Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(9): 20755–20789.
DOI:10.3934/math.20231058
Received: 17 January 2023
Revised: 19 June 2023
Accepted: 20 June 2023
Published: 29 June 2023

Research article

Computational modeling of financial crime population dynamics under different fractional operators

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Abstract: This paper presents an analysis and numerical simulation of financial crime population dynamics using fractional order calculus and Newton's polynomial. The dynamics of financial crimes are modeled as a fractional-order system, which is then solved using numerical methods based on Newton's polynomial. The results of the simulation provide insights into the behavior of financial crime populations over time, including the stability and convergence of the systems. The study provides a new approach to understanding financial crime populations and has potential applications in developing effective strategies for combating financial crimes. Fractional derivatives are commonly applied in many interdisciplinary fields of science because of its effectiveness in understanding and analyzing complicated phenomena. In this work, a mathematical model for the population dynamics of financial crime with fractional derivatives is reformulated and analyzed. A fractional-order financial crime model using the new Atangana-Baleanu-Caputo (ABC) derivative is introduced. The reproduction number for financial crime is calculated. In addition, the relative significance of model parameters is also determined by sensitivity analysis. The existence and uniqueness of the solution in consideration of the ABC derivative are discussed. A number of conditions are established for the existence and Ulam-Hyers stability of financial crime equilibria. A numerical scheme is presented for the proposed model, starting with the Caputo-Fabrizio fractional derivative, followed by the Caputo and Atangana-Baleanu fractional derivatives. Finally, we solve the models with fractal-fractional derivatives.

Keywords: financial crime; stability; reproduction number; fractional modeling; numerical schemes
Mathematics Subject Classification: 34A08, 49J15, 65P99

1. Introduction

A country's political and socioeconomic systems can be negatively affected by financial crime and its dynamic spread in a given population. Crimes against property (such as financial crimes) can be defined as unlawful conversions of another's properties for one's own gain [1,2]. Most often, it involves non-violent and subtle methods of gaining financial gain from unsuspecting individuals. Financial crime involves several components, as described by Pickett and Pickett [2], such as trickery, deliberate actions, breach of trust, hidden truths and monetary harm, among other components.

A number of financial crimes are prevalent, including fraud of advance fees (obtaining money under false pretenses), embezzlement (misappropriating or directing public funds), cybercrime (an internet-based scam), bribery, money laundering (the process of converting illegal proceeds into legal forms), extortion, forgery, stealing and the issuing of fake checks, which are all examples of fraud. The above examples and other related crimes have been categorized into four groups: corruption, fraud, theft and manipulation [3]. There are several reasons why financial crimes occur, such as a lack of income, joblessness, low salaries, avarice, high living expenses, social pressure, a desire to get rich fast, a desire for power, idleness, ineffective criminal investigations and prosecutions and a lack of serious punishments for criminal behavior. There are wide-ranging effects of financial crimes on individuals, organizations, companies and nations. Some of these effects include bankruptcies, increased wealth concentration among criminals, crime rates rising, instability in the economy, trauma-related suicides, a reduction in foreign direct investment in the affected countries, a decrease in the attractiveness of legitimate businesses, a decrease in foreign direct investment in the affected countries, decreased confidence in economies where financial crimes prosper, unrest and decreased moral standards. Through the use of mathematical concepts and language, mathematical modeling can describe and simplify physical phenomena or real-world systems [4]. Modeling aids in the transformation of a complex real-world system with many interconnected variables into a more simplified model in order to demonstrate the effects of these variables on the system's dynamics. To gain a better understanding of crime dynamics in a community, a variety of models have been developed and investigated.

To gain a changing perspective on how varying intervention parameters affect criminality, Zhao et al. [5] analyzed the stability of poverty-crime dynamics using a system of ordinary differential equations. An interaction model was presented by Nuño et al. [6] to analyze the prey-predator dynamics of three societal types: owners, criminals and security guards. Moreover, Nuño et al. [7] presented a study on society with three major groups, which were the poor, the rich and cheaters, and they outlined their evolution over time in a dynamical system. By considering the immigration and exodus statistics of vulnerable populations and criminals, Shukla et al. [8], using a mathematical model, examined how technology plays a factor in combating social forms in a dynamic population. A series of dynamic systems models of crime, imprisonment and recidivism, using only abstract transition parameters, were proposed and analyzed by McMillon et al. [9]. A study of crime as a social epidemic process using a dynamic system has been introduced by González-Parra et al. [10]. Srivastav et al. [11] studied crime dynamics using a nonlinear mathematical model based on simple mass-action type incidents and constant recruitment and death types of a population. Recently, using a mathematical model with optimal control measures, Akanni et al. [4] developed and analyzed a population dynamics model for financial crime. Ibrahim et al. [12] conducted a study based on an age-structured paradigm to examine how correctional interventions affect criminal gang dynamics in Nigeria. The literature review encompasses several studies related to mathematical modeling and understanding the dynamics of crime, corruption, poverty and the interaction between law enforcement and criminal activities. Athithan et al. [13] has focused on the mathematical modeling and optimal control of corruption dynamics. Roslan et al. [14] presented a mathematical model that examines the dynamics of poverty, the poor and crime in West Malaysia. Chaharborj et al. [15] has proposed a dynamic economic model to understand criminal activity within the framework of criminal law. Nyabadza et al. [16] developed a mathematical model to investigate the role of correctional services in relation to gangs. Sooknanan et al. [17] introduced a modified predator-prey model to analyze the interaction between police and gangs. Finally, Manasevich et al. [18] examined the global existence of solutions for a chemotaxis-type system that arises in crime modeling. Collectively, these studies contribute to our understanding of crime dynamics and provide insights into developing effective strategies for crime prevention and control.

As a tool for studying dynamical systems, fractional calculus (FC) has become increasingly important. In FC, differentiation and integration are generalized to non-integer orders. A variety of disciplines have applied FC in their research. Modeling fractional-order differential equations gives deeper insight into a disease. Several mathematical models have been proposed and studied for various diseases; for instance, Goyal et al. [19] proposed an efficient technique for modeling the spread of Lassa hemorrhagic fever in pregnant women using a time-fractional model. Gao et al. [20] presented a new approach utilizing the Mittag-Leffler function to describe a deadly disease in pregnant women. Alqahtani et al. [21] conducted a dynamical analysis of a bio-ethanol production model using a generalized nonlocal operator in the Caputo sense. Agarwal and Singh [22] modeled the transmission dynamics of the Nipah virus using a fractional-order approach. Zarin et al. [23] analyzed a fractional COVID-19 epidemic model under the Caputo operator. Lastly, Zarin et al. [24] studied the fractional-order dynamics of a Chagas-HIV epidemic model considering different fractional operators. These studies contribute to the understanding and analysis of epidemic dynamics using FC.

Moreover, various studies have put forth fractional operators encompassing both singular and nonsingular kernels [25–30]. Extensive research has been conducted on these topics and their applications, as evidenced by recent publications [31–38]. In the realm of mathematical modeling, there has been a surge in studies focusing on social issues, particularly those related to criminal matters, utilize (FC). For instance, Bansal et al. proposed a fractional-order crime transmission model, extending it to a delayed model by incorporating a time-delay coefficient to account for the temporal gap between an individual's offense and the corresponding judgment [39]. Pritam et al. examined a fractional-order mathematical model of crime transmission with memory properties by considering the influence of previous inputs when predicting the crime growth rate [40]. In a groundbreaking study by Partohaghighi et al., fractional-order crime systems were devised and contrasted using Caputo-Fabrizio, Caputo and Atangana-Baleanu-Caputo (ABC) derivatives. This research incorporated genuine initial conditions specific to subgroups within the USA. To derive approximate solutions for the proposed models, the researchers developed numerical techniques [41]. Furthermore, Rahman et al. conducted an investigation exploring the dynamics of a fractional mathematical model concerning serial killing, employing the Mittag-Leffler kernel. They utilized the iterative fractional-order Adams-Bashforth approach to find an approximate solution and conducted numerical simulations to assess various control strategies at different fractional orders [42].

In light of previous studies, we find that the topic of the fractional-order ABC derivative of the financial crime epidemic model has not been addressed before; therefore, in this work, a mathematical model for the population dynamics of financial crime with fractional derivatives is reformulated and analyzed. The paper is arranged as follows. Section 2 is devoted to providing definitions of differential and integral operators. In Section 3, the nonlinear ordinary differential equation model is formulated and the basic reproduction number is derived. Section 4 examines the local stability of the crime-free equilibrium. In Section 5, the sensitivity coefficients for the parameters of the model are calculated. Section 6 focuses on the fractional-order model and investigates the existence and uniqueness of solutions for the ABC model. In Section 7, a number of conditions are established for the existence and Ulam-Hyers stability of financial crime equilibria. Section 8 involves a numerical scheme for the fractional-order model. Concluding remarks are given in Section 9.

2. Preliminaries

Below are some definitions of differential and integral operators [43]:
Caputo fractional derivative:

$$ {}_{0}^{C}\mathbb{D}_{t}^{\delta}\mathcal{F}(t) = \frac{1}{\Gamma(1-\delta)}\int_{0}^{t}\frac{d}{d\Phi}f(\Phi)(t-\Phi)^{-\delta}d\Phi. \tag{2.1}$$

Caputo-Fabrizio fractional derivative [43]:

$$ {}_{0}^{CF}\mathbb{D}_{t}^{\delta}\mathcal{F}(t) = \frac{M(\delta)}{1-\delta}\int_{0}^{t}\frac{d}{d\Phi}f(\Phi)\exp\left[-\frac{\delta}{1-\delta}(t-\Phi)\right]d\Phi. \tag{2.2}$$

Atangana-Baleanu fractional derivative:

$$ {}^{ABC}\mathbb{D}_{t}^{\delta}\mathcal{F}(t) = \frac{AB(\delta)}{1-\delta}\int_{0}^{t}\frac{d}{d\Phi}f(\Phi)E_{x}\left[-\frac{\delta}{1-\delta}(t-\Phi)^{\delta}\right]d\Phi. \tag{2.3}$$

The fractal-fractional derivatives are given by [43]

$$\begin{aligned} {}_{0}^{FFP}\mathbb{D}_{t}^{\delta,\kappa}\mathcal{F}(t) &= \frac{1}{\Gamma(1-\delta)}\frac{d}{dt^{\kappa}}\int_{0}^{t}f(\Phi)(t-\Phi)^{-\delta}d\Phi \quad \text{(with power-law kernel)},\\ {}_{0}^{FFE}\mathbb{D}_{t}^{\delta,\kappa}\mathcal{F}(t) &= \frac{M(\delta)}{1-\delta}\frac{d}{dt^{\kappa}}\int_{0}^{t}f(\Phi)\exp\left[-\frac{\delta}{1-\delta}(t-\Phi)\right]d\Phi \quad \text{(with exponential decay)},\\ 0^{FFM}\mathbb{D}_{t}^{\delta,\kappa}\mathcal{F}(t) &= \frac{AB(\delta)}{1-\delta}\frac{d}{dt^{\kappa}}\int_{0}^{t}f(\Phi)E_{\delta}\left[-\frac{\delta}{1-\delta}(t-\Phi)^{\delta}\right]d\Phi \quad \text{(with Mittag-Leffler kernel)}, \end{aligned} \tag{2.4}$$

where

$$\frac{d\mathcal{F}(t)}{dt^{\kappa}} = \lim_{t\to t_1}\frac{\mathcal{F}(t)-f(t_1)}{t^{2-\kappa}-t_1^{2-\kappa}}(2-\kappa).$$

The fractal-fractional integrals are as below [43].

$$\begin{aligned} {}_{0}^{FFP}J_{t}^{\delta,\kappa}\mathcal{F}(t) &= \frac{1}{\Gamma(\delta)}\int_{0}^{t}(t-\Phi)^{\delta-1}\Phi^{1-\kappa}f(\Phi)d\Phi \quad \text{(with power-law kernel)},\\ {}_{0}^{FFE}J_{t}^{\delta,\kappa}\mathcal{F}(t) &= \frac{1-\delta}{M(\delta)}t^{1-\kappa}\mathcal{F}(t) + \frac{\delta}{M(\delta)}\int_{0}^{t}\Phi^{1-\kappa}f(\Phi)d\Phi \quad \text{(exponential decay)},\\ {}_{0}^{FFM}J_{t}^{\delta,\kappa}\mathcal{F}(t) &= \frac{1-\delta}{AB(\delta)}t^{1-\kappa}\mathcal{F}(t) + \frac{\delta}{AB(\delta)\Gamma(\delta)}\int_{0}^{t}(t-\Phi)^{\delta-1}\Phi^{1-\kappa}f(\Phi)d\Phi \quad \text{(Mittag-Leffler kernel)}. \end{aligned} \tag{2.5}$$

3. Model formulation

Mathematical modeling is a powerful tool for understanding the dynamics of financial crime populations. It allows researchers to identify patterns and relationships in the data and make predictions about future trends. This can help law enforcement agencies to allocate resources more effectively, and to target their interventions where they are most needed. There are many different mathematical models used to study financial crime, including agent-based models, network models and time-series models. These models are used to investigate a range of topics, including money laundering, fraud and cybercrime. The precision of these models is contingent upon the caliber of the accessible data and the veracity of the foundational assumptions. In this paper, we revisit the mathematical model of financial crime populations [4] by introducing FC; the employed model adheres to the classical framework, characterized by the following ordinary differential equations:

$$\begin{cases} \dfrac{d\mathbb{K}}{dt} = \eta - \beta\mathbb{K}(t)\mathbb{L}(t) - (\rho+\eta)\mathbb{K}(t), \\ \dfrac{d\mathbb{L}}{dt} = \beta\mathbb{K}(t)\mathbb{L}(t) + \omega(1-\rho)\mathbb{P}(t) - (\varphi+\rho+\eta)\mathbb{L}(t) + \tau\lambda\mathbb{M}(t), \\ \dfrac{d\mathbb{M}}{dt} = \varphi\mathbb{L}(t) - (\lambda+\sigma+\eta)\mathbb{M}(t), \\ \dfrac{d\mathbb{P}}{dt} = \sigma\mathbb{M}(t) - (\omega+\eta)\mathbb{P}(t), \\ \dfrac{d\mathbb{Q}}{dt} = (1-\tau)\lambda\mathbb{M}(t) + \rho(\mathbb{K}(t) + \mathbb{L}(t) + \omega\mathbb{P}(t)) - \eta\mathbb{Q}(t), \end{cases} \tag{3.1}$$

where

$$\mathbb{N}(t) = \mathbb{K} + \mathbb{L} + \mathbb{M} + \mathbb{P} + \mathbb{Q}.$$

Table 1. Descriptions of the parameters.

Parameters	Discriptions
η	Rate of recruitment and removal in the susceptible individuals
β	Rate of influence
ρ	The conversion rate to honest individuals
φ	Financial criminal prosecution rate per capita (of financial criminals)
λ	Rate of discharges and acquittals from prosecutions
τ	Proportion of discharge rate from prosecution
σ	Rate of transition to prison
ω	Rate of freedom from prison

In model (3.1), $\mathbb{N}(t)$ represents the total population at time t, divided into five categories: susceptible $\mathbb{K}(t)$ (individuals susceptible to financial criminal activities), financial criminal $\mathbb{L}(t)$ (individuals who commit financial crimes or are financially dishonest), those under prosecution $\mathbb{M}(t)$ (trial defendants), those jailed $\mathbb{P}(t)$ (incarcerated for financial crime) and those who are honest $\mathbb{Q}(t)$ (individuals without financial criminal history and unaffected by financial criminal activities). Assuming that the population growth (births) and population loss (natural deaths) rates happen at a per capita rate m, then $\mathbb{N}(t) = \mathbb{N}$ [44,45]. Furthermore, the description of the model parameters are given in Table 1.

3.1. Basic reproduction number

For model (3.1), there exists a positively invariant set $\Upsilon = \left\{(\mathbb{K},\mathbb{L},\mathbb{M},\mathbb{P},\mathbb{Q}) \in \mathbb{R}^5_+ \middle| 0 \le \mathbb{N} \le \frac{\sigma}{\eta}\right\}$. In addition, all solutions of $(\mathbb{K},\mathbb{L},\mathbb{M},\mathbb{P},\mathbb{Q}) \in \mathbb{R}^5_+$ are contained in Υ for all $t > 0$.

Model (3.1) has a criminal-free equilibrium (CFE), where the population is free of financial crime, denoted by $\check{D}^0$ and given by

$$\check{D}^0 = (\mathbb{K}_0,\mathbb{L}_0,\mathbb{M}_0,\mathbb{P}_0,\mathbb{Q}_0,) = \left(\frac{\eta}{\rho+\eta}, 0, 0, 0, \frac{\rho}{\rho+\eta}\right).$$

Let us define a matrix $\hat{J}$ for the Jacobian matrix of the criminal compartment of model (3.1) by the following:

$$\hat{J} = \begin{pmatrix} \beta\mathbb{K}(t) - (\varphi+\rho+\eta) & \tau\lambda & \omega(1-\rho) & \varphi \\ \varphi & -(\lambda+\sigma+\eta) & 0 & 0 \\ 0 & \sigma & -(\omega+\eta) & \end{pmatrix}.$$

Put $\hat{J}$ such that $\hat{J} = \hat{U} - \hat{V}$; we get

$$\hat{U} = \begin{pmatrix} \beta\mathbb{K}(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{V} = \begin{pmatrix} (\varphi+\rho+\eta) & -\tau\lambda & -\omega(1-\rho) \\ -\varphi & (\lambda+\sigma+\eta) & 0 \\ 0 & -\sigma & (\omega+\eta) \end{pmatrix},$$

where the matrix $\hat{U}$ contains the new criminal terms; $\hat{V}$ represents the exchange of crime from one compartment to another. Thus, R_0, which is the spectral radius of a matrix $\hat{U}\hat{V}^{-1}$, is obtained as

$$R_0 = \frac{y_1 y_2 y_3}{y y_1 y_2 - \varphi\left(\tau r y_2 \sigma + y_4\right)},$$

where

$$y = \varphi + \rho + \eta, \quad y_1 = r + \sigma + \eta, \quad y_2 = \omega + \eta,$$
$$y_3 = \beta\eta/(\rho + \eta), \quad y_4 = \omega(1 - \rho).$$

Through simplification, we find that $y y_1 y_2 > \varphi\left(\sigma y_4 + \tau r y_2\right)$. R_0 represents the financial crime reproduction number and is an indicator of the potential spreading of financial crime among naive or susceptible populations.

4. Local stability

Theorem 1. *If $R_0 < 1$, then $\check{D}^0$ of the system (3.1) is locally asymptotically stable.*

Proof. The Jacobian matrix of the system (3.1) at E^0 is given by

$$\hat{J}^{|0|} = \begin{pmatrix} -(\beta C + (\theta + \mu)) & -\beta S & 0 & 0 & 0 \\ \beta C & \beta S - (\phi + \theta + \mu) & \tau\gamma & \omega(1-\theta) & 0 \\ 0 & \phi & -(\gamma + \sigma + \mu) & 0 & 0 \\ 0 & 0 & \sigma & -(\omega + \mu) & 0 \\ \theta & \theta & (1-\tau)\gamma & \theta\omega & -\mu \end{pmatrix}. \tag{4.1}$$

The characteristic equation of the matrix $\hat{J}^{|0|}$ at $\check{D}^0$ is calculated as follows:

$$f(\xi) = (\xi + \eta)(\xi^4 + \varpi_1\xi^3 + \varpi_2\xi^2 + \varpi_3\xi + \varpi_4.$$
$$(\xi + \eta)(\xi^4 + \varpi_1\xi^3 + \varpi_2\xi^2 + \varpi_3\xi + \varpi_4 = 0.$$

Here, $\varpi_1 = \beta\mathbb{K}_0 + y + y_1 + y_2 + y_3$,
$\varpi_2 = \beta(y_2 + y_3)\mathbb{K}_0 + y_1y_2 + y_2y_3 + yy_2 + y_1y_3 + yy_3 + (yy_1 - \tau\varphi\lambda)(1 - R_0)$,
$\varpi_3 = \beta(y_2y_3\mathbb{K}_0 + \tau\varphi\lambda) + y_1y_2y_3 + yy_2y_3 + (1 - R_0)(y_2 + y_3)(yy_1 - \tau\varphi\lambda)$,
$\varpi_4 = \beta y_1y_3\mathbb{K}_0 + (1 - R_0)y_2y_3(yy_1 - \tau\varphi\lambda)$.
Note that the one eigenvalue of the characteristic equation of $\hat{J}^{|0|}$ is negative. i.e., $\xi_1 = -\eta$.

The Routh-Hurwitz criteria for polynomials of degree, $m = 4$ are [46]

$$\varpi_1 > 0, \varpi_3 > 0, \varpi_4 > 0 \text{ and } \varpi_1\varpi_2\varpi_3 > \varpi_3^2 + \varpi_1^2\varpi_4.$$

Then, in accordance with the Routh-Hurwitz criteria, all roots of $f(\xi)$ have negative real parts. Hence, $\check{D}^0$ is locally asymptotically stable. This shows that the CFE $\check{D}^0$ is asymptotically stable if $R_0 < 1$.

5. Sensitivity analysis

In mathematical epidemiology, many parameters are shrouded in variability and uncertainty, making it necessary to conduct sampling and sensitivity analysis. Through this process, it is possible to determine which parameter has the most significant impact on the model's output and informs decision-making on disease control measures. SaSAT, or Sampling and Sensitivity Analysis Tools, is a software designed specifically for this purpose, as outlined in [47]. Sensitivity analysis identifies the parameters that are most effective in curbing crime spread. Even though forward sensitivity analysis becomes tedious for complex biological models, it is an essential component of phenomenon modeling. Ecologists and epidemiologists have taken an active interest in R_0 sensitivity analysis.

Definition 1. *The normalized forward sensitivity index of R_0 that depends differentiability on a parameter $\varkappa$ is defined as*

$$\Upsilon_\varkappa = \frac{\varkappa}{R_0}\frac{\partial R_0}{\partial \varkappa}.$$

There are three common methods for calculating the sensitivity indices: (i) via direct differentiation, (ii) by using Latin hypercube sampling and (iii) by linearizing the system (3.1) and then computing the linear algebraic equations obtained. We will utilize the direct differentiation method since it provides analytical expressions for the indices. Besides showing us the effects of several factors involved with financial crime, the indices also supply us with crucial information about the comparative variation between R_0 and different parameters. Therefore, it helps to develop control methods [26].

$$\begin{aligned}
\Upsilon_{\eta}^{R_0} &= \frac{\partial R_0}{\partial \eta}\frac{\eta}{R_0} = 0.01, \Upsilon_{\beta}^{R_0} = \frac{\partial R_0}{\partial \beta}\frac{\beta}{R_0} = 1.00,\\
\Upsilon_{\rho}^{R_0} &= \frac{\partial R_0}{\partial \rho}\frac{\rho}{R_0} = -0.80, \Upsilon_{\varphi}^{R_0} = \frac{\partial R_0}{\partial \varphi}\frac{\varphi}{R_0} = 0.19,\\
\Upsilon_{\lambda}^{R_0} &= \frac{\partial R_0}{\partial \lambda}\frac{\lambda}{R_0} = -0.49, \Upsilon_{\tau}^{R_0} = \frac{\partial R_0}{\partial \tau}\frac{\tau}{R_0} = 0.78,\\
\Upsilon_{\sigma}^{R_0} &= \frac{\partial R_0}{\partial \sigma}\frac{\sigma}{R_0} = 0.11, \Upsilon_{\omega}^{R_0} = \frac{\partial R_0}{\partial \omega}\frac{\omega}{R_0} = 0.75.
\end{aligned} \tag{5.1}$$

The primary determinant of the model's sensitivity is the conversion rate into the honest class, with the influence rate among other parameters following closely. This indicates that the population free from financial crime relies on decreasing the influence rate for susceptible individuals and increasing the conversion rate into the honest class. Equation (5.1) reveals that the most sensitive parameter is represented by ρ, which signifies the conversion rate to the honest class. Given that $\Upsilon_{\rho}^{R_0} = -0.80$, a 10% increase (or decrease) in ρ leads to a corresponding decrease (or increase) of 12.4% in the crime reproduction number, R_0. Consequently, a corrective intervention strategy is imperative to mitigate the prevalence of financial crime in the population. The influence rate, β, emerges as the second most sensitive parameter, as evidenced by the positive index. As demonstrated in Eq (5.1), a 10% increase (or decrease) in β results in a 10% increase (or decrease) in R_0. Therefore, a preventive intervention strategy should be implemented to hinder the propagation of financial crime in the population. The interpretation of the sensitivity indexes for the remaining parameters follows the same pattern as that of ρ and β. The graphical depiction of the relationship between R_0 and these sensitive parameters is illustrated in Figures 1 and 2.

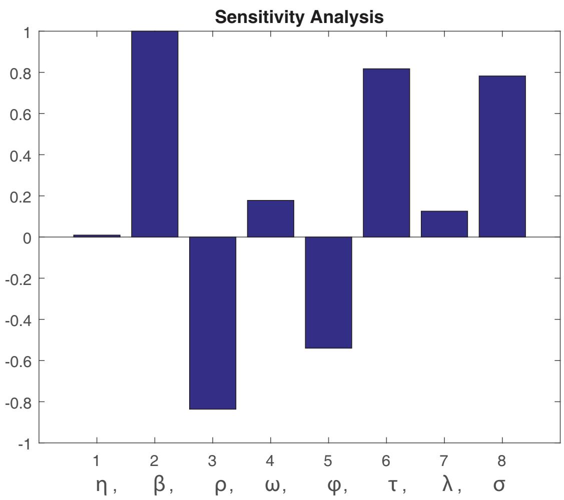
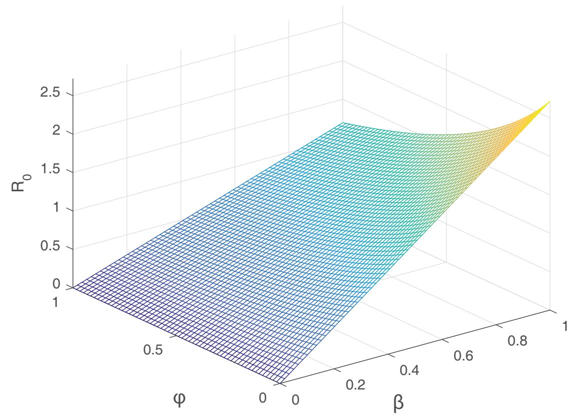
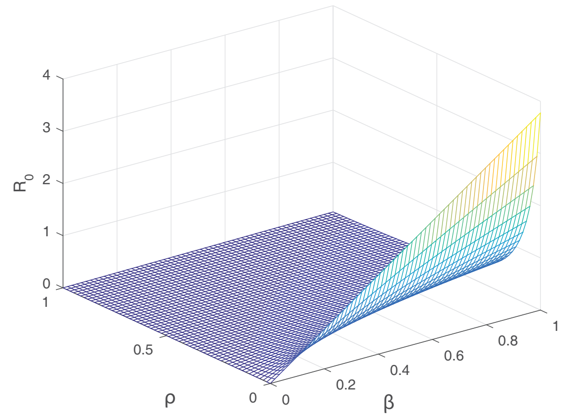


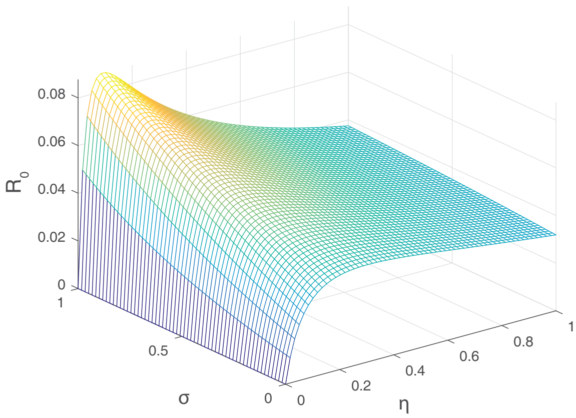
Figure 1. The sensitivities of the model parameters that impact the basic reproduction number of model (3.1).



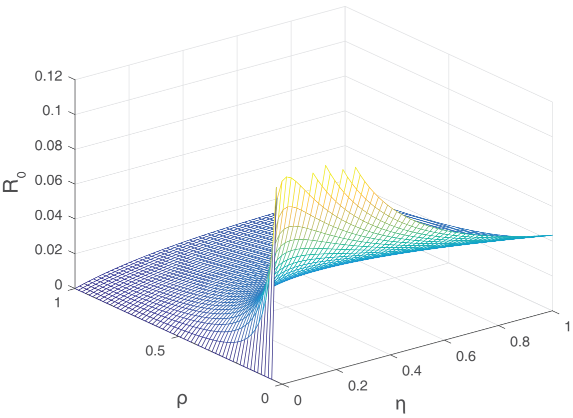
(a) Sensitivity analysis of R_0 to β and φ



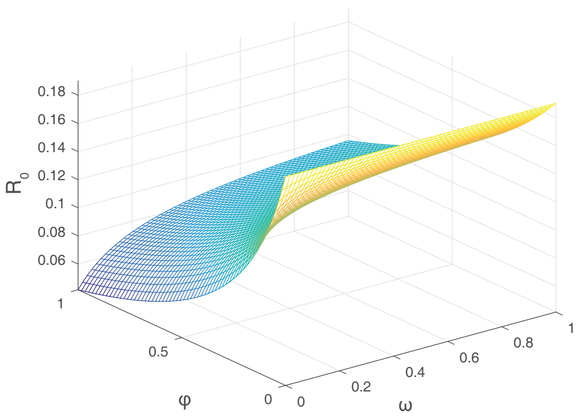
(b) Sensitivity analysis of R_0 to β and ρ



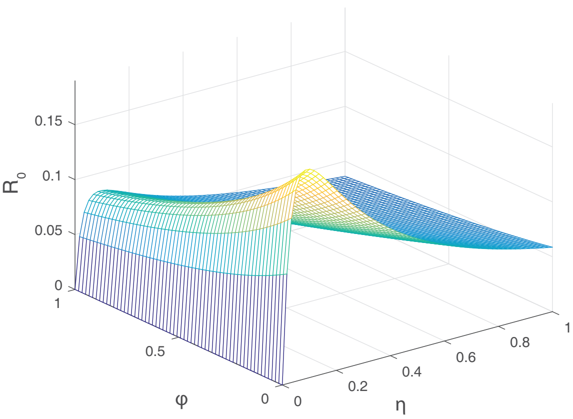
(c) Sensitivity analysis of R_0 to η and σ



(d) Sensitivity analysis of R_0 to η and ρ



(e) Sensitivity analysis of R_0 to ω and φ



(f) Sensitivity analysis of R_0 to η and φ

Figure 2. R_0 versus different sensitive parameters.

6. Fractional-order model

The mathematical modeling of real-world problems has garnered recent attention among researchers, as evidenced by studies conducted by various authors [48–54]. To the best of our knowledge, no work has been conducted on the fractional-order financial crime epidemic model taking the ABC fractional derivative . Hence, based on model (3.1), we introduce a fractional-order financial crime model using the new ABC derivative [55]:

$$\begin{cases} {}^{\mathrm{ABC}}\mathbb{D}^{\delta}_{0,t}[\mathbb{K}(t)] = \{\eta - \beta\mathbb{K}(t)\mathbb{L}(t) - (\rho+\eta)\mathbb{K}(t)\}, \\ {}^{\mathrm{ABC}}\mathbb{D}^{\delta}_{0,t}[\mathbb{L}(t)] = \{\beta\mathbb{K}(t)\mathbb{L}(t) + \omega(1-\rho)\mathbb{P}(t) - (\varphi+\rho+\eta)\mathbb{L}(t) + \tau\lambda\mathbb{M}(t)\}, \\ {}^{\mathrm{ABC}}\mathbb{D}^{\delta}_{0,t}[\mathbb{M}(t)] = \{\varphi\mathbb{L}(t) - (\lambda+\sigma+\eta)\mathbb{M}(t)\}, \\ {}^{\mathrm{ABC}}\mathbb{D}^{\delta}_{0,t}[\mathbb{P}(t)] = \{\sigma\mathbb{M}(t) - (\omega+\eta)\mathbb{P}(t)\}, \\ {}^{\mathrm{ABC}}\mathbb{D}^{\delta}_{0,t}[\mathbb{Q}(t)] = \{(1-\tau)\lambda\mathbb{M}(t) + \rho(\mathbb{K}(t)+\mathbb{L}(t)+\omega\mathbb{P}(t)) - \eta\mathbb{Q}(t)\}, \end{cases} \tag{6.1}$$

subject to the following initial conditions:

$$\mathbb{K}(0) = \mathbb{K}_0, \quad \mathbb{L}(0) = \mathbb{L}_0, \quad \mathbb{M}(0) = \mathbb{M}_0, \quad \mathbb{P}(0) = \mathbb{P}_0, \quad \mathbb{Q}(0) = \mathbb{Q}_0 \geq 0.$$

6.1. Existence and uniqueness of solutions for the fractional-order model

Below, we examine the existence and uniqueness of the solution of ABC model (6.1). Let $X(\mathcal{T})$ be a Banach space on interval $\mathcal{T} = [0,\Lambda]$, where it is a continuous real-valued function that has the sup norm property $X = B(\mathcal{T}) \times B(\mathcal{T}) \times B(\mathcal{T}) \times B(\mathcal{T}) \times B(\mathcal{T})$ with norm $||(\mathbb{K},\mathbb{L},\mathbb{M},\mathbb{P},\mathbb{Q})|| = ||\mathbb{K}|| + ||\mathbb{L}|| + ||\mathbb{M}|| + ||\mathbb{P}|| + ||\mathbb{Q}||$, where $||\mathbb{K}|| = \sup_{t\in\mathcal{T}} |\mathbb{K}(t)|, ||\mathbb{L}|| = \sup_{t\in\mathcal{T}} |\mathbb{L}(t)|, ||\mathbb{M}|| = \sup_{t\in\mathcal{T}} |\mathbb{M}(t)|, ||\mathbb{P}|| = \sup_{t\in\mathcal{T}} |\mathbb{P}(t)|, ||\mathbb{Q}|| = \sup_{t\in\mathcal{T}} |\mathbb{Q}(t)|$. After applying the ABC fractional integral operator to both sides of Eq (6.1), we get

$$\begin{cases} \mathbb{K}(t) - \mathbb{K}(0) = {}^{\mathrm{ABC}}\ \mathbb{D}^{\delta}_{0,t}[\mathbb{K}(t)]\,\{\eta - \beta\mathbb{K}(t)\mathbb{L}(t) - (\rho+\eta)\mathbb{K}(t)\}, \\ \mathbb{L}(t) - \mathbb{L}(0) = {}^{\mathrm{ABC}}\ \mathbb{D}^{\delta}_{0,t}[\mathbb{L}(t)]\,\{\beta\mathbb{K}(t)\mathbb{L}(t) + \omega(1-\rho)\mathbb{P}(t)\}, \\ \mathbb{M}(t) - \mathbb{M}(0) = {}^{\mathrm{ABC}}\ \mathbb{D}^{\delta}_{0,t}[\mathbb{M}(t)]\,\{\varphi\mathbb{L}(t) - (\lambda+\sigma+\eta)\mathbb{M}(t)\}, \\ \mathbb{P}(t) - \mathbb{P}(0) = {}^{\mathrm{ABC}}\ \mathbb{D}^{\delta}_{0,t}[\mathbb{P}(t)]\,\{\sigma\mathbb{M}(t) - (\omega+\eta)\mathbb{P}(t)\}, \\ \mathbb{Q}(t) - \mathbb{Q}(0) = {}^{\mathrm{ABC}}\ \mathbb{D}^{\delta}_{0,t}[\mathbb{Q}(t)]\,\{(1-\tau)\lambda\mathbb{M}(t) + \rho(\mathbb{K}(t)+\mathbb{L}(t)+\omega\mathbb{P}(t)) - \eta\mathbb{Q}(t)\}. \end{cases} \tag{6.2}$$

Now, Definition (2.3) gives us

$$\begin{cases} \mathbb{K}(t) - \mathbb{K}(0) = \dfrac{1-\delta}{B(\delta)}\alpha_1(\delta,t,\mathbb{K}(t)) + \dfrac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_1(\delta,\varsigma,\mathbb{K}(\varsigma))d\varsigma, \\ \mathbb{L}(t) - \mathbb{L}(0) = \dfrac{1-\delta}{B(\delta)}\alpha_2(\delta,t,\mathbb{L}(t)) + \dfrac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_2(\delta,\varsigma,\mathbb{L}(\varsigma))d\varsigma, \\ \mathbb{M}(t) - \mathbb{M}(0) = \dfrac{1-\delta}{B(\delta)}\alpha_3(\delta,t,\mathbb{M}(t)) + \dfrac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_3(\delta,\varsigma,\mathbb{M}(\varsigma))d\varsigma, \\ \mathbb{P}(t) - \mathbb{P}(0) = \dfrac{1-\delta}{B(\delta)}\alpha_4(\delta,t,\mathbb{P}(t)) + \dfrac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_4(\delta,\varsigma,\mathbb{P}(\varsigma))d\varsigma, \\ \mathbb{Q}(t) - \mathbb{Q}(0) = \dfrac{1-\delta}{B(\delta)}\alpha_5(\delta,t,\mathbb{Q}(t)) + \dfrac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_5(\delta,\varsigma,\mathbb{Q}(\varsigma))d\varsigma, \end{cases} \tag{6.3}$$

where

$$\begin{aligned}
&\alpha_1(\delta, t, \mathbb{K}(t)) = \eta - \beta\mathbb{K}(t)\mathbb{L}(t) - (\rho + \eta)\mathbb{K}(t),\\
&\alpha_2(\delta, t, \mathbb{L}(t)) = \beta\mathbb{K}(t)\mathbb{L}(t) + \omega(1 - \rho)\mathbb{P}(t) - (\varphi + \rho + \eta)\mathbb{L}(t) + \tau\lambda\mathbb{M}(t),\\
&\alpha_3\,(\delta, t, \mathbb{M}(t)) = \varphi\mathbb{L}(t) - (\lambda + \sigma + \eta)\mathbb{M}(t),\\
&\alpha_4\,(\delta, t, \mathbb{P}(t)) = \sigma\mathbb{M}(t) - (\omega + \eta)\mathbb{P}(t),\\
&\alpha_5(\delta, t, \mathbb{Q}(t)) = (1 - \tau)\lambda\mathbb{M}(t) + \rho(\mathbb{K}(t) + \mathbb{L}(t) + \omega\mathbb{P}(t)) - \eta\mathbb{Q}(t).
\end{aligned}$$

The symbols $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 must be valid for the Lipschitz condition only if $\mathbb{K}(t), \mathbb{L}(t), \mathbb{M}(t), \mathbb{P}(t)$ and $\mathbb{Q}(t)$ have an upper bound; we reach

$$\|\alpha_1(\delta, t, \mathbb{K}(t)) - \alpha_1\,(\delta, t, \mathbb{K}^*(t))\| = \|-\,(\beta\mathbb{L} + (\rho + \eta))\,(\mathbb{K}(t) - \mathbb{K}^*(t))\|\,.$$

Taking into account $\vartheta_1 := \|-\,(\beta\mathbb{L} + (\rho + \eta))\|$, one reaches

$$\|\alpha_1(\delta, t, \mathbb{K}(t)) - \alpha_1\,(\delta, t, \mathbb{K}^*(t))\| \le \vartheta_1\,\|\mathbb{K}(t) - \mathbb{K}^*(t)\|\,. \tag{6.4}$$

Also, we can get

$$\begin{aligned}
&\|\alpha_2(\delta, t, \mathbb{L}(t)) - \alpha_2\,(\delta, t, \mathbb{L}^*(t))\| \le \vartheta_2\,\|(\mathbb{L}(t) - \mathbb{L}^*(t)\|\,,\\
&\|\alpha_3\,(\delta, t, \mathbb{M}(t)) - \alpha_3\,(\delta, t, \mathbb{M}^*(t))\| \le \vartheta_3\,\|\mathbb{M}(t) - \mathbb{M}^*(t)\|\,,\\
&\|\alpha_4\,(\delta, t, \mathbb{P}(t)) - \alpha_4\,(\delta, t, \mathbb{P}^*(t))\| \le \vartheta_4\,\|\mathbb{P}(t) - \mathbb{P}^*(t)\|\,,\\
&\|\alpha_5(\delta, t, \mathbb{Q}(t)) - \alpha_5\,(\delta, t, \mathbb{Q}^*(t))\| \le \vartheta_5\,\|\mathbb{Q}(t) - \mathbb{Q}^*(t)\|\,,
\end{aligned} \tag{6.5}$$

where

$$\begin{aligned}
\vartheta_2 &= \|-\,(\beta\mathbb{K})\|\,,\\
\vartheta_3 &= \|-\,(\lambda + \sigma + \eta)\|\,,\\
\vartheta_4 &= \|-\,(\omega + \eta)\|\,,\\
\vartheta_5 &= \|-\,(\eta)\|\,;
\end{aligned}$$

as a result, Lipschitz's condition holds. By continuing in a recursive manner, Eq (6.3) becomes

$$\begin{aligned}
\mathbb{K}_m(t) - \mathbb{K}(0) =& \frac{1-\delta}{B(\delta)}\alpha_1(\delta, t, \mathbb{K}_{m-1}(t)) + \frac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_1(\delta, \varsigma, \mathbb{K}_{m-1}(\varsigma))d\varsigma,\\
\mathbb{L}_m(t) - \mathbb{L}(0) =& \frac{1-\delta}{B(\delta)}\alpha_2(\delta, t, \mathbb{L}_{m-1}(t)) + \frac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_2(\delta, \varsigma, \mathbb{L}_{m-1}(\varsigma))d\varsigma,\\
\mathbb{M}_m(t) - \mathbb{M}(0) =& \frac{1-\delta}{B(\delta)}\alpha_3(\delta, t, \mathbb{M}_{m-1}(t)) + \frac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_3(\delta, \varsigma, \mathbb{M}_{m-1}(\varsigma))d\varsigma,\\
\mathbb{P}_m(t) - \mathbb{P}(0) =& \frac{1-\delta}{B(\delta)}\alpha_4(\delta, t, \mathbb{P}_{m-1}(t)) + \frac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_4(\delta, \varsigma, \mathbb{P}_{m-1}(\varsigma))d\varsigma,\\
\mathbb{Q}_m(t) - \mathbb{Q}(0) =& \frac{1-\delta}{B(\delta)}\alpha_5(\delta, t, \mathbb{Q}_{m-1}(t)) + \frac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_5(\delta, \varsigma, \mathbb{Q}_{m-1}(\varsigma))d\varsigma.
\end{aligned} \tag{6.6}$$

Taking $\mathbb{K}_0(t) = \mathbb{K}(0), \mathbb{L}_0(t) = \mathbb{L}(0), \mathbb{M}_0(t) = \mathbb{M}(0), \mathbb{P}_0(t) = \mathbb{P}(0)$ and $\mathbb{Q}_0(t) = \mathbb{Q}(0)$ into account, the difference of consecutive terms gives

$$\mathbb{S}_{\mathbb{K},m}(t) = \mathbb{K}_m(t) - \mathbb{K}_{m-1}(t) = \frac{1-\delta}{B(\delta)}(\alpha_1(\delta, t, \mathbb{K}_{m-1}(t)) - \alpha_1(\delta, t, \mathbb{K}_{m-2}(t)))$$

$$
\begin{aligned}
&+\frac{\delta}{B(\delta)\Gamma(\delta)}\int_0^t (t-\varsigma)^{\delta-1}\left(\alpha_1\left(\delta,\varsigma,\mathbb{K}_{m-1}(\varsigma)\right)-\alpha_1\left(\delta,\varsigma,\mathbb{K}_{m-2}(\varsigma)\right)\right)d\varsigma,\\
\mathbb{S}_{\mathbb{L},m}(t) =&\mathbb{L}_m(t)-\mathbb{L}_{m-1}(t)=\frac{1-\delta}{B(\delta)}(\alpha_2(\delta,t,\mathbb{L}_{m-1}(t))-\alpha_2(\delta,t,\mathbb{L}_{m-2}(t)))\\
&+\frac{\delta}{B(\delta)\Gamma(\delta)}\int_0^l (t-\varsigma)^{\delta-1}\left(\alpha_2\left(\delta,\varsigma,\mathbb{L}_{m-1}(\varsigma)\right)-\alpha_2\left(\delta,\varsigma,\mathbb{L}_{m-2}(\varsigma)\right)\right)d\varsigma,\\
\mathbb{S}_{\mathbb{M},m}(t) =&\mathbb{M}_m(t)-\mathbb{M}_{m-1}(t)=\frac{1-\delta}{B(\delta)}(\alpha_3(\delta,t,\mathbb{M}_{m-1}(t))-\alpha_3(\delta,t,\mathbb{M}_{m-2}(t)))\\
&+\frac{\delta}{B(\delta)\Gamma(\delta)}\int_0^t (t-\varsigma)^{\delta-1}\left(\alpha_3\left(\delta,\varsigma,\mathbb{M}_{m-1}(\varsigma)\right)-\alpha_3\left(\delta,\varsigma,\mathbb{M}_{m-2}(\varsigma)\right)\right)d\varsigma,\\
\mathbb{S}_{\mathbb{P},m}(t) =&\mathbb{P}_m(t)-\mathbb{P}_{m-1}(t)=\frac{1-\delta}{B(\delta)}(\alpha_4(\delta,t,\mathbb{P}_{m-1}(t))-\alpha_4(\delta,t,\mathbb{P}_{m-2}(t)))\\
&+\frac{\delta}{B(\delta)\Gamma(\delta)}\int_0^t (t-\varsigma)^{\delta-1}\left(\alpha_4\left(\delta,\varsigma,\mathbb{P}_{m-1}(\varsigma)\right)-\alpha_4\left(\delta,\varsigma,\mathbb{P}_{m-2}(\varsigma)\right)\right)d\varsigma,\\
\mathbb{S}_{\mathbb{Q},m}(t) =&\mathbb{Q}_m(t)-\mathbb{Q}_{m-1}(t)=\frac{1-\delta}{B(\delta)}(\alpha_5(\delta,t,\mathbb{Q}_{m-1}(t))-\alpha_5(\delta,t,\mathbb{Q}_{m-2}(t)))\\
&+\frac{\delta}{B(\delta)\Gamma(\delta)}\int_0^t (t-\varsigma)^{\delta-1}\left(\alpha_5\left(\delta,\varsigma,\mathbb{Q}_{m-1}(\varsigma)\right)-\alpha_5\left(\delta,\varsigma,\mathbb{Q}_{m-2}(\varsigma)\right)\right)d\varsigma.
\end{aligned}
\tag{6.7}
$$

It is important to note that $\mathbb{K}_m(t)=\sum_{i=0}^m \mathbb{S}_{\mathbb{K},i}(t)$, $\mathbb{L}_m(t)=\sum_{i=0}^m \mathbb{S}_{\mathbb{L},i}(t)$, $\mathbb{M}_m(t)=\sum_{i=0}^m \mathbb{S}_{\mathbb{M},i}(t)$, $\mathbb{P}_m(t)=\sum_{i=0}^m \mathbb{S}_{\mathbb{P},i}(t)$ and $\mathbb{Q}_m(t)=\sum_{i=0}^m \mathbb{S}_{\mathbb{Q},i}(t)$. Furthermore, by using Eqs (6.4)–(6.5) and assuming that $\mathbb{S}_{\mathbb{K},m-1}(t)=\mathbb{K}_{m-1}(t)-\mathbb{K}_{m-2}(t)$, $\mathbb{S}_{\mathbb{L},m-1}(t)=\mathbb{L}_{m-1}(t)-\mathbb{L}_{m-2}(t)$, $\mathbb{S}_{\mathbb{M},m-1}(t)=\mathbb{M}_{m-1}(t)-\mathbb{M}_{m-2}(t)$, $\mathbb{S}_{\mathbb{P},m-1}(t)=\mathbb{P}_{m-1}(t)-\mathbb{P}_{m-2}(t)$ and $\mathbb{S}_{\mathbb{Q},m-1}(t)=\mathbb{Q}_{m-1}(t)-\mathbb{Q}_{m-2}(t)$, we get

$$
\begin{aligned}
&\left\|\mathbb{S}_{\mathbb{K},m}(t)\right\|\le \tfrac{1-\delta}{B(\delta)}\vartheta_1\left\|\mathbb{S}_{\mathbb{K},m-1}(t)\right\|\tfrac{\delta}{B(\delta)\Gamma(\delta)}\vartheta_1\times\int_0^t (t-\varsigma)^{\delta-1}\left\|\mathbb{S}_{\mathbb{K},m-1}(\varsigma)\right\|d\varsigma,\\
&\left\|\mathbb{S}_{\mathbb{L},m}(t)\right\|\le \tfrac{1-\delta}{B(\delta)}\vartheta_2\left\|\mathbb{S}_{\mathbb{L},m-1}(t)\right\|\tfrac{\delta}{B(\delta)\Gamma(\delta)}\vartheta_2\times\int_0^t (t-\varsigma)^{\delta-1}\left\|\mathbb{S}_{\mathbb{L},m-1}(\varsigma)\right\|d\varsigma,\\
&\left\|\mathbb{S}_{\mathbb{M},m}(t)\right\|\le \tfrac{1-\delta}{B(\delta)}\vartheta_3\left\|\mathbb{S}_{\mathbb{M},m-1}(t)\right\|\tfrac{\delta}{B(\delta)\Gamma(\delta)}\vartheta_3\times\int_0^t (t-\varsigma)^{\delta-1}\left\|\mathbb{S}_{\mathbb{M},m-1}(\varsigma)\right\|d\varsigma,\\
&\left\|\mathbb{S}_{\mathbb{P},m}(t)\right\|\le \tfrac{1-\delta}{B(\delta)}\vartheta_4\left\|\mathbb{S}_{\mathbb{P},m-1}(t)\right\|\tfrac{\delta}{B(\delta)\Gamma(\delta)}\vartheta_4\times\int_0^t (t-\varsigma)^{\delta-1}\left\|\mathbb{S}_{\mathbb{P},m-1}(\varsigma)\right\|d\varsigma,\\
&\left\|\mathbb{S}_{\mathbb{Q},m}(t)\right\|\le \tfrac{1-\delta}{B(\delta)}\vartheta_5\left\|\mathbb{S}_{\mathbb{Q},m-1}(t)\right\|\tfrac{\delta}{B(\delta)\Gamma(\delta)}\vartheta_5\times\int_0^t (t-\varsigma)^{\delta-1}\left\|\mathbb{S}_{\mathbb{Q},m-1}(\varsigma)\right\|d\varsigma.
\end{aligned}
\tag{6.8}
$$

Theorem 2. *Suppose that the following condition is valid:*

$$
\frac{1-\delta}{B(\delta)}\vartheta_i+\frac{\delta}{B(\delta)\Gamma(\delta)}\Lambda^{\delta}\vartheta_i<1, i=1,2,\dots,5. \tag{6.9}
$$

Then, model (6.1) has a unique solution for $t\in[0,\Lambda]$.

Proof. Clearly $\mathbb{K}(t),\mathbb{L}(t),\mathbb{M}(t),\mathbb{P}(t)$ and $\mathbb{Q}(t)$ are bounded functions. Moreover, as shown by Eqs (6.4) and (6.5), the symbols $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ and α_5 hold for the Lipchitz condition. Consequently, by utilizing

Eq (6.8) in conjunction with a recursive assumption, we can establish

$$\begin{aligned}
&\left\|\mathbb{S}_{\mathbb{K},m}(t)\right\| \leq \|\mathbb{K}_0(t)\|\left(\tfrac{1-\delta}{B(\delta)}\vartheta_1 + \tfrac{\delta\Lambda^\delta}{B(\delta)\Gamma(\delta)}\vartheta_1\right)^m \\
&\left\|\mathbb{S}_{\mathbb{L},m}(t)\right\| \leq \|\mathbb{L}_0(t)\|\left(\tfrac{1-\delta}{B(\delta)}\vartheta_3 + \tfrac{\delta\Lambda^\delta}{B(\delta)\Gamma(\delta)}\vartheta_2\right)^m \\
&\left\|\mathbb{S}_{\mathbb{M},m}(t)\right\| \leq \|\mathbb{M}_0(t)\|\left(\tfrac{1-\delta}{B(\delta)}\vartheta_3 + \tfrac{\delta\Lambda^\delta}{B(\delta)\Gamma(\delta)}\vartheta_3\right)^m \\
&\left\|\mathbb{S}_{\mathbb{P},m}(t)\right\| \leq \|\mathbb{P}_0(t)\|\left(\tfrac{1-\delta}{B(\delta)}\vartheta_4 + \tfrac{\delta\Lambda^\delta}{B(\delta)\Gamma(\delta)}\vartheta_4\right)^m \\
&\left\|\mathbb{S}_{\mathbb{Q},m}(t)\right\| \leq \|\mathbb{Q}_0(t)\|\left(\tfrac{1-\delta}{B(\delta)}\vartheta_5 + \tfrac{\delta\Lambda^\delta}{B(\delta)\Gamma(\delta)}\vartheta_5\right)^m .
\end{aligned} \tag{6.10}$$

This implies that sequences exist that satisfy $\left\|\mathbb{S}_{\mathbb{K},m}(t)\right\| \to 0$, $\left\|\mathbb{S}_{\mathbb{L},m}(t)\right\| \to 0$, $\left\|\mathbb{S}_{\mathbb{M},m}(t)\right\| \to 0$, $\left\|\mathbb{S}_{\mathbb{P},m}(t)\right\| \to 0$, $\left\|\mathbb{S}_{\mathbb{Q},m}(t)\right\| \to 0$ as $m \to \infty$. Furthermore, through the utilization of Eq (6.10) and by applying the triangle inequality, we can derive an expression for any given value of k:

$$\begin{aligned}
&\|\mathbb{K}_{m+k}(t) - \mathbb{K}_m(t)\| \leq \textstyle\sum_{j=m+1}^{m+k} Y_1^j = \frac{Y_1^{m+1} - Y_1^{m+k+1}}{1-Y_1} \\
&\|\mathbb{L}_{m+k}(t) - \mathbb{L}_m(t)\| \leq \textstyle\sum_{j=m+1}^{m+k} Y_2^j = \frac{Y_2^{m+1} - Y_2^{m+k+1}}{1-Y_2} \\
&\|\mathbb{M}_{m+k}(t) - \mathbb{M}_m(t)\| \leq \textstyle\sum_{j=m+1}^{m+k} Y_3^j = \frac{Y_3^{m+1} - Y_3^{m+k+1}}{1-Y_3} \\
&\|\mathbb{P}_{m+k}(t) - \mathbb{P}_m(t)\| \leq \textstyle\sum_{j=m+1}^{m+k} Y_4^j = \frac{Y_4^{m+1} - Y_4^{m+k+1}}{1-Y_4} \\
&\|\mathbb{Q}_{m+k}(t) - \mathbb{Q}_m(t)\| \leq \textstyle\sum_{i=m+1}^{m+k} Y_5^j = \frac{Y_5^{m+1} - Y_5^{m+k+1}}{1-Y_5},
\end{aligned} \tag{6.11}$$

with $Y_i = \frac{1-\delta}{B(\delta)}\vartheta_i + \frac{\delta}{B(\delta)\Gamma(\delta)}\Lambda^\delta\vartheta_i < 1$ by hypothesis. As well, it can be shown that the proposed model, with various fractional derivatives, has a unique solution.

7. Hyers-Ulam stability

The main advantage of Hyers-Ulam stability for dynamical systems is its ability to provide a guarantee of stability even when exact solutions are difficult or impossible to obtain. Hyers-Ulam stability, also known as the Hyers-Ulam-Rassias stability, is a concept in mathematical analysis that deals with the stability of functional equations. In the context of dynamical systems, it allows us to analyze the behavior of a system without explicitly solving the system's equations of motion. Instead, it provides an approximate solution or an estimate of the solution's behavior based on small perturbations in the initial conditions or the system's parameters. This advantage is particularly valuable when dealing with complex or nonlinear systems where finding exact solutions may be mathematically intractable. Hyers-Ulam stability provides a framework to assess the stability of these systems, even when explicit solutions are elusive. It offers a practical and robust approach to understanding the long-term behavior and sensitivity of dynamical systems to small perturbations. By employing the principles of Hyers-Ulam stability, researchers and engineers can gain insights into the stability and robustness of dynamical systems, which is crucial for a wide range of applications, including physics, engineering, biology, economics and many other fields.

Definition 2. *[26] The ABC fractional integral system given by Eq (6.3) is said to be Hyers-Ulam-stable if there exist constants $\gamma_i > 0$, $i \in \mathbf{N}^5$ satisfying the following: For every $\theta_i > 0, i \in \mathbf{N}^5$, when*

$$
\begin{aligned}
|\mathbb{K}(t) - &\frac{1-\delta}{B(\delta)}\alpha_1(\delta, t, \mathbb{K}(t)) + \frac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_1(\delta, \varsigma, \mathbb{K}(\varsigma))d\varsigma| \leq \theta_1,\\
|\mathbb{L}(t) - &\frac{1-\delta}{B(\delta)}\alpha_2(\delta, t, \mathbb{L}(t)) + \frac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_2(\delta, \varsigma, \mathbb{L}(\varsigma))d\varsigma| \leq \theta_2,\\
|\mathbb{M}(t) - &\frac{1-\delta}{B(\delta)}\alpha_3(\delta, t, \mathbb{M}(t)) + \frac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_3(\delta, \varsigma, \mathbb{M}(\varsigma))d\varsigma| \leq \theta_3,\\
|\mathbb{P}(t) - &\frac{1-\delta}{B(\delta)}\alpha_4(\delta, t, \mathbb{P}(t)) + \frac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_4(\delta, \varsigma, \mathbb{P}(\varsigma))d\varsigma| \leq \theta_4,\\
|\mathbb{Q}(t) - &\frac{1-\delta}{B(\delta)}\alpha_5(\delta, t, \mathbb{Q}(t)) + \frac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_5(\delta, \varsigma, \mathbb{Q}(\varsigma))d\varsigma| \leq \theta_5,
\end{aligned}
$$

there exist $(\dot{\mathbb{K}}(t), \dot{\mathbb{L}}(t), \dot{\mathbb{M}}(t), \dot{\mathbb{P}}(t), \dot{\mathbb{Q}}(t))$ *satisfying*

$$
\begin{aligned}
\dot{\mathbb{K}}(t) =& \frac{1-\delta}{B(\delta)}\alpha_1(\delta, t, \mathbb{K}(t)) + \frac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_1(\delta, \varsigma, \dot{\mathbb{K}}(\varsigma))d\varsigma,\\
\dot{\mathbb{L}}(t) =& \frac{1-\delta}{B(\delta)}\alpha_2(\delta, t, \mathbb{L}(t)) + \frac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_2(\delta, \varsigma, \dot{\mathbb{L}}(\varsigma))d\varsigma,\\
\dot{\mathbb{M}}(t) =& \frac{1-\delta}{B(\delta)}\alpha_3(\delta, t, \mathbb{M}(t)) + \frac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_3(\delta, \varsigma, \dot{\mathbb{M}}(\varsigma))d\varsigma,\\
\dot{\mathbb{P}}(t) =& \frac{1-\delta}{B(\delta)}\alpha_4(\delta, t, \mathbb{P}(t)) + \frac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_4(\delta, \varsigma, \dot{\mathbb{P}}(\varsigma))d\varsigma,\\
\dot{\mathbb{Q}}(t) =& \frac{1-\delta}{B(\delta)}\alpha_5(\delta, t, \mathbb{Q}(t)) + \frac{\delta}{B(\delta)\Gamma(\delta)} \times \int_0^t (t-\varsigma)^{\delta-1}\alpha_5(\delta, \varsigma, \dot{\mathbb{Q}}(\varsigma))d\varsigma
\end{aligned}
$$

such that

$$
\begin{aligned}
&|\mathbb{K}(t) - \dot{\mathbb{K}}(t)| \leq \gamma_1\theta_1, \quad |\mathbb{L}(t) - \dot{\mathbb{L}}(t)| \leq \gamma_2\theta_2, \quad |\mathbb{M}(t) - \dot{\mathbb{M}}(t)| \leq \gamma_3\theta_3, \quad |\mathbb{P}(t) - \dot{\mathbb{P}}(t)| \leq \gamma_4\theta_4,\\
&|\mathbb{Q}(t) - \dot{\mathbb{Q}}(t)| \leq \gamma_5\theta_5.
\end{aligned}
$$

Theorem 3. *With assumption $\mathcal{T}$, the suggested model of fractional order (6.2) is Hyers-Ulam-stable.*

Proof. By Theorem 2, the proposed ABC fractional model (6.2) has a unique solution $(\mathbb{K}(t), \mathbb{L}(t), \mathbb{M}(t), \mathbb{P}(t), \mathbb{Q}(t))$ satisfying the equations of system (6.3). Thus, we get

$$
\begin{aligned}
\|\mathbb{K}(t) - \dot{\mathbb{K}}(t)\| \leq& \frac{1-\delta}{B(\delta)}\left\|\alpha_1(\delta, t, \mathbb{K}(t)) - \alpha_1(\delta, t, \dot{\mathbb{K}}(t))\right\|\\
&+ \frac{\delta}{B(\delta)\Gamma(\delta)}\int_0^t (t-\varsigma)^{\delta-1}\left\|\alpha_1(\delta, t, \mathbb{K}(t)) - \alpha_1(\delta, t, \dot{\mathbb{K}}(t))\right\| d\varsigma\\
\leq& \left[\frac{1-\delta}{B(\delta)} + \frac{\delta}{B(\delta)\Gamma(\delta)}\right]\mathbb{S}_1\|\mathbb{K} - \dot{\mathbb{K}}\|,
\end{aligned}
$$

$$\begin{aligned}\|\mathbb{L}(t)-\dot{\mathbb{L}}(t)\| &\le \frac{1-\delta}{B(\delta)}\left\|\alpha_2(\delta,t,\mathbb{L}(t))-\alpha_2(\delta,t,\dot{\mathbb{L}}(t))\right\| \\ &+ \frac{\delta}{B(\delta)\Gamma(\delta)}\int_0^t (t-\varsigma)^{\delta-1}\left\|\alpha_2(\delta,t,\mathbb{L}(t))-\alpha_2(\delta,t,\dot{\mathbb{L}}(t))\right\| d\varsigma \\ &\le \left[\frac{1-\delta}{B(\delta)}+\frac{\delta}{B(\delta)\Gamma(\delta)}\right]\mathbb{S}_2\|\mathbb{L}-\dot{\mathbb{L}}\|,\end{aligned}$$

$$\begin{aligned}\|\mathbb{M}(t)-\dot{\mathbb{M}}(t)\| &\le \frac{1-\delta}{B(\delta)}\left\|\alpha_3(\delta,t,\mathbb{M}(t))-\alpha_3(\delta,t,\dot{\mathbb{M}}(t))\right\| \\ &+ \frac{\delta}{B(\delta)\Gamma(\delta)}\int_0^t (t-\varsigma)^{\delta-1}\left\|\alpha_3(\delta,t,\mathbb{M}(t))-\alpha_3(\delta,t,\dot{\mathbb{M}}(t))\right\| d\varsigma \\ &\le \left[\frac{1-\delta}{B(\delta)}+\frac{\delta}{B(\delta)\Gamma(\delta)}\right]\mathbb{S}_3\|\mathbb{M}-\dot{\mathbb{M}}\|,\end{aligned}$$

$$\begin{aligned}\|\mathbb{P}(t)-\dot{\mathbb{P}}(t)\| &\le \frac{1-\delta}{B(\delta)}\left\|\alpha_4(\delta,t,\mathbb{P}(t))-\alpha_4(\delta,t,\dot{\mathbb{P}}(t))\right\| \\ &+ \frac{\delta}{B(\delta)\Gamma(\delta)}\int_0^t (t-\varsigma)^{\delta-1}\left\|\alpha_4(\delta,t,\mathbb{P}(t))-\alpha_4(\delta,t,\dot{\mathbb{P}}(t))\right\| d\varsigma \\ &\le \left[\frac{1-\delta}{B(\delta)}+\frac{\delta}{B(\delta)\Gamma(\delta)}\right]\mathbb{S}_4\|\mathbb{P}-\dot{\mathbb{P}}\|,\end{aligned}$$

$$\begin{aligned}\|\mathbb{Q}(t)-\dot{\mathbb{Q}}(t)\| &\le \frac{1-\delta}{B(\delta)}\left\|\alpha_5(\delta,t,\mathbb{Q}(t))-\alpha_5(\delta,t,\dot{\mathbb{Q}}(t))\right\| \\ &+ \frac{\delta}{B(\delta)\Gamma(\delta)}\int_0^t (t-\varsigma)^{\delta-1}\left\|\alpha_5(\delta,t,\mathbb{Q}(t))-\alpha_5(\delta,t,\dot{\mathbb{Q}}(t))\right\| d\varsigma \\ &\le \left[\frac{1-\delta}{B(\delta)}+\frac{\delta}{B(\delta)\Gamma(\delta)}\right]\mathbb{S}_5\|\mathbb{Q}-\dot{\mathbb{Q}}\|.\end{aligned}$$

Taking $\upsilon_i = \mathbb{S}_i$ and $\mathcal{S}_i = \frac{1-\delta}{B(\delta)} + \frac{\delta}{B(\delta)\Gamma(\delta)}$, this implies

$$\|\mathbb{K}(t)-\dot{\mathbb{K}}(t)\| \le \upsilon_1\mathcal{S}_1. \tag{7.1}$$

In a similar manner, we have

$$\begin{cases}\|\mathbb{L}(t)-\dot{\mathbb{L}}(t)\| \le \upsilon_2\mathcal{S}_2,\\ \|\mathbb{M}(t)-\dot{\mathbb{M}}(t)\| \le \upsilon_3\mathcal{S}_3,\\ \|\mathbb{P}(t)-\dot{\mathbb{P}}(t)\| \le \upsilon_4\mathcal{S}_4,\\ \|\mathbb{Q}(t)-\dot{\mathbb{Q}}(t)\| \le \upsilon_5\mathcal{S}_5.\end{cases} \tag{7.2}$$

Using Eqs (7.1) and (7.2), the ABC fractional integral system (6.3) is Hyers-Ulam-stable and, consequently, the ABC fractional-order model (6.2) is Hyers-Ulam stable. This completes the proof. □

8. Numerical scheme

Newton's polynomial numerical method, named after Sir Isaac Newton, is a powerful technique for approximating the values of functions based on a set of data points. Developed in the 17th century, it revolutionized the field of numerical analysis. This method involves constructing a polynomial function that passes through the given data points, allowing for the estimation of intermediate values. The advantages of Newton's polynomial lie in its simplicity and efficiency. It provides a straightforward approach to interpolation and extrapolation, enabling accurate predictions and computations. Additionally, Newton's polynomial can be easily implemented and computed using finite differences, making it widely applicable in various scientific and engineering disciplines. Its historical significance and practical advantages make Newton's polynomial an essential tool in numerical analysis. Based on the Newton polynomial [56, 57], a numerical scheme for the underlying model is presented in this section. The method is structured as follows: in the first step, we will discuss the Caputo-Fabrizio fractional derivative, and then the Caputo and Atangana-Baleanu fractional derivative. In the final step, we will calculate the models with a fractal-fractional derivative:

$$\begin{cases} {}^{CF}_{0}\mathcal{D}^{\delta}_{t}\mathbb{K} = \eta - (\beta(\mathbb{K}(t)\mathbb{L}(t))) - (\rho + \eta)\,\mathbb{K}(t), \\ {}^{CF}_{0}\mathcal{D}^{\delta}_{t}\mathbb{L} = \beta\mathbb{K}(t)\mathbb{L}(t) + \omega(1-\rho)\mathbb{P}(t) - (\varphi + \rho + \eta)\mathbb{L}(t) + \tau\lambda\mathbb{M}(t), \\ {}^{CF}_{0}\mathcal{D}^{\delta}_{t}\mathbb{M} = \varphi\mathbb{L}(t) - (\lambda + \sigma + \eta)\mathbb{M}(t), \\ {}^{CF}_{0}\mathcal{D}^{\delta}_{t}\mathbb{P} = \sigma\mathbb{M}(t) - (\omega + \eta)\mathbb{P}(t), \\ {}^{CF}_{0}\mathcal{D}^{\delta}_{t}\mathbb{Q} = (1-\tau)\lambda\mathbb{M}(t) + \rho\,(\mathbb{K}(t) + \mathbb{L}(t) + \omega\mathbb{P}(t)) - \eta\mathbb{Q}(t). \end{cases} \tag{8.1}$$

The above equation is written as follows for simplicity:

$$\begin{cases} {}^{CF}_{0}\mathcal{D}^{\delta}_{t}\mathbb{K} = \mathbb{K}^{*}(t, \mathbb{K}, \mathbb{L}, \mathbb{M}, \mathbb{P}, \mathbb{Q}), \\ {}^{CF}_{0}\mathcal{D}^{\delta}_{t}\mathbb{L} = \mathbb{L}^{*}(t, \mathbb{K}, \mathbb{L}, \mathbb{M}, \mathbb{P}, \mathbb{Q}), \\ {}^{CF}_{0}\mathcal{D}^{\delta}_{t}\mathbb{M} = \mathbb{M}^{*}(t, \mathbb{M}, \mathbb{L}, \mathbb{M}, \mathbb{P}, \mathbb{Q}), \\ {}^{CF}_{0}\mathcal{D}^{\delta}_{t}\mathbb{P} = \mathbb{P}^{*}(t, \mathbb{K}, \mathbb{L}, \mathbb{M}, \mathbb{P}, \mathbb{Q}), \\ {}^{CF}_{0}\mathcal{D}^{\delta}_{t}\mathbb{Q} = \mathbb{Q}^{*}(t, \mathbb{K}, \mathbb{L}, \mathbb{M}, \mathbb{P}, \mathbb{Q}). \end{cases} \tag{8.2}$$

Applying fractional integrals with an exponential kernel and putting the Newton polynomial into these equations, the model can be solved as follows:

$$\begin{aligned} \mathbb{K}^{\nu+1} &= \mathbb{K}^{\nu} + \frac{1-\delta}{M(\delta)}\begin{bmatrix} \mathbb{K}^{*}(t_{\nu}, \mathbb{K}^{\nu}, \mathbb{L}^{\nu}, \mathbb{M}^{\nu}, \mathbb{P}^{\nu}, \mathbb{Q}^{\nu},) \\ -\mathbb{K}^{*}(t_{\nu-1}, \mathbb{K}^{\nu-1}, \mathbb{L}^{\nu-1}, \mathbb{M}^{\nu}, \mathbb{P}^{\nu-1}, \mathbb{Q}^{\nu-1}) \end{bmatrix} \\ &+ \frac{\delta}{M(\delta)}\begin{Bmatrix} \frac{23}{12}\mathbb{K}^{*}(t_{\nu}, \mathbb{K}^{\nu}, \mathbb{L}^{\nu}, \mathbb{M}^{\nu}, \mathbb{P}^{\nu}, \mathbb{Q}^{\nu},)\Delta t \\ -\frac{4}{3}\mathbb{K}^{*}(t_{\nu-1}, \mathbb{K}^{\nu-1}, \mathbb{L}^{\nu-1}, \mathbb{M}^{\nu}, \mathbb{P}^{\nu-1}, \mathbb{Q}^{\nu-1})\Delta t \\ +\frac{5}{12}\mathbb{K}^{*}(t_{\nu-2}, \mathbb{K}^{\nu-2}, \mathbb{L}^{\nu-2}, \mathbb{M}^{\nu-2}, \mathbb{P}^{\nu-2}, \mathbb{Q}^{\nu-2})\Delta t, \end{Bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbb{L}^{\nu+1} &= \mathbb{L}^{\nu} + \frac{1-\delta}{M(\delta)}\begin{bmatrix} \mathbb{L}^{*}(t_{\nu}, \mathbb{K}^{\nu}, \mathbb{L}^{\nu}, \mathbb{M}^{\nu}, \mathbb{P}^{\nu}, \mathbb{Q}^{\nu},) \\ -\mathbb{L}^{*}(t_{\nu-1}, \mathbb{K}^{\nu-1}, \mathbb{L}^{\nu-1}, \mathbb{M}^{\nu}, \mathbb{P}^{\nu-1}, \mathbb{Q}^{\nu-1}) \end{bmatrix} \\ &+ \frac{\delta}{M(\delta)}\begin{Bmatrix} \frac{23}{12}\mathbb{L}^{*}(t_{\nu}, \mathbb{K}^{\nu}, \mathbb{L}^{\nu}, \mathbb{M}^{\nu}, \mathbb{P}^{\nu}, \mathbb{Q}^{\nu},)\Delta t \\ -\frac{4}{3}\mathbb{L}^{*}(t_{\nu-1}, \mathbb{K}^{\nu-1}, \mathbb{L}^{\nu-1}, \mathbb{M}^{\nu}, \mathbb{P}^{\nu-1}, \mathbb{Q}^{\nu-1})\Delta t \\ +\frac{5}{12}\mathbb{L}^{*}(t_{\nu-2}, \mathbb{K}^{\nu-2}, \mathbb{L}^{\nu-2}, \mathbb{M}^{\nu-2}, \mathbb{P}^{\nu-2}, \mathbb{Q}^{\nu-2})\Delta t, \end{Bmatrix} \end{aligned}$$

$$\mathbb{M}^{v+1} = \mathbb{M}^{v} + \frac{1-\delta}{M(\delta)}\begin{bmatrix} \mathbb{M}^{*}(t_v, \mathbb{M}^{v}, \mathbb{L}^{v}, \mathbb{M}^{v}, \mathbb{P}^{v}, \mathbb{Q}^{v},) \\ -\mathbb{M}^{*}(t_{v-1}, \mathbb{K}^{v-1}, \mathbb{L}^{v-1}, \mathbb{M}^{v}, \mathbb{P}^{v-1}, \mathbb{Q}^{v-1}) \end{bmatrix}$$
$$+ \frac{\delta}{M(\delta)}\begin{Bmatrix} \frac{23}{12}\mathbb{P}^{*}(t_v, \mathbb{K}^{v}, \mathbb{L}^{v}, \mathbb{M}^{v}, \mathbb{P}^{v}, \mathbb{Q}^{v},)\Delta t \\ -\frac{4}{3}\mathbb{M}^{*}(t_{v-1}, \mathbb{K}^{v-1}, \mathbb{L}^{v-1}, \mathbb{M}^{v}, \mathbb{P}^{v-1}, \mathbb{Q}^{v-1})\Delta t \\ +\frac{5}{12}\mathbb{M}^{*}(t_{v-2}, \mathbb{K}^{v-2}, \mathbb{L}^{v-2}, \mathbb{M}^{v-2}, \mathbb{P}^{v-2}, \mathbb{Q}^{v-2})\Delta t, \end{Bmatrix}$$

$$\mathbb{P}^{v+1} = \mathbb{P}^{v} + \frac{1-\delta}{M(\delta)}\begin{bmatrix} \mathbb{P}^{*}(t_v, \mathbb{M}^{v}, \mathbb{L}^{v}, \mathbb{M}^{v}, \mathbb{P}^{v}, \mathbb{Q}^{v},) \\ -\mathbb{P}^{*}(t_{v-1}, \mathbb{K}^{v-1}, \mathbb{L}^{v-1}, \mathbb{M}^{v}, \mathbb{P}^{v-1}, \mathbb{Q}^{v-1}) \end{bmatrix}$$
$$+ \frac{\delta}{M(\delta)}\begin{Bmatrix} \frac{23}{12}\mathbb{P}^{*}(t_v, \mathbb{K}^{v}, \mathbb{L}^{v}, \mathbb{M}^{v}, \mathbb{P}^{v}, \mathbb{Q}^{v},)\Delta t \\ -\frac{4}{3}\mathbb{P}^{*}(t_{v-1}, \mathbb{K}^{v-1}, \mathbb{L}^{v-1}, \mathbb{M}^{v}, \mathbb{P}^{v-1}, \mathbb{Q}^{v-1})\Delta t \\ +\frac{5}{12}\mathbb{P}^{*}(t_{v-2}, \mathbb{K}^{v-2}, \mathbb{L}^{v-2}, \mathbb{M}^{v-2}, \mathbb{P}^{v-2}, \mathbb{Q}^{v-2})\Delta t, \end{Bmatrix}$$

$$\mathbb{Q}^{v+1} = \mathbb{Q}^{v} + \frac{1-\delta}{M(\delta)}\begin{bmatrix} \mathbb{Q}^{*}(t_v, \mathbb{M}^{v}, \mathbb{L}^{v}, \mathbb{M}^{v}, \mathbb{P}^{v}, \mathbb{Q}^{v},) \\ -\mathbb{Q}^{*}(t_{v-1}, \mathbb{K}^{v-1}, \mathbb{L}^{v-1}, \mathbb{M}^{v}, \mathbb{P}^{v-1}, \mathbb{Q}^{v-1}) \end{bmatrix}$$
$$+ \frac{\delta}{M(\delta)}\begin{Bmatrix} \frac{23}{12}\mathbb{Q}^{*}(t_v, \mathbb{K}^{v}, \mathbb{L}^{v}, \mathbb{M}^{v}, \mathbb{P}^{v}, \mathbb{Q}^{v},)\Delta t \\ -\frac{4}{3}\mathbb{Q}^{*}(t_{v-1}, \mathbb{K}^{v-1}, \mathbb{L}^{v-1}, \mathbb{M}^{v}, \mathbb{P}^{v-1}, \mathbb{Q}^{v-1})\Delta t \\ +\frac{5}{12}\mathbb{Q}^{*}(t_{v-2}, \mathbb{K}^{v-2}, \mathbb{L}^{v-2}, \mathbb{M}^{v-2}, \mathbb{P}^{v-2}, \mathbb{Q}^{v-2})\Delta t. \end{Bmatrix}$$

The numerical scheme for the case of Mittag-Leffler is derived as follows.

$$\mathbb{K}^{v+1} = \frac{1-\delta}{AB(\delta)} + \mathbb{K}^{*}(t_v, \mathbb{K}^{v}, \mathbb{L}^{v}, \mathbb{M}^{v}, \mathbb{P}^{v}, \mathbb{Q}^{v})$$
$$+ \frac{\delta(\Delta t)^{\delta}}{AB(\delta)\Gamma(\delta+1)}\sum_{u=2}^{v}\mathbb{K}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2})\Pi$$
$$+ \frac{\delta(\Delta t)^{\delta}}{AB(\delta)\Gamma(\delta+2)}\sum_{u=2}^{v}\begin{bmatrix} \mathbb{K}^{*}(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -\mathbb{K}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix}\Sigma$$
$$+ \frac{\delta(\Delta t)^{\delta}}{2AB(\delta)\Gamma(\delta+3)}\sum_{u=2}^{v}\begin{Bmatrix} \mathbb{K}^{*}(t_{u}, \mathbb{K}^{u}, \mathbb{L}^{u}, \mathbb{M}^{u}, \mathbb{P}^{u}, \mathbb{Q}^{u}) \\ -2\mathbb{K}^{*}(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +\mathbb{K}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{Bmatrix}\Delta,$$

$$\mathbb{L}^{v+1} = \frac{1-\delta}{AB(\delta)} + \mathbb{L}^{*}(t_v, \mathbb{K}^{v}, \mathbb{L}^{v}, \mathbb{M}^{v}, \mathbb{P}^{v}, \mathbb{Q}^{v})$$
$$+ \frac{\delta(\Delta t)^{\delta}}{AB(\delta)\Gamma(\delta+1)}\sum_{u=2}^{v}\mathbb{L}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2})\Pi$$
$$+ \frac{\delta(\Delta t)^{\delta}}{AB(\delta)\Gamma(\delta+2)}\sum_{u=2}^{v}\begin{bmatrix} \mathbb{L}^{*}(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -\mathbb{L}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix}\Sigma$$
$$+ \frac{\delta(\Delta t)^{\delta}}{2AB(\delta)\Gamma(\delta+3)}\sum_{u=2}^{v}\begin{Bmatrix} \mathbb{L}^{*}(t_{u}, \mathbb{K}^{u}, \mathbb{L}^{u}, \mathbb{M}^{u}, \mathbb{P}^{u}, \mathbb{Q}^{u}) \\ -2\mathbb{L}^{*}(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +\mathbb{L}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{Bmatrix}\Delta,$$

$$
\begin{aligned}
\mathbb{M}^{v+1} = {} & \frac{1-\delta}{AB(\delta)} + \mathbb{M}^*(t_v, \mathbb{K}^v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v) \\
& + \frac{\delta(\Delta t)^\delta}{AB(\delta)\Gamma(\delta+1)} \sum_{u=2}^{v} \mathbb{M}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2})\Pi \\
& + \frac{\delta(\Delta t)^\delta}{AB(\delta)\Gamma(\delta+2)} \sum_{u=2}^{v} \begin{bmatrix} \mathbb{M}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -\mathbb{M}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix} \Sigma \\
& + \frac{\delta(\Delta t)^\delta}{2AB(\delta)\Gamma(\delta+3)} \sum_{u=2}^{v} \left\{ \begin{matrix} \mathbb{M}^*(t_u, \mathbb{K}^u, \mathbb{L}^u, \mathbb{M}^u, \mathbb{P}^u, \mathbb{Q}^u) \\ -2\mathbb{M}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +\mathbb{M}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{matrix} \right\} \Delta,
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}^{v+1} = {} & \frac{1-\delta}{AB(\delta)} + \mathbb{P}^*(t_v, \mathbb{K}^v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v) \\
& + \frac{\delta(\Delta t)^\delta}{AB(\delta)\Gamma(\delta+1)} \sum_{u=2}^{v} \mathbb{P}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2})\Pi \\
& + \frac{\delta(\Delta t)^\delta}{AB(\delta)\Gamma(\delta+2)} \sum_{u=2}^{v} \begin{bmatrix} \mathbb{P}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -\mathbb{P}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix} \Sigma \\
& + \frac{\delta(\Delta t)^\delta}{2AB(\delta)\Gamma(\delta+3)} \sum_{u=2}^{v} \left\{ \begin{matrix} \mathbb{P}^*(t_u, \mathbb{K}^u, \mathbb{L}^u, \mathbb{M}^u, \mathbb{P}^u, \mathbb{Q}^u) \\ -2\mathbb{P}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +\mathbb{P}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{matrix} \right\} \Delta,
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{Q}^{v+1} = {} & \frac{1-\delta}{AB(\delta)} + \mathbb{Q}^*(t_v, \mathbb{K}^v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v) \\
& + \frac{\delta(\Delta t)^\delta}{AB(\delta)\Gamma(\delta+1)} \sum_{u=2}^{v} \mathbb{Q}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2})\Pi \\
& + \frac{\delta(\Delta t)^\delta}{AB(\delta)\Gamma(\delta+2)} \sum_{u=2}^{v} \begin{bmatrix} \mathbb{Q}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -\mathbb{Q}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix} \Sigma \\
& + \frac{\delta(\Delta t)^\delta}{2AB(\delta)\Gamma(\delta+3)} \sum_{u=2}^{v} \left\{ \begin{matrix} \mathbb{Q}^*(t_u, \mathbb{K}^u, \mathbb{L}^u, \mathbb{M}^u, \mathbb{P}^u, \mathbb{Q}^u) \\ -2\mathbb{Q}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +\mathbb{Q}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{matrix} \right\} \Delta,
\end{aligned}
$$

where

$$
\Delta = \begin{bmatrix} (v-u+1)^\delta \begin{bmatrix} 2(v-u)^2 + (3\delta+10)(v-u) \\ +2\delta^2 + 9\delta + 12 \end{bmatrix} \\ -(v-u)^\delta \begin{bmatrix} 2(v-u)^2 + (5\delta+10)(v-u) \\ +6\delta^2 + 18\delta + 12 \end{bmatrix} \end{bmatrix},
$$

$$
\Sigma = \begin{bmatrix} (v-u+1)^\delta (v-u+3+2\delta) \\ -(v-u)^\delta (v-u+3+3\delta) \end{bmatrix}, \Pi = [(v-u+1)^\delta - (v-u)^\delta].
$$

Finally, a numerical approximation is obtained by utilizing the Caputo derivative.

$$\begin{aligned}
\mathbb{K}^{v+1} &= \frac{(\Delta t)^{\delta}}{\Gamma(\delta+1)} \sum_{u=2}^{v} \mathbb{K}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2})\Pi \\
&+ \frac{(\Delta t)^{\delta}}{\Gamma(\delta+2)} \sum_{u=2}^{v} \begin{bmatrix} \mathbb{K}^{*}(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -\mathbb{K}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix} \Sigma \\
&+ \frac{(\Delta t)^{\delta}}{2\Gamma(\delta+3)} \sum_{u=2}^{v} \begin{Bmatrix} \mathbb{K}^{*}(t_{u}, \mathbb{K}^{u}, \mathbb{L}^{u}, \mathbb{M}^{u}, \mathbb{P}^{u}, \mathbb{Q}^{u}) \\ -2\mathbb{K}^{*}(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +\mathbb{K}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{Bmatrix} \Delta,
\end{aligned}$$

$$\begin{aligned}
\mathbb{L}^{v+1} &= \frac{(\Delta t)^{\delta}}{\Gamma(\delta+1)} \sum_{u=2}^{v} \mathbb{L}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2})\Pi \\
&+ \frac{(\Delta t)^{\delta}}{\Gamma(\delta+2)} \sum_{u=2}^{v} \begin{bmatrix} \mathbb{L}^{*}(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -\mathbb{L}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix} \Sigma \\
&+ \frac{(\Delta t)^{\delta}}{2\Gamma(\delta+3)} \sum_{u=2}^{v} \begin{Bmatrix} \mathbb{L}^{*}(t_{u}, \mathbb{K}^{u}, \mathbb{L}^{u}, \mathbb{M}^{u}, \mathbb{P}^{u}, \mathbb{Q}^{u}) \\ -2\mathbb{L}^{*}(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +\mathbb{L}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{Bmatrix} \Delta,
\end{aligned}$$

$$\begin{aligned}
\mathbb{M}^{v+1} &= \frac{(\Delta t)^{\delta}}{\Gamma(\delta+1)} \sum_{u=2}^{v} \mathbb{M}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2})\Pi \\
&+ \frac{(\Delta t)^{\delta}}{\Gamma(\delta+2)} \sum_{u=2}^{v} \begin{bmatrix} \mathbb{M}^{*}(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -\mathbb{M}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix} \Sigma \\
&+ \frac{(\Delta t)^{\delta}}{2\Gamma(\delta+3)} \sum_{u=2}^{v} \begin{Bmatrix} \mathbb{M}^{*}(t_{u}, \mathbb{K}^{u}, \mathbb{L}^{u}, \mathbb{M}^{u}, \mathbb{P}^{u}, \mathbb{Q}^{u}) \\ -2\mathbb{M}^{*}(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +\mathbb{M}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{Bmatrix} \Delta,
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}^{v+1} &= \frac{(\Delta t)^{\delta}}{\Gamma(\delta+1)} \sum_{u=2}^{v} \mathbb{P}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2})\Pi \\
&+ \frac{(\Delta t)^{\delta}}{\Gamma(\delta+2)} \sum_{u=2}^{v} \begin{bmatrix} \mathbb{P}^{*}(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -\mathbb{P}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix} \Sigma \\
&+ \frac{(\Delta t)^{\delta}}{2\Gamma(\delta+3)} \sum_{u=2}^{v} \begin{Bmatrix} \mathbb{P}^{*}(t_{u}, \mathbb{K}^{u}, \mathbb{L}^{u}, \mathbb{M}^{u}, \mathbb{P}^{u}, \mathbb{Q}^{u}) \\ -2\mathbb{P}^{*}(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +\mathbb{P}^{*}(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{Bmatrix} \Delta,
\end{aligned}$$

$$
\begin{aligned}
\mathbb{Q}^{v+1} &= \frac{(\Delta t)^{\delta}}{\Gamma(\delta+1)} \sum_{u=2}^{v} \mathbb{Q}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2})\Pi \\
&+ \frac{(\Delta t)^{\delta}}{\Gamma(\delta+2)} \sum_{u=2}^{v} \begin{bmatrix} \mathbb{Q}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -\mathbb{Q}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix} \Sigma \\
&+ \frac{(\Delta t)^{\delta}}{2\Gamma(\delta+3)} \sum_{u=2}^{v} \begin{Bmatrix} \mathbb{Q}^*(t_{u}, \mathbb{K}^{u}, \mathbb{L}^{u}, \mathbb{M}^{u}, \mathbb{P}^{u}, \mathbb{Q}^{u}) \\ -2\mathbb{Q}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +\mathbb{Q}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{Bmatrix} \Delta.
\end{aligned}
$$

In the present study, we investigate our model incorporating fractal-fractional operators. The analysis commences by introducing the Caputo-Fabrizio fractal-fractional derivative.

$$
\begin{cases}
{}^{\mathrm{FFE}}_{0}\mathcal{D}_t^{\delta,\kappa}\mathbb{K} = \mathbb{K}^*(t, \mathbb{K}, \mathbb{L}, \mathbb{M}, \mathbb{P}, \mathbb{Q}), \\
{}^{\mathrm{FFE}}_{0}\mathcal{D}_t^{\delta,\kappa}\mathbb{L} = \mathbb{L}^*(t, \mathbb{K}, \mathbb{L}, \mathbb{M}, \mathbb{P}, \mathbb{Q}), \\
{}^{\mathrm{FFE}}_{0}\mathcal{D}_t^{\delta,\kappa}\mathbb{M} = \mathbb{M}^*(t, \mathbb{K}, \mathbb{L}, \mathbb{M}, \mathbb{P}, \mathbb{Q}), \\
{}^{\mathrm{FFE}}_{0}\mathcal{D}_t^{\delta,\kappa}\mathbb{P} = \mathbb{P}^*(t, \mathbb{K}, \mathbb{L}, \mathbb{M}, \mathbb{P}, \mathbb{Q}), \\
{}^{\mathrm{FFE}}_{0}\mathcal{D}_t^{\delta,\kappa}\mathbb{Q} = \mathbb{Q}^*(t, \mathbb{K}, \mathbb{L}, \mathbb{M}, \mathbb{P}, \mathbb{Q}).
\end{cases} \tag{8.3}
$$

The following results come from applying a fractal-fractional integral to model (8.3) with an exponential kernel:

$$
\begin{aligned}
\mathbb{K}^{v+1} &= \mathbb{K}^{v} + \frac{1-\delta}{M(\delta)} \begin{bmatrix} t_v^{1-\kappa}\mathbb{K}^*(t_v, \mathbb{K}^v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v) \\ -t_{v-1}^{1-\kappa}\mathbb{K}^*(t_{v-1}, \mathbb{K}^{v-1}, \mathbb{L}^{v-1}, \mathbb{M}^{v}, \mathbb{P}^{v-1}, \mathbb{Q}^{v-1}) \end{bmatrix} \\
&+ \frac{\delta}{M(\delta)} \begin{Bmatrix} \frac{23}{12}t_v^{1-\kappa}\mathbb{K}^*(t_v, \mathbb{K}^v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v)\Delta t \\ -\frac{4}{3}t_{v-1}^{1-\kappa}\mathbb{K}^*(t_{v-1}, \mathbb{K}^{v-1}, \mathbb{L}^{v-1}, \mathbb{M}^{v}, \mathbb{P}^{v-1}, \mathbb{Q}^{v-1})\Delta t \\ +\frac{5}{12}t_{v-2}^{1-\kappa}\mathbb{K}^*(t_{v-2}, \mathbb{K}^{v-2}, \mathbb{L}^{v-2}, \mathbb{M}^{v-2}, \mathbb{P}^{v-2}, \mathbb{Q}^{v-2})\Delta t, \end{Bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{L}^{v+1} &= \mathbb{L}^{v} + \frac{1-\delta}{M(\delta)} \begin{bmatrix} t_v^{1-\kappa}\mathbb{L}^*(t_v, \mathbb{K}^v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v) \\ -t_{v-1}^{1-\kappa}\mathbb{L}^*(t_{v-1}, \mathbb{K}^{v-1}, \mathbb{L}^{v-1}, \mathbb{M}^{v}, \mathbb{P}^{v-1}, \mathbb{Q}^{v-1}) \end{bmatrix} \\
&+ \frac{\delta}{M(\delta)} \begin{Bmatrix} \frac{23}{12}t_v^{1-\kappa}\mathbb{L}^*(t_v, \mathbb{K}^v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v)\Delta t \\ -\frac{4}{3}t_{v-1}^{1-\kappa}\mathbb{L}^*(t_{v-1}, \mathbb{K}^{v-1}, \mathbb{L}^{v-1}, \mathbb{M}^{v}, \mathbb{P}^{v-1}, \mathbb{Q}^{v-1})\Delta t \\ +\frac{5}{12}t_{v-2}^{1-\kappa}\mathbb{L}^*(t_{v-2}, \mathbb{K}^{v-2}, \mathbb{L}^{v-2}, \mathbb{M}^{v-2}, \mathbb{P}^{v-2}, \mathbb{Q}^{v-2})\Delta t, \end{Bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{M}^{v+1} &= \mathbb{M}^{v} + \frac{1-\delta}{M(\delta)} \begin{bmatrix} t_v^{1-\kappa}\mathbb{M}^*(t_v, \mathbb{K}^v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v) \\ -t_{v-1}^{1-\kappa}\mathbb{M}^*(t_{v-1}, \mathbb{K}^{v-1}, \mathbb{L}^{v-1}, \mathbb{M}^{v}, \mathbb{P}^{v-1}, \mathbb{Q}^{v-1}) \end{bmatrix} \\
&+ \frac{\delta}{M(\delta)} \begin{Bmatrix} \frac{23}{12}t_v^{1-\kappa}\mathbb{M}^*(t_v, \mathbb{K}^v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v)\Delta t \\ -\frac{4}{3}t_{v-1}^{1-\kappa}\mathbb{M}^*(t_{v-1}, \mathbb{K}^{v-1}, \mathbb{L}^{v-1}, \mathbb{M}^{v}, \mathbb{P}^{v-1}, \mathbb{Q}^{v-1})\Delta t \\ +\frac{5}{12}t_{v-2}^{1-\kappa}\mathbb{M}^*(t_{v-2}, \mathbb{K}^{v-2}, \mathbb{L}^{v-2}, \mathbb{M}^{v-2}, \mathbb{P}^{v-2}, \mathbb{Q}^{v-2})\Delta t, \end{Bmatrix}
\end{aligned}
$$

$$\mathbb{P}^{v+1} = \mathbb{P}^v + \frac{1-\delta}{M(\delta)}\begin{bmatrix} t_v^{1-\kappa}\mathbb{P}^*(t_v, \mathbb{K}^v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v) \\ -t_{v-1}^{1-\kappa}\mathbb{P}^*(t_{v-1}, \mathbb{K}^{v-1}, \mathbb{L}^{v-1}, \mathbb{M}^v, \mathbb{P}^{v-1}, \mathbb{Q}^{v-1}) \end{bmatrix} + \frac{\delta}{M(\delta)}\begin{Bmatrix} \frac{23}{12}t_v^{1-\kappa}\mathbb{P}^*(t_v, \mathbb{K}^v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v)\Delta t \\ -\frac{4}{3}t_{v-1}^{1-\kappa}\mathbb{P}^*(t_{v-1}, \mathbb{K}^{v-1}, \mathbb{L}^{v-1}, \mathbb{M}^v, \mathbb{P}^{v-1}, \mathbb{Q}^{v-1})\Delta t \\ +\frac{5}{12}t_{v-2}^{1-\kappa}\mathbb{P}^*(t_{v-2}, \mathbb{K}^{v-2}, \mathbb{L}^{v-2}, \mathbb{M}^{v-2}, \mathbb{P}^{v-2}, \mathbb{Q}^{v-2})\Delta t, \end{Bmatrix}$$

$$\mathbb{Q}^{v+1} = \mathbb{Q}^v + \frac{1-\delta}{M(\delta)}\begin{bmatrix} t_v^{1-\kappa}\mathbb{Q}^*(t_v, \mathbb{K}^v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v) \\ -t_{v-1}^{1-\kappa}\mathbb{Q}^*(t_{v-1}, \mathbb{K}^{v-1}, \mathbb{L}^{v-1}, \mathbb{M}^v, \mathbb{P}^{v-1}, \mathbb{Q}^{v-1}) \end{bmatrix} + \frac{\delta}{M(\delta)}\begin{Bmatrix} \frac{23}{12}t_v^{1-\kappa}\mathbb{Q}^*(t_v, \mathbb{K}^v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v)\Delta t \\ -\frac{4}{3}t_{v-1}^{1-\kappa}\mathbb{Q}^*(t_{v-1}, \mathbb{K}^{v-1}, \mathbb{L}^{v-1}, \mathbb{M}^v, \mathbb{P}^{v-1}, \mathbb{Q}^{v-1})\Delta t \\ +\frac{5}{12}t_{v-2}^{1-\kappa}\mathbb{Q}^*(t_{v-2}, \mathbb{K}^{v-2}, \mathbb{L}^{v-2}, \mathbb{M}^{v-2}, \mathbb{P}^{v-2}, \mathbb{Q}^{v-2})\Delta t. \end{Bmatrix}$$

We obtain the following numerical scheme for the Mittag-Leffler kernel:

$$\begin{aligned}\mathbb{K}^{v+1} &= \frac{1-\delta}{AB(\delta)}t_v^{1-\kappa}\mathbb{K}^*(t_v, \mathbb{K}^v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v) \\ &+ \frac{\delta(\Delta t)^\delta}{AB(\delta)\Gamma(\delta+1)}\sum_{u=2}^{v} t_{u-2}^{1-\kappa}\mathbb{K}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2})\Pi \\ &+ \frac{\delta(\Delta t)^\delta}{AB(\delta)\Gamma(\delta+2)}\sum_{u=2}^{v}\begin{bmatrix} t_{u-1}^{1-\kappa}\mathbb{K}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -t_{u-2}^{1-\kappa}\mathbb{K}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix}\Sigma \\ &+ \frac{\delta(\Delta t)^\delta}{2AB(\delta)\Gamma(\delta+3)}\sum_{u=2}^{v}\begin{Bmatrix} t_u^{1-\kappa}\mathbb{K}^*(t_u, \mathbb{K}^u, \mathbb{L}^u, \mathbb{M}^u, \mathbb{P}^u, \mathbb{Q}^u) \\ -2t_{u-1}^{1-\kappa}\mathbb{K}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +t_{u-2}^{1-\kappa}\mathbb{K}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{Bmatrix}\Delta,\end{aligned}$$

$$\begin{aligned}\mathbb{L}^{v+1} &= \frac{1-\delta}{AB(\delta)}t_v^{1-\kappa}\mathbb{L}^*(t_v, \mathbb{K}v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v) \\ &+ \frac{\delta(\Delta t)^\delta}{AB(\delta)\Gamma(\delta+1)}\sum_{u=2}^{v} t_{u-2}^{1-\kappa}\mathbb{L}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2})\Pi \\ &+ \frac{\delta(\Delta t)^\delta}{AB(\delta)\Gamma(\delta+2)}\sum_{u=2}^{v}\begin{bmatrix} t_{u-1}^{1-\kappa}\mathbb{L}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -t_{u-2}^{1-\kappa}\mathbb{L}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix}\Sigma \\ &+ \frac{\delta(\Delta t)^\delta}{2AB(\delta)\Gamma(\delta+3)}\sum_{u=2}^{v}\begin{Bmatrix} t_u^{1-\kappa}\mathbb{L}^*(t_u, \mathbb{K}^u, \mathbb{L}^u, \mathbb{M}^u, \mathbb{P}^u, \mathbb{Q}^u) \\ -2t_{u-1}^{1-\kappa}\mathbb{L}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +t_{u-2}^{1-\kappa}\mathbb{L}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{Bmatrix}\Delta,\end{aligned}$$

$$
\begin{aligned}
\mathbb{M}^{v+1} = & \frac{1-\delta}{AB(\delta)} t_v^{1-\kappa} \mathbb{M}^*(t_v, \mathbb{K}^v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v) \\
& + \frac{\delta(\Delta t)^\delta}{AB(\delta)\Gamma(\delta+1)} \sum_{u=2}^{v} t_{u-2}^{1-\kappa} \mathbb{M}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \Pi \\
& + \frac{\delta(\Delta t)^\delta}{AB(\delta)\Gamma(\delta+2)} \sum_{u=2}^{v} \begin{bmatrix} t_{u-1}^{1-\kappa} \mathbb{M}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -t_{u-2}^{1-\kappa} \mathbb{M}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix} \Sigma \\
& + \frac{\delta(\Delta t)^\delta}{2AB(\delta)\Gamma(\delta+3)} \sum_{u=2}^{v} \begin{Bmatrix} t_u^{1-\kappa} \mathbb{M}^*(t_u, \mathbb{K}^u, \mathbb{L}^u, \mathbb{M}^u, \mathbb{P}^u, \mathbb{Q}^u) \\ -2t_{u-1}^{1-\kappa} \mathbb{M}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +t_{u-2}^{1-\kappa} \mathbb{M}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{Bmatrix} \Delta,
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}^{v+1} = & \frac{1-\delta}{AB(\delta)} t_v^{1-\kappa} \mathbb{P}^*(t_v, \mathbb{K}^v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v) \\
& + \frac{\delta(\Delta t)^\delta}{AB(\delta)\Gamma(\delta+1)} \sum_{u=2}^{v} t_{u-2}^{1-\kappa} \mathbb{P}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \Pi \\
& + \frac{\delta(\Delta t)^\delta}{AB(\delta)\Gamma(\delta+2)} \sum_{u=2}^{v} \begin{bmatrix} t_{u-1}^{1-\kappa} \mathbb{P}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -t_{u-2}^{1-\kappa} \mathbb{P}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix} \Sigma \\
& + \frac{\delta(\Delta t)^\delta}{2AB(\delta)\Gamma(\delta+3)} \sum_{u=2}^{v} \begin{Bmatrix} t_u^{1-\kappa} \mathbb{P}^*(t_u, \mathbb{K}^u, \mathbb{L}^u, \mathbb{M}^u, \mathbb{P}^u, \mathbb{Q}^u) \\ -2t_{u-1}^{1-\kappa} \mathbb{P}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +t_{u-2}^{1-\kappa} \mathbb{P}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{Bmatrix} \Delta,
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{Q}^{v+1} = & \frac{1-\delta}{AB(\delta)} t_v^{1-\kappa} \mathbb{Q}^*(t_v, \mathbb{K}^v, \mathbb{L}^v, \mathbb{M}^v, \mathbb{P}^v, \mathbb{Q}^v) \\
& + \frac{\delta(\Delta t)^\delta}{AB(\delta)\Gamma(\delta+1)} \sum_{u=2}^{v} t_{u-2}^{1-\kappa} \mathbb{Q}*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \Pi \\
& + \frac{\delta(\Delta t)^\delta}{AB(\delta)\Gamma(\delta+2)} \sum_{u=2}^{v} \begin{bmatrix} t_{u-1}^{1-\kappa} \mathbb{Q}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -t_{u-2}^{1-\kappa} \mathbb{Q}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix} \Sigma \\
& + \frac{\delta(\Delta t)^\delta}{2AB(\delta)\Gamma(\delta+3)} \sum_{u=2}^{v} \begin{Bmatrix} t_u^{1-\kappa} \mathbb{Q}^*(t_u, \mathbb{K}^u, \mathbb{L}^u, \mathbb{M}^u, \mathbb{P}^u, \mathbb{Q}^u) \\ -2t_{u-1}^{1-\kappa} \mathbb{Q}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +t_{u-2}^{1-\kappa} \mathbb{Q}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{Bmatrix} \Delta.
\end{aligned}
$$

We obtain the following numerical scheme for the power-law kernel:

$$
\begin{aligned}
\mathbb{K}^{v+1} = & \frac{(\Delta t)^\delta}{\Gamma(\delta+1)} \sum_{u=2}^{v} t_{u-2}^{1-\kappa} \mathbb{K}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \Pi \\
& + \frac{(\Delta t)^\delta}{\Gamma(\delta+2)} \sum_{u=2}^{v} \begin{bmatrix} t_{u-1}^{1-\kappa} \mathbb{K}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -t_{u-2}^{1-\kappa} \mathbb{K}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix} \Sigma \\
& + \frac{(\Delta t)^\delta}{2\Gamma(\delta+3)} \sum_{u=2}^{v} \begin{Bmatrix} t_u^{1-\kappa} \mathbb{K}^*(t_u, \mathbb{K}^u, \mathbb{L}^u, \mathbb{M}^u, \mathbb{P}^u, \mathbb{Q}^u) \\ -2t_{u-1}^{1-\kappa} \mathbb{K}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +t_{u-2}^{1-\kappa} \mathbb{K}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{Bmatrix} \Delta,
\end{aligned}
$$

$$\mathbb{L}^{v+1} = \frac{(\Delta t)^{\delta}}{\Gamma(\delta+1)} \sum_{u=2}^{v} t_{u-2}^{1-\kappa} \mathbb{L}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2})\Pi$$
$$+ \frac{(\Delta t)^{\delta}}{\Gamma(\delta+2)} \sum_{u=2}^{v} \begin{bmatrix} t_{u-1}^{1-\kappa} \mathbb{L}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -t_{u-2}^{1-\kappa} \mathbb{L}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix} \Sigma$$
$$+ \frac{(\Delta t)^{\delta}}{2\Gamma(\delta+3)} \sum_{u=2}^{v} \begin{Bmatrix} t_{u}^{1-\kappa} \mathbb{L}^*(t_{u}, \mathbb{K}^{u}, \mathbb{L}^{u}, \mathbb{M}^{u}, \mathbb{P}^{u}, \mathbb{Q}^{u}) \\ -2t_{u-1}^{1-\kappa} \mathbb{L}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +t_{u-2}^{1-\kappa} \mathbb{L}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{Bmatrix} \Delta,$$

$$\mathbb{M}^{v+1} = \frac{(\Delta t)^{\delta}}{\Gamma(\delta+1)} \sum_{u=2}^{v} t_{u-2}^{1-\kappa} \mathbb{M}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2})\Pi$$
$$+ \frac{(\Delta t)^{\delta}}{\Gamma(\delta+2)} \sum_{u=2}^{v} \begin{bmatrix} t_{u-1}^{1-\kappa} \mathbb{M}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -t_{u-2}^{1-\kappa} \mathbb{M}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix} \Sigma$$
$$+ \frac{(\Delta t)^{\delta}}{2\Gamma(\delta+3)} \sum_{u=2}^{v} \begin{Bmatrix} t_{u}^{1-\kappa} \mathbb{M}^*(t_{u}, \mathbb{K}^{u}, \mathbb{L}^{u}, \mathbb{M}^{u}, \mathbb{P}^{u}, \mathbb{Q}^{u}) \\ -2t_{u-1}^{1-\kappa} \mathbb{M}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +t_{u-2}^{1-\kappa} \mathbb{M}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{Bmatrix} \Delta,$$

$$\mathbb{P}^{v+1} = \frac{(\Delta t)^{\delta}}{\Gamma(\delta+1)} \sum_{u=2}^{v} t_{u-2}^{1-\kappa} \mathbb{P}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2})\Pi$$
$$+ \frac{(\Delta t)^{\delta}}{\Gamma(\delta+2)} \sum_{u=2}^{v} \begin{bmatrix} t_{u-1}^{1-\kappa} \mathbb{P}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -t_{u-2}^{1-\kappa} \mathbb{P}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix} \Sigma$$
$$+ \frac{(\Delta t)^{\delta}}{2\Gamma(\delta+3)} \sum_{u=2}^{v} \begin{Bmatrix} t_{u}^{1-\kappa} \mathbb{P}^*(t_{u}, \mathbb{K}^{u}, \mathbb{L}^{u}, \mathbb{M}^{u}, \mathbb{P}^{u}, \mathbb{Q}^{u}) \\ -2t_{u-1}^{1-\kappa} \mathbb{P}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +t_{u-2}^{1-\kappa} \mathbb{P}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{Bmatrix} \Delta,$$

$$\mathbb{Q}^{v+1} = \frac{(\Delta t)^{\delta}}{\Gamma(\delta+1)} \sum_{u=2}^{v} t_{u-2}^{1-\kappa} \mathbb{Q}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2})\Pi$$
$$+ \frac{(\Delta t)^{\delta}}{\Gamma(\delta+2)} \sum_{u=2}^{v} \begin{bmatrix} t_{u-1}^{1-\kappa} \mathbb{Q}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ -t_{u-2}^{1-\kappa} \mathbb{Q}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{bmatrix} \Sigma$$
$$+ \frac{(\Delta t)^{\delta}}{2\Gamma(\delta+3)} \sum_{u=2}^{v} \begin{Bmatrix} t_{u}^{1-\kappa} \mathbb{Q}^*(t_{u}, \mathbb{K}^{u}, \mathbb{L}^{u}, \mathbb{M}^{u}, \mathbb{P}^{u}, \mathbb{Q}^{u}) \\ -2t_{u-1}^{1-\kappa} \mathbb{Q}^*(t_{u-1}, \mathbb{K}^{u-1}, \mathbb{L}^{u-1}, \mathbb{M}^{u-1}, \mathbb{P}^{u-1}, \mathbb{Q}^{u-1}) \\ +t_{u-2}^{1-\kappa} \mathbb{Q}^*(t_{u-2}, \mathbb{K}^{u-2}, \mathbb{L}^{u-2}, \mathbb{M}^{u-2}, \mathbb{P}^{u-2}, \mathbb{Q}^{u-2}) \end{Bmatrix} \Delta.$$

8.1. Numerical results and discussions

In this section of the article, our objective is to obtain an approximate solution for the non-integer-order financial crime model (6.1) using the Newton polynomial method. The simulation was conducted within a time interval ranging from 0 to 50 steps, employing MATLAB 2019. The parameter values for the system were set as follows: $\eta = 0.12; \beta = 0.65; \rho = 0.05; w = 0.75; \varphi = 0.2; \tau = 0.65; \lambda =$

$0.43; \sigma = 0.67$ [4], and these values were utilized for the graphical representation. Numerical simulations were carried out for different orders of the fractional derivative, denoted as δ. The results demonstrate that the non-integer-order fractional derivative leads to favorable outcomes in terms of controlling the corrupt individuals. The dynamics of each class in the system (6.1) are depicted in Figure 3 for various values of δ, such as $0.90, 0.85, 0.80, 0.75, 0.70, 0.65, 0.60, 0.55, 0.50$. From Figure 3a, it is observed that the number of susceptible individuals in the population decreases more rapidly, with a decay occurring in the fractional order δ. Conversely, in Figure 3b, the magnitude of financial criminals (corrupt individuals) decreases with a decay in the fractional order δ. Similarly, Figure 3c represents the dynamics of individuals under prosecution, showing a decrease in this population within the first 5 days, followed by a slow increase toward the stationary point as the fractional order δ is increased. Furthermore, the population of honest individuals is depicted in Figure 3d, indicating an increase as the fractional order δ is raised. Similarly, Figure 3e demonstrates a decrease in the population of jailed individuals, with a decay in the fractional order δ. Moreover, similar trends are observed in the simulation results of other fractional operators. Figures 4 and 5 represent the simulation results of model (6.1), as obtained by using the Caputo-Fabrizio and Caputo operations, respectively. Similarly, Figures 6–8 display the simulation results for model (6.1) using the fractal and fractional operators.

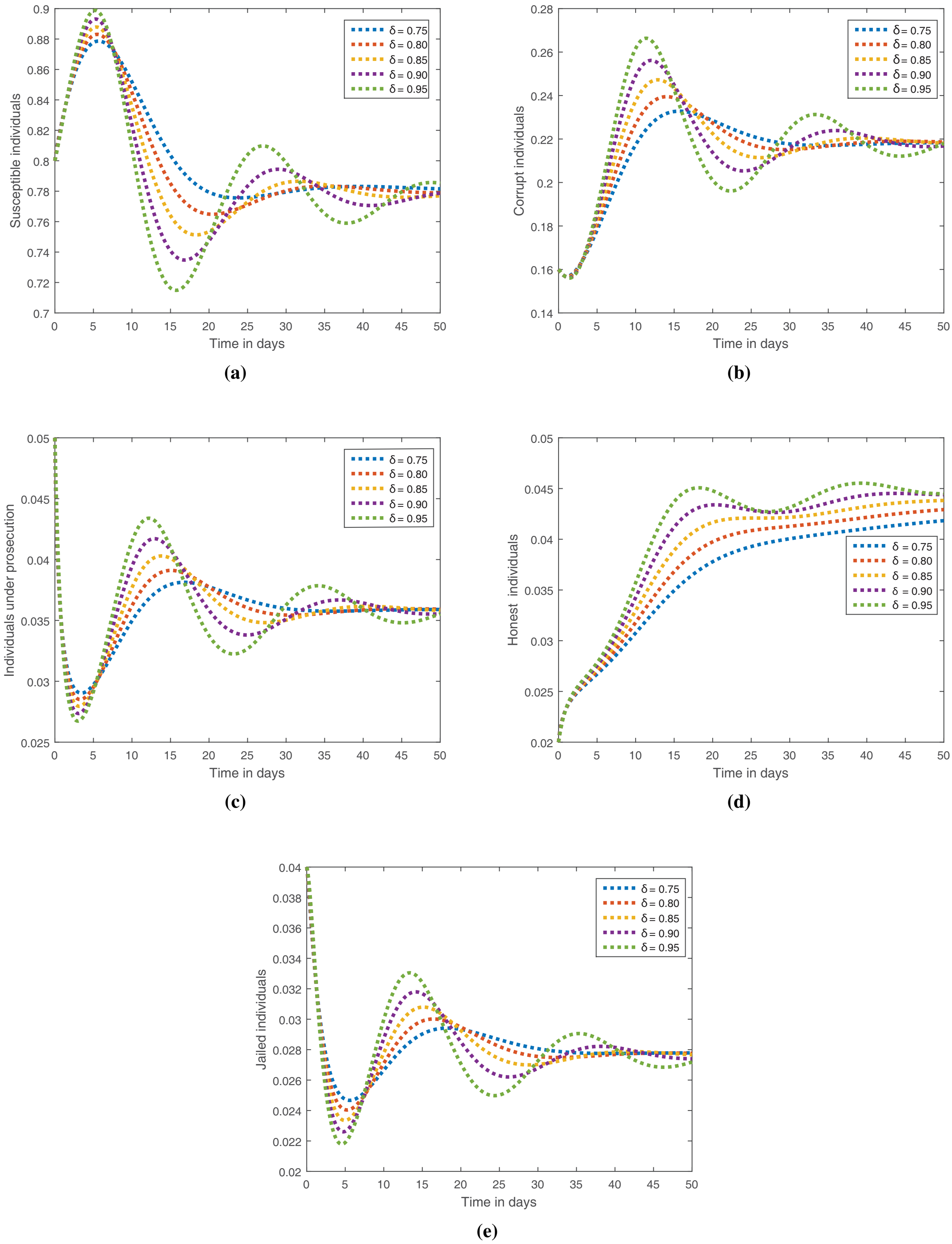


Figure 3. Graphs illustrating the behavior of each component for the ABC version of model (6.1) at different values of δ.

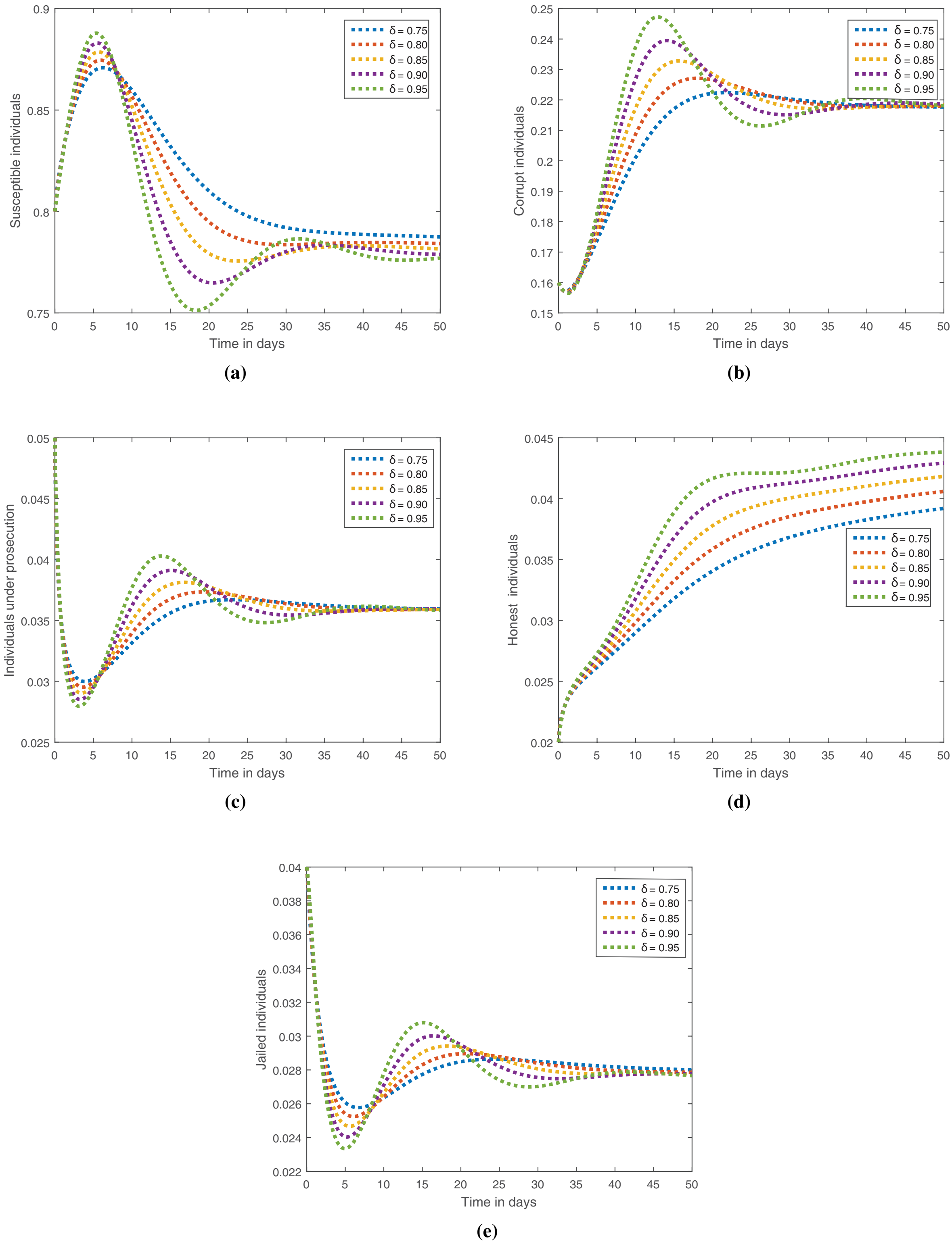


Figure 4. Graphs illustrating the behavior of each component for the Caputo-Fabrizio version of model (6.1) at different values of δ.

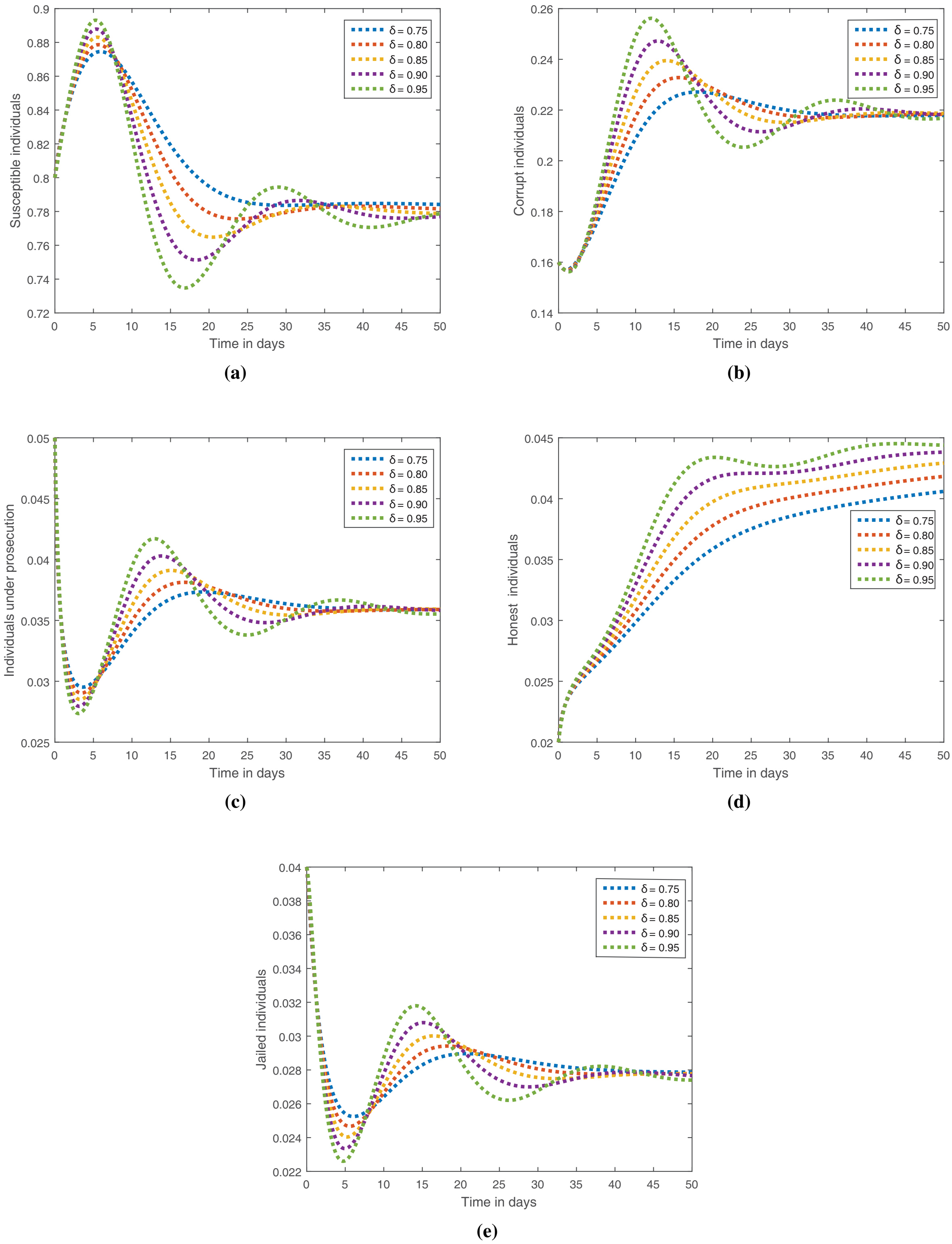


Figure 5. Graphs illustrating the behavior of each component for the Caputo version of model (6.1) at different values of δ.

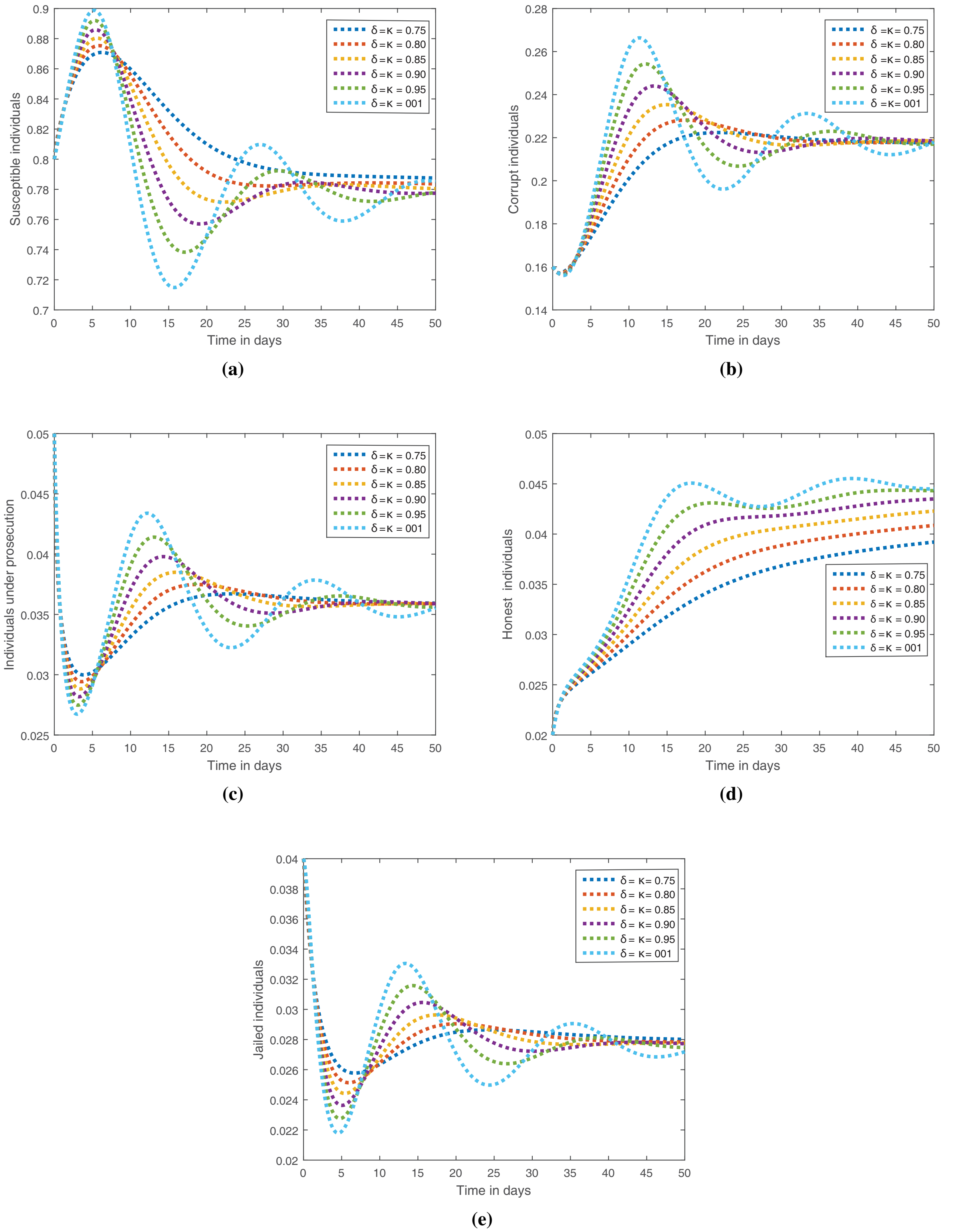


Figure 6. Graphs illustrating the behavior of each component for the fractal and fractional version of model (6.1) with the Mittag-Leffler kernel and different fractional orders δ and fractal dimensions κ.

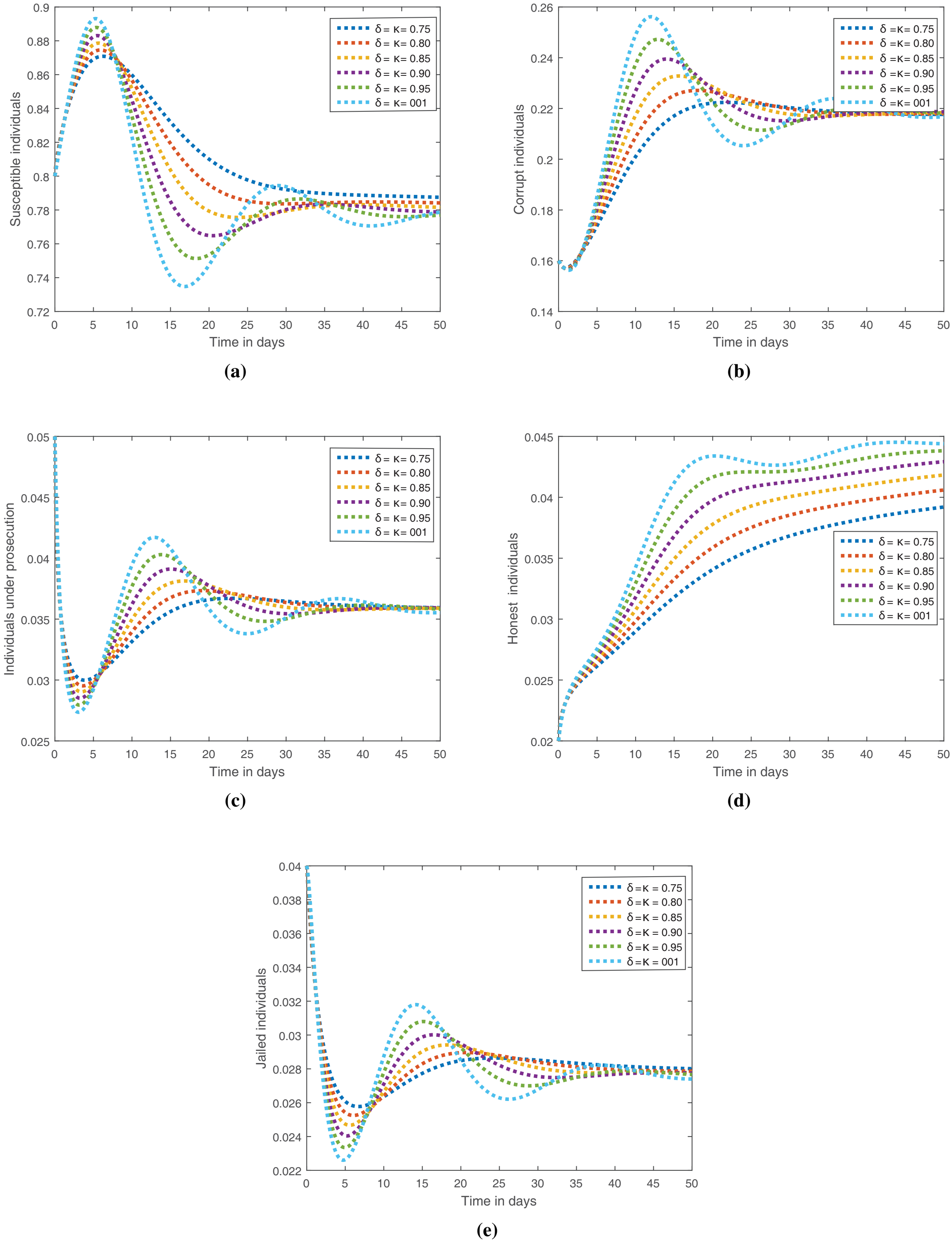


Figure 7. Graphs illustrating the behavior of each component for the fractal and fractional version of model (6.1) with the power-law kernel and different fractional orders δ and fractal dimensions κ.

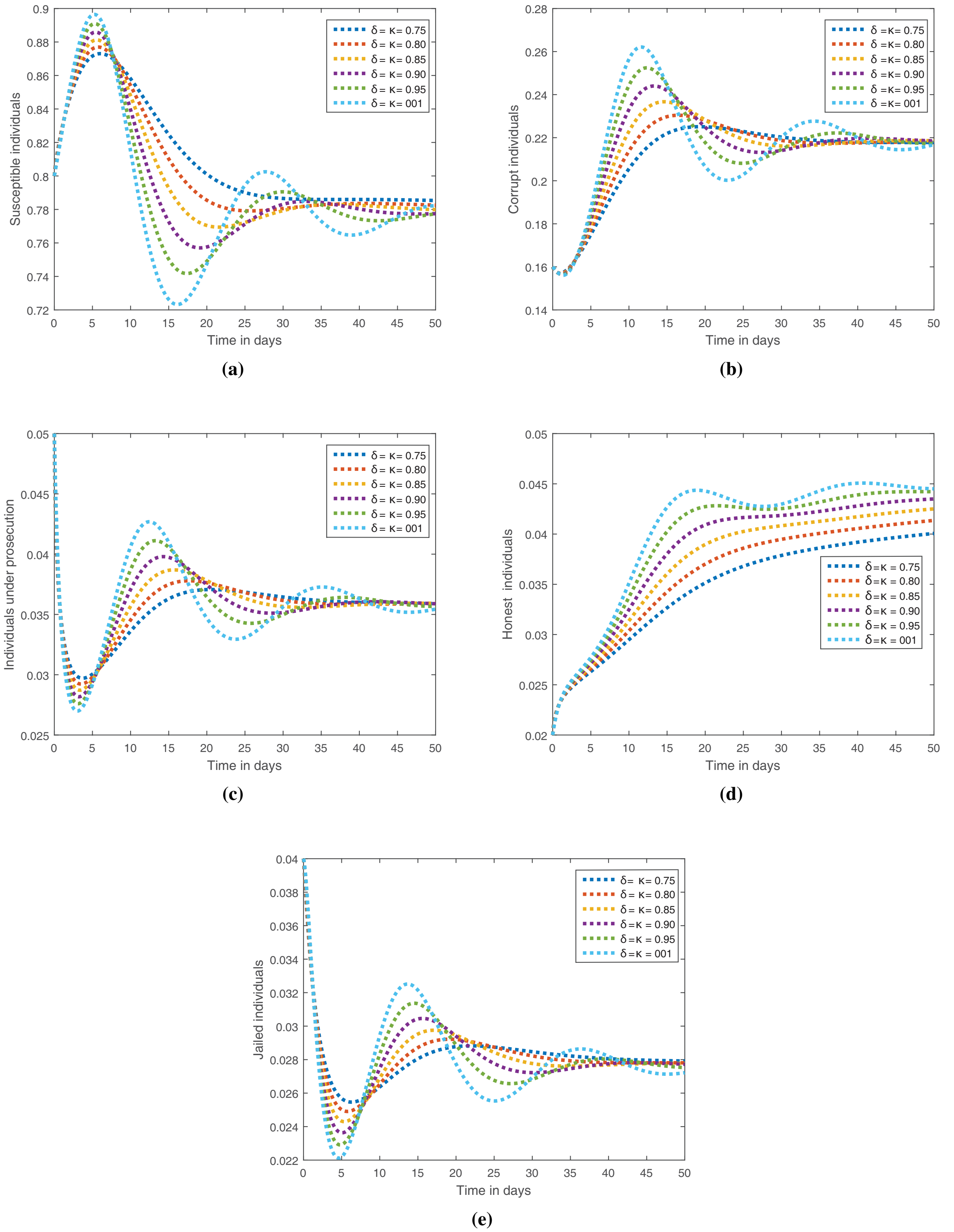


Figure 8. Graphs illustrating the behavior of each component for the fractal and fractional version of model (6.1) with the exponential decay kernel and different fractional orders δ and fractal dimensions κ.

9. Conclusions

In this study, we have redefined the financial crime model by incorporating a fractional derivative. The key focus was on determining the basic reproduction number (R_0), establishing a feasible region and identifying a CFE point ($\check{D}^0$). It has been mathematically proven that $\check{D}^0$ exhibits local asymptotic stability when R_0 is less than one. Subsequently, the model was fractionalized by using the Atangana-Baleanu fractional derivative in the Caputo sense. The existence and uniqueness of the solution, along with the Ulam-Hyers stability, were rigorously demonstrated. To solve the model numerically, a novel numerical scheme proposed by Atangana and Seda was employed. This scheme, based on the Newton polynomial, is known for its enhanced accuracy compared to the Lagrange polynomial utilized in the Adams-Bashforth method. Additionally, we explored the numerical solutions of the same model by using other fractional derivatives, including the Caputo-Fabrizio, Caputo and fractal-fractional derivatives, as well as incorporating power law, exponential decay and Mittag-Leffler kernels. Future research could focus on incorporating real-world data on financial crime incidents to enhance the accuracy and applicability of the computational models. By using empirical data, the models can be calibrated and validated, leading to more reliable predictions and insights into the dynamics of financial crime populations. Second, we compared different fractional operators. The study currently focuses on the use of fractional operators such as Newton's polynomial, Caputo-Fabrizio, Caputo and Atangana-Baleanu operators. Future research could explore the comparative performance and suitability of these operators in modeling financial crime dynamics. This comparative analysis can provide insights into the strengths and limitations of different fractional operators and guide researchers in selecting the most appropriate operator for specific modeling scenarios.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University, Abha, Saudi Arabia, for funding this work through the Research Group Project under grant number (RGP.2/27/44). The first author appreciates the support provided by Petchra Pra Jom Klao Ph.D. Research Scholarship through grant No. (50/2565), by King Mongkut's University of Technology Thonburi, Thailand.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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