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Analytical solutions for the model of moderately thick plates by symplectic elasticity approach

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Abstract: A general moderately thick rectangular plate model is proposed and its analytical solutions are obtained by using the symplectic elasticity approach (SEA). First, the equilibrium equations of the model are transformed into a Hamiltonian dual equation and the eigenvalues and eigenvectors of the corresponding Hamiltonian operators are calculated. Furthermore, the symplectic orthogonality and the completeness of eigenvectors are proved, and the analytical solutions of the problem are presented based on boundary conditions. The feasibility of the proposed framework and the effectiveness of the SEA are verified by numerical examples of the bending problems of moderately thick rectangular plates on the different elastic foundations and the free vibration problem of moderately thick rectangular plates.

Keywords: moderately thick plate; symplectic elasticity approach; Hamiltonian operator; completeness; analytical solution
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1. Introduction

The moderately thick plate is an important material that can adapt to high pressure, high temperature and strong radiation environments, often used in construction engineering, machinery manufacturing, container manufacturing and so on [1, 2]. In recent years, many achievements have been made in numerical methods of moderately thick plate problems. There are some typical numerical methods, such as the meshless local radial point interpolation method [3], discrete singular convolution methods [4], spectral element method [5], meshless local Petrov-Galerkin method [6], differential quadrature element method [7], finite integral transform method [8] and chaotic radial basis function method [9]. Compared with numerical methods, the analytical method seems more accurate, but it is difficult to construct. Through the efforts of many scholars, the analytical method has made great progress. The common analytical methods are integral transformation [10, 11],

semi-inverse method [12], infinite series method [13] and Green's functions [14]. Traditionally, some analytical solutions always need to eliminate the unknown functions by various methods in the range of a class of variables so as to obtain a partial differential equation with higher order, which will make separating variables and other mathematical methods impossible. In order to avoid this phenomenon, academician Zhong put forward the SEA in the 1990s. The essence of SEA is the method of separation of variables based on Hamiltonian system, which does not need to assume the form of the solution in advance. The main feature of the approach is that the solution procedure is conducted in the symplectic space with dual variables rather than in the Euclidean space with one kind of variables, which is rational to solve the equations of elasticity [15, 16]. Within the framework of the Hamiltonian system, the analytical solutions of the considered problem can be obtained by using the complete expansion of symplectic eigenvectors without any prior assumptions of the solution forms, which shows the distinctive advantage of the SEA.

SEA needs to transform the equations of elasticity into a proper Hamiltonian system and can uniformly solve the state vector, including as many unknown functions, such as the physical displacements and forces, as possible. Nowadays, the SEA has been extensively studied by some scholars. Xu et al. [17] presented the fracture analysis of fractional two-dimensional viscoelastic media by SEA. Zhao et al. [18] utilized SEA on the wave propagation problem for periodic structures with uncertainty. Qiao et al. [19] considered the plane elasticity problems of two-dimensional octagonal quasicrystals by SEA. Zhou et al. [20] studied new buckling solutions of moderately thick rectangular plates by the symplectic superposition method within the Hamiltonian system framework. Li et al. [21] explored new analytic free vibration solutions of doubly curved shallow shells. Su et al. [22] obtained an analytical free vibration solution of fully free orthotropic rectangular thin plates on two-parameter elastic foundations by the symplectic superposition method. In the last thirty years, the application of SEA has been extended continuously in the electromagnetic elastic solid problem [23], non-Lévy-type functionally graded rectangular plates [24], special mechanical problems in mine engineering [25], buckling of regular and auxetic honeycombs [26], free vibration of non-Lévy-type porous FGM rectangular plates [27], the free vibration of edge-cracked thick rectangular plates [28], the buckling problem of CNT reinforced composite rectangular plates [29] and so on.

In this paper, based on the idea of [30], the mechanical problems of rectangular moderately thick plates are analyzed and a unified framework for a class of rectangular moderately thick plates under the simply supported boundary conditions at opposite sides is given. The model considered in this paper contains free parameters, including the bending, buckling and free vibration problems of the plates. We start from the basic equations of the rectangular moderately thick plates, guide the equations to the Hamiltonian system by introducing appropriate state vector and solve the eigenvalue problem of the Hamiltonian operators. More importantly, we prove the completeness of the eigenfunction system, which lays a theoretical foundation for our general solution. For the applications of the model, based on the obtained general solutions, rectangular moderately thick plates under Winkler type foundation and Pasternak type two-parameter foundation are numerically simulated, which have not been solved by SEA so far. The free vibration problem of moderately thick plates with fully simply supported is also discussed. The satisfactory agreement further confirms the stability of the present method.

2. The model of the moderately thick plates and SEA

2.1. Basic equations and Hamiltonian system

For the problems of rectangular moderately thick plates, we propose the following general model:

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = q_1(x, y), \tag{2.1a}$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = q_2(x, y), \tag{2.1b}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \alpha_1 \frac{\partial^2 w}{\partial x^2} + \beta_1 \frac{\partial^2 w}{\partial y^2} + \gamma w = q_3(x, y), \tag{2.1c}$$

where q_1, q_2, q_3 are loads, α_1, β_1, γ are free parameters, M_x, M_y, M_{xy}, Q_x, Q_y, w are the bending moments, torsional moment, shear forces and the transverse modal displacement, respectively.

The internal forces of the plates can be presented as

$$M_x = -D(\frac{\partial \psi_x}{\partial x} + v\frac{\partial \psi_y}{\partial y}), \tag{2.2a}$$

$$M_y = -D(\frac{\partial \psi_y}{\partial y} + v\frac{\partial \psi_x}{\partial x}), \tag{2.2b}$$

$$M_{xy} = -\frac{D(1-v)}{2}(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}), \tag{2.2c}$$

$$Q_x = C(\frac{\partial w}{\partial x} - \psi_x), \tag{2.2d}$$

$$Q_y = C(\frac{\partial w}{\partial y} - \psi_y), \tag{2.2e}$$

where $C = \frac{5Eh}{12(1+v)}$, $D = \frac{Eh^3}{12(1-v^2)}$ are the elastic constants, in which E denotes the modulus of elasticity, h represents the thickness of the plate and v is Poisson's ratio. From (2.1a)–(2.2e), the model can be transformed into the following form:

$$(-D\frac{\partial^2}{\partial x^2} - \frac{D(1-v)}{2}\frac{\partial^2}{\partial y^2} + C)\psi_x - \frac{D(1+v)}{2}\frac{\partial^2}{\partial x \partial y}\psi_y - C\frac{\partial}{\partial x}w = q_1(x, y), \tag{2.3a}$$

$$(-D\frac{\partial^2}{\partial y^2} - \frac{D(1-v)}{2}\frac{\partial^2}{\partial x^2} + C)\psi_y - \frac{D(1+v)}{2}\frac{\partial^2}{\partial x \partial y}\psi_x - C\frac{\partial}{\partial y}w = q_2(x, y), \tag{2.3b}$$

$$-C\frac{\partial}{\partial x}\psi_x - C\frac{\partial}{\partial y}\psi_y + (\alpha\frac{\partial^2}{\partial x^2} + \beta\frac{\partial^2}{\partial y^2} + \gamma)w = q_3(x, y), \tag{2.3c}$$

where $\alpha = \alpha_1 + C, \beta = \beta_1 + C$.

Setting $V_x = \alpha\frac{\partial w}{\partial x} - C\psi_x$ and combing (2.2a)–(2.3c), we can obtain the following matrix equation:

$$\frac{\partial \mathbf{U}(x, y)}{\partial x} = \mathbf{H}U(x, y) + \mathbf{f}(x, y). \tag{2.4}$$

It is easy to verify that $\mathbf{H} = \begin{pmatrix} \mathbf{0} & \mathbf{T} \\ \mathbf{S} & \mathbf{0} \end{pmatrix}$ satisfies $\mathbf{H}^* = \mathbf{JHJ}$ in which $\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{0} \end{pmatrix}$, $\mathbf{I}_3$ is a 3×3 unit matrix. So $\mathbf{H}$ is a Hamiltonian operator matrix, in which

$$
\mathbf{T} = \begin{pmatrix} -\frac{1}{D} & -\nu\frac{\partial}{\partial y} & 0 \\ \nu\frac{\partial}{\partial y} & D\left(\nu^2 - 1\right)\frac{\partial^2}{\partial y^2} + C & -C\frac{\partial}{\partial y} \\ 0 & C\frac{\partial}{\partial y} & -\beta\frac{\partial^2}{\partial y^2} - \gamma \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} \frac{C^2}{\alpha} - C & \frac{\partial}{\partial y} & \frac{C}{\alpha} \\ -\frac{\partial}{\partial y} & -\frac{2}{D(\nu-1)} & 0 \\ \frac{C}{\alpha} & 0 & \frac{1}{\alpha} \end{pmatrix},
$$

$\mathbf{f}(x, y) = (0, -q_2(x, y), q_3(x, y), q_1(x, y), 0, 0)^T$, $\mathbf{U}(x, y) = (\psi_x, -M_{xy}, V_x, M_x, \psi_y, w)^T$.

Next, we continue to use the SEA to solve the Hamiltonian system (2.4).

2.2. Eigenvalues and eigenvectors of H

To solve the Hamiltonian system, the homogeneous system is first considered:

$$
\frac{\partial \mathbf{U}(x, y)}{\partial x} = \mathbf{H}\mathbf{U}(x, y). \tag{2.5}
$$

Using the separation of variables method, assuming $U(x, y) = X(x)Y(y)$, we have

$$
\frac{\mathrm{d}X(x)}{\mathrm{d}x} = \mu X(x), \tag{2.6}
$$

$$
\mathbf{H}Y(y) = \mu \mathbf{Y}(y), \tag{2.7}
$$

with eigenvalue μ and eigenvector $\mathbf{Y}(y)$.

The considered model is solved in the rectangular region $\{(x, y) \mid 0 \leq x \leq a, 0 \leq y \leq b\}$ and the boundary conditions are simply supported at $y = 0$ and $y = b$ as

$$
w = \psi_x = M_y = 0. \tag{2.8}
$$

The eigenvalue problems (2.6) and (2.7) are actually quite complicated to compute. For the sake of brevity below, let $\zeta_n = \frac{n\pi}{b}$ and

$$
P_n = \sqrt{\frac{C^4 - 2\alpha C^3 + C^2\left(\alpha^2 + 2D\zeta_n^2(\alpha - \beta) + 2\gamma D\right) + 2\alpha CD\left(\zeta_n^2(\alpha - \beta) + \gamma\right) + D^2\left(\zeta_n^2(\alpha - \beta) + \gamma\right)^2}{D^2}},
$$

$n \in \mathbb{N}$, where $\mathbb{N}$ stands for the set of positive integers.

With the help of symbolic software, the corresponding results may be divided into the following three cases.

Case 1. $\mu_1 = \frac{\sqrt{2}\sqrt{-C}}{\sqrt{D(\nu-1)}}, \mu_2 = -\mu_1$ are eigenvalues of $\mathbf{H}$ and the corresponding eigenvectors are

$$
\mathbf{Y}_1 = (0, C, 0, 0, \mu_1, 0)^T, \mathbf{Y}_2 = (0, C, 0, 0, \mu_2, 0)^T,
$$

where the superscript T signifies the transpose of a vector.

Case 2. When $P_n \neq 0$, $\mathbf{H}$ has six simple eigenvalues, which can be expressed as

$$
\mu_{n1} = \frac{\sqrt{D(\nu-1)\zeta_n^2 - 2C}}{\sqrt{D(\nu-1)}}, \mu_{n2} = -\mu_{n1};
$$

$$
\mu_{n3} = \sqrt{-\frac{C^2 + D\left(\zeta_n^2(-(\alpha + \beta)) + P_n + \gamma\right) - \alpha C}{2\alpha D}}, \mu_{n4} = -\mu_{n3}
$$

and

$$\mu_{n5} = \sqrt{\frac{-C^2 + D\left(\zeta_n^2(\alpha + \beta) + P_n - \gamma\right) + \alpha C}{2\alpha D}}, \ \mu_{n6} = -\mu_{n5} \ (n \in \mathbb{N}).$$

The corresponding eigenvectors with respect to μ_{ni} can be rewritten as

$$\mathbf{Y}_{n,i} = \begin{pmatrix} \sin(y\zeta_n) \\ -\frac{D(\nu-1)\cos(y\zeta_n)\left(C^2\left(\zeta_n^2+\mu_{ni}^2\right)-C\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)+D\left(\nu\zeta_n^2+\mu_{ni}^2\right)\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)\right)}{\zeta_n\left(2C^2+D(\nu+1)\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)\right)} \\ \frac{C\sin(y\zeta_n)\left(-2C^2+2\alpha C-\gamma D(\nu+1)+D\zeta_n^2(\alpha(-\nu)+\alpha+\beta\nu+\beta)-2\alpha D\mu_{ni}^2\right)}{2C^2+D(\nu+1)\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)} \\ \frac{D\sin(y\zeta_n)\left(2C^2(\nu-1)\mu_{ni}^2-2C\nu\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)+D(\nu-1)\left(\nu\zeta_n^2+\mu_{ni}^2\right)\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)\right)}{\mu_{ni}\left(2C^2+D(\nu+1)\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)\right)} \\ \frac{\cos(y\zeta_n)\left(2C^2\mu_{ni}^2-2C\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)+D\left((\nu-1)\zeta_n^2+2\mu_{ni}^2\right)\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)\right)}{\zeta_n\mu_{ni}\left(2C^2+D(\nu+1)\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)\right)} \\ \frac{C\sin(y\zeta_n)\left(2C+D(\nu-1)\left(\mu_{ni}^2-\zeta_n^2\right)\right)}{\mu_{ni}\left(2C^2+D(\nu+1)\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)\right)} \end{pmatrix},$$

$(i = 1, 2, \cdots, 6; n \in \mathbb{N})$.

Case 3. When $P_n = 0$, $\mathbf{H}$ contains two single eigenvalues:

$$\mu_{n1} = \frac{\sqrt{D(\nu-1)\zeta_n^2 - 2C}}{\sqrt{D(\nu-1)}}, \mu_{n2} = -\mu_{n1},$$

and two double eigenvalues:

$$\mu_{n3} = \sqrt{\frac{C(\alpha - C) + D\zeta_n^2(\alpha + \beta) - \gamma D}{2\alpha D}}, \mu_{n4} = -\mu_{n3} \ (n \in \mathbb{N}).$$

In this case, the basic eigenvectors corresponding to μ_{n1} $(i = 1, 2, 3, 4; n \in \mathbb{N})$ are

$$\mathbf{Y}_{n,i}^0 = \begin{pmatrix} \sin(y\zeta_n) \\ -\frac{D(\nu-1)\cos(y\zeta_n)\left(C^2\left(\zeta_n^2+\mu_{ni}^2\right)-C\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)+D\left(\nu\zeta_n^2+\mu_{ni}^2\right)\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)\right)}{\zeta_n\left(2C^2+D(\nu+1)\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)\right)} \\ \frac{C\sin(y\zeta_n)\left(-2C^2+2\alpha C-\gamma D(\nu+1)+D\zeta_n^2(\alpha(-\nu)+\alpha+\beta\nu+\beta)-2\alpha D\mu_{ni}^2\right)}{2C^2+D(\nu+1)\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)} \\ \frac{D\sin(y\zeta_n)\left(2C^2(\nu-1)\mu_{ni}^2-2C\nu\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)+D(\nu-1)\left(\nu\zeta_n^2+\mu_{ni}^2\right)\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)\right)}{\mu_{ni}\left(2C^2+D(\nu+1)\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)\right)} \\ \frac{\cos(y\zeta_n)\left(2C^2\mu_{ni}^2-2C\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)+D\left((\nu-1)\zeta_n^2+2\mu_{ni}^2\right)\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)\right)}{\zeta_n\mu_{ni}\left(2C^2+D(\nu+1)\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)\right)} \\ \frac{C\sin(y\zeta_n)\left(2C+D(\nu-1)\left(\mu_{ni}^2-\zeta_n^2\right)\right)}{\mu_{ni}\left(2C^2+D(\nu+1)\left(\gamma-\beta\zeta_n^2+\alpha\mu_{ni}^2\right)\right)} \end{pmatrix}.$$

As for the situation of double eigenvalues, there exist first-Jordan form eigenvectors. Solving γ from $P_n = 0$ yields that

$$\gamma = \gamma_{n1} = \frac{-2\sqrt{\alpha}C^{\frac{3}{2}} - C^2 - \alpha C - \alpha D\zeta_n^2 + \beta D\zeta_n^2}{D}$$

and

$$\gamma = \gamma_{n2} = \frac{2\sqrt{\alpha}C^{\frac{3}{2}} - C^2 - \alpha C - \alpha D\zeta_n^2 + \beta D\zeta_n^2}{D}$$

for further simplification. According to $\mathbf{H}Y_{n,i}^1(y) = \mu_n \mathbf{Y}_{n,i}^1(y) + \mathbf{Y}_{n,i}^0(y)$, $(i = 3, 4, n \in \mathbb{N})$. The first-order Jordan eigenvectors are simplified as follows.

Case 3.1. If $\gamma = \gamma_{n1} = \frac{-2\sqrt{\alpha}C^{\frac{3}{2}} - C^2 - \alpha C - \alpha D\zeta_n^2 + \beta D\zeta_n^2}{D}$, then

$$\mathbf{Y}_{n,3}^1 = \begin{pmatrix} \dfrac{\sin(y\zeta_n)\left(-C\left(2\sqrt{\alpha}+\sqrt{C}\right)-2\sqrt{\alpha}D\zeta_n^2\right)}{C^{\frac{3}{2}}\sqrt{\frac{\frac{C^{\frac{3}{2}}}{\sqrt{\alpha}}+C+D\zeta_n^2}{D}}} \\ \dfrac{D(\nu-1)\zeta_n\cos(y\zeta_n)\left(C^{\frac{3}{2}}+2\sqrt{\alpha}C+2\sqrt{\alpha}D\zeta_n^2\right)}{C^{\frac{3}{2}}\sqrt{\frac{\frac{C^{\frac{3}{2}}}{\sqrt{\alpha}}+C+D\zeta_n^2}{D}}} \\ \dfrac{\sin(y\zeta_n)\left(C^{\frac{3}{2}}+\sqrt{\alpha}C+2\sqrt{\alpha}D\zeta_n^2\right)}{\sqrt{C}\sqrt{\frac{\frac{C^{\frac{3}{2}}}{\sqrt{\alpha}}+C+D\zeta_n^2}{D}}} \\ \dfrac{2\sqrt{\alpha}D\sin(y\zeta_n)\left(C+(D-D\nu)\zeta_n^2\right)}{C^{\frac{3}{2}}} \\ -\dfrac{2\sqrt{\alpha}D\zeta_n\cos(y\zeta_n)}{C^{\frac{3}{2}}} \\ 0 \end{pmatrix}, \mathbf{Y}_{n,4}^1 = \begin{pmatrix} \dfrac{\sin(y\zeta_n)\left(C^{\frac{3}{2}}+2\sqrt{\alpha}C+2\sqrt{\alpha}D\zeta_n^2\right)}{C^{\frac{3}{2}}\sqrt{\frac{\frac{C^{\frac{3}{2}}}{\sqrt{\alpha}}+C+D\zeta_n^2}{D}}} \\ -\dfrac{D(\nu-1)\zeta_n\cos(y\zeta_n)\left(C^{\frac{3}{2}}+2\sqrt{\alpha}C+2\sqrt{\alpha}D\zeta_n^2\right)}{C^{\frac{3}{2}}\sqrt{\frac{\frac{C^{\frac{3}{2}}}{\sqrt{\alpha}}+C+D\zeta_n^2}{D}}} \\ \dfrac{\sin(y\zeta_n)\left(-C\left(\sqrt{\alpha}+\sqrt{C}\right)-2\sqrt{\alpha}D\zeta_n^2\right)}{\sqrt{C}\sqrt{\frac{\frac{C^{\frac{3}{2}}}{\sqrt{\alpha}}+C+D\zeta_n^2}{D}}} \\ \dfrac{2\sqrt{\alpha}D\sin(y\zeta_n)\left(C+(D-D\nu)\zeta_n^2\right)}{C^{\frac{3}{2}}} \\ -\dfrac{2\sqrt{\alpha}D\zeta_n\cos(y\zeta_n)}{C^{\frac{3}{2}}} \\ 0 \end{pmatrix}.$$

Case 3.2. If $\gamma = \gamma_{n2} = \frac{2\sqrt{\alpha}C^{\frac{3}{2}} - C^2 - \alpha C - \alpha D\zeta_n^2 + \beta D\zeta_n^2}{D}$, we have

$$\mathbf{Y}_{n,3}^1 = \begin{pmatrix} \dfrac{\sin(y\zeta_n)\left(-C\left(-2\sqrt{\alpha}+\sqrt{C}\right)+2\sqrt{\alpha}D\zeta_n^2\right)}{C^{\frac{3}{2}}\sqrt{\frac{-\frac{C^{\frac{3}{2}}}{\sqrt{\alpha}}+C+D\zeta_n^2}{D}}} \\ \dfrac{D(1-\nu)\zeta_n\cos(y\zeta_n)\left(-C^{\frac{3}{2}}+2\sqrt{\alpha}C+2\sqrt{\alpha}D\zeta_n^2\right)}{C^{\frac{3}{2}}\sqrt{\frac{-\frac{C^{\frac{3}{2}}}{\sqrt{\alpha}}+C+D\zeta_n^2}{D}}} \\ \dfrac{\sin(y\zeta_n)\left(C^{\frac{3}{2}}-\sqrt{\alpha}C-2\sqrt{\alpha}D\zeta_n^2\right)}{\sqrt{C}\sqrt{\frac{-\frac{C^{\frac{3}{2}}}{\sqrt{\alpha}}+C+D\zeta_n^2}{D}}} \\ -\dfrac{2\sqrt{\alpha}D\sin(y\zeta_n)\left(C+(D-D\nu)\zeta_n^2\right)}{C^{\frac{3}{2}}} \\ \dfrac{2\sqrt{\alpha}D\zeta_n\cos(y\zeta_n)}{C^{\frac{3}{2}}} \\ 0 \end{pmatrix}, \mathbf{Y}_{n,4}^1 = \begin{pmatrix} \dfrac{\sin(y\zeta_n)\left(C^{\frac{3}{2}}-2\sqrt{\alpha}C-2\sqrt{\alpha}D\zeta_n^2\right)}{C^{\frac{3}{2}}\sqrt{\frac{-\frac{C^{\frac{3}{2}}}{\sqrt{\alpha}}+C+D\zeta_n^2}{d}}} \\ \dfrac{D(\nu-1)\zeta_n\cos(y\zeta_n)\left(-C^{\frac{3}{2}}+2\sqrt{\alpha}C+2\sqrt{\alpha}D\zeta_n^2\right)}{C^{\frac{3}{2}}\sqrt{\frac{-\frac{C^{\frac{3}{2}}}{\sqrt{\alpha}}+C+D\zeta_n^2}{D}}} \\ \dfrac{\sin(y\zeta_n)\left(-C\left(-\sqrt{\alpha}+\sqrt{C}\right)+2\sqrt{\alpha}D\zeta_n^2\right)}{\sqrt{C}\sqrt{\frac{-\frac{C^{\frac{3}{2}}}{\sqrt{\alpha}}+C+D\zeta_n^2}{D}}} \\ -\dfrac{2\sqrt{\alpha}D\sin(y\zeta_n)\left(C+(D-D\nu)\zeta_n^2\right)}{C^{\frac{3}{2}}} \\ \dfrac{2\sqrt{\alpha}D\zeta_n\cos(y\zeta_n)}{C^{\frac{3}{2}}} \\ 0 \end{pmatrix}.$$

2.3. Properties of eigenvectors

After calculating the eigenvalues and eigenvectors, we continue to study the properties of the eigenvectors. And for the analytical solution of this kind of moderately thick rectangular plates, the choice of space is very important, and here we choose the Hilbert space $\mathcal{Y} := L^2(0, b) \oplus L^2(0, b) \oplus L^2(0, b) \oplus L^2(0, b) \oplus L^2(0, b) \oplus L^2(0, b)$, where $L^2(0, b)$ is the space of all square integrable functions. For $L(y), R(y) \in \mathcal{Y}$, the symplectic inner product is defined as

$$\langle L(y), R(y) \rangle = \int_0^b L(y)^T \mathbf{J} R(y)\, \mathrm{d}y,$$

where $\mathbf{J}$ is the 6×6 symplectic unit matrix.

2.3.1. When the eigenvalues are simple

Symplectic orthogonality of eigenvectors lays a foundation for the expansion of eigenvectors. The following lemma can be obtained via straightforward calculation.

Lemma 1. *The system of eigenvectors* $\{Y_1(y), Y_2(y)\} \cup \{Y_{n,i}(y) \mid n \in \mathbb{N}, i = 1, 2, \cdots, 6\}$ *of H satisfies the following symplectic orthogonality relations:*

$$\langle Y_1(y), Y_2(y) \rangle = \begin{cases} \dfrac{2\sqrt{2}b(-C)^{\frac{3}{2}}}{\sqrt{D(v-1)}}, & n = m, \\ 0, & n \neq m; \end{cases}$$

$$\langle Y_{n,3}(y), Y_{m,4}(y) \rangle = \begin{cases} A_{34,n}, & n = m, \\ 0, & n \neq m; \end{cases}$$

$$\langle Y_{n,5}(y), Y_{m,6}(y) \rangle = \begin{cases} A_{56,n}, & n = m, \\ 0, & n \neq m; \end{cases}$$

$$\langle Y_{n,1}(y), Y_{m,2}(y) \rangle = \begin{cases} \dfrac{-bC\sqrt{D(v-1)\zeta_n^2 - 2C}}{\sqrt{D(v-1)}\zeta_n^2}, & n = m, \\ 0, & n \neq m, \end{cases}$$

where the constants $A_{34,n}$ and $A_{56,n}$ are presented in Appendix A. Other eigenvectors satisfy the following symplectic orthogonal relation:

$$\langle Y_{n,1}(y), Y_{m,3}(y) \rangle = \langle Y_{n,1}(y), Y_{m,4}(y) \rangle = \langle Y_{n,1}(y), Y_{m,5}(y) \rangle = \langle Y_{n,1}(y), Y_{m,6}(y) \rangle$$
$$= \langle Y_{n,2}(y), Y_{m,3}(y) \rangle = \langle Y_{n,2}(y), Y_{m,4}(y) \rangle = \langle Y_{n,2}(y), Y_{m,5}(y) \rangle = \langle Y_{n,2}(y), Y_{m,6}(y) \rangle$$
$$= \langle Y_{n,3}(y), Y_{m,5}(y) \rangle = \langle Y_{n,3}(y), Y_{m,6}(y) \rangle = \langle Y_{n,4}(y), Y_{m,5}(y) \rangle = \langle Y_{n,4}(y), Y_{m,6}(y) \rangle$$
$$= \langle Y_{n,i}(y), Y_{m,i}(y) \rangle = 0, n, m \in \mathbb{N}.$$

Based on the conjugate symplectic orthogonality of eigenvectors, we can obtain the following completeness theorem of eigenvectors, which provides the theoretical basis for the expression of general solutions.

Theorem 1. *The system of eigenvectors* $\{Y_1(y), Y_2(y)\} \cup \{Y_{n,i}(y)\}_{n=1}^{+\infty}$ $(i = 1, 2, \cdots, 6)$ *of H is complete in $\mathcal{Y}$.*

Proof. For any $\mathbf{F}(y) = (f_1(y), f_2(y), f_3(y), f_4(y), f_5(y), f_6(y))^T \in \mathcal{Y}$, it is sufficient to prove that there exist constant sequences $\{C_{n,i}\}_{n=1}^{+\infty}$, $(i = 1, 2, \cdots, 6)$, C_1 and C_2 such that

$$\mathbf{F}(y) = C_1 \mathbf{Y}_1 + C_2 \mathbf{Y}_2 + \sum_{n=1}^{+\infty} \left(\sum_{i=1}^{6} C_{n,i} \mathbf{Y}_{n,i}(y) \right). \tag{2.9}$$

Applying Lemma 1, we can take

$$C_{n,1} = \frac{\langle \mathbf{Y}_{n,2}(y), \mathbf{F}(y) \rangle}{\langle \mathbf{Y}_{n,2}(y), \mathbf{Y}_{n,1}(y) \rangle}, C_{n,2} = \frac{\langle \mathbf{Y}_{n,1}(y), \mathbf{F}(y) \rangle}{\langle \mathbf{Y}_{n,1}(y), \mathbf{Y}_{n,2}(y) \rangle}, C_{n,3} = \frac{\langle \mathbf{Y}_{n,4}(y), \mathbf{F}(y) \rangle}{\langle \mathbf{Y}_{n,4}(y), \mathbf{Y}_{n,3}(y) \rangle},$$

$$C_{n,4} = \frac{\langle \mathbf{Y}_{n,3}(y), \mathbf{F}(y) \rangle}{\langle \mathbf{Y}_{n,3}(y), \mathbf{Y}_{n,4}(y) \rangle}, C_{n,5} = \frac{\langle \mathbf{Y}_{n,6}(y), \mathbf{F}(y) \rangle}{\langle \mathbf{Y}_{n,6}(y), \mathbf{Y}_{n,5}(y) \rangle}, C_{n,6} = \frac{\langle \mathbf{Y}_{n,5}(y), \mathbf{F}(y) \rangle}{\langle \mathbf{Y}_{n,5}(y), \mathbf{Y}_{n,6}(y) \rangle}, \tag{2.10}$$

$$C_1 = \frac{\langle \mathbf{Y}_2(y), \mathbf{F}(y) \rangle}{\langle \mathbf{Y}_2(y), \mathbf{Y}_1(y) \rangle}, C_2 = \frac{\langle \mathbf{Y}_1(y), \mathbf{F}(y) \rangle}{\langle \mathbf{Y}_1(y), \mathbf{Y}_2(y) \rangle},$$

and then

$$C_1\mathbf{Y}_1 + C_2\mathbf{Y}_2 + \sum_{n=1}^{+\infty}(\sum_{i=1}^{6} C_{n,i}\mathbf{Y}_{n,i}(y))$$

$$= \begin{pmatrix} 0 \\ \frac{1}{b}\int_0^b f_2(y)\mathrm{d}y \\ 0 \\ 0 \\ \frac{1}{b}\int_0^b f_5(y)\mathrm{d}y \\ 0 \end{pmatrix} + \sum_{n=1}^{+\infty} \begin{pmatrix} \frac{2}{b}(\int_0^b f_1(y)\sin\frac{n\pi y}{b}\mathrm{d}y)\sin\frac{n\pi y}{b} \\ \frac{2}{b}(\int_0^b f_2(y)\cos\frac{n\pi y}{b}\mathrm{d}y)\cos\frac{n\pi y}{b} \\ \frac{2}{b}(\int_0^b f_3(y)\sin\frac{n\pi y}{b}\mathrm{d}y)\sin\frac{n\pi y}{b} \\ \frac{2}{b}(\int_0^b f_4(y)\sin\frac{n\pi y}{b}\mathrm{d}y)\sin\frac{n\pi y}{b} \\ \frac{2}{b}(\int_0^b f_5(y)\cos\frac{n\pi y}{b}\mathrm{d}y)\cos\frac{n\pi y}{b} \\ \frac{2}{b}(\int_0^b f_6(y)\sin\frac{n\pi y}{b}\mathrm{d}y)\sin\frac{n\pi y}{b} \end{pmatrix}.$$

Clearly, $\sum_{n=1}^{+\infty} \frac{2}{b}(\int_0^b f_i(y)\sin\frac{n\pi y}{b}\mathrm{d}y)\sin\frac{n\pi y}{b}$ $(i = 1, 3, 4, 6)$ and

$$\frac{1}{b}\int_0^b f_i(y)\mathrm{d}y + \sum_{n=1}^{+\infty} \frac{2}{b}(\int_0^b f_i(y)\cos\frac{n\pi y}{b}\mathrm{d}y)\cos\frac{n\pi y}{b} \quad (i = 2, 5)$$

are the corresponding Fourier series expansions of $f_i(y)$ based on the orthogonal systems $\{\sin\frac{n\pi y}{b}\}_{n=1}^{+\infty}$ and $\{\cos\frac{n\pi y}{b}\}_{n=0}^{+\infty}$ in $L^2(0, b)$, respectively. Thus, $\mathbf{F}(y)$ can be expressed by (2.9) and the proof is completed. $\square$

2.3.2. When the eigenvalues are double

Similar to the case where the eigenvalues are single, we first discuss the orthogonal conjugations of eigenvectors. These relations play a crucial role in proving the symplectic eigenvectors expansion theorem.

Lemma 2. *The system of eigenvectors* $\{\mathbf{Y}_1(y), \mathbf{Y}_2(y)\} \cup \{Y_{n,i}^0(y), Y_{n,j}^1(y) \mid n \in \mathbb{N}; i = 1, 2, 3, 4; j = 3, 4\}$ *of* $\mathbf{H}$ *have the following symplectic orthogonality relations:*

$$\langle Y_{n,1}^0(y), Y_{m,3}^0(y)\rangle = \langle Y_{n,1}^0(y), Y_{m,3}^1(y)\rangle = \langle Y_{n,1}^0(y), Y_{m,4}^0(y)\rangle = \langle Y_{n,1}^0(y), Y_{m,4}^1(y)\rangle$$
$$= \langle Y_{n,2}^0(y), Y_{m,3}^0(y)\rangle = \langle Y_{n,2}^0(y), Y_{m,3}^1(y)\rangle = \langle Y_{n,2}^0(y), Y_{m,4}^0(y)\rangle = \langle Y_{n,2}^0(y), Y_{m,4}^1(y)\rangle$$
$$= \langle Y_{n,3}^0(y), Y_{m,3}^1(y)\rangle = \langle Y_{n,3}^0(y), Y_{m,4}^0(y)\rangle = \langle Y_{n,4}^0(y), Y_{m,4}^1(y)\rangle = \langle Y_{n,i}^0(y), Y_{m,i}^0(y)\rangle$$
$$= 0, \quad n, m \in \mathbb{N}.$$

When $\gamma = \gamma_{n1} = \frac{-2\sqrt{\alpha}C^{\frac{3}{2}}-C^2-\alpha C-\alpha D\zeta_n^2+\beta D\zeta_n^2}{D}$, *we have*

$$\langle \mathbf{Y}_1(y), \mathbf{Y}_2(y)\rangle = \begin{cases} \dfrac{2\sqrt{2}b(-C)^{\frac{3}{2}}}{\sqrt{D}(\nu-1)}, & n = m, \\ 0, & n \neq m; \end{cases}$$

$$\langle Y_{n,3}^0(y), Y_{m,4}^1(y)\rangle = \begin{cases} \dfrac{2\sqrt{\alpha}bD}{\sqrt{C}}, & n = m, \\ 0, & n \neq m; \end{cases}$$

$$\langle Y_{n,3}^1(y), Y_{m,4}^0(y)\rangle = \begin{cases} -\frac{2\sqrt{\alpha}bD}{\sqrt{C}}, & n = m, \\ 0, & n \neq m; \end{cases}$$

$$\langle Y_{n,1}^0(y), Y_{m,2}^0(y)\rangle = \begin{cases} \frac{-bC\sqrt{D(\nu-1)\zeta_n^2 - 2C}}{\sqrt{D(\nu-1)\zeta_n^2}}, & n = m, \\ 0, & n \neq m; \end{cases}$$

$$\langle Y_{n,3}^1(y), Y_{m,4}^1(y)\rangle = \begin{cases} -\frac{2b\left(\sqrt{\alpha}C^{\frac{3}{2}}D + 2\alpha CD + 2\alpha D^2\zeta_n^2\right)}{C^2\sqrt{\frac{C^{\frac{3}{2}}}{\sqrt{\alpha}} + C + D\zeta_n^2}}, & n = m, \\ 0, & n \neq m. \end{cases}$$

If $\gamma = \gamma_{n2} = \frac{2\sqrt{\alpha}C^{\frac{3}{2}} - C^2 - \alpha C - \alpha D\zeta_n^2 + \beta D\zeta_n^2}{D}$, *then*

$$\langle Y_1(y), Y_2(y)\rangle = \begin{cases} \frac{2\sqrt{2}b(-C)^{\frac{3}{2}}}{\sqrt{D(\nu-1)}}, & n = m, \\ 0, & n \neq m; \end{cases}$$

$$\langle Y_{n,3}^0(y), Y_{m,4}^1(y)\rangle = \begin{cases} -\frac{2\sqrt{\alpha}bD}{\sqrt{C}}, & n = m, \\ 0, & n \neq m; \end{cases}$$

$$\langle Y_{n,3}^1(y), Y_{m,4}^0(y)\rangle = \begin{cases} \frac{2\sqrt{\alpha}bD}{\sqrt{C}}, & n = m, \\ 0, & n \neq m; \end{cases}$$

$$\langle Y_{n,1}^0(y), Y_{m,2}^0(y)\rangle = \begin{cases} \frac{-bC\sqrt{D(\nu-1)\zeta_n^2 - 2C}}{\sqrt{D(\nu-1)\zeta_n^2}}, & n = m, \\ 0, & n \neq m; \end{cases}$$

$$\langle Y_{n,3}^1(y), Y_{m,4}^1(y)\rangle = \begin{cases} \frac{2b\left(\sqrt{\alpha}C^{\frac{3}{2}}D - 2\alpha CD - 2\alpha D^2\zeta_n^2\right)}{C^2\sqrt{\frac{-C^{\frac{3}{2}}}{\sqrt{\alpha}} + C + D\zeta_n^2}}, & n = m, \\ 0, & n \neq m. \end{cases}$$

Based on Lemma 2, the following completeness theorem is obtained.

Theorem 2. *The system of eigenvectors* $\{Y_1(y), Y_2(y)\} \cup \{Y_{n,i}^0(y)\} \cup \{Y_{n,j}^1(y)\}$, $(i = 1, 2, 3, 4; j = 3, 4; n \in \mathbb{N})$ *of* H *is complete in* $\mathcal{Y}$.

Proof. It is similar to the proof of Theorem 1 that we just have to take the sequences:

$$C_{n,1}^0 = \frac{\langle Y_{n,2}^0(y), F(y)\rangle}{\langle Y_{n,2}^0(y), Y_{n,1}^0(y)\rangle}, C_{n,2}^0 = \frac{\langle Y_{n,1}^0(y), F(y)\rangle}{\langle Y_{n,1}^0(y), Y_{n,2}^0(y)\rangle}, C_{n,3}^1 = \frac{\langle Y_{n,4}^0(y), F(y)\rangle}{\langle Y_{n,4}^0(y), Y_{n,3}^1(y)\rangle},$$

$$C_{n,4}^1 = \frac{\langle Y_{n,3}^0(y), F(y)\rangle}{\langle Y_{n,3}^0(y), Y_{n,4}^1(y)\rangle}, C_1 = \frac{\langle Y_2(y), F(y)\rangle}{\langle Y_2(y), Y_1(y)\rangle}, C_2 = \frac{\langle Y_1(y), F(y)\rangle}{\langle Y_1(y), Y_2(y)\rangle}, \quad (2.11)$$

$$C_{n,3}^0 = \frac{\langle Y_{n,4}^1(y), F(y)\rangle - C_{n,3}^1\langle Y_{n,4}^1(y), Y_{n,3}^1(y)\rangle}{\langle Y_{n,4}^1(y), Y_{n,3}^0(y)\rangle}, C_{n,4}^0 = \frac{\langle Y_{n,3}^1(y), F(y)\rangle - C_{n,4}^1\langle Y_{n,3}^1(y), Y_{n,4}^1(y)\rangle}{\langle Y_{n,3}^1(y), Y_{n,4}^0(y)\rangle}.$$

It is easy to calculate that

$$C_1\mathbf{Y}_1 + C_2\mathbf{Y}_2 + \sum_{n=1}^{+\infty}(\sum_{i=1}^{4} C_{n,i}^0\mathbf{Y}_{n,i}^0(y) + \sum_{j=3}^{4} C_{n,j}^1\mathbf{Y}_{n,j}^1(y))$$

$$= \begin{pmatrix} 0 \\ \frac{1}{b}\int_0^b f_2(y)\mathrm{d}y \\ 0 \\ 0 \\ \frac{1}{b}\int_0^b f_5(y)\mathrm{d}y \\ 0 \end{pmatrix} + \sum_{n=1}^{+\infty} \begin{pmatrix} \frac{2}{b}(\int_0^b f_1(y)\sin\frac{n\pi y}{b}\mathrm{d}y)\sin\frac{n\pi y}{b} \\ \frac{2}{b}(\int_0^b f_2(y)\cos\frac{n\pi y}{b}\mathrm{d}y)\cos\frac{n\pi y}{b} \\ \frac{2}{b}(\int_0^b f_3(y)\sin\frac{n\pi y}{b}\mathrm{d}y)\sin\frac{n\pi y}{b} \\ \frac{2}{b}(\int_0^b f_4(y)\sin\frac{n\pi y}{b}\mathrm{d}y)\sin\frac{n\pi y}{b} \\ \frac{2}{b}(\int_0^b f_5(y)\cos\frac{n\pi y}{b}\mathrm{d}y)\cos\frac{n\pi y}{b} \\ \frac{2}{b}(\int_0^b f_6(y)\sin\frac{n\pi y}{b}\mathrm{d}y)\sin\frac{n\pi y}{b} \end{pmatrix}.$$

As a result,

$$\mathbf{F}(y) = C_1\mathbf{Y}_1 + C_2\mathbf{Y}_2 + \sum_{n=1}^{+\infty}(\sum_{i=1}^{4} C_{n,i}^0\mathbf{Y}_{n,i}^0(y) + \sum_{j=3}^{4} C_{n,j}^1\mathbf{Y}_{n,j}^1(y)). \tag{2.12}$$

□

2.4. General solution

This section will give the general solutions of the Hamiltonian system under simply supported boundary conditions at $y = 0$ and $y = b$ based on the proven completeness theorems.

2.4.1. When eigenvalues are simple

The general solution of (2.5) can be represented in the form:

$$\mathbf{U}(x,y) = X_1(x)\mathbf{Y}_1(y) + X_2(x)\mathbf{Y}_2(y) + \sum_{n=1}^{+\infty}(\sum_{i=1}^{6}(X_{n,i}(x)\mathbf{Y}_{n,i}(y))). \tag{2.13}$$

The nonhomogeneous term $\mathbf{f}(x,y)$ expands to

$$\mathbf{f}(x,y) = \theta_1(x)\mathbf{Y}_1(y) + \theta_2(x)\mathbf{Y}_2(y) + \sum_{n=1}^{+\infty}(\sum_{i=1}^{6}(\theta_{n,i}(x)\mathbf{Y}_{n,i}(y))). \tag{2.14}$$

From the above formulas and (2.6), we obtain

$$X_1(x) = \Psi_1 e^{\mu_1 x} + \int_0^x \theta_1(t)e^{\mu_1(x-t)}\mathrm{d}t,$$

$$X_2(x) = \Psi_2 e^{\mu_2 x} + \int_0^x \theta_2(t)e^{\mu_2(x-t)}\mathrm{d}t,$$

$$X_{n,i}(x) = \Psi_{ni} e^{\mu_{ni} x} + \int_0^x \theta_{n,i}(t)e^{\mu_{ni}(x-t)}\mathrm{d}t, \tag{2.15}$$

where $\Psi_1, \Psi_2, \Psi_{ni}$ are undetermined constants, $\theta_1, \theta_2, \theta_{n,i}$ can be expressed in a form similar to (2.10) via the symplectic inner product of $\mathbf{f}(x,y)$ and Y. In particular, if $q_k(x,y)$ ($k = 1, 2, 3$) is the uniformly

distributed load acting on the plate, which means that it is a constant independent of x and of y. If $q_k(x, y)$ $(k = 1, 2, 3)$ represents a concentrated load of strength B acting on the plate, then it can be expressed as $B\delta(x - x_0, y - y_0)$ in which $\delta(x - x_0, y - y_0)$ denotes the δ function.

The general solutions of ψ_y and w can be obtained by taking the fifth and sixth components of (2.13), respectively.

$$
\begin{aligned}
\psi_y = {}& \left[\left(\Psi_1 e^{\mu_1 x} + \int_0^x \int_0^y \frac{q_2(x,y)}{2bC} e^{\mu_1(x-t)} \mathrm{d}y \mathrm{d}t \right) - \left(\Psi_2 e^{\mu_2 x} + \int_0^x \int_0^y \frac{q_2(x,y)}{2bC} e^{\mu_2(x-t)} \mathrm{d}y \mathrm{d}t \right) \right] \mu_1 \\
& + \sum_{n=1}^{+\infty} \left\{ \left[\left(\int_0^x e^{\mu_{n1}(x-t)} \theta_{n,1}(t) \, \mathrm{d}t + \Psi_{n1} e^{\mu_{n1} x} \right) - \left(\int_0^x e^{-\mu_{n1}(x-t)} \theta_{n,2}(t) \, \mathrm{d}t + \Psi_{n2} e^{-\mu_{n1} x} \right) \right] \right. \\
& \frac{\cos(y\zeta_n) \left[2C^2 \mu_{n1}^2 - 2C\left(\alpha\mu_{n1}^2 + \gamma - \beta\zeta_n^2\right) + D\left(2\mu_{n1}^2 + (\nu-1)\zeta_n^2\right)\left(\alpha\mu_{n1}^2 + \gamma - \beta\zeta_n^2\right) \right]}{\mu_{n1}\zeta_n \left(2C^2 + D(\nu+1)\left(\alpha\mu_{n1}^2 + \gamma - \beta\zeta_n^2\right)\right)} \\
& + \left[\left(\int_0^x e^{\mu_{n3}(x-t)} \theta_{n,3}(t) \, \mathrm{d}t + e^{\mu_{n3} x} \Psi_{n3} \right) - \left(\int_0^x e^{-\mu_{n3}(x-t)} \theta_{n,4}(t) \, \mathrm{d}t + \Psi_{n4} e^{-\mu_{n3} x} \right) \right] \\
& \frac{\cos(y\zeta_n) \left[2C^2 \mu_{n3}^2 - 2C\left(\alpha\mu_{n3}^2 + \gamma - \beta\zeta_n^2\right) + D\left(2\mu_{n3}^2 + (\nu-1)\zeta_n^2\right)\left(\alpha\mu_{n3}^2 + \gamma - \beta\zeta_n^2\right) \right]}{\mu_{n3}\zeta_n \left(2C^2 + D(\nu+1)\left(\alpha\mu_{n3}^2 + \gamma - \beta\zeta_n^2\right)\right)} \\
& + \left[\left(\int_0^x e^{\mu_{n5}(x-t)} \theta_{n,5}(t) \, \mathrm{d}t + \Psi_{n5} e^{\mu_{n5} x} \right) - \left(\int_0^x e^{-\mu_{n5}(x-t)} \theta_{n,6}(t) \, \mathrm{d}t + \Psi_{n6} e^{-\mu_{n5} x} \right) \right] \\
& \left. \frac{\cos(y\zeta_n) \left[2C^2 \mu_{n5}^2 - 2C\left(\alpha\mu_{n5}^2 + \gamma - \beta\zeta_n^2\right) + D\left(2\mu_{n5}^2 + (\nu-1)\zeta_n^2\right)\left(\alpha\mu_{n5}^2 + \gamma - \beta\zeta_n^2\right) \right]}{\mu_{n5}\zeta_n \left(2C^2 + D(\nu+1)\left(\alpha\mu_{n5}^2 + \gamma - \beta\zeta_n^2\right)\right)} \right\},
\end{aligned} \tag{2.16}
$$

$$
\begin{aligned}
w = {}& \sum_{n=1}^{+\infty} \left\{ \left[\left(\int_0^x e^{\mu_{n3}(x-t)} \Psi_{n,3}(t) \, dt + \Psi_{n3} e^{\mu_{n3} x} \right) - \left(\int_0^x e^{-\mu_{n3}(x-t)} \theta_{n,4}(t) \, dt + \Psi_{n4} e^{-\mu_{n3} x} \right) \right] \right. \\
& \frac{C \sin(y\zeta_n) \left(2C + D(\nu-1)\left(\mu_{n3}^2 - \zeta_n^2\right) \right)}{\mu_{n3} \left(2C^2 + D(\nu+1)\left(\alpha\mu_{n3}^2 + \gamma - \beta\zeta_n^2\right) \right)} + \frac{C \sin(y\zeta_n) \left(2C + D(\nu-1)\left(\mu_{n5}^2 - \zeta_n^2\right) \right)}{\mu_{n5} \left(2C^2 + D(\nu+1)\left(\alpha\mu_{n5}^2 + \gamma - \beta\zeta_n^2\right) \right)} \times \\
& \left. \left[\left(\int_0^x e^{\mu_{n5}(x-t)} \theta_{n,5}(t) \, dt + \Psi_{n5} e^{\mu_{n5} x} \right) - \left(\int_0^x e^{-\mu_{n5}(x-t)} \theta_{n,6}(t) \, dt + \Psi_{n6} e^{-\mu_{n5} x} \right) \right] \right\}.
\end{aligned} \tag{2.17}
$$

2.4.2. When eigenvalues are double

Compared to the case where the eigenvalues are simple, we combine (2.6) with (2.12) to obtain:

$$
X_1(x) = Q_1 e^{\mu_1 x} + \int_0^x \Theta_1(t) e^{\mu_1(x-t)} \mathrm{d}t,
$$

$$
X_2(x) = Q_2 e^{\mu_2 x} + \int_0^x \Theta_2(t) e^{\mu_2(x-t)} \mathrm{d}t,
$$

$$
X_{n,1}^0(x) = Q_{n,1}^0 e^{\mu_{n1}(x)} + \int_0^x \Theta_{n,1}^0(t) e^{\mu_{n1}(x-t)} \mathrm{d}t,
$$

$$
X_{n,2}^0(x) = Q_{n,2}^0 e^{\mu_{n2}(x)} + \int_0^x \Theta_{n,2}^0(t) e^{\mu_{n2}(x-t)} \mathrm{d}t,
$$

$$X_{n,3}^1(x) = Q_{n,3}^1 e^{\mu_{n3}(x)} + \int_0^x \Theta_{n,3}^0(t)e^{\mu_{n3}(x-t)}\mathrm{d}t,$$

$$X_{n,4}^1(x) = Q_{n,4}^1 e^{\mu_{n4}(x)} + \int_0^x \Theta_{n,4}^0(t)e^{\mu_{n4}(x-t)}\mathrm{d}t,$$

$$X_{n,3}^0(x) = (Q_{n,3}^0 + Q_{n,3}^1 x)e^{\mu_{n3}(x)} + \int_0^x \Theta_{n,3}^0(t)e^{\mu_{n3}(x-t)}\mathrm{d}t + \int_0^x \int_0^t \Theta_{n,3}^1(t)e^{\mu_{n3}(x-\tau)}\mathrm{d}\tau\mathrm{d}t,$$

$$X_{n,4}^0(x) = (Q_{n,4}^0 + Q_{n,4}^1 x)e^{\mu_{n4}(x)} + \int_0^x \Theta_{n,4}^0(t)e^{\mu_{n4}(x-t)}\mathrm{d}t + \int_0^x \int_0^t \Theta_{n,4}^1(t)e^{\mu_{n4}(x-\tau)}\mathrm{d}\tau\mathrm{d}t,$$

where $Q_1, Q_2, Q_{n,i}^0, Q_{n,j}^1$ ($i = 1, 2, 3, 4$; $j = 3, 4$) are undetermined constants, $\Theta_{1,2}, \Theta_{n,i}^0, \Theta_{n,j}^1$ can be computed similarly as (2.11) by symplectic inner product between $\mathbf{f}(x, y)$ and the eigenvectors.

Then, $\mathbf{U}(x, y)$ can be given by

$$\mathbf{U}(x, y) = X_1(x)\mathbf{Y}_1(y) + X_2(x)\mathbf{Y}_2(y) + \sum_{n=1}^{+\infty}\left(\sum_{i=1}^{4}(X_{n,i}^0(x)\mathbf{Y}_{n,i}^0(y)) + \sum_{j=3}^{4}(X_{n,j}^1(x)\mathbf{Y}_{n,j}^1(y))\right). \tag{2.18}$$

As for the case of double eigenvalues, in order to obtain a simple form of ψ_y and w, a classification discussion is carried out.

(i)When $\gamma = \gamma_{n1} = \frac{-2\sqrt{\alpha}C^{\frac{3}{2}} - C^2 - \alpha C - \alpha D\zeta_n^2 + \beta D\zeta_n^2}{D}$, we have

$$\psi_y = \left[\left(Q_1 e^{\mu_1 x} + \int_0^x \Theta_1(t)e^{\mu_1(x-t)}\mathrm{d}t\right) - \left(Q_2 e^{\mu_2 x} + \int_0^x \Theta_2(t)e^{-\mu_1(x-t)}\mathrm{d}t\right)\right]\frac{\sqrt{-2C}}{\sqrt{D(\nu-1)}}$$

$$+ \sum_{n=1}^{+\infty}\left\{\left[e^{\mu_{n1} x}Q_{n,1}^0 + \int_0^x e^{\mu_{n1}(x-t)}\Theta_{n,1}^0(t)\,\mathrm{d}t - \left(e^{\mu_{n2} x}Q_{n,2}^0 + \int_0^x e^{\mu_{n2}(x-t)}\Theta_{n,2}^0(t)\,\mathrm{d}t\right)\right]$$

$$\left(\frac{\cos(y\zeta_n)\sqrt{D(\nu-1)\zeta_n^2 - 2C}}{\sqrt{D(\nu-1)}\zeta_n}\right) + \left[\left(e^{\mu_{n3} x}\left(xQ_{n,3}^1 + Q_{n,3}^0\right) + \int_0^x e^{\mu_{n3}(x-t)}\Theta_{n,3}^0(t)\,\mathrm{d}t + \right.\right.$$

$$\left.\int_0^x \int_0^t e^{\mu_{n3}(x-\tau)}\Theta_{n,3}^1(t)\mathrm{d}\tau\mathrm{d}t\right) - \left(e^{\mu_{n4} x}\left(xQ_{n,4}^1 + Q_{n,4}^0\right) + \int_0^x \int_0^t e^{\mu_{n4}(x-\tau)}\Theta_{n,4}^1(t)\,\mathrm{d}\tau\mathrm{d}t\right.$$

$$\left.\left.+ \int_0^x e^{\mu_{n4}(x-t)}\Theta_{n,4}^9(t)\mathrm{d}t\right)\right]\left(\frac{\zeta_n\cos(y\zeta_n)}{\sqrt{\frac{C^{\frac{3}{2}}}{\sqrt{\alpha}} + C + D\zeta_n^2}{D}}}\right) + \left[(e^{\mu_{n3} x}Q_{n,3}^1 + \int_0^x e^{\mu_{n3}(x-t)}\Theta_{n,3}^0(t)\,\mathrm{d}t)\right.$$

$$\left.+ (e^{\mu_{n4} x}Q_{n,4}^1 + \int_0^x e^{\mu_{n4}(x-t)}\Theta_{n,4}^0(t)\,\mathrm{d}t)\right]\left(-\frac{2\sqrt{\alpha}D\zeta_n\cos(y\zeta_n)}{C^{\frac{3}{2}}}\right)\right\},$$

$$w = \sum_{n=1}^{+\infty}\left\{\left[e^{\mu_{n3} x}\left(xQ_{n,3}^1 + Q_{n,3}^0\right) + \int_0^x \int_0^t e^{\mu_{n3}(x-\tau)}\Theta_{n,3}^1(t)\mathrm{d}\tau\mathrm{d}t + \int_0^x e^{\mu_{n3}(x-t)}\Theta_{n,3}^0(t)\,\mathrm{d}t\right]\right.$$

$$\left.- \left[e^{\mu_{n4} x}\left(xQ_{n,4}^1 + Q_{n,4}^0\right) + \int_0^x \int_0^t e^{\mu_{n4}(x-\tau)}\Theta_{n,4}^1(t)\mathrm{d}\tau\mathrm{d}t + \int_0^x e^{\mu_{n4}(x-t)}\Theta_{n,4}^0(t)\,\mathrm{d}t\right]\right.$$

$$\times \left\{ -\frac{\sqrt{C} \sin(y\zeta_n)}{\sqrt{\alpha} \sqrt{\frac{C^{\frac{3}{2}}}{\sqrt{\alpha}} + C + D\zeta_n^2}} \right\} \Bigg\} .$$

(ii)When $\gamma = \gamma_{n2} = \frac{2\sqrt{\alpha}C^{\frac{3}{2}} - C^2 - \alpha C - \alpha D\zeta_n^2 + \beta D\zeta_n^2}{D}$, we can see that

$$\psi_y = \left[\left(Q_1 e^{\mu_1 x} + \int_0^x \Theta_1(t) e^{\mu_1(x-t)} \, dt \right) - \left(Q_2 e^{\mu_2 x} + \int_0^x \Theta_2(t) e^{\mu_2(x-t)} \, dt \right) \right] \frac{\sqrt{-2C}}{\sqrt{D(\nu-1)}}$$

$$+ \sum_{n=1}^{+\infty} \left\{ \left[e^{\mu_{n1}x} Q_{n,1}^0 + \int_0^x e^{\mu_{n1}(x-t)} \Theta_{n,1}^0(t) \, dt - \left(e^{\mu_{n2}x} Q_{n,2}^0 + \int_0^x e^{\mu_{n2}(x-t)} \Theta_{n,2}^0(t) \, dt \right) \right] \right.$$

$$\left(\frac{\cos(y\zeta_n) \sqrt{D(\nu-1)\zeta_n^2 - 2C}}{\sqrt{D(\nu-1)}\zeta_n} \right) + \left[\left(e^{\mu_{n3}x} \left(x Q_{n,3}^1 + Q_{n,3}^0 \right) + \int_0^x e^{\mu_{n3}(x-t)} \Theta_{n,3}^0(t) \, dt \right. \right.$$

$$+ \int_0^x \int_0^t e^{\mu_{n3}(x-\tau)} \Theta_{n,3}^1(t) d\tau dt \bigg) - \left(e^{\mu_{n4}x} \left(x Q_{n,4}^1 + Q_{n,4}^0 \right) + \int_0^x e^{\mu_{n4}(x-t)} \Theta_{n,4}^0(t) \, dt \right.$$

$$+ \int_0^x \int_0^t e^{\mu_{n4}(x-\tau)} \Theta_{n,4}^1(t) d\tau dt \bigg) \bigg] \left(\frac{\zeta_n \cos(y\zeta_n)}{\sqrt{\frac{-\frac{C^{\frac{3}{2}}}{\sqrt{\alpha}} + C + D\zeta_n^2}{D}}} \right) + \left(\frac{2\sqrt{\alpha}D\zeta_n \cos(y\zeta_n)}{C^{\frac{3}{2}}} \right)$$

$$\times \left[\left(e^{\mu_{n3}x} Q_{n,3}^1 + \int_0^x e^{\mu_{n3}(x-t)} \Theta_{n,3}^0(t) \, dt \right) + \left(e^{\mu_{n4}x} Q_{n,4}^1 + \int_0^x e^{\mu_{n4}(x-t)} \Theta_{n,4}^0(t) \, dt \right) \right] \bigg] \bigg\},$$

$$w = \sum_{n=1}^{+\infty} \left\{ \left[\left(e^{\mu_{n3}x} \left(x Q_{n,3}^1 + Q_{n,3}^0 \right) + \int_0^x \int_0^t e^{\mu_{n3}(x-\tau)} \Theta_{n,3}^1(t) d\tau dt + \int_0^x e^{\mu_{n3}(x-t)} \Theta_{n,3}^0(t) \, dt \right) \right. \right.$$

$$- \left(e^{\mu_{n4}x} \left(x Q_{n,4}^1 + Q_{n,4}^0 \right) + \int_0^x \int_0^t e^{\mu_{n4}(x-\tau)} \Theta_{n,4}^1(t) d\tau dt + \int_0^x e^{\mu_{n4}(x-t)} \Theta_{n,4}^0(t) \, dt \right) \right]$$

$$\times \left\{ \frac{\sqrt{C} \sin(y\zeta_n)}{\sqrt{\alpha} \sqrt{\frac{-\frac{C^{\frac{3}{2}}}{\sqrt{\alpha}} + C + D\zeta_n^2}{D}}} \right\} \Bigg\} .$$

3. Applications

The general solution of a kind of rectangular moderately thick plates has been obtained, which can be applied to some specific mechanical problems, such as bending, buckling, free vibration, etc. For example, when $\alpha = \beta = C, \gamma = q_1(x,y) = q_2(x,y) = 0, q_3(x,y) = -q$, the model reduces to the bending problem of moderately thick plates with two opposite edges simply supported [31]; when $\alpha = C + N_x, \beta = C + N_y, \gamma = q_1(x,y) = q_2(x,y) = q_3(x,y) = 0$, it represents the document [20] in discussing the buckling problem of the moderately thick rectangular plates; the reference [32] discusses a free vibration problem of rectangular moderately thick plates, corresponding to $\alpha = \beta = C, \gamma = \rho h \varpi^2$

(ϖ represents circular frequency) and $q_1(x,y) = q_2(x,y) = q_3(x,y) = 0$. For these specific problems, although they may correspond to different boundary conditions, most of them can be obtained by the general solution in this paper.

So far, SEA has not been applied to the problems of moderately thick rectangular plates with elastic foundations. In this section, the solutions to two elastic foundation problems are discussed first based on the general solutions of the considered model and the numerical results are compared with those in the literature.

Example 1. *Consider a uniformly loaded and simply supported square plate resting on a Winkler-type elastic foundation with side length a. In this case, $q_1(x,y) = q_2(x,y) = 0$, $q_3(x,y) = q$, $\alpha_1 = \beta_1 = 0$ and $\gamma = k$, where q denotes the uniform load and k is the subgrade reaction operator for Winkler-type foundation [33–35], the corresponding equilibrium has the following form,*

$$\begin{cases} \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \\ \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0, \\ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + kw = q. \end{cases} \tag{3.1}$$

According to (2.17), the form of w is given in detail:

$$w = \lim_{N\to+\infty} \sum_{n=1}^{N} \left[\frac{e^{\mu_{n3}(-x)}\left((e^{\mu_{n3}x}-1)^2\theta_{n,3} + \mu_{n3}\Psi_{n3}e^{2\mu_{n3}x} - \mu_{n3}\Psi_{n4}\right)}{\mu_{n3}} \times \frac{\sqrt{2}\sin(y\zeta_n)\left(4C^2 - D(\nu-1)(k+P_1)\right)}{(4C^2 + D(\nu+1)(k-P_1))\sqrt{\frac{2C\zeta_n^2-k-P_1}{C}}} \right.$$
$$\left. + \frac{e^{\mu_{n5}(-x)}\left((e^{\mu_{n5}x}-1)^2\theta_{n,5} + \mu_{n5}\Psi_{n5}e^{2\mu_{n5}x} - \mu_{n5}\Psi_{n6}\right)}{\mu_{n5}} \times \frac{\sqrt{2}\sin(y\zeta_n)\left(4C^2 - D(\nu-1)(k-P_1)\right)}{(4C^2 + D(\nu+1)(k+P_1))\sqrt{\frac{2C\zeta_n^2-k+P_1}{C}}} \right],$$

where $\mu_{n3} = \sqrt{-\frac{-2C\zeta_n^2+k+P_1}{2C}}, \mu_{n5} = \sqrt{\frac{2C\zeta_n^2-k+P_1}{2C}}, P_1 = \sqrt{k\left(\frac{4C^2}{D}+k\right)},$

$$\theta_{n,3} = -\left[2C^2q\zeta_n(\cos(b\zeta_n)-1)\left(4C^2 - D(\nu-1)(k+P_1)\right)\left(4C^2 + D(\nu+1)(k-P_1)\right)\right]/$$
$$\left\{bD\left[32C^5(k-P_1)\zeta_n^2 + 16C^4k^2(\nu-1) - 8C^3D\zeta_n^2\left(k^2\left(\nu^2-2\right)+2kP_1+(1-2\nu)P_1^2\right)\right.\right.$$
$$\left.\left. +8C^2Dk(\nu-1)\left(k^2-P_1^2\right) + D^2(\nu-1)\left(k^2-P_1^2\right)^2 - 2CD^2\left(\nu^2-1\right)(k-P_1)^2(k+P_1)\zeta_n^2\right]\right\},$$

$$\theta_{n,5} = -\left[2C^2q\zeta_n(\cos(b\zeta_n)-1)\left(4C^2 - D(\nu-1)(k-P_1)\right)\left(4C^2 + D(\nu+1)(k+P_1)\right)\right]/$$
$$\left\{bD\left[32C^5(k+P_1)\zeta_n^2 + 16C^4k^2(\nu-1) - 8C^3D\zeta_n^2\left(k^2\left(\nu^2-2\right)-2kP_1+(1-2\nu)P_1^2\right)\right.\right.$$
$$\left.\left. +8C^2Dk(\nu-1)\left(k^2-P_1^2\right) + D^2(\nu-1)\left(k^2-P_1^2\right)^2 - 2CD^2\left(\nu^2-1\right)(k-P_1)(k+P_1)^2\zeta_n^2\right]\right\}.$$

Herein, $\Psi_{n,i}$ are determined by the simply supported conditions at $x = 0$ and $x = a$. The constants $E = 2.06\times10^{11}pa$, $\nu = 0.3$, $q = 6.89\times10^7pa$, $a = 1.016m$ are taken from [33–35]. The corresponding results are shown in Table 1.

In Table 1, we only take the first 20 items of the series to achieve a better fitting effect with the data in the reference [33–35], which indicates that it is satisfactory to use the SEA to solve this moderately thick rectangular plate with single parameter foundation. Table 2 presents the data convergence

analysis for the central deflection w when $\frac{h}{a} = 0.3$ with the different parameters k. It can be seen that convergence of the series is satisfactory, and it confirms the completeness theorem of the eigenvector system discussed above in this paper.

Table 1. Central deflection *w* of SEA and comparison with others.

k $(10^5 N/m^2)$	h/a	w (mm)			
		[33]	[34]	[35]	Present
54.25	0.5	–	–	–	0.2758
	0.3	0.8153	0.8154	0.8588	0.8157
	0.2	2.2896	2.2896	2.3546	2.2858
	0.1	15.9992	15.987	16.1275	15.8836
	0.05	117.463	118.32	117.94	118.2312
542.5	0.5	–	–	–	0.2759
	0.3	0.8125	0.8129	0.8555	0.8162
	0.2	2.2670	2.2688	2.3307	2.2696
	0.1	14.9535	15.023	15.0655	15.0431
	0.05	77.6112	78.535	77.999	78.0131
5425	0.5	–	–	–	0.2763
	0.3	0.7846	0.7858	0.8247	0.8195
	0.2	2.0625	2.0615	2.1150	2.0843
	0.1	9.0424	9.0472	9.0833	9.0525
	0.05	17.668	17.921	17.703	17.8997

Table 2. Convergence analysis of central deflection *w* when $h/a = 0.3$.

k $(10^5 N/m^2)$	w (mm)							
N	1	5	8	10	15	20	25	30
54.25	0.8362	0.8169	0.8155	0.8162	0.8159	0.8157	0.8157	0.8157
542.5	0.8362	0.8173	0.8158	0.8165	0.8162	0.8162	0.8162	0.8162
5425	0.8397	0.8205	0.8191	0.8197	0.8195	0.8195	0.8195	0.8195

Example 2. *The equilibrium equations of the bending problem of rectangular moderately thick plates with Pasternak type two-parameter elastic foundation are [36, 37]:*

$$\begin{cases} \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \\ \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0, \\ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + G_f(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}) - k_f w = -q. \end{cases}$$

For comparison with the data in [36, 37], a square plate with the side length of a is considered under fully simply supported conditions. This is the special case of our model when $q_1(x, y) = q_2(x, y) = 0$, $q_3(x, y) = -q$, $\alpha = \beta = C + G_f$, $\gamma = -k_f$. Furthermore, we take the general solution of the fifth and sixth components of $\mathbf{U}(x, y)$, and the specific forms of the deflection w and the bending moment M_x are as follows:

$$w = -\sum_{n=1}^{+\infty}\left[\frac{e^{\mu_{n3}(-x)}\left((e^{\mu_{n3}x}-1)^2\theta_{n,3}+\mu_{n3}\Psi_{n3}e^{2\mu_{n3}x}-\mu_{n3}\Psi_{n4}\right)}{\mu_{n3}}\times\right.$$

$$\frac{C\sin(y\zeta_n)\left(4C^2+C(\nu+3)G_f+D(\nu-1)\left(k_f-P_{2,n}\right)\right)}{\left(C+G_f\right)\left(4C^2+C(\nu+1)G_f-D(\nu+1)\left(k_f+P_{2,n}\right)\right)\sqrt{\frac{CG_f+Dk_f-DP_{2,n}}{2CD+2DG_f}+\zeta_n^2}}$$

$$+\frac{e^{\mu_{n5}(-x)}\left((e^{\mu_{n5}x}-1)^2\theta_{n,5}+\mu_{n5}\Psi_{n5}e^{2\mu_{n5}x}-\mu_{n5}\Psi_{n6}\right)}{\mu_{n5}}\times$$

$$\left.\frac{C\sin(y\zeta_n)\left(4C^2+C(\nu+3)G_f+D(\nu-1)\left(k_f+P_{2,n}\right)\right)}{\left(C+G_f\right)\left(4C^2+C(\nu+1)G_f-D(\nu+1)\left(k_f-P_{2,n}\right)\right)\sqrt{\frac{CG_f+Dk_f+DP_{2,n}}{2CD+2DG_f}+\zeta_n^2}}\right],$$

$$M_x = -\sum_{n=1}^{+\infty}\left\{\frac{\sin(y\zeta_n)\left(\Psi_{n1}De^{\mu_{n1}x}-\Psi_{n2}De^{-\mu_{n1}x}\right)}{\mu_{n1}\left(2C^2+D(\nu+1)\left(\left(C+G_f\right)\left(-\left(\zeta_n^2-\mu_{n1}^2\right)\right)-k_f\right)\right)}\times\left[2C^2\mu_{n1}^2(\nu-1)+2C\nu\times\right.\right.$$

$$\left.\left(\left(C+G_f\right)(\zeta_n^2-\mu_{n1}^2)-k_f\right)+D(\nu-1)\left(\mu_{n1}^2+\nu\zeta_n^2\right)\left(\left(C+G_f\right)\left(-\left(\zeta_n^2-\mu_{n1}^2\right)\right)-k_f\right)\right]$$

$$+\frac{\sin(y\zeta_n)\,De^{-\mu_{n3}x}\left((e^{\mu_{n3}x}-1)^2\theta_{n,3}+\mu_{n3}\left(\Psi_{n3}e^{2\mu_{n3}x}-\Psi_{n4}\right)\right)}{\mu_{n3}^2\left(2C^2+D(\nu+1)\left(\left(C+G_f\right)\left(-\left(\zeta_n^2-\mu_{n3}^2\right)\right)-k_f\right)\right)}\times\left[2C^2\mu_{n3}^2(\nu-1)\right.$$

$$\left.-2C\nu\left(\left(C+G_f\right)\left(-\left(\zeta_n^2-\mu_{n3}^2\right)\right)-k_f\right)-D(\nu-1)\left(\mu_{n3}^2+\nu\zeta_n^2\right)\left(\left(C+G_f\right)\left(\zeta_n^2-\mu_{n3}^2\right)\right.\right.$$

$$\left.\left.+k_f\right)\right]+\frac{\sin(y\zeta_n)\,De^{-\mu_{n5}x}\left((e^{\mu_{n5}x}-1)^2\theta_{n,5}+\mu_{n5}\left(\Psi_{n5}e^{2\mu_{n5}x}-\Psi_{n6}\right)\right)}{\mu_{n5}^2\left(2C^2+D(\nu+1)\left(\left(C+G_f\right)\left(-\left(\zeta_n^2-\mu_{n5}^2\right)\right)-k_f\right)\right)}\times\left[2C^2\mu_{n5}^2\times\right.$$

$$\left.\left.(\nu-1)-2C\nu\left(\left(C+G_f\right)\left(-\left(\zeta_n^2-\mu_{n5}^2\right)\right)-k_f\right)-D(\nu-1)\left(\mu_{n5}^2+\nu\zeta_n^2\right)\left(\left(C+G_f\right)\times\right.\right.\right.$$

$$\left.\left.\left.\left(\left(\zeta_n^2-\mu_{n5}^2\right)+k_f\right)\right]\right\},$$

in which

$$P_{2,n} = \sqrt{\frac{C^2\left(G_f^2-4Dk_f\right)-2CDG_fk_f+D^2k_f^2}{D^2}},$$

$$\mu_{n3} = \sqrt{\frac{CG_f+Dk_f-DP_{2,n}}{2CD+2DG_f}+\zeta_n^2},$$

$$\mu_{n5} = \sqrt{\frac{CG_f+Dk_f+DP_{2,n}}{2CD+2DG_f}+\zeta_n^2},$$

$\theta_{n,3}$ *and* $\theta_{n,5}$ *are displayed in Appendix B.*

Taking the first 20 terms of the above formula, we calculate the center deflection w and bending moment M_x corresponding to different values of h/a and G_f. The relevant results are listed in Table 3 and compared with the data in the references. The reference [36] uses the grid method of 19×19, and the reference [37] uses the analytic trigonometric series method. Compared with them, it is shown that the SEA in the presented paper is effective.

Table 3. The center deflection w and bending moment M_x of fully simply supported square plate under two-parameter foundation, where $k_f = \frac{200D}{a^4}$, $v = 0.25$.

h/a		$G_f = \frac{5D}{a^2}$		$G_f = \frac{20D}{a^2}$	
		$w \times (\frac{10^3 D}{qa^4})$	$M_x \times (\frac{10^2}{qa^2})$	$w \times (\frac{10^3 D}{qa^4})$	$M_x \times (\frac{10^2}{qa^2})$
0.005	Present	2.264014	2.418051	1.567611	1.612293
	[36]	2.264013	2.417782	1.567607	1.612873
	[37]	2.263888	2.417870	1.567556	1.612893
0.100	Present	2.313681	2.361804	1.587336	1.569424
	[36]	2.313650	2.361704	1.587351	1.569177
	[37]	2.303082	2.352117	1.581634	1.563309
0.200	Present	2.449645	2.207744	1.641966	1.449790
	[36]	2.449532	2.207587	1.641869	1.450101
	[37]	2.440886	2.198786	1.637296	1.444638

Example 3. *For the free vibration of fully simply supported moderately thick rectangular plates, its governing equations are as follows [38, 39]:*

$$\begin{cases} \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \\ \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0, \\ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \rho \omega^2 w = 0, \end{cases}$$

where ρ is the mass per unit area of the plate and ω is the natural frequency of the plates.

By analyzing the boundary conditions and letting the determinant of the coefficients be equal to zero, the frequency equation of the free vibration problem is given by

$$\frac{1}{4\mu_{n5}^2} \left(C^2 D^2 \left(e^{2a\mu_{n1}} - 1 \right) \left(e^{2a\mu_{n3}} - 1 \right) \left(e^{2a\mu_{n5}} - 1 \right) \left(\mu_{n1}^2 - \mu_{n3}^2 \right)^2 e^{-a(\mu_{n1} + \mu_{n3} + \mu_{n5})}$$

$$\left(D(v-1)\zeta_n^2 - 2C \right)^2 \left(2C\mu_{n5}^2 + D(v-1)\zeta_n^2 \left(\zeta_n^2 - \mu_{n5}^2 \right) \right) \right) = 0. \tag{3.2}$$

To justify the frequency equation, we give the numerical results for the lowest six natural frequency parameters $\bar{k} = \frac{\rho \omega^2 b^4}{D\pi^4}$ in Table 4. Compared with [38] and [39], the satisfactory agreement further confirms the stability of the present model and method.

Table 4. The lowest six natural frequency parameters $\bar{k}$ of fully simply supported moderately thick rectangular plates, where $\delta_b = \frac{\pi^2 D}{b^2 C}$, $\vartheta = \frac{b}{a}$ and $\nu = 0.3$.

δ_b	Modes	ϑ 0.2 [38]	[39]	Present	0.6 [38]	[39]	Present	1 [38]	[39]	Present	2 [38]	[39]	Present
0.05	1	1.028	1.028	1.0281	1.732	1.732	1.7318	3.636	3.636	3.6364	20.00	20.00	20.0000
	2	1.272	1.272	1.2718	5.306	5.306	5.3062	20.00	20.00	20.0000	45.71	45.71	45.7143
	3	1.732	1.732	1.7318	14.83	14.83	14.8330	45.71	45.71	45.7143	102.4	102.4	102.4240
	4	-	-	2.4858	-	-	15.6072	-	-	66.6667	-	-	156.2160
	5	-	-	3.6364	-	-	23.2654	-	-	102.4240	-	-	200.0000
	6	-	-	5.3062	-	-	34.1537	-	-	156.2160	-	-	277.7780
0.10	1	0.980	0.980	0.9797	1.628	1.628	1.6282	3.333	3.333	3.3333	16.67	16.67	16.6667
	2	1.206	1.206	1.2057	4.786	4.786	4.7858	16.67	16.67	16.6667	35.56	35.56	35.5556
	3	1.628	1.628	1.6282	12.63	12.63	12.6247	35.56	35.56	35.5556	73.48	73.48	73.4783
	4	-	-	2.3107	-	-	13.2379	-	-	50.0000	-	-	107.0370
	5	-	-	3.3333	-	-	19.1668	-	-	73.4783	-	-	133.3330
	6	-	-	4.7859	-	-	27.2657	-	-	107.0370	-	-	178.5710
0.20	1	0.895	0.895	0.89536	1.454	1.454	1.4541	2.857	2.857	2.8571	12.50	12.50	12.5000
	2	1.092	1.092	1.0922	4.001	4.001	4.0011	12.50	12.50	12.5000	24.62	24.62	24.6154
	3	1.454	1.454	1.4541	9.728	9.728	9.7281	24.62	24.62	24.6154	46.94	46.94	46.9444
	4	-	-	2.0253	-	-	10.1547	-	-	33.3333	-	-	65.6818
	5	-	-	2.8571	-	-	14.1732	-	-	46.9444	-	-	80.0000
	6	-	-	4.0011	-	-	19.4293	-	-	65.6818	-	-	104.1670
0.40	1	0.764	0.764	0.7638	1.198	1.198	1.1979	2.222	2.222	2.2222	8.333	8.333	8.3333
	2	0.919	0.919	0.9191	3.013	3.013	3.0130	8.333	8.333	8.3333	15.24	15.24	15.2381
	3	1.198	1.198	1.1979	6.668	6.668	6.6683	15.24	15.24	15.2381	27.26	27.26	27.2581
	4	-	-	1.6242	-	-	6.9277	-	-	20.0000	-	-	37.0513
	5	-	-	2.2222	-	-	9.3179	-	-	27.2581	-	-	44.4444
	6	-	-	3.0130	-	-	12.3374	-	-	37.0513	-	-	56.8182
0.60	1	0.666	0.666	0.6660	1.019	1.019	1.0185	1.818	1.818	1.8182	6.250	6.250	6.2500
	2	0.793	0.793	0.7934	2.416	2,416	2.4162	6.250	6.250	6.2500	11.03	11.03	11.0345
	3	1.019	1.019	1.0185	5.073	5.073	5.0727	11.03	11.03	11.0345	19.21	19.21	19.2045
	4	-	-	1.3557	-	-	5.2571	-	-	14.2857	-	-	25.8036
	5	-	-	1.8182	-	-	6.9403	-	-	19.2045	-	-	30.7692
	6	-	-	2.4162	-	-	9.0383	-	-	25.8036	-	-	39.0625
0.80	1	0.590	0.590	0.5904	0.886	0.886	0.8858	1.538	1.538	1.5385	5.000	5.000	5.0000
	2	0.698	0.698	0.6979	2.017	2.017	2.0168	5.000	5.000	5.0000	8.649	8.649	8.6487
	3	0.886	0.886	0.8858	4.093	4.093	4.0933	8.649	8.649	8.6487	14.83	14.83	14.8246
	4	-	-	1.1633	-	-	4.2357	-	-	11.1111	-	-	19.7945
	5	-	-	1.5385	-	-	5.5295	-	-	14.8246	-	-	23.5294
	6	-	-	2.0168	-	-	7.1313	-	-	19.7945	-	-	29.7619
1.00	1	0.530	0.530	0.5302	0.784	0.784	0.7837	1.333	1.333	1.3333	4.167	4.167	4.1667
	2	0.623	0.623	0.6230	1.731	1.731	1.7307	4.167	4.167	4.1667	7.111	7.111	7.1111
	3	0.784	0.784	0.7837	3.431	3.431	3.4308	7.111	7.111	7.1111	12.07	12.07	12.0714
	4	-	-	1.0188	-	-	3.5466	-	-	9.0909	-	-	16.0556
	5	-	-	1.3333	-	-	4.5953	-	-	12.0710	-	-	19.0476
	6	-	-	1.7307	-	-	5.8889	-	-	16.0560	-	-	24.0385

4. Conclusions

In this paper, the model for a class of moderately thick rectangular plates is considered, which includes the bending, free vibration and buckling problems of plates and is expected to be widely used in functionally graded material problems. Based on the equilibrium equations of the model, the relationship between force and displacement, the Hamiltonian system and the corresponding

Hamiltonian operator are derived, which provides the possibility for the applications of SEA and the symplectic superposition method. Based on the calculation of eigenvalues and the analysis of the properties of symplectic eigenvectors, the general solution under the simply supported boundary conditions of opposite edges is given by using the proved completeness theorem. In addition, the general solution is applied to two kinds of elastic foundation problems of moderately thick rectangular plates, which have not been solved by the SEA before. At the same time, the higher order and more accurate frequency parameters are solved for the free vibration problem of moderately thick rectangular plates with fully simply supported. The satisfactory numerical results confirm the effectiveness and universality of our model. It should be noted that the buckling problem of the plates is solved similarly to the free vibration problem except that the buckling factor is included in the values of α_1 and β_1 in the model.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. S. P. Timoshenko, S.Woinowsky-Krieger, *Theory of Plates and Shells*, New York: McGraw-Hill College, 1959.

2. A. C. Ugural, *Plates and Shells: Theory and Analysis*, Boca Raton: CRC Press, 2018.

3. P. Xia, S. Y. Long, H. X. Cui, G. Y. Li, The static and free vibration analysis of a nonhomogeneous moderately thick plate using the meshless local radial point interpolation method, *Eng. Anal. Bound. Elem.*, **33** (2009), 770–777. https://doi.org/10.1016/j.enganabound.2009.01.001

4. Ö. Civalek, Three-dimensional vibration, buckling and bending analyses of thick rectangular plates based on discrete singular convolution method, *Int. J. Mech. Sci.*, **49** (2007), 752–765. https://doi.org/10.1016/j.ijmecsci.2006.10.002

5. F. Abad, J. Rouzegar, Exact wave propagation analysis of moderately thick Levy-type plate with piezoelectric layers using spectral element method, *Thin-Walled Struct.*, **141** (2019), 319–331. https://doi.org/10.1016/j.tws.2019.04.007

6. P. Xia, S. Y. Long, X. Wei, An analysis for the elasto-plastic problem of the moderately thick plate using the meshless local Petrov-Galerkin method, *Eng. Anal. Bound. Elem.*, **35** (2011), 908–914. https://doi.org/10.1016/j.enganabound.2011.02.006

7. J. B. Han, K. W. Liew, Static analysis of Mindlin plates: the differential quadrature element method (DQEM), *Comput. Methods Appl. Mech. Eng.*, **177** (1999), 51–75. https://doi.org/10.1016/S0045-7825(99)00371-0

8. S. Ullah, H. Y. Wang, X. R. Zheng, J. H. Zhang, Y. Zhong, R. Li, New analytic buckling solutions of moderately thick clamped rectangular plates by a straightforward finite integral transform method, *Arch. Appl. Mech.*, **89** (2019), 1885–1897. https://doi.org/10.1007/s00419-019-01549-6

9. E. Bahmyari, Free and forced vibration analysis of moderately thick plates with uncertain material properties using the Chaotic Radial Basis Function, *Eng. Anal. Bound. Elem.*, **106** (2019), 349–358. https://doi.org/10.1016/j.enganabound.2019.04.035

10. B. Tian, R. Li, Y. Zhong, Integral transform solutions to the bending problems of moderately thick rectangular plates with all edges free resting on elastic foundation, *Appl. Math. Model.*, **39** (2015), 128–136. https://doi.org/10.1016/j.apm.2014.05.012

11. J. H. Zhang, J. L. Lu, S. Ullah, Y. Y. Gao, D. H. Zhao, Buckling analysis of rectangular thin plates with two opposite edges free and others rotationally restrained by finite Fourier integral transform method, *Z. Angew. Math. Mech.*, **101** (2021), e202000153. https://doi.org/10.1002/zamm.202000153

12. J. H. He, Semi-inverse method and generalized variational principles with multi-variables in elasticity, *Appl. Math. Mech.*, **21** (2000), 797–808. https://doi.org/10.1007/BF02428378

13. S. Kshirsagar, K. Bhaskar, Accurate and elegant free vibration and buckling studies of orthotropic rectangular plates using untruncated infinite series, *J. Sound. Vib.*, **314** (2008), 837–850. https://doi.org/10.1016/j.jsv.2008.01.013

14. A. D. Kerr, M. A. El-Sibaie, Green's functions for continuously supported plates, *J. Appl. Math. Phys.*, **40** (1989), 15–38. https://doi.org/10.1007/BF00945307

15. W. A. Yao, W. X. Zhong, C. W. Lim, *Symplectic Elasticity*, Singapore: World Scientific, 2009.

16. C. W. Lim, X. S. Xu, Symplectic elasticity: theory and applications, *Appl. Mech. Rev.*, **63** (2010), 050802. http://doi.org./10.1115/1.4003700

17. C. H. Xu, S. Leng, Z. H. Zhou, X. S. Xu, Z. C. Deng, Accurate and straightforward symplectic approach for fracture analysis of fractional viscoelastic media, *Appl. Math. Mech.*, **43** (2022), 403–416. https://doi.org/10.1007/s10483-022-2825-8

18. Y. Zhao, H. Liang, Y. W. Zhang, J. H. Lin, Symplectic approach on the wave propagation problem for periodic structures with uncertainty, *Acta Mech. Solida Sin.,* **382** (2019), 287–297. https://doi.org/10.1007/s10338-019-00084-9

19. Y. F. Qiao, G. L. Hou, A. Chen, Symplectic approach for plane elasticity problems of two-dimensional octagonal quasicrystals, *Appl. Math. Comput.*, **400** (2021), 126043. https://doi.org/10.1016/j.amc.2021.126043

20. C. Zhou, D. Q. An, J. Y. Zhou, Z. X. Wang, R. Li, On new buckling solutions of moderately thick rectangular plates by the sympletic superposition method within the Hamiltonian-system framwork, *Appl. Math. Model.*, **94** (2021), 226–241. https://doi.org/10.1016/j.apm.2021.01.020

21. R. Li, C. Zhou, X. R. Zheng, On new analytic free vibration solutions of doubly curved shallow shells by the symplectic superposition method within the Hamiltonian-system framework, *J. Vib. Acoust.*, **143** (2021), 011002. https://doi.org/10.1115/1.4047701

22. X. Su, E. Bai, Analytical free vibration solutions of fully free orthotropic rectangular thin plates on two-parameter elastic foundations by the symplectic superposition method, *J. Vib. Control*, **28** (2022), 3–16. https://doi.org/10.1177/1077546320967823

23. W. Yao, X. Li, Symplectic dual system in plane magnetoelectroelastic solids, *Appl. Math. Mech.*, **27** (2006), 195–205. http://dx.doi.org/10.1007/s10483-006-0207-z

24. S. Xiong, C. Zhou, X. Zheng, D. An, D. Xu, Z. Hu, et al., New analytic thermal buckling solutions of non-Lévy-type functionally graded rectangular plates by the symplectic superposition method, *Acta Mech.*, **233** (2022), 2955–2968. https://doi.org/10.1007/s00707-022-03258-8

25. Z. Y. Jiang, L. Jiang, G. Q. Zhou, Symplectic elasticity analysis of stress in surrounding rock of elliptical tunnel, *KSCE J. Civil Eng.*, **24** (2020), 3119–3130. https://doi.org/10.1007/s12205-020-1810-7

26. X. Hou, S. Zhou, Z. Cheng, Z. Zhu, Y. Wang, Z. Deng, Buckling of regular and auxetic honeycombs under a general macroscopic stress state in symplectic system, *Appl. Math. Model.*, **109** (2022), 318–340. https://doi.org/10.1016/j.apm.2022.04.033

27. Z. Y. Hu, Y, Q. Shi, S. J. Xiong, X. R. Zheng, R. Li, New analytic free vibration solutions of non-Lévy-type porous FGM rectangular plates within the symplectic framework, *Thin Wall Struct.*, **185** (2023), 110609. https://doi.org/10.1016/j.tws.2023.110609

28. Z. Y. Hu, Z. F. Ni, D. Q. An, Y. M. Chen, R. Li, Hamiltonian system-based analytical solutions for the free vibration of edge-cracked thick rectangular plates, *Appl. Math. Model.*, **117** (2023), 451–478. https://doi.org/10.1016/j.apm.2022.12.036

29. Z. Y. Hu, C. Zhou, Z. F. Ni, X. Q. Lin, R. Li, New symplectic analytic solutions for buckling of CNT reinforced composite rectangular plates, *Compos. Struct.*, **303** (2023), 116361. https://doi.org/10.1016/j.compstruct.2022.116361

30. Y. F. Qiao, G. L. Hou, A. Chen, A complete symplectic approach for a class of partial differential equations arising from the elasticity, *Appl. Math. Model.*, **89** (2021), 1124–1139. https://doi.org/10.1016/j.apm.2020.08.002

31. Y. Zhong, R. Li, Y. M. Liu, B. Tian, On new symplectic approach for exact bending solutions of moderately thick rectangular plates with two opposite edges simply supported, *Int. J. Solids Struct.*, **46** (2009), 2506–2513. https://doi.org/10.1016/j.ijsolstr.2009.02.001

32. R. Li, P. C. Wang, Y. R. Xue, X. Guo, New analytic solutions for free vibration of rectangular thick plates with an edge free, *Int. J. Mech. Sci.*, **131-132** (2017), 179–190. https://doi.org/10.1016/j.ijmecsci.2017.07.002

33. D. Frederick, Thick rectangular plates on elastic foundation, *Trans. American Soci. Civil Eng.*, **122** (1957), 1069–1085. https://doi.org/10.1061/TACEAT.0007435

34. Q. H. Qin, Hybrid-Trefftz finite element method for Reissner plates on an elastic foundation, *Comput. Methods Appl. Mech. Eng.*, **122** (1995), 379–392. https://doi.org/10.1016/0045-7825(94)00730-b

35. H. S. Shen, Nonlinear bending of simply supported rectangular Reissner Mindlin plates under transverse and in-plane loads and resting on elastic foundations, *Eng. Struct.*, **22** (2000), 847–856. https://doi.org/10.1016/s0141-0296(99)00044-9

36. J. B. Han, K. M. Liew, Numerical differential quadrature method for Reissner/Mindlin plates on two-parameter foundations, *Int. J. Mech. Sci.*, **39** (2000), 977–989. https://doi.org/10.1016/j.ijmecsci.2006.10.002

37. J. G. Wang, X. X. Wang, M. K. Huang, Fundamental solutions and boundary integral equations for Reissner's plates on two parameter foundations, *Int. J. Solids Struct.*, **29** (1992), 1233–1239. https://doi.org/10.1016/0020-7683(92)90234-K

38. Knowledge Management System of Institue of Mechanics, CAS, Bending, Stabilization and Vibration of Sandwich Plate Shell, in Chinese, 1977. Available from: http://dspace.imech.ac.cn/handle/311007/15281 .

39. R. Li, Y. Zhong, On new symplectic approach for exact free vibration solutions of moderately thick rectangular plates with two opposite edges simply supported, *Int. J. Eng. Appl. Sci.*, **1** (2009), 13–28.

Appendix

Appendix A

$$
\begin{aligned}
A_{34,n} = &- \Big(\Big(b\Big((1-\nu)\Big(C^4 - 2\alpha C^3 + C^2\Big(\alpha^2 + 2\gamma D + D\zeta_n^2(\alpha(\nu-1)-2\beta)\Big) + CD\alpha(2\gamma+ \\
&\zeta_n^2(\alpha(-\nu)+\alpha-2\beta))\Big) - D^2\Big(-\gamma + (\beta-\alpha)\zeta_n^2 + P_n\Big)\Big(\gamma + \zeta_n^2(-(\alpha\nu+\beta)) + P_n\Big)\Big) \\
&\Big(C^4 - 2\alpha C^3 + C^2\Big(\alpha^2 + 2\gamma D + 2D\zeta_n^2(\alpha(\nu-2)-\beta)\Big) + 2\alpha CD\Big(\gamma - \zeta_n^2(\alpha\nu+\beta)\Big) \\
&- D^2\Big(-\gamma + (\beta-\alpha)\zeta_n^2 + P_n\Big)\Big(\gamma + \zeta_n^2(-(2\alpha\nu+\alpha+\beta)) + P_n\Big)\Big) + \alpha D\zeta_n^2\Big(C^2(\nu- \\
&3) + D(\nu+1)\Big(-\gamma + (\beta-\alpha)\zeta_n^2 + P_n\Big) - \alpha C(\nu+1)\Big)\Big(3C^4(\nu-1) + \alpha C^3(2-6\nu) \\
&+ D^2(\nu-1)\Big(-\gamma + (\beta-\alpha)\zeta_n^2 + P_n\Big)\Big(-\gamma + \zeta_n^2(2\alpha\nu+\alpha+\beta) - P_n\Big) - 2\alpha CD(P_n \\
&- 2\gamma\nu + \zeta_n^2(\alpha((\nu-2)\nu-1) + 2\beta\nu) + \nu P_n) + C^2\Big(\alpha^2(3\nu+1) + 2D(\nu-1)(2\gamma+ \\
&\zeta_n^2(\alpha(\nu-1)-2\beta) + P_n)\Big)\Big) + 4C^2 D\alpha\zeta_n^2\Big(C^2(\nu-1) - \alpha C(\nu+3) + D(\nu-1)(\gamma+ \\
&(\alpha-\beta)\zeta_n^2 + P_n))(C^2 - \alpha C + D\Big(\nu\Big(\gamma + (\alpha-\beta)\zeta_n^2\Big) - P_n\Big)\Big)\Big) \Big/ \Big(\sqrt{2}\alpha^2 D\zeta_n^2\Big(C^2 \\
&(\nu-3) - \alpha C(\nu+1) + D(\nu+1)\Big(-\gamma + (\beta-\alpha)\zeta_n^2 + P_n\Big)\Big)^2 (C(\alpha-C) - D(P_n+ \\
&\alpha + C(\nu+1) + D(\nu+1)\Big(\gamma + (\alpha-\beta)\zeta_n^2 + P_n\Big)\Big)^2 ,
\end{aligned}
$$

$$
\begin{aligned}
A_{56,n} = &\Big(b\Big((\nu-1)\Big(-C^4 + 2\alpha C^3 + C^2\Big(-\alpha^2 - 2\gamma D + 2D\zeta_n^2(\beta-\alpha(\nu-2))\Big)\Big) + 2\alpha CD \\
&\Big(\zeta_n^2(\alpha\nu+\beta) - \gamma\Big) + D^2\Big(\gamma + (\alpha-\beta)\zeta_n^2 + P_n\Big)\Big(-\gamma - \zeta_n^2(2\alpha\nu+\alpha+\beta) + P_n\Big)\Big) \\
&\Big(-C^4 + 2\alpha C^3 - C^2\Big(\alpha^2 + 2\gamma D + D\zeta_n^2(\alpha(\nu-1)-2\beta)\Big) + D^2\Big(\gamma + (\alpha-\beta)\zeta_n^2
\end{aligned}
$$

$$+ P_n)(-\gamma + \zeta_n^2(\alpha\nu + \beta) + P_n) + \alpha CD\left(\zeta_n^2(\alpha(\nu - 1) + 2\beta) - 2\gamma\right)\right) - 4C^2\alpha\left(C^2\right.$$

$$(\nu - 1) - \alpha C(\nu + 3) - D(\nu - 1)\left(-\gamma + (\beta - \alpha)\zeta_n^2 + P_n\right)\right)\left(C^2 - \alpha C + D\right.$$

$$\left(\nu\left(\gamma + (\alpha - \beta)\zeta_n^2\right) + P_n\right)\right) + \alpha\left(-C^2(\nu - 3) + \alpha C(\nu + 1) + D(\nu + 1)(P_n + \gamma +\right.$$

$$(\alpha - \beta)\zeta_n^2)\right)\left(3C^4(\nu - 1) + \alpha C^3(2 - 6\nu) - D^2(\nu - 1)\left(\gamma + (\alpha - \beta)\zeta_n^2 + P_n\right)\right.$$

$$(-\gamma + \zeta_n^2(2\alpha\nu + \alpha + \beta) + P_n) + 2CD\alpha\left(2\gamma\nu + \zeta_n^2(\alpha(1 - (\nu - 2)\nu) - 2\beta\nu) + P_n\right.$$

$$\nu + P_n) + C^2\left(\alpha^2(3\nu + 1) - 2D(\nu - 1)\left(-2\gamma + \zeta_n^2(\alpha(-\nu) + \alpha + 2\beta) + P_n\right)\right)\right)\right)\right)$$

$$\Bigg/\Bigg(\sqrt{\frac{-C^2 + \alpha C + D\left(-\gamma + (\alpha + \beta)\zeta_n^2 + P_n\right)}{\alpha D}}\,\sqrt{2}D\zeta_n^2\alpha^2\left(C(\nu + 1) - C^2(\nu - 3)\right.$$

$$\alpha + C(\nu + 1) + D(\nu + 1)\left(\gamma + (\alpha - \beta)\zeta_n^2 + P_n\right)\Bigg)^2.$$

Appendix B

$$\theta_{n,3} = -\left(CD(C + G_f)q(4C^2 + CG_f(\nu + 1) - D(\nu + 1)(k_f + P_{2,n}))\left(4C^2 + D\left(k_f - P_{2,n}\right)\right.\right.$$

$$C(\nu + 3)G_f + (\nu - 1)\right)\zeta_n\sqrt{\frac{2\left(CG_f + D\left(k_f - P_{2,n}\right)\right)}{D\left(C + G_f\right)} + 4\zeta_n^2(\cos(b\zeta_n) - 1)}\Bigg/$$

$$\left(b\sqrt{\frac{CG_f + Dk_f - DP_{2,n}}{2CD + 2DG_f}} + \zeta_n^2\left(4C^2D\left(4C^2 + C(\nu + 3)G_f + D(\nu - 1)\left(k_f - P_{2,n}\right)\right)\right.\right.$$

$$\left(C + G_f\right)\zeta_n^2\left(CG_f + D\left(\nu k_f + P_{2,n}\right)\right) + (1 - \nu)\left(-4C^2Dk_f + C^2G_f^2 - 2CDG_fk_f +\right.$$

$$D^2(k_f^2 - P_{2,n}^2) + D\zeta_n^2\left(C + G_f\right)\left(-4C^2 - C(\nu + 1)G_f + D(\nu + 1)\left(k_f + P_{2,n}\right)\right)\right)\left(C^2G_f^2\right.$$

$$+ D^2(k_f^2 - P_{2,n}^2) - 4C^2Dk_f - 2CDG_fk_f + 2D(C + G_f)\left(-4C^2 - C(\nu + 1)G_f + D\times\right.$$

$$(\nu + 1)\left(k_f + P_{2,n}\right)\right)\zeta_n^2 + D\zeta_n^2\left(C + G_f\right)\left(-4C^2 - C(\nu + 1)G_f + D(\nu + 1)\left(k_f + P_{2,n}\right)\right)$$

$$\left(C^3(2 - 6\nu)\left(C + G_f\right) + 3C^4(\nu - 1) + D^2(k_f + P_{2,n})(\nu - 1)\left(2\left(C + G_f\right) + k_f - P_{2,n}\times\right.\right.$$

$$(\nu + 1)\zeta_n^2\right) - 2CD\left(C + G_f\right)\left(\left(\nu^2 - 1\right)\zeta_n^2\left(C + G_f\right) + 2\nu k_f + \nu P_{2,n} + P_{2,n}\right) + \left(\left(C + G_f\right)^2\right.$$

$$2D(\nu - 1)\left((\nu - 3)\zeta_n^2\left(C + G_f\right) - 2k_f + P_{2,n}\right) + (3\nu + 1)\right)C^2\right)\right)\right),$$

$$\theta_{n,5} = \left(C(C + G_f)q(4C^2 + CG_f(\nu + 1) - D(\nu + 1)(k_f + P_{2,n}))\left(4C^2 + D\left(k_f + P_{2,n}\right)\right.\right.$$

$$C(\nu + 3)G_f + (\nu - 1)\right)\zeta_n\sqrt{\frac{2\left(CG_f + D\left(k_f + P_{2,n}\right)\right)}{D\left(C + G_f\right)} + 4\zeta_n^2(\cos(b\zeta_n) - 1)}\Bigg/$$

$$\left(b\zeta_n\sqrt{\frac{CG_f + Dk_f + DP_{2,n}}{2CD + 2DG_f}} + \zeta_n^2\left(-4C^2\left(C + G_f\right)(CG_f + D\nu k_f - DP_{2,n})\left(4C^2 +\right.\right.\right.$$

$$+1\, C(\nu + 3)G_f + D(\nu - 1)\left(k_f + P_{2,n}\right)\right) + \frac{1}{D\zeta_n^2}(\nu - 1)\left(4C^2Dk_f - C^2G_f^2 + 2CDG_fk_f\right.$$

$$+ D^2 P_{2,n}^2 - D^2 k_f^2 + D\zeta_n^2 \left(C + G_f\right)\left(4C^2 + C(\nu + 1)G_f - D(\nu + 1)\left(k_f - P_{2,n}\right)\right)\right) \times$$

$$\left(-C^2 G_f^2 + 4C^2 D k_f + 2CDG_f k_f - D^2 k_f^2 + D^2 P_{2,n}^2 + 2D(C + G_f)\left(C(\nu + 1)G_f + \right.\right.$$

$$4C^2 - D(\nu + 1)\left(k_f - P_{2,n}\right)\right)\zeta_n^2\right) + \left(C + G_f\right)\left(4C^2 + C(\nu)G_f - D(\nu + 1)\left(k_f - P_{2,n}\right)\right)$$

$$\left(4C^3\left(G_f - 2D(\nu - 1)\zeta_n^2\right) + D^2(\nu - 1)\left(k_f - P_{2,n}\right)\left(2(\nu + 1)G_f \zeta_n^2 + k_f + P_{2,n}\right) + C^2\right.$$

$$\left(-2D(\nu - 1)(\nu + 5)G_f \zeta_n^2 + 4D\left(-2\nu k_f + k_f + P_{2,n}\right) + (3\nu + 1)G_f^2\right) + 2CDG_f \times$$

$$\left(-2\nu k_f + \nu P_{2,n} + P_{2,n}\right) - \left(\nu^2 - 1\right)\zeta_n^2\left(D\left(P_{2,n} - k_f\right) + G_f^2\right)\right)\right)\right).$$

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