The index of strong rotundity

Francisco Javier García-Pacheco*

Department of Mathematics, College of Engineering, University of Cadiz, Avda. de la Universidad 10, Puerto Real 11519, Spain

* Correspondence: Email: garcia.pacheco@uca.es.

Abstract: The index of strong rotundity is introduced. This index is used to determine how far an element of the unit sphere of a real Banach space is from being a strongly exposed point of the unit ball. This index is computed for Hilbert spaces. Characterizations of the set of rotund points and the set of smooth points are provided for a better understanding of the construction of the index of strong rotundity. Finally, applications to the stereographic projection are provided.

Keywords: exposed point; rotund point; smooth point; Banach space; Hilbert space
Mathematics Subject Classification: 46B20

1. Introduction

The irruption of the moduli of convexity and smoothness [7] in the literature of the geometry of the real Banach spaces was a huge revolution that brought strong implications to longstanding open problems such as the Banach-Mazur conjecture for rotations or the fixed-point property problem. The main purpose of this manuscript is to introduce an index that measures a convexity stronger than strict convexity but weaker than uniform convexity. All Banach spaces considered throughout this manuscript will be over the reals. A point $x$ in the unit sphere $S_X$ of a Banach space $X$ is said to be a strongly exposed point of the unit ball $B_X$ if there exists $x^*$ in the unit sphere $S_{X^*}$ of the dual space $X^*$ verifying the following property: If $(x_n)_{n \in \mathbb{N}} \subseteq B_X$ is such that $(x^*(x_n))_{n \in \mathbb{N}}$ converges to 1, then $(x_n)_{n \in \mathbb{N}}$ converges to $x$. The functional $x^*$ is said to strongly expose $x$ on $B_X$ and it is trivial that $(x, x^*) \in \Pi_X$, where $\Pi_X := \{(x, x^*) \in S_X \times S_{X^*} : x^*(x) = 1\}$. We will let

$$\Pi^e_X := \{(x, x^*) \in S_X \times S_{X^*} : x^* \text{ strongly exposes } x \text{ on } B_X\}.$$

A weaker notion than a strongly exposed point is that of an exposed point. A point $x \in S_X$ is said to be an exposed point of $B_X$ if there exists $x^* \in S_{X^*}$ in such a way that $(x^*)^{-1}(\{1\}) \cap B_X = \{x\}$. This time the functional $x^*$ is called a supporting functional that exposes $x$ on $B_X$. We will let $\Pi^e_X := \{(x, x^*) \in S_X \times S_{X^*} : x^* \text{ exposes } x \text{ on } B_X\}.$
\[(x, x') \in S_X \times S_{X^*} : x' \text{ exposes } x \text{ on } B_X\]. Observe that \(\Pi_X^{\text{ex}} \subseteq \Pi_X^c \subseteq \Pi_X\). Another trivial fact is the following: for every surjective linear isometry \(T\) between Banach spaces \(X, Y\) and for every \((x, x') \in S_X \times S_{X^*}\), we have that \((x, x') \in \Pi_X \text{ or } \Pi_X^c \text{ or } \Pi_X^{\text{ex}}\) if and only if \((T(x), T^*(x')) \in \Pi_Y \text{ or } \Pi_Y^c \text{ or } \Pi_Y^{\text{ex}}\), respectively. In other words, the previous notions are invariant under surjective linear isometries. Another geometrical notion employed in this manuscript is that of a rotund point. A point \(x \in S_X\) in the unit sphere of a Banach space \(X\) is said to be a rotund point of the unit ball of \(X\) if \(x\) is contained in no non-trivial segment of the unit sphere, in other words, \([x]\) is a maximal proper face of \(B_X\). The set of rotund points of \(B_X\) is denoted by \(\text{rot}(B_X)\). In view of the Hahn–Banach Separation Theorem, the set of rotund points can be described as follows:

\[
\text{rot}(B_X) = \{ x \in S_X : \text{if } x' \in S_{X^*} \text{ is so that } (x, x') \in \Pi_X, \text{ then } (x, x') \in \Pi_X^c \}.
\]

We refer the reader to [2, 3] for a wider perspective on the above concepts and some other geometrical properties related with renormings. The duality mapping [4, 5] of a Banach space \(X\) is the set-valued map defined as

\[
J : X \to \mathcal{P}(X^*) \quad x \mapsto J(x) := \{ x' \in X^* : \| x' \| = \| x \| \text{ and } x'(x) = \| x' \| \| x \| \}.
\]

A point \(x\) in the unit sphere \(S_X\) of \(X\) is said to be a smooth point [8] of the unit ball \(B_X\) of \(X\) provided that \(J(x)\) is a singleton. The subset of smooth points of \(B_X\) is typically denoted by \(\text{smo}(B_X)\). Rotund points and smooth points are dual notions.

The main goal of this manuscript is to introduce an index that measures accurately how far a point in the unit sphere is from being a strongly exposed point of the unit ball. For this, we first establish characterizations of the set of rotund points and the set of smooth points. Then we introduce the index of strong rotundity and show the most basic properties related to such index. We compute the index of strong rotundity of a Hilbert space. Finally, we construct a new set of pairs contained in \(\Pi_X^c\) for which the stereographic projection [13, 14] in a Banach space is a homeomorphism.

### 2. Results

We will begin by providing a new characterization of the set of rotund points of the unit ball of a Banach space. As usual, if \(X\) is a vector space and \(x, y \in X\), then \(\text{st}(x, y) := x + \mathbb{R}(y - x)\) is the straight line passing through \(x, y\) and \([x, y] := x + [0, 1](y - x)\) is the segment joining \(x, y\).

**Theorem 2.1.** For a Banach space \(X\), \(\text{rot}(B_X) = \{ x \in S_X : \forall y \in S_X \quad \text{st}(x, y) \cap S_X = \{ x, y \} \}.

**Proof.** We will prove both inclusions in two simple steps.

\[\subseteq \text{ Let } x \in \text{rot}(B_X). \text{ Fix an arbitrary } y \in S_X \setminus \{x\}. \text{ If there exists } z \in \text{st}(x, y) \cap S_X \text{ different from } x, y, \text{ then we end up with three points aligned in the unit sphere, so the whole segment containing } x, y, z \text{ lies entirely in the unit sphere. However, this contradicts the fact that } \{x\} \text{ is a maximal proper face of } B_X.\]

\[\supseteq \text{ Conversely, let } x \in S_X \text{ satisfying that } \text{st}(x, y) \cap S_X = \{ x, y \} \text{ for all } y \in S_X. \text{ If } x \not\in \text{rot}(B_X), \text{ then we can find } y \in S_X \setminus \{x\} \text{ such that } [x, y] \subseteq S_X, \text{ contradicting that } \text{st}(x, y) \cap S_X = \{ x, y \} \text{ since } [x, y] \subseteq S_X.\]
The next result is a characterization of the set of smooth points in similar terms as the previous theorem. However, a technical lemma is needed first.

**Lemma 2.1.** Let $X$ be a Banach space. Let $x \in S_X$ and consider a straight line $L \subseteq X$ containing $x$ such that $L \cap (S_X \setminus \{x\}) = \emptyset$. Then $L \cap U_X = \emptyset$, where $U_X$ is the open unit ball of $X$.

**Proof.** Suppose on the contrary that there exists $u \in L \cap U_X$. Consider the continuous function

$$(−∞,0] \to \mathbb{R}, \quad t \mapsto ||u+t(x−u)||.$$  

If $t = 0$, then $||u|| < 1$. Note that

$$\frac{1+||u||}{||x−u||} ≥ 1 > ||u||,$$

therefore, if $t < −\frac{1+||u||}{||x−u||}$, then

$$||u+t(x−u)|| ≥ ||u||−||t(x−u)|| = ||u||−|t||x−u|| = |t||x−u||−||u|| > 1.$$  

Bolzano’s Theorem assures the existence of $s ∈ (−∞,0)$ such that $||u+s(x−u)|| = 1$, that is, $u+s(x−u) \in L \cap S_X$, meaning that $u+s(x−u) = x$, hence $(1−s)u = (1−s)x$, so $u = x$. This is a contradiction because $||u|| < 1 = ||x||$.

**Theorem 2.2.** For a Banach space $X$ with dim$(X) ≥ 2$,

$$\text{smo}(B_X) = \left\{ x ∈ S_X : \exists x^* ∈ S_{X^*} \ (x, x^*) ∈ Π_X \text{ and } ∀ z ∈ (x^*)^{-1}(\{1\}) \text{ st}(−x, z) ∩ (S_X \setminus \{−x\}) ≠ \emptyset \right\}.$$  

**Proof.** We will prove both inclusions in two steps.

1. Let $x ∈ \text{smo}(B_X)$. There exists $x^* ∈ S_{X^*}$ satisfying that $J(x) = \{x^*\}$. Fix an arbitrary $z ∈ (x^*)^{-1}(\{1\})$. Suppose on the contrary that $\text{st}(−x, z) ∩ (S_X \setminus \{−x\}) = \emptyset$. Note that, in this case, $||z|| > 1$ because if $||z|| = 1$, then $z ∈ \text{st}(−x, z) ∩ (S_X \setminus \{−x\})$. By bearing in mind Lemma 2.1, $\text{st}(−x, z) ∩ U_X = \emptyset$. The Hahn-Banach Separation Theorem allows the existence of $y^* ∈ S_{X^*}$ such that $\text{st}(−x, z) ⊆ (y^*)^{-1}(\{1\})$. Then $y^*(−x) = 1$, meaning that $−y^*(x) = 1$, so $−y^* ∈ J(x) = \{x^*\}$, reaching the contradiction that $y^*(z) = 1$ and $y^*(z) = −x^*(z) = −1$.

2. Take $x ∈ S_X$ for which there exists $x^* ∈ J(x)$ satisfying that $\text{st}(−x, z) ∩ (S_X \setminus \{−x\}) ≠ \emptyset$ for all $z ∈ (x^*)^{-1}(\{1\})$. Suppose on the contrary that $x ∈ \text{smo}(B_X)$. There exists a 2-dimensional subspace $Y$ of $X$ containing $x$ for which $x$ is not a smooth point of $B_Y$. Let $J_Y : Y → Π(Y^*)$ denote the dual mapping of $Y$. Notice that $J_Y(x)$ is a nontrivial segment that lies entirely in $S_{Y^*}$. Thus we can write $J_Y(x) = [a^*, b^*]$ where $a^* ≠ b^*$ both are in $S_{Y^*}$. Notice that $B_Y ≤ (x^*)^{-1}(−1,1) ∩ (a^*)^{-1}([−1,1]) ∩ (b^*)^{-1}([−1,1])$. Also, $x^*|_Y ∈ J_Y(x) = [a^*, b^*]$, therefore, either $a^* ≠ x^*|_Y$ or $b^* ≠ x^*|_Y$. Let us assume without any loss of generality that $a^* ≠ x^*|_Y$. Then $(a^*)^{-1}(−1,1)$ is not parallel to $(x^*)^{-1}(\{1\})$, hence the straight line $(a^*)^{-1}(−1,1)$ intersects $(x^*)^{-1}(\{1\})$. Choose any $z ∈ (x^*)^{-1}(\{1\})$ with $(a^*)(z) < −1$. On the one hand, $\text{st}(−x, z) ⊆ Y$, thus $\text{st}(−x, z) ∩ (S_X \setminus \{−x\}) ⊆ \text{st}(−x, z) ∩ (S_Y \setminus \{−x\})$. On the other hand, we will prove that $\text{st}(−x, z) ∩ (S_Y \setminus \{−x\}) = \emptyset$. Indeed, pick any $t ∈ \mathbb{R} \setminus \{0\}$ and distinguish the following two cases:

- $t > 0$. In this case, $a^*(tz + (1−t)(−x)) = t(a^*(z) + 1) − 1 < −1$. Since $B_Y ≤ (a^*)^{-1}([−1,1])$, we conclude that $tz + (1−t)(−x) ∉ B_Y$.
Lemma 2.2. Let $X$ be a Banach space. Let $\varepsilon$ of rotundity. Now, the following lemma unveils the most basic properties satisfied by the index of the unit ball. Later on, we will relate the index of strong rotundity to the modulus of local uniform rotundity. The index of strong rotundity will serve, among other things, to characterize the strongly exposed points of the unit ball. Later on, we will relate the index of strong rotundity to the modulus of local uniform rotundity. Next, we introduce next the index of strong rotundity of a Banach space. First, we recall that, for a Banach space $X$, $U_X$ stands for the open unit ball, and $U_X(x, \varepsilon)$ denotes the open ball of center $x \in X$ and radius $\varepsilon \geq 0$.

Definition 2.1. (Index of strong rotundity) Let $X$ be a Banach space. The local index of strong rotundity of $X$ at $(x, x^*) \in \Pi_X$ is defined as

$$
\eta_X(\cdot, (x, x^*)) : [0, 2] \to [0, 2],
$$

$$
\varepsilon \mapsto \eta_X(\varepsilon, (x, x^*)) := d((x^*)^{-1}(1)), B_X \setminus U_X(x, \varepsilon).
$$

The index of strong rotundity of $X$ at is defined as

$$
\eta_X : [0, 2] \to [0, 2],
$$

$$
\varepsilon \mapsto \eta_X(\varepsilon) := \inf \{ \eta_X(\varepsilon, (x, x^*)) : (x, x^*) \in \Pi_X \}.
$$

The index of strong rotundity will serve, among other things, to characterize the strongly exposed points of the unit ball. Later on, we will relate the index of strong rotundity to the modulus of local uniform rotundity. Now, the following lemma unveils the most basic properties satisfied by the index of rotundity.

Lemma 2.2. Let $X$ be a Banach space. Let $(x, x^*) \in \Pi_X$. Then:

1. $\eta_X(0, (x, x^*)) = 0$.
2. $(x^*)^{-1}([-1, 1]) \cap B_X \subseteq B_X \setminus U_X(x, 2)$, hence $\eta_X(2, (x, x^*)) \leq 2$.
3. If $(x^*)^{-1}([-1, 1]) \cap B_X = B_X \setminus U_X(x, 2)$, then $\eta_X(2, (x, x^*)) = 2$.
4. If $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq 2$, then $\eta_X(\varepsilon_1, (x, x^*)) \leq \eta_X(\varepsilon_2, (x, x^*))$.
5. If $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq 2$, then $\eta_X(\varepsilon_1) \leq \eta_X(\varepsilon_2)$.
6. $\Pi_X^\varepsilon = \{(x, x^*) \in \Pi_X : \forall \varepsilon \in (0, 2] \ \eta_X(\varepsilon, (x, x^*)) > 0 \}$.

Proof. We will only prove the second, third, and last items.

2. For every $z \in (x^*)^{-1}([-1, 1]) \cap B_X$, $2 \geq \|z - x\| \geq |x^*(z) - x^*(x)| = 2$, meaning that $z \in B_X \setminus U_X(x, 2)$.

3. On the one hand, if $u \in (x^*)^{-1}([1, 1])$ and $v \in (x^*)^{-1}([-1, 1]) \cap B_X$, then $|u - v| \geq |x^*(u) - x^*(v)| = |x^*(u) - x^*(v)| = 2$, thus $d\left( (x^*)^{-1}([1, 1]), (x^*)^{-1}([-1, 1]) \cap B_X \right) = 2$. On the other hand,

$$
2 \geq \eta_X(2, (x, x^*)) := d\left( (x^*)^{-1}([1, 1]), B_X \setminus U_X(x, \varepsilon) \right) = d\left( (x^*)^{-1}([1, 1]), (x^*)^{-1}([-1, 1]) \cap B_X \right) = 2.
$$

6. Fix an arbitrary $(x, x^*) \in \Pi_X^\varepsilon$. Suppose on the contrary that there exists $\varepsilon \in (0, 2)$ for which $\eta_X(\varepsilon, (x, x^*)) = 0$. Then we can find two sequences $(y_n)_{n \in \mathbb{N}} \subseteq (x^*)^{-1}([1, 1])$ and $(x_n)_{n \in \mathbb{N}} \subseteq B_X \setminus U_X(x, \varepsilon)$ such that $|y_n - x_n| \to 0$ as $n \to \infty$. Notice that $x^*(x_n) \to 1$ as $n \to \infty$ but $(x_n)_{n \in \mathbb{N}}$ does not converge to $x$ because $x_n \notin U_X(x, \varepsilon)$ for each $n \in \mathbb{N}$, contradicting that $(x, x^*) \in \Pi_X^\varepsilon$. Conversely, take any $(x, x^*) \in \Pi_X$ satisfying that $\eta_X(\varepsilon, (x, x^*)) > 0$ for all $\varepsilon \in (0, 2]$. Assume to
the contrary that \((x, x') \not\in \Pi_X^\circ\). There exists a sequence \((x_n)_{n \in \mathbb{N}} \subseteq B_X\) in such a way that \(x^*(x_n) \to 1\) as \(n \to \infty\) but \((x_n)_{n \in \mathbb{N}}\) does not converge to \(x\). By passing to an appropriate subsequence \((x_m)_{m \in \mathbb{N}}\), we can find \(\varepsilon \in (0, 2]\) such that \(\|x_m - x\|_X \geq \varepsilon\) for all \(k \in \mathbb{N}\). Notice that \((x^*(x_m))_{m \in \mathbb{N}}\) still converges to 1, so, by assuming that \(x^*(x_m) \neq 0\) for all \(k \in \mathbb{N}\), we have that \(\|\frac{x_m}{x^*(x_m)} - x\|_X \to 0\), meaning that \(\eta_X(x, (x, x^*)) = d\left((x^*)^{-1}(\{1\}), B_X \setminus U_X(x, \varepsilon)\right) = 0\), which contradicts our initial assumption.

Notice that, under the settings of Lemma 2.2, it does not always hold that \((x^*)^{-1}([-1]) \cap B_X = B_X \setminus U_X(x, 2)\).

**Proposition 2.1.** Let \(X\) be a Banach space. Let \((x, x^*) \in \Pi_X^\circ\). If \(x \notin \text{rot}(B_X)\), then \((x^*)^{-1}([-1]) \cap B_X \subseteq B_X \setminus U_X(x, 2)\).

**Proof.** In accordance with Lemma 2.2(2), \((x^*)^{-1}([-1]) \cap B_X \subseteq B_X \setminus U_X(x, 2)\). Since \((x, x^*) \in \Pi_X^\circ\), we have that \((x^*)^{-1}(\{1\}) \cap B_X = \{x\}\), so \((x^*)^{-1}([-1]) \cap B_X = \{-x\}\). Since \(x \notin \text{rot}(B_X)\), there exists a non-trivial segment of the unit sphere containing \(x\), that is, there exists \(y \in S_X \setminus \{x\}\) with \(\|\frac{x+y}{2}\| = 1\), in other words, \(\|x+y\| = 2\), meaning that \(-y \in B_X \setminus U_X(x, 2)\). Finally, \(-y \notin \{-x\} = (x^*)^{-1}([-1]) \cap B_X\). As a consequence, \((x^*)^{-1}([-1]) \cap B_X \subseteq B_X \setminus U_X(x, 2)\).

In \(\ell_\infty\), we will let \(1\) denote the constant sequence of general term 1. Also, \((\varepsilon_n)_{n \in \mathbb{N}}\) will denote the sequence of canonical unit vectors, that is, \(\varepsilon_n(m) = \delta_{nm}\) for all \(n, m \in \mathbb{N}\). And

\[
\delta_n : \ell_\infty \to \mathbb{R}, \quad x \mapsto \delta_n(x) := x(n)
\]

(2.3)
is the \(n^{th}\)-coordinate functional. Observe that \((1, \delta_1), (e_1, \delta_1) \in \Pi_{\ell_\infty}\).

**Proposition 2.2.** \(\eta_{\ell_\infty}(x, (1, \delta_1)) = 0\) for all \(x \in [0, 2]\) and \(\eta_{\ell_\infty}(2, (e_1, \delta_1)) = 2\).

**Proof.** According to Lemma 2.2(4), it only suffices to prove that \(\eta_{\ell_\infty}(2, (1, \delta_1)) = 0\). For this, we will show that \((\delta_1)^{-1}(\{1\}) \cap (B_{\ell_\infty} \setminus U_{\ell_\infty}(1, 2)) \neq \emptyset\). Indeed,

\[
\{-1, -1, -1, \ldots\} \in (\delta_1)^{-1}(\{1\}) \cap (B_{\ell_\infty} \setminus U_{\ell_\infty}(1, 2)).
\]

On the other hand, \(B_{\ell_\infty} \setminus U_{\ell_\infty}(e_1, 2) = (\delta_1)^{-1}([-1]) \cap B_{\ell_\infty}\), thus

\[
\eta_{\ell_\infty}(2, (e_1, \delta_1)) = d\left((\delta_1)^{-1}(\{1\}), B_{\ell_\infty} \setminus U_{\ell_\infty}(e_1, 2)\right) = d\left((\delta_1)^{-1}(\{1\}), (\delta_1)^{-1}([-1]) \cap B_{\ell_\infty}\right) = 2
\]
in view of Lemma 2.2(3).

Recall [1,6] that, given a Banach space \(X\) with unit sphere \(S_X\) and unit ball \(B_X\), a closed subspace \(Y \subseteq X\) is said to be an \(L^p\)-summand subspace of \(X\), where \(1 \leq p \leq \infty\), if \(Y\) is \(L^p\)-complemented in \(X\), that is, there exists a closed subspace \(Z \subseteq X\) such that \(X = Y \oplus_p Z\), in the sense that \(\|y + z\|_p = \|y\|_p + \|z\|_p\) for all \(y \in Y\) and all \(z \in Z\). A point \(x \in X\) is said to be an \(L^p\)-summand vector of \(X\) provided that \(\mathbb{R}x\) is an \(L^p\)-summand subspace of \(X\). In accordance with [1], every unit \(L^2\)-summand vector is a round point as well as a smooth point of the unit ball.

**Theorem 2.3.** Let \(X\) be a Banach space with \(\dim(X) \geq 2\). For every \((x, x^*) \in \Pi_X\) such that \(x\) is an \(L^2\)-summand vector of \(X\) and for every \(\varepsilon \in [0, 2]\),

\[
\eta_X(x, (x, x^*)) = \frac{\varepsilon^2}{2}.
\]
Proof. First off, notice that \( \ker(x^*) \) is the \( L^2 \)-complement of \( \mathbb{R} x \). For every \( z \in (x^*)^{-1}(\{1\}) \) and every \( y \in B_X \setminus U_X(x, \varepsilon) \), \( \|z - y\| \geq |x^*(z - y)| = |x^*(z) - x^*(y)| = 1 - x^*(y) \). Fix an arbitrary \( y \in B_X \setminus U_X(x, \varepsilon) \) and write \( y = x^*(y)x + m_y \) with \( m_y \in \ker(x^*) \). Notice that \( x^*(y)^2 + \|m_y\|^2 \leq 1 \) and \( (1 - x^*(y))^2 + \|m_y\|^2 = \|x - y\|^2 \geq \varepsilon^2 \). Then \( (1 - x^*(y))^2 \geq \varepsilon^2 - \|m_y\|^2 \geq \varepsilon^2 + x^*(y)^2 - 1 \). In other words, \( (1 - x^*(y))^2 + 1 - x^*(y)^2 \geq \varepsilon^2 \), that is, \( 2 - 2x^*(y) \geq \varepsilon^2 \). Going back to the beginning, for every \( z \in (x^*)^{-1}(\{1\}) \) and every \( y \in B_X \setminus U_X(x, \varepsilon) \), \( \|z - y\| \geq 1 - x^*(y) \geq \frac{\varepsilon^2}{2} \), which implies that \( \eta_X(\varepsilon, (x, x^*)) \geq \frac{\varepsilon^2}{2} \). Finally, since \( \dim(X) \geq 2 \), we can take any \( y \in S_X \cap S_X(x, \varepsilon) \) and \( z := x + m_y \), where \( m_y := y - x^*(y)x \). Simply notice that \( \|z - y\| = \|x + m_y\| = \|x + y - x^*(y)x - y\| = \|x - x^*(y)x\| = 1 - x^*(y) = \frac{\varepsilon^2}{2} \).

Corollary 2.1. If \( X \) is a Hilbert space with \( \dim(X) \geq 2 \), then \( \eta_X(\varepsilon) = \frac{\varepsilon^2}{2} \).

Proof. By bearing in mind [6], a Banach space is a Hilbert space if and only every unit vector is an \( L^2 \)-summand vector. Therefore, by applying Theorem 2.3, \( \eta_X(\varepsilon, (x, x^*)) = \frac{\varepsilon^2}{2} \) for every \( (x, x^*) \in \Pi_X \), obtaining the desired result.

Another index can be defined which lies in between the local modulus of convexity and the index of strong rotundity. Indeed, if \( X \) is a Banach space, then we define at \( (x, x^*) \in \Pi_X \) the following function:

\[
u_X(\cdot, (x, x^*)) : [0, 2] \to [0, 2], \quad \varepsilon \mapsto \nu_X(\varepsilon, (x, x^*)) := \inf \{1 - x^*(y) : \|y\| \leq 1, \|x - y\| \geq \varepsilon\}.
\]  

(2.4)

We remind the reader that the modulus of local convexity [12] at \( x \in S_X \) is given by

\[
\delta_X(\cdot, x) : [0, 2] \to [0, 1], \quad \varepsilon \mapsto \delta_X(\varepsilon, x) := \inf \{1 - \|x + \varepsilon y\| : \|y\| \leq 1, \|x - y\| \geq \varepsilon\}.
\]

(2.5)

Theorem 2.4. Let \( X \) be a Banach space. For every \( (x, x^*) \in \Pi_X \) and every \( \varepsilon \in [0, 2] \),

\[
2\delta_X(\varepsilon, x) \leq \nu_X(\varepsilon, (x, x^*)) \leq \eta_X(\varepsilon, (x, x^*)).
\]

Proof. On the one hand, for every \( z \in (x^*)^{-1}(\{1\}) \) and every \( y \in B_X \setminus U_X(x, \varepsilon) \),

\[
\|z - y\| \geq x^*(z - y) = 1 - x^*(y) \geq \nu_X(\varepsilon, (x, x^*)).
\]

As a consequence, \( \eta_X(\varepsilon, (x, x^*)) \geq \nu_X(\varepsilon, (x, x^*)) \). On the other hand, for every \( y \in B_X \setminus U_X(x, \varepsilon) \),

\[
\delta_X(\varepsilon, x) \leq 1 - \left\| \frac{x + y}{2} \right\| \leq 1 - x^*\left(\frac{x + y}{2}\right) = 1 - \frac{1 + x^*(y)}{2} = \frac{1 - x^*(y)}{2}.
\]

Therefore, \( \delta_X(\varepsilon, x) \leq \frac{1}{2} \nu_X(\varepsilon, (x, x^*)) \).

Let us finally tackle some applications to the stereographic projection. According to [9, 10], if \( X \) is a Banach space and \( (x, x^*) \in \Pi_X \), then

\[
S_X \setminus \{-x\} \to (x^*)^{-1}(\{1\}), \quad y \mapsto -x + \frac{y^2}{x^*(y) + 1}(y + x)
\]

(2.6)

is a well-defined and continuous function known as stereographic projection.
Definition 2.2. (Stereographic projection pair) Let $X$ be a Banach space. Let $(x, x^*) \in \Pi_X$. We will say that $(x, x^*)$ is a stereographic projection pair provided that the following conditions are satisfied:

- \(\text{st}(-x, y) \cap S_X = \{-x, y\}\) for all \(y \in S_X\).
- \(\text{st}(-x, z) \cap (S_X \setminus \{-x\}) \neq \emptyset\) for every \(z \in (x^*)^{-1}(\{1\})\).
- If \((y_j)_{j \in \mathbb{N}} \subseteq S_X \setminus \{-x\}\) is a sequence converging to \(-x\), then \(\left\| \frac{y_j + x}{x^*(y_j)} + 1 \right\| \to \infty\) as \(j \to \infty\).

The set of stereographic projection pairs will be denoted by \(\Pi^p_X\).

According to Theorem 2.1, the first condition in the above definition is equivalent to the fact that \(-x \in \text{rot}(B_X)\). The second condition is equivalent to the fact that \(x \in \text{sмо}(B_X)\) in view of Theorem 2.2. As a consequence, if \((x, x^*) \in \Pi^p_X\), then \(x \in \text{rot}(B_X) \cap \text{sмо}(B_X)\), hence \((x, x^*) \in \Pi_X\). Nevertheless, observe that the last condition of the previous definition is an unusual geometrical property in the sense that it only works for sequences in the unit sphere, but not for sequences in the unit ball. Indeed, if we take \(y_j := t_j x\) for all \(j \in \mathbb{N}\), where \((t_j)_{j \in \mathbb{N}} \subseteq (-1, 1)\) converges to \(-1\), then

\[
\left\| \frac{y_j + x}{x^*(y_j) + 1} = \left\| \frac{t_j x + x}{t_j + 1} \right\| = 1
\]

for all \(j \in \mathbb{N}\).

By bearing in mind [9, Lemma 2.1], if \(x \in S_X\) is an \(L^2\)-summand vector of a Banach space \(X\) and \(x^* \in S_{X^*}\), satisfies that \(\ker(x^*)\) is the \(L^2\)-complement of \(\mathbb{R}_x\), then \((x, x^*)\) is a stereographic projection pair. Since a Banach space is a Hilbert space if and only every unit vector is an \(L^2\)-summand vector [6], we obtain the following theorem whose proof we omit.

Theorem 2.5. If \(X\) is a Hilbert space, then \(\Pi_X = \Pi^p_X\).

The following theorem generalizes and improves [9, Lemma 2.1]. First, a technical remark is needed, which is a simple limit from a Calculus course.

Remark 2.1. For each \(p \in (1, \infty)\), \(\lim_{x \to -1^+} \frac{1 - (-x)^p}{(1 + x)^p} = +\infty\).

Theorem 2.6. Let \(X\) be a Banach space with \(\text{dim}(X) \geq 2\). Let \((x, x^*) \in \Pi_X\). If \(x\) is an \(L^p\)-summand vector of \(X\), for \(1 < p < \infty\), and \(\ker(x^*)\) is the \(L^p\)-complement of \(\mathbb{R}_x\), then \((x, x^*)\) is a stereographic projection pair.

Proof. In the first place, \(x\) is an \(L^p\)-summand vector of any 2-dimensional subspace containing it. Therefore, any 2-dimensional subspace containing \(x\) is linearly isometric to \(\ell_p^2\), which is rotund and smooth. Since rotund points and smooth points are 2-dimensional properties, we conclude that \(x \in \text{rot}(B_X) \cap \text{sмо}(B_X)\). By applying Theorem 2.1 and Theorem 2.2, we conclude that the first two conditions of Definition 2.2 are satisfied. Let us prove the third condition. Take any sequence \((y_j)_{j \in \mathbb{N}} \subseteq S_X \setminus \{-x\}\) converging to \(-x\). Let us write \(y_j = x^*(y_j) x + m_j\), where \(m_j \in \ker(x^*)\). Since \((x^*(y_j))_{j \in \mathbb{N}}\) converges to \(-1\), we may assume that \(-1 < x^*(y_j) < 0\) for all \(j \in \mathbb{N}\). Notice then that \(1 = (-x^*(y_j))^p + \|m_j\|^p\) for all \(j \in \mathbb{N}\). Then

\[
\left\| \frac{y_j + x}{x^*(y_j) + 1} \right\|^p = \frac{(1 + x^*(y_j))^p + \|m_j\|^p}{(x^*(y_j) + 1)^p} = \frac{(1 + x^*(y_j))^p}{(x^*(y_j) + 1)^p} + \frac{1 - (-x^*(y_j))^p}{(x^*(y_j) + 1)^p} \to \infty
\]

as \(j \to \infty\) in view of Remark 2.1.
The final result in this manuscript shows that stereographic projection pairs make possible that stereographic projections in Banach spaces be homeomorphisms, improving [9, Theorem 2.2]. However, let us recall first the following well-known topological fact [11].

**Remark 2.2.** Let $X, Y$ be topological spaces. Let $f : X \to Y$ be injective. Let $x \in X$. Suppose that for every net $(x_i)_{i \in I} \subseteq X$ such that $(f(x_i))_{i \in I}$ converges to $f(x)$, there exists a subnet $(z_j)_{j \in J}$ of $(x_i)_{i \in I}$ convergent to $x$. Then $f^{-1} : f(X) \to X$ is continuous at $f(x)$.

One can easily understand that, under the settings of Remark 2.2, if $X, Y$ are both first countable, then Remark 2.2 remains true if we switch nets with sequences.

**Theorem 2.7.** Let $X$ be a Banach space. If $(x, x^*) \in \Pi_X^p$, then the stereographic projection (2.6) is an homeomorphism.

**Proof.** First off, let us denote by $\phi$ to the stereographic projection (2.6). We already know that $\phi$ is well defined, continuous, and $\phi(y) \in \mathrm{st}(-x, y)$ for all $y \in S_X \setminus \{x\}$. Let us check now that $\phi$ is surjective. Fix an arbitrary $z \in (x^*)^{-1}(\{1\})$. If $z = x$, then $\phi(x) = x$. So let us assume that $z \neq x$. Since $(x, x^*) \in \Pi_X^p$, by definition we have that $\mathrm{st}(-x, z) \cap (S_X \setminus \{x\}) \neq \emptyset$. Let $u \in \mathbb{R} \setminus \{0\}$ such that $y := -x + u(z + x) \in S_X \setminus \{x\}$. We will show that $\phi(y) = z$. Indeed

$$\phi(y) = -x + 2 \frac{y + x}{x^*(y) + 1} = -x + 2 \frac{-x + u(z + x) + x}{x^*(-x + u(z + x)) + 1} = -x + 2 \frac{u(z + x)}{2u} = -x + (z + x) = z.$$  

Next step is to prove that $\phi$ is one-to-one. Indeed, take $y_1, y_2 \in S_X \setminus \{x\}$ with $\phi(y_1) = \phi(y_2)$. Then $y_2 = -x + \frac{x^*(y_2) + 1}{x^*(y_1) + 1}(y_1 + x) \in \mathrm{st}(-x, y_1) \cap S_X = \{x, y_1\}$, meaning that $y_1 = y_2$. Let us finally prove that $\phi^{-1}$ is continuous. We will rely on Remark 2.2 for sequences. Fix an arbitrary $y \in S_X \setminus \{x\}$. Take a sequence $(y_i)_{i \in \mathbb{N}} \subseteq S_X \setminus \{x\}$ such that $(\phi(y_i))_{i \in \mathbb{N}}$ converges to $\phi(y)$. We will show the existence of a subsequence $(y_{i_j})_{j \in \mathbb{N}}$ convergent to $y$. Indeed, there exists a subsequence $(y_{i_j})_{j \in \mathbb{N}}$ such that $(x^*(y_{i_j}))_{j \in \mathbb{N}}$ is convergent to some $r \in [-1, 1]$. Then $(\phi(y_{i_j}))_{j \in \mathbb{N}}$ converges to $\phi(y)$. This is equivalent to saying that $(\frac{y_{i_j} + x}{x^*(y_{i_j}) + 1})_{j \in \mathbb{N}}$ converges to $\frac{y + x}{x^*(y) + 1}$. Since $(x^*(y_{i_j}) + 1)_{j \in \mathbb{N}}$ is convergent to $r + 1$, we conclude that $(x^*(y_{i_j}) + 1)_{j \in \mathbb{N}}$ converges to $(r + 1) - x$, in other words, $(y_{i_j} + x)_{j \in \mathbb{N}}$ converges to $(r + 1) - x$. Since otherwise we obtain that $(y_{i_j})_{j \in \mathbb{N}}$ converges to $-x$, reaching the contradiction that $\frac{|y_{i_j} + x|}{|x^*(y_{i_j})|} \to \infty$ as $j \to \infty$ by bearing in mind that $(x, x^*) \in \Pi_X^p$. As a consequence, $-x + (r + 1) \frac{y + x}{x^*(y) + 1} \in \mathrm{st}(-x, y) \cap S_X = \{x, y\}$, that is, either $-x + (r + 1) \frac{y + x}{x^*(y) + 1} = -x$ or $-x + (r + 1) \frac{y + x}{x^*(y) + 1} = y$. If $-x + (r + 1) \frac{y + x}{x^*(y) + 1} = -x$, then $y = -x$, which is impossible since $y \in S_X \setminus \{x\}$. Thus, $-x + (r + 1) \frac{y + x}{x^*(y) + 1} = y$. By relying on Remark 2.2, we conclude that $\phi^{-1}$ is continuous at $b$.

### 3. Conclusions

The use of indices in the literature of Geometry of Banach Spaces has always been very useful to determine the exact shape of the unit ball. The most known indices are the modulus of convexity, the modulus of smoothness, the index of rotundity, and the Bishop-Phelps-Bollobás index, among others.
The index of strong rotundity is a novel concept introduced in this manuscript. This index serves to determine how far a point of the unit sphere is from being an strongly exposed point. As expected, Hilbert spaces have a very particular index of strong rotundity. Finally, applications to the stereographic projection are provided, in particular, a novel set of pairs (the stereographic projection pairs) are introduced in this manuscript which guarantee that the stereographic projection is an homeomorphism.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was funded by Ministerio de Ciencia, Innovación y Universidades: PGC-101514-B-I00 (Métodos analíticos en Simetrías, Teoría de Control y Operadores); and Consejería de Universidad, Investigación e Innovación de la Junta de Andalucía: FEDER-UCA18-105867 (Dispositivos electrónicos para la estimulación magnética transcraneal), ProyExcel00780 (Operator Theory: An interdisciplinary approach), and ProyExcel01036 (Multifísica y optimización multiobjetivo de estimulación magnética transcraneal).

Conflict of interest

The author declares that there is no conflict of interest.

References


©2023 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)