbf{V}\mathbf{x}(\mathbf{t} - \varsigma) + \mathbf{x}^T\mathbf{Q}\Delta\mathbf{V}\mathbf{x}(\mathbf{t} - \varsigma) + \mathbf{x}^T\mathbf{Q}\mathbf{C}\varphi(\mathbf{x}, t)\\
&+ \mathbf{x}^T\mathbf{Q}\Delta\mathbf{C}\mathbf{sat}(\mathbf{K}\mathbf{x}) + \mathbf{x}^T\mathbf{Q}\mathbf{d}(\mathbf{x}, \mathbf{x}(\mathbf{t} - \varsigma))\\
\leq& \mathbf{x}^T\mathbf{Q}\mathbf{U}_{cl}\mathbf{x} + \epsilon_1^{-1}\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x} + \epsilon_1\mathbf{x}^T\Delta\mathbf{U}^T\Delta\mathbf{U}\mathbf{x} + \epsilon_2^{-1}\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x} + \epsilon_2\mathbf{x}(\mathbf{t} - \varsigma)^T\mathbf{V}^T\mathbf{V}\mathbf{x}(\mathbf{t} - \varsigma)\\
&+ \epsilon_3^{-1}\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x} + \epsilon_3\mathbf{x}(\mathbf{t} - \varsigma)^T\Delta\mathbf{V}^T\Delta\mathbf{V}\mathbf{x}(\mathbf{t} - \varsigma) + \epsilon_4^{-1}\mathbf{x}^T\mathbf{Q}CC^T\mathbf{Q}^T\mathbf{x} + \epsilon_4\varphi(\mathbf{x}, t)^T\varphi(\mathbf{x}, t)\\
&+ \epsilon_5^{-1}\mathbf{x}^T\mathbf{Q}\Delta\mathbf{C}\Delta\mathbf{C}^T\mathbf{Q}^T\mathbf{x} + \epsilon_5\mathbf{sat}(\mathbf{K}\mathbf{x})^T\mathbf{sat}(\mathbf{K}\mathbf{x}) + \epsilon_6^{-1}\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x}\\
&+ \epsilon_6\mathbf{d}(\mathbf{x}, \mathbf{x}(\mathbf{t} - \varsigma))^T\mathbf{d}(\mathbf{x}, \mathbf{x}(\mathbf{t} - \varsigma)).
\end{aligned} \tag{3.50}$$

Then, using Assumptions 2.6 and 2.7, we have

$$\begin{aligned}
\mathcal{T}_t^\alpha G \leq& \mathbf{x}^T\mathbf{Q}\mathbf{U}_{cl}\mathbf{x} + \epsilon_1^{-1}\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x} + \epsilon_1\mathbf{x}^T\alpha^2\mathbf{x} + \epsilon_2^{-1}\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x} + \epsilon_2\mathbf{x}(\mathbf{t} - \varsigma)^T\mathbf{V}^T\mathbf{V}\mathbf{x}(\mathbf{t} - \varsigma)\\
&+ \epsilon_3^{-1}\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x} + \epsilon_3\mathbf{x}(\mathbf{t} - \varsigma)^T\beta^2\mathbf{x}(\mathbf{t} - \varsigma) + \epsilon_4^{-1}\mathbf{x}^T\mathbf{Q}CC^T\mathbf{Q}^T\mathbf{x} + \epsilon_4\mathbf{x}^T l_\varphi^2\mathbf{x}\\
&+ \epsilon_5^{-1}\mathbf{x}^T\mathbf{Q}\gamma^2\mathbf{Q}^T\mathbf{x} + \epsilon_5\mathbf{x}^T(l_\varphi + ||K||)^2\mathbf{x} + \epsilon_6^{-1}\mathbf{x}^T\mathbf{Q}\mathbf{Q}^T\mathbf{x} + \epsilon_6(l_1||\mathbf{x}||^2 + l_2||\mathbf{x}(\mathbf{t} - \varsigma)||_\varsigma^2 + l_3)\\
=& \mathbf{x}^T(\mathbf{Q}\mathbf{U}_{cl} + (\epsilon_1^{-1} + \epsilon_2^{-1} + \epsilon_3^{-1} + \epsilon_6^{-1} + \epsilon_5^{-1}\gamma^2)\mathbf{Q}\mathbf{Q}^T + \epsilon_4^{-1}\mathbf{Q}CC^T\mathbf{Q}^T\\
&+ (\epsilon_1\alpha^2 + \epsilon_4 l_\varphi^2 + \epsilon_6 l_1 + \epsilon_5(l_\varphi + ||K||)^2)\mathbf{I})\mathbf{x}\\
&+ \mathbf{x}(\mathbf{t} - \varsigma)^T(\epsilon_2\mathbf{V}^T\mathbf{V} + \epsilon_3\beta^2\mathbf{I} + \epsilon_6 l_2\mathbf{I})\mathbf{x}(\mathbf{t} - \varsigma) + \epsilon_6 l_3.
\end{aligned} \tag{3.51}$$

This together with (3.44) and (3.45), we have

$$\mathcal{T}_t^\alpha G \leq -\hat{\varrho}_3||\mathbf{x}||^2 + \hat{\varrho}_4||\mathbf{x}(\mathbf{t} - \varsigma)||_\varsigma^2 + \epsilon_6 l_3. \tag{3.52}$$

Then, with the help of (3.32), (3.46) and (3.52), one can apply Theorem 3.1 to conclude that the closed-loop system (3.18) is robustly exponentially ultimately bounded and the solution obeys (3.47). The proof is completed.

Remark 3.9. *Taking $\mathbf{W} = \mathbf{Q}^{-1}$ in Theorem 3.8, the boundedness condition becomes*

$$\begin{aligned}
&\mathbf{U}_{cl}\mathbf{W} + (\epsilon_1^{-1} + \epsilon_2^{-1} + \epsilon_3^{-1} + \epsilon_6^{-1} + \epsilon_5^{-1}\gamma^2)\mathbf{I} + \epsilon_4^{-1}CC^T\\
&+ (\epsilon_1\alpha^2 + \epsilon_4 l_\varphi^2 + \epsilon_6 l_1 + \epsilon_5(l_\varphi + ||K||)^2 + \hat{\varrho}_3)\mathbf{W}\mathbf{I}\mathbf{W}^{-1} \leq 0,\\
&\epsilon_2\mathbf{V}^T\mathbf{V} + \epsilon_3\beta^2\mathbf{I} + \epsilon_6 l_2\mathbf{I} + \hat{\varrho}_4\mathbf{I} \leq 0,\\
&\lambda_{\min}(\mathbf{W})\hat{\varrho}_3 - \lambda_{\max}(\mathbf{W})\hat{\varrho}_4 > 0.
\end{aligned}$$

Remark 3.10. *Take Theorem 3.8 for example, the design procedure of the controller is as follows:*

(1) Calculate l_i, $i = 1,2,3$ from Assumption 2.6, and calculate α, β and γ from Assumption 2.7.
(2) Choose constants $\epsilon_i > 0$, $i = 1,...,6$ from Lemma 2.5.
(3) Compute $\mathbf{QU}_{cl}$, $\mathbf{QQ}^T$ $\mathbf{Q}CC^T\mathbf{Q}^T$, $\mathbf{V}^T\mathbf{V}$, $\lambda_{\min}(\mathbf{Q})$ and $\lambda_{\max}(\mathbf{Q})$.
(4) Choose $\hat{\varrho}_4 > 0$ which satisfies (3.45).
(5) Choose $\hat{\varrho}_4 > 0$ which satisfies (3.46).
(6) Select suitable controller matrix $\mathbf{K}$ such that (3.45) holds.

Remark 3.11. *Based on the Lyapunov method, some sufficient conditions of the stability for a class of fractional-order systems under input saturation has been derived in [8]. Obviously, their conditions cannot be used to verify the stability and boundedness of the system (2.4). In fact, the conditions in [8] are limited to stability and are not valid for boundedness. On the other hand, the conditions in [8] are limited to Caputo fractional-order systems and are not suitable for conformable fractional-order systems.*

Remark 3.12. *Although some effective methods for studying stability and boundedness have been proposed for conformable fractional-order systems [26, 27], these results are ineffective to investigate the stability and boundedness of (2.4) since time delays and input saturation were ignored in [26, 27].*

4. Examples

In the current section, two examples are provided to illustrate the effectiveness of the main results.

Example 4.1. Consider the following fractional-order delay systems:

$$\begin{bmatrix} \mathcal{T}_{t_0}^{0.9}\mathbf{x}_1 \\ \mathcal{T}_{t_0}^{0.9}\mathbf{x}_2 \end{bmatrix} = \left\{ \begin{bmatrix} -12.5 & 0 \\ 0 & -13.5 \end{bmatrix} + \Delta\mathbf{U} \right\} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \left\{ \begin{bmatrix} 1.5 & 0.5 \\ -0.7 & 1.3 \end{bmatrix} + \Delta\mathbf{V} \right\} \begin{bmatrix} \mathbf{x}_1(t-2) \\ \mathbf{x}_2(t-2) \end{bmatrix} + \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \Delta\mathbf{C} \right\} \mathbf{sat}(\mathbf{u}) + \mathbf{d}(\mathbf{x}, \mathbf{x}(\mathbf{t} - \varsigma)), \tag{4.1}$$

where

$$\Delta\mathbf{U} = 0.5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sin(t), \ \Delta\mathbf{V} = 0.2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cos(t), \ \Delta\mathbf{C} = 0.3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(t),$$
$$\mathbf{d}(\mathbf{x}, \mathbf{x}(\mathbf{t} - \varsigma)) = 0.5\mathbf{x} + 0.5\mathbf{x}(\mathbf{t} - \mathbf{2}) + 2.$$

Let $\mathbf{Q} = \mathbf{I}_2$. Based on Remark 3.10, one can check that all the conditions of Theorem 3.8 are satisfied by taking $\epsilon_1 = \epsilon_4 = \epsilon_6 = 1$, $\epsilon_2 = 0.5$, $\epsilon_3 = 10$, $\epsilon_5 = 0.1$, $\varrho_3 = 3.55$, $\vartheta = 0.05$, $\varrho_4 = 2.85$ and $\mathbf{K} = [-1, -1]$. According to Theorem 3.8, the robust exponential boundedness of closed-loop system (4.1) is reached. The time response of the closed-loop system (4.1) with initial conditions is shown in Figure 1.

Example 4.2. Consider the following fractional-order delay systems:

$$\begin{bmatrix} \mathcal{T}_{t_0}^{0.95}\mathbf{x}_1 \\ \mathcal{T}_{t_0}^{0.95}\mathbf{x}_2 \end{bmatrix} = \left\{ \begin{bmatrix} -1.5 & -1 \\ -1 & -1.5 \end{bmatrix} + \Delta\mathbf{U} \right\} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \left\{ \begin{bmatrix} 0.05 & 0 \\ 0 & 0.04 \end{bmatrix} + \Delta\mathbf{V} \right\} \begin{bmatrix} \mathbf{x}_1(t-2) \\ \mathbf{x}_2(t-2) \end{bmatrix} + \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \Delta\mathbf{C} \right\} \mathbf{sat}(\mathbf{u}), \tag{4.2}$$

where

$$\Delta\mathbf{U} = 0.05\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\sin(t),\ \Delta\mathbf{V} = 0.04\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\cos(t),\ \Delta\mathbf{C} = 0.03\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos(t).$$

One can check that all the conditions of Theorem 3.7 are satisfied by taking $\mathbf{Q} = \mathbf{I}_2$, $\epsilon_1 = \epsilon_2 = \epsilon_3 = 10$, $\epsilon_4 = 0.5$, $\epsilon_5 = 0.1$, $\varrho_3 = 0.35$, $\vartheta = 0.05$, $\varrho_4 = 0.05$ and $\mathbf{K} = [-1, -1]$. According to Theorem 3.7, the robust exponential stability of the closed-loop system (4.2) is reached. The time response of the closed-loop system (4.2) with initial conditions is shown in Figure 2.

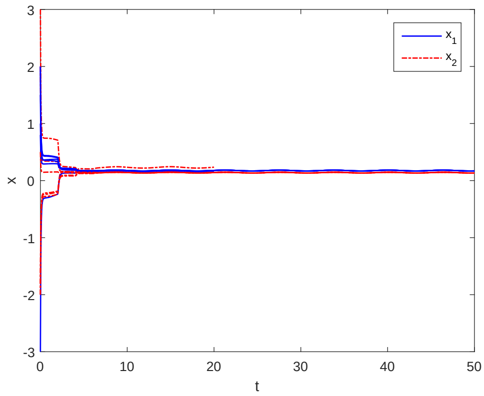

Figure 1. The trajectories of $\mathbf{x}_1$ and $\mathbf{x}_2$ of system (4.1).

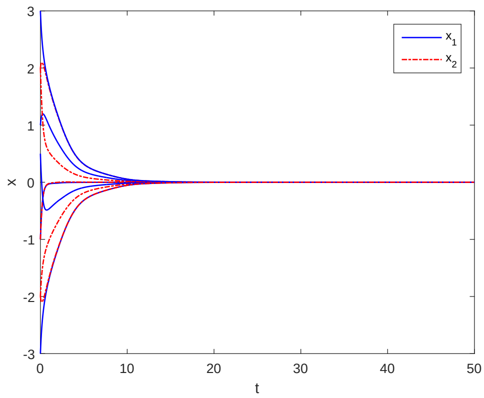

Figure 2. The trajectories of $\mathbf{x}_1$ and $\mathbf{x}_2$ of system (4.2).

5. Conclusions

In this article, we considered uncertain conformable fractional-order delay systems under input saturation. The Lyapunov boundedness theorem for conformable fractional-order delay systems was proposed by the the fractional comparison principle. Using the Lyapunov boundedness theorem, some sufficient conditions for robust stability and boundedness of the systems were presented. Two examples were given to show the validity of the obtained results. Considering that time delays sometimes appear in the derivative of the state, we will extend the results of this article to the neutral case in future work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

No potential conflict of interest was reported by the author.

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