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Research article

Regularity and abundance on semigroups of transformations preserving an equivalence relation on an invariant set

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Abstract: Let $T(X)$ be the full transformation semigroup on a nonempty set X. For an equivalence relation E on X and a nonempty subset Y of X, let

$$\overline{S}_E(X,Y) = \{\alpha \in T(X) : \forall x, y \in Y, (x,y) \in E \Rightarrow (x\alpha, y\alpha) \in E, x\alpha, y\alpha \in Y\}.$$

Then $\overline{S}_E(X,Y)$ is a subsemigroup of $T(X)$ consisting of all full transformations that leave Y and the equivalence relation E on Y invariant. In this paper, we show that $\overline{S}_E(X,Y)$ is not regular in general and determine all its regular elements. Then we characterize relations $\mathcal{L}$, $\mathcal{L}^*$, $\mathcal{R}$ and $\mathcal{R}^*$ on $\overline{S}_E(X,Y)$ and apply these characterizations to obtain the abundance on such semigroup.

Keywords: transformation semigroup; Green's relations; abundant semigroup
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1. Introduction

Let X be an arbitrary nonempty set. The full transformation semigroup on X, denoted by $T(X)$, is the semigroup consisting of all mappings from X to X under the operation of composition of functions. It is well-known that $T(X)$ is a regular semigroup (see [3], for details). Moreover, every semigroup can be embedded in $T(X)$ for some appropriate set X.

For a fixed nonempty subset Y of X, let

$$S(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}.$$

Then $S(X, Y)$ is a semigroup of total transformations of X which leave a subset Y of X invariant. In 1975, Symons [17] described the automorphism group of this semigroup. In 2005, Nenthein, Youngkhong and Kemprasit [7] showed that the semigroup $S(X, Y)$ is regular if and only if $X = Y$ or Y contains exactly one element, and $R = \{\alpha \in S(X, Y) : X\alpha \cap Y = Y\alpha\}$ is the set of all regular elements of $S(X, Y)$. In addition, they determined the number of regular elements in $S(X, Y)$ when X is a finite set. In 2011, Honyam and Sanwong [4] characterized when $S(X, Y)$ is isomorphic to $T(Z)$ for some set Z and proved that every semigroup A can be embedded in $S(A^1, A)$. They also described Green's relations of the semigroup $S(X, Y)$, its group $\mathcal{H}$-classes, and its ideals. In 2013, Choomanee, Honyam and Sanwong [2] described left regular, right regular and intra-regular elements of $S(X, Y)$ and considered the relationships between these elements. Furthermore, they found the number of left regular elements of $S(X, Y)$ when X is a finite set. In [14], Sun and Wang studied the natural partial order in $S(X, Y)$. They determined when two elements of $S(X, Y)$ are related, found the elements which are compatible and described the maximal elements, the minimal elements and the greatest lower bound of two elements. Also, they showed that the semigroup $S(X, Y)$ is abundant. In [13], Sun L. and Sun J. investigated all the elements in the semigroup $S(X, Y)$ which are left compatible with respect to the natural partial order. In [1], Chinram and Baupradist characterized left magnifying elements and right magnifying elements of semigroups of transformations with invariant set.

Let E be an equivalence relation on a set X. Consider the following subset of $T(X)$:

$$T_E(X) = \{\alpha \in T(X) : \forall x, y \in X,\ (x, y) \in E \text{ implies } (x\alpha, y\alpha) \in E\}.$$

It is obvious that if E is a non-trivial equivalence relation, then $T_E(X)$ is a proper subsemigroup of $T(X)$ and if E is the identity or universal relation, then $T_E(X)$ and $T(X)$ are identical. In 1994 and 1996, Pei discussed α-congruences and some regular subsemigroup inducing a certain lattice on $T_E(X)$ [9, 10]. In 2005, Pei [8] investigated regularity of elements and Green's relations on the semigroup $T_E(X)$. In [11], Pei determined the rank of the homeomorphism group and considered the rank of $T_E(X)$ when X is a finite set and each class of the equivalence E has the same cardinality. Furthermore, he also studied the rank of $\Gamma(X)$, the semigroup of all closed function α on a topological space X for which E classes form a basis. In 2008, Sun, Pei and Cheng [15] characterized the natural partial order on the semigroup $T_E(X)$. The compatibility of multiplication and all compatible elements were investigated. Moreover, they found maximal, minimal and covering elements with respect to the order. In 2011, Pei and Zhou [12] considered the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on the semigroup $T_E(X)$ and the equivalence relations E under which $T_E(X)$ becomes abundant. In 2019, Sun [16] investigated the left and right compatibility with respect to the natural partial order on $T_E(X)$. Kaewnoi, Petapirak and Chinram studied left and right magnifying elements of the semigroup $T_E(X)$ in [5].

In our work, we introduce a new transformation subsemigroup of $T(X)$ by letting Y be a fixed nonempty subset of X and define

$$\overline{S}_E(X, Y) = \{\alpha \in T(X) : \forall x, y \in Y, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E, x\alpha, y\alpha \in Y\}.$$

Then $\overline{S}_E(X, Y)$ is a generalization of all aforementioned semigroups. In Section 3, we describe all regular elements on $\overline{S}_E(X, Y)$ and give necessary and sufficient conditions for $\overline{S}_E(X, Y)$ to be regular. In Section 4, we characterize relations $\mathcal{L}$, $\mathcal{L}^*$, $\mathcal{R}$ and $\mathcal{R}^*$ on $\overline{S}_E(X, Y)$. Consequently we prove that $\overline{S}_E(X, Y)$ is always left abundant but not right abundant, in general. Also, we determine the conditions for $\overline{S}_E(X, Y)$ to be abundant.

2. Preliminary and notation

This section provides some basic properties of $\overline{S}_E(X, Y)$ and notation which will be used throughout the paper. In addition, [3] is suggested for more basic concepts in semigroup theory.

Let X be a nonempty set and E an equivalence relation on X. Hereafter, we denote by X/E the quotient set of X by E, i.e., the set of all E-classes. By $A \in X/E$, we mean that A is an equivalence class of E that can be viewed both as an element of X/E and as a subset of X. Let $\mathcal{A}$ and $\mathcal{B}$ be two collections of nonempty subsets of X. Then $\mathcal{A}$ is said to refine $\mathcal{B}$, if for each $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $A \subseteq B$. For $\alpha, \beta \in T(X)$, the composition of α and β, denoted by $\alpha\beta$, is a mapping obtained by performing first α and then β. The notation $x\alpha$ means the image of x under α and $X\alpha = \{x\alpha : x \in X\}$ is the range of α. Moreover, $\pi(\alpha)$ is a partition of X induced by α, i.e.,

$$\pi(\alpha) = \{x\alpha^{-1} : x \in X\alpha\},$$

where $x\alpha^{-1}$ is the inverse image of x under α. Note that by a slight abuse of notation, we write $P\alpha = x$, where $P = x\alpha^{-1} \in \pi(\alpha)$. Particularly,

$$\varepsilon(\alpha) = \{A\alpha^{-1} : A \in X/E \text{ and } A \cap X\alpha \neq \emptyset\},$$

where $A\alpha^{-1} = \bigcup\{x\alpha^{-1} : x \in A \cap X\alpha\}$. Obviously, $\pi(\alpha)$ refines $\varepsilon(\alpha)$. For a nonempty subset Y of X, we define restrictions to Y of $\pi(\alpha)$ and $\varepsilon(\alpha)$ by

$$\pi_Y(\alpha) = \{P \in \pi(\alpha) : P \cap Y \neq \emptyset\}$$

and

$$\varepsilon_Y(\alpha) = \{(A \cap Y)\alpha^{-1} : A \in X/E \text{ and } A \cap X\alpha \cap Y \neq \emptyset\}.$$

Let Y be a nonempty subset of a set X. Consider

$$\overline{S}_E(X, Y) = \{\alpha \in T(X) : \forall x, y \in Y, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E, x\alpha, y\alpha \in Y\}.$$

We can see that $\overline{S}_E(X, Y)$ is a subsemigroup of $T(X)$ and $S(X, Y)$ and id_X, the identity map on X, belongs to $\overline{S}_E(X, Y)$. We first present relationships between $\overline{S}_E(X, Y)$ and some famous transformation semigroups as follows:

Proposition 2.1. *$\overline{S}_E(X, Y) \subseteq T_E(X)$ if and only if either one of the following hold:*

(1) $E = X \times X$;

(2) for each $z \in X \setminus Y$, $a \in X$, $(z, a) \in E$ implies $z = a$.

Consequently, $\overline{S}_E(X, Y) = T_E(X)$ if and only if $Y = X$.

Proof. Assume that $E \neq X \times X$ and there exists $(z, a) \in E$, such that $z \in X \setminus Y$, $a \in X$ in which $z \neq a$. Since $E \neq X \times X$, $|X/E| \geq 2$, and so there exists $A \in X/E$ such that $z \notin A$.

Case 1. $A \cap Y \neq \emptyset$. Let $b \in A \cap Y$ and $\alpha : X \to X$ be defined by

$$x\alpha = \begin{cases} z, & \text{if } x = z, \\ b, & \text{otherwise.} \end{cases}$$

Let $x, y \in Y$ and $(x, y) \in E$. Then $(x\alpha, y\alpha) = (b, b) \in E$, which implies $\alpha \in \overline{S}_E(X, Y)$. However, $\alpha \notin T_E(X)$ since $(z\alpha, a\alpha) = (z, b) \notin E$.

Case 2. $A \cap Y = \emptyset$. Choose $b \in A$ and $c \in Y$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b, & \text{if } x = z, \\ c, & \text{otherwise.} \end{cases}$$

Let $x, y \in Y$ and $(x, y) \in E$. Then $(x\alpha, y\alpha) = (c, c) \in E$, which implies $\alpha \in \overline{S}_E(X, Y)$. However, $\alpha \notin T_E(X)$ since $(z\alpha, a\alpha) = (b, c) \notin E$.

The converse is obvious in the case of $E = X \times X$. Assume $E \neq X \times X$ and (2) holds. Let $\alpha \in \overline{S}_E(X, Y)$ and let $x, y \in X$ be such that $(x, y) \in E$. If $x \in Y$, then $y \in Y$. Thus $(x\alpha, y\alpha) \in E$. If $x \notin Y$, then $x = y$ by (2) and so $(x\alpha, y\alpha) \in E$. Hence $\alpha \in T_E(X)$.

Consequently, if $Y \subsetneq X$, there exists $z \in X \setminus Y$ and $X_z \in T_E(X)$, where X_z is a constant map on X with $X\alpha = \{z\}$. Clearly, $\alpha \notin \overline{S}_E(X, Y)$ and so $T_E(X) \neq \overline{S}_E(X, Y)$. □

Proposition 2.2. $\overline{S}_E(X, Y) = S(X, Y)$ *if and only if* $Y \times Y \subseteq E$ *or* $(Y \times Y) \cap E = id_Y$*, where* id_Y *is the identity relation on* Y*.*

Proof. Assume that $Y \times Y \nsubseteq E$ and $(Y \times Y) \cap E \neq id_Y$. Then there exist distinct $a, b \in Y$ such that $(a, b) \notin E$ and there exist distinct $c, d \in Y$ such that $(c, d) \in E$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a, & \text{if } x = c, \\ b, & \text{if } x = d, \\ x, & \text{otherwise.} \end{cases}$$

Then $\alpha \in S(X, Y)$, but $\alpha \notin \overline{S}_E(X, Y)$ since $(c\alpha, d\alpha) = (a, b) \notin E$.

Assume that $Y \times Y \subseteq E$ or $(Y \times Y) \cap E = id_Y$. Clearly, $\overline{S}_E(X, Y) \subseteq S(X, Y)$. Let $\alpha \in S(X, Y)$ and $x, y \in Y$ be such that $(x, y) \in E$. Then $x\alpha, y\alpha \in Y$. If $Y \times Y \subseteq E$, then $(x\alpha, y\alpha) \in E$. If $(Y \times Y) \cap E = id_Y$, then $x = y$, and so $(x\alpha, y\alpha) \in E$. Hence, $\alpha \in \overline{S}_E(X, Y)$. □

3. Regularity on $\overline{S}_E(X, Y)$

Recall that, for an element a in a semigroup S, a is said to be regular if there exists $x \in S$ such that $a = axa$. In particular, if all elements of S are regular, then S is called a regular semigroup. We begin this section by characterizing all regular elements in $\overline{S}_E(X, Y)$. To do this, the following lemma is a crucial tool.

Lemma 3.1. *Let* $\alpha \in \overline{S}_E(X, Y)$*. Then, for each* $A \in X/E$*, there exists* $B \in X/E$ *such that* $(A \cap Y)\alpha \subseteq B \cap Y \neq \emptyset$*.*

Proof. Let $A \in X/E$. It is clear in the case of $A \cap Y = \emptyset$. Let $x \in A \cap Y$. Then there exists $B \in X/E$ such that $x\alpha \in B$ and so $B \cap Y \neq \emptyset$. To show $(A \cap Y)\alpha \subseteq B \cap Y$, we let $y \in A \cap Y$. Since x and y belong to the same partition, $(x, y) \in E$. Since $x, y \in Y$ and $\alpha \in \overline{S}_E(X, Y)$, $(x\alpha, y\alpha) \in E$ and $y\alpha \in Y$. Hence, $y\alpha \in B \cap Y$, as required. □

Theorem 3.1. *Let* $\alpha \in \overline{S}_E(X, Y)$*. Then* α *is regular if and only if, for each* $A \in X/E$*, there exists* $B \in X/E$ *such that* $B \cap Y \neq \emptyset$ *and* $A \cap X\alpha \cap Y \subseteq (B \cap Y)\alpha$*.*

Proof. Assume α is regular and let $A \in X/E$. Then $\alpha = \alpha\beta\alpha$ for some $\beta \in \overline{S}_E(X, Y)$. From Lemma 3.1, there exists $B \in X/E$ such that $(A \cap Y)\beta \subseteq B \cap Y \neq \emptyset$. For each $x \in A \cap X\alpha \cap Y$, we have $x = a\alpha$ for some $a \in X$, and so $x = a\alpha = a\alpha\beta\alpha = x\beta\alpha \in (B \cap Y)\alpha$. Hence, $A \cap X\alpha \cap Y \subseteq (B \cap Y)\alpha$.

Conversely, assume that the condition hold. For each $A \in X/E$ in which $A \cap Y \neq \emptyset$, by the assumption, we can fix $B_A \in X/E$ such that $B_A \cap Y \neq \emptyset$ and $A \cap X\alpha \cap Y \subseteq (B_A \cap Y)\alpha$. Let $y_A \in B_A \cap Y$ be fixed. Consider $x \in A \cap Y$. If $x \in A \cap Y \cap X\alpha$, then we choose $x' \in B_A \cap Y$ such that $x'\alpha = x$. If $x \in (A \cap Y) \setminus X\alpha$, then we set $x' = y_A$. For each $x \in X\alpha \setminus Y$, we choose $x' \in X$ such that $x'\alpha = x$. Now, define $\beta : X \to X$ by

$$x\beta = \begin{cases} x', & \text{if } x \in X\alpha \cup Y, \\ x, & \text{otherwise.} \end{cases}$$

To show $\beta \in \overline{S}_E(X, Y)$, let $x, y \in Y$ in which $(x, y) \in E$. It is clear that $x\beta, y\beta \in Y$ and x and y belong to the same equivalence class, say A. Then $x\beta = x'$ and $y\beta = y'$ are both in B_A, that is, $(x\beta, y\beta) \in E$. To show $\alpha = \alpha\beta\alpha$, let $x \in X$. Then $x\alpha\beta\alpha = (x\alpha)\beta\alpha = (x\alpha)'\alpha = x\alpha$. This completes the proof. □

Theorem 3.2. *Let $\emptyset \neq Y \subseteq X$. Then $\overline{S}_E(X, Y)$ is a regular semigroup if and only if one of the following conditions hold:*

(1) $|Y| = 1$.

(2) $Y = X$ *and* $E = X \times X$.

(3) $Y = X$ *and* $E = id_X$.

Proof. Assume that all conditions are false.
Case 1. $|Y| \neq 1$ and $Y \neq X$. Then there exist distinct $a, b \in Y$ and $X \setminus Y \neq \emptyset$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b, & \text{if } x \in Y, \\ a, & \text{otherwise.} \end{cases}$$

It is clear that $\alpha \in \overline{S}_E(X, Y)$. Let $A \in X/E$ such that $a \in A$. For each $B \in X/E$, we have $(B \cap Y)\alpha \subseteq Y\alpha = \{b\}$. Since $a \in A \cap Y \cap X\alpha$, we obtain $A \cap X\alpha \cap Y \nsubseteq (B \cap Y)\alpha$ for all $B \in X/E$. By Theorem 3.1, α is not regular.
Case 2. $|Y| \neq 1$, $Y = X$, $E \neq X \times X$ and $E \neq id_X$. Then there exists $A \in X/E$ such that $A \neq X$ and $|A| \geq 2$. Let a and b be two distinct elements in A. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a, & \text{if } x \in A, \\ b, & \text{otherwise.} \end{cases}$$

It is clear that $\alpha \in \overline{S}_E(X, Y)$ and $(B \cap X)\alpha = B\alpha$ is a singleton set, for all $B \in X/E$. Since $a, b \in A \cap X\alpha \cap X$, we get $A \cap X\alpha \cap X \nsubseteq (B \cap X)\alpha$. By Theorem 3.1, α is not regular.

Conversely, assume one of three aforementioned conditions holds. Let $\alpha \in \overline{S}_E(X, Y)$. If $|Y| = 1$, then we let $Y = \{y\}$. Hence, there exists $B \in X/E$ such that $y \in B$. In this case, $A \cap X\alpha \cap Y \subseteq \{y\} = (B \cap Y)\alpha$ for all $A \in X/E$, and so α is regular. For the case of $Y = X$ and $E = X \times X$, we have X is exactly one equivalence class in X/E and $X \cap Y \cap X\alpha = X\alpha = (X \cap Y)\alpha$. This implies α is regular. Finally, if $Y = X$ and $X = id_X$, then each equivalence class in X/E is a singleton set of elements in $X = Y$. Consider $\{x\} \in X/E$. If $x \notin X\alpha$, then $\{x\} \cap Y \cap X\alpha = \emptyset \subseteq (\{x\} \cap Y)\alpha$. If $x \in X\alpha$, then $x = x'\alpha$ for some

$x' \in X$. Hence, $\{x\} \cap X \cap X\alpha = \{x\} = (\{x'\} \cap X)\alpha$, and so α is regular. Therefore, $\overline{S}_E(X, Y)$ is regular, as required. □

If $X = Y$, then $\overline{S}_E(X, Y) = T_E(X)$, and we have the following corollaries, which first appeared in [8].

Corollary 3.1. *[8] $\alpha \in T_E(X)$ is regular if and only if for each $A \in X/E$, there exists $B \in X/E$ such that $A \cap X\alpha \subseteq B\alpha$.*

Corollary 3.2. *[8] $T_E(X)$ is regular if and only if $E = id_X$ or $X \times X$.*

4. Abundance of $\overline{S}_E(X, Y)$

For any semigroup S, denote by S^1 the semigroup obtained from S by adjoining an identity if S has no identity and $S^1 = S$ if S is a monoid. For any $a, b \in S$, define

$$a\,\mathcal{L}\,b \text{ if and only if } S^1a = S^1b,$$

or equivalently, $a\,\mathcal{L}\,b$ if and only if $a = xb, b = ya$ for some $x, y \in S^1$. Dually, define

$$a\,\mathcal{R}\,b \text{ if and only if } aS^1 = bS^1,$$

or equivalently, $a\,\mathcal{R}\,b$ if and only if $a = bx, b = ay$ for some $x, y \in S^1$. Moreover,

$$a\,\mathcal{L}^*\,b \text{ if and only if } a\,\mathcal{L}\,b \text{ in some oversemigroup of } S.$$

Analogously,

$$a\,\mathcal{R}^*\,b \text{ if and only if } a\,\mathcal{R}\,b \text{ in some oversemigroup of } S.$$

It is well-known that all four aforementioned relations are equivalence relations on S in which $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$. Particularly, if S is a regular semigroup, then $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{R} = \mathcal{R}^*$. In addition, S is said to be left abundant if each $\mathcal{L}^*$-class contains an idempotent. Right abundant semigroup is defined dually. If S is both left and right abundant, then S is called an abundant semigroup. Since all $\mathcal{L}$-classes and $\mathcal{R}$-classes of any regular semigroups always contain at least one idempotent, all regular semigroups are abundant.

In fact, $\overline{S}_E(X, Y)$ is not regular in general. Here, we study the relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{L}^*$ and $\mathcal{R}^*$ on $\overline{S}_E(X, Y)$ and apply the results to describe the condition for $\overline{S}_E(X, Y)$ to be abundant.

Theorem 4.1. *Let $\alpha, \beta \in \overline{S}_E(X, Y)$. Then $\alpha = \gamma\beta$ for some $\gamma \in \overline{S}_E(X, Y)$ if and only if $X\alpha \subseteq X\beta$ and, for each $A \in X/E$, there exists $B \in X/E$ such that $(A \cap Y)\alpha \subseteq (B \cap Y)\beta$. Consequently, $\alpha\,\mathcal{L}\,\beta$ if and only if $X\alpha = X\beta$ and, for each $A \in X/E$, there exist $B, C \in X/E$ such that $(A \cap Y)\alpha \subseteq (B \cap Y)\beta$ and $(A \cap Y)\beta \subseteq (C \cap Y)\alpha$.*

Proof. Assume $\alpha = \gamma\beta$ for some $\gamma \in \overline{S}_E(X, Y)$. Clearly, $X\alpha = (X\gamma)\beta \subseteq X\beta$. Let $A \in X/E$. By Lemma 3.1, there exists $B \in X/E$ such that $(A \cap Y)\gamma \subseteq B \cap Y$. Hence, $(A \cap Y)\alpha = (A \cap Y)\gamma\beta \subseteq (B \cap Y)\beta$.

Conversely, assume the conditions hold. For each $A \in X/E$, we fix $A' \in X/E$ such that $(A \cap Y)\alpha \subseteq (A' \cap Y)\beta$. Let $x \in X$. Then there exists $A_x \in X/E$ such that $x \in A_x$. If $x \in Y$, we choose $x' \in A'_x \cap Y$ such that $x\alpha = x'\beta$. If $x \notin Y$, then there exists $x' \in X$ such that $x\alpha = x'\beta$. Define $\gamma : X \to X$ by $x\gamma = x'$ for all $x \in X$. To show $\gamma \in \overline{S}_E(X, Y)$, let $x, y \in Y$ in which $(x, y) \in E$. Then x, y belong to the same equivalence class in X/E, say A. Hence $x', y' \in A' \cap Y$. Therefore, $(x\gamma, y\gamma) = (x', y') \in E$ and $x\gamma, y\gamma \in Y$. For each $x \in X$, we have $x\gamma\beta = x'\beta = x\alpha$. Hence $\alpha = \gamma\beta$. This completes the proof. □

Theorem 4.2. *Let $\alpha, \beta \in \overline{S}_E(X, Y)$. Then $\alpha = \beta\gamma$ for some $\gamma \in \overline{S}_E(X, Y)$ if and only if $\pi(\beta)$ refines $\pi(\alpha)$ and $\varepsilon_Y(\beta)$ refines $\varepsilon_Y(\alpha)$. Consequently, $\alpha \mathcal{R} \beta$ if and only if $\pi(\beta) = \pi(\alpha)$ and $\varepsilon_Y(\beta) = \varepsilon_Y(\alpha)$.*

Proof. Assume that $\alpha = \beta\gamma$ for some $\gamma \in \overline{S}_E(X, Y)$. It is clear that $\pi(\beta)$ refines $\pi(\alpha)$. To show that $\varepsilon_Y(\beta)$ refines $\varepsilon_Y(\alpha)$, let $U \in \varepsilon_Y(\beta)$. Then $U = (A \cap Y)\beta^{-1}$ and $A \cap X\beta \cap Y \neq \emptyset$ for some $A \in X/E$. Thus $U\beta \subseteq A \cap Y$. Since $\gamma \in \overline{S}_E(X, Y)$, we get $(A \cap Y)\gamma \subseteq B \cap Y \neq \emptyset$ for some $B \in X/E$. This implies $U\alpha = U\beta\gamma \subseteq (A \cap Y)\gamma \subseteq B \cap Y$. Hence, $U \subseteq (B \cap Y)\alpha^{-1} \in \varepsilon_Y(\alpha)$, as required.

On the other hand, assume the conditions hold. For each $x \in X\beta$, choose $x' \in X$ such that $x = x'\beta$. For each $A \in X/E$ in which $A \cap X\beta \cap Y \neq \emptyset$, choose $x_A \in A \cap X\beta \cap Y$. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} x'\alpha, & \text{if } x \in X\beta, \\ x'_A\alpha, & \text{if } x \in A \cap Y \setminus X\beta \text{ and } A \cap X\beta \cap Y \neq \emptyset \text{ and } A \in X/E, \\ x, & \text{otherwise.} \end{cases}$$

To show $\gamma \in \overline{S}_E(X, Y)$, let $x, y \in Y$ and $(x, y) \in E$. Then there exists $A \in X/E$ such that $x, y \in A$. Clearly, $x\gamma, y\gamma \in Y$. If $A \cap X\beta \cap Y = \emptyset$, then $(x\gamma, y\gamma) = (x, y) \in E$. If $A \cap X\beta \cap Y \neq \emptyset$, then $(x, x_A), (y, x_A) \in E$. Assume $x \in X\beta$ and $y \notin X\beta$ (the other cases can be proved similar). Then $x, x_A \in A \cap X\beta \cap Y$, and so $x\beta^{-1}, x_A\beta^{-1} \subseteq (A \cap Y)\beta^{-1} \in \varepsilon_Y(\beta)$. Since $\varepsilon_Y(\beta) = \varepsilon_Y(\alpha)$, $(A \cap Y)\beta^{-1} = (B \cap Y)\alpha^{-1}$ for some $B \in X/E$ such that $B \cap Y \cap X\alpha \neq \emptyset$. Since $x' \in x\beta^{-1}$ and $x'_A \in x_A\beta^{-1}$, we get $x', x'_A \in (A \cap Y)\beta^{-1} = (B \cap Y)\alpha^{-1}$. Hence, $x'\alpha, x'_A\alpha \in B$ which yields $(x\gamma, y\gamma) = (x'\alpha, x'_A\alpha) \in E$. Let $x \in X$ be such that $x\beta = y$. Since $y'\beta = y$, $x, y' \in y\beta^{-1} \in \pi(\beta) = \pi(\alpha)$, we obtain $x\alpha = y'\alpha$. This implies that $x\beta\gamma = y\gamma = y'\alpha = x\alpha$ and so $\alpha = \beta\gamma$. □

Next, we provide necessary and sufficient conditions for any two elements of $\overline{S}_E(X, Y)$ to be $\mathcal{L}^*$-related and $\mathcal{R}^*$-related. To prove the results, we need the following three lemmas.

Lemma 4.1. *[6] Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*

(1) $a \mathcal{L}^* b$.

(2) For all $x, y \in S^1$, $ax = ay$ if and only if $bx = by$.

Lemma 4.2. *[6] Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*

(1) $a \mathcal{R}^* b$.

(2) For all $x, y \in S^1$, $xa = ya$ if and only if $xb = yb$.

Lemma 4.3. *[3] Let $\alpha, \beta \in T(X)$.*

(1) $\alpha \mathcal{L} \beta$ if and only if $X\alpha = X\beta$.

(2) $\alpha \mathcal{R} \beta$ if and only if $\pi(\alpha) = \pi(\beta)$.

Theorem 4.3. *Let $\alpha, \beta \in \overline{S}_E(X, Y)$. Then $\alpha \mathcal{L}^* \beta$ if and only if $X\alpha = X\beta$.*

Proof. Assume that $X\alpha = X\beta$. By Lemma 4.3(1), $\alpha \mathcal{L} \beta$ in $T(X)$. As $T(X)$ is an oversemigroup of $\overline{S}_E(X, Y)$ then (by definition of $\mathcal{L}^*$) $\alpha \mathcal{L}^* \beta$.

Conversely, assume $\alpha \mathcal{L}^* \beta$. Let $A \in X/E$. If $A \cap X\alpha \cap Y \neq \emptyset$, then we choose $x_A \in A \cap X\alpha \cap Y$. If $A \cap X\alpha \cap Y = \emptyset$, then we choose $x_A \in X\alpha \cap Y$. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} x, & \text{if } x \in X\alpha, \\ x_A, & \text{if } x \in A \setminus X\alpha \text{ for some } A \in X/E. \end{cases}$$

Let $x, y \in Y$ be such that $(x, y) \in E$. Then $x, y \in A$ for some $A \in X/E$. If $A \cap X\alpha \cap Y \neq \emptyset$, then $x, y, x_A \in A \cap Y$, which implies $(x\gamma, y\gamma) \in E$ and $x\gamma, y\gamma \in Y$. If $A \cap X\alpha \cap Y = \emptyset$, then $x, y \notin X\alpha$. Hence, $(x\gamma, y\gamma) = (x_A, x_A) \in E$ and $x\gamma, y\gamma \in Y$. Thus $\gamma \in \overline{S}_E(X, Y)$ in which $\alpha\gamma = \alpha = \alpha id_X$. By Lemma 4.1, $\beta\gamma = \beta id_X = \beta$. This implies that $X\beta = X\beta\gamma \subseteq X\gamma = X\alpha$. By the same argument, we can show that $X\alpha \subseteq X\beta$. Therefore, $X\alpha = X\beta$, as required. □

Notice that the element γ, as define in the proof of Theorem 4.3, is an idempotent and $X\gamma = X\alpha$. Therefore, an arbitrary $\mathcal{L}^*$-class of $\overline{S}_E(X, Y)$ contains an idempotent and we obtain the following:

Theorem 4.4. *The semigroup $\overline{S}_E(X, Y)$ is left abundant.*

Theorem 4.5. *Let $\alpha, \beta \in \overline{S}_E(X, Y)$. Then $\alpha\mathcal{R}^*\beta$ if and only if $\pi(\alpha) = \pi(\beta)$.*

Proof. Assume that $\pi(\alpha) = \pi(\beta)$. By Lemma 4.3(2), $\alpha\mathcal{R}\beta$ in $T(X)$ and so $\alpha\mathcal{R}^*\beta$.

Conversely, assume $\alpha\mathcal{R}^*\beta$. Let $a, b \in X$. To show $a\alpha = b\alpha$ if and only if $a\beta = b\beta$, we first assume, for the if part, that $a\alpha = b\alpha$.

Case 1. $a \in Y$ or $b \in Y$. Without loss of generality, we assume that $b \in Y$. Since X/E is a partition of X, there exists $A \in X/E$ such that $a \in A$. If $A \cap Y = Y$ or $a \notin Y$, then we let $Z = Y \setminus \{a\}$. If $A \cap Y \neq Y$ and $a \in Y$, then we set $Z = Y \setminus A$. Define $\gamma, \delta : X \to X$ by

$$x\gamma = \begin{cases} b, & \text{if } x \in Z, \\ a, & \text{otherwise,} \end{cases}$$

and $x\delta = b$ for all $x \in X$. Clearly, $\gamma, \delta \in \overline{S}_E(X, Y)$ and $\gamma\alpha = \delta\alpha$. By Lemma 4.2, we obtain $\gamma\beta = \delta\beta$ and so $a\beta = a\gamma\beta = a\delta\beta = b\beta$.

Case 2. $a, b \notin Y$. Choose $c \in Y$. Define $\gamma, \delta : X \to X$ by

$$x\gamma = \begin{cases} c, & \text{if } x \in Y, \\ a, & \text{otherwise,} \end{cases}$$

and

$$x\delta = \begin{cases} c, & \text{if } x \in Y, \\ a, & \text{if } x = a, \\ b, & \text{otherwise.} \end{cases}$$

Clearly, $\gamma, \delta \in \overline{S}_E(X, Y)$ and $\gamma\alpha = \delta\alpha$. By Lemma 4.2, we obtain $\gamma\beta = \delta\beta$ and so $a\beta = b\gamma\beta = b\delta\beta = b\beta$.

The only part can be proved similar. Therefore, $\pi(\alpha) = \pi(\beta)$, as required. □

However, $\overline{S}_E(X, Y)$ is probably not right abundant. Consider $X = \{1, 2, 3, 4, 5, 6\}$, E is an equivalence relation on X such that $X/E = \{\{1, 2, 3\}, \{4, 5\}, \{6\}\}$ and

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 2 & 2 & 3 & 3 \end{pmatrix} \in \overline{S}_E(X, X).$$

We can see that there is no idempotent $\gamma \in \overline{S}_E(X, X)$ such that $\pi(\gamma) = \pi(\alpha)$. Hence, the $\mathcal{R}^*$-class containing α has no idempotent.

In the last part of this paper, we will present the the conditions for being abundant of $\overline{S}_E(X, Y)$.

Theorem 4.6. *$\overline{S}_E(X, Y)$ is abundant if and only if $|\{A \in X/E : |A \cap Y| \geq 2\}| \leq 1$ or $|A \cap Y| < 3$ for all $A \in X/E$.*

Proof. Assume that there exist $A, B \in X/E$ such that $|A \cap Y| \geq 3$, $|B \cap Y| \geq 2$ and $A \neq B$. Choose $a_1, a_2, a_3 \in A \cap Y$ and $b_1, b_2 \in B \cap Y$ that are all distinct elements. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a_1, & \text{if } x = a_3, \\ a_2, & \text{if } x = b_1, \\ a_3, & \text{if } x = b_2, \\ x, & \text{otherwise.} \end{cases}$$

Then $\alpha \in \overline{S}_E(X, Y)$ and $\{\{a_1, a_3\}, \{a_2, b_1\}, \{b_2\}\} \subseteq \pi(\alpha)$. Let $\gamma \in \overline{S}_E(X, Y)$ be such that $\gamma^2 = \gamma$. We will show that $(\alpha, \gamma) \notin \mathcal{R}^*$. Suppose that $(\alpha, \gamma) \in \mathcal{R}^*$. Then $\pi(\alpha) = \pi(\gamma)$. Since $\gamma^2 = \gamma$, we have $b_2\gamma = b_2 \in B$. From $(b_1, b_2) \in E$, we get $b_1\gamma, b_2\gamma \in B$. Hence $a_2\gamma = b_1\gamma \in B$. From $(a_1, a_2) \in E$, we obtain $(a_1\gamma, a_2\gamma) \in E$. Thus $a_1\gamma \in B$. Since $a_1\gamma \in \{a_1, a_3\}$, we have $\{a_1, a_3\} \cap B \neq \emptyset$ which is a contradition. Therefore $(\alpha, \gamma) \notin \mathcal{R}^*$. Hence $\overline{S}_E(X, Y)$ is not abundant.

Conversely, assume $|\{A \in X/E : |A \cap Y| \geq 2\}| \leq 1$ or $|A \cap Y| < 3$ for all $A \in X/E$.

Case 1. $|A \cap Y| < 3$ for all $A \in X/E$. Let $\alpha \in \overline{S}_E(X, Y)$. For each $A \in X/E$ such that $A \cap X\alpha \cap Y \neq \emptyset$, let $R_A = \{x\alpha^{-1} : x \in A \cap X\alpha \cap Y\}$. By assumption, we have $|R_A| \leq 2$. Now, let $A \in X/E$ be such that $A \cap X\alpha \cap Y \neq \emptyset$ and $R_A \nsubseteq \pi_{B\cap Y}(\alpha)$ for all $B \in X/E$. For each $P \in R_A$, if $P \cap Y \neq \emptyset$, then fix $x_P \in P \cap Y$. If $P \cap Y = \emptyset$, then fix $x_P \in P$. Next, let $A \in X/E$ be such that $A \cap X\alpha \cap Y \neq \emptyset$ and there exists $B \in X/E$ such that $R_A \subseteq \pi_{B\cap Y}(\alpha)$. Choose $A' \in X/E$ such that $R_A \subseteq \pi_{A'\cap Y}(\alpha)$. Hence $P \cap A' \cap Y \neq \emptyset$ for all $P \in R_A$. We fix $x_P \in P \cap A' \cap Y$ for all $P \in R_A$. Finally, for $P \in \pi(\alpha)$ such that $P = x\alpha^{-1}$ and $x \in X\alpha \setminus Y$, fix $x_P \in P$. Define $\gamma : X \to X$ by $x\gamma = x_P$ where $P \in \pi(\alpha)$ and $x \in P$. Let $x \in X$. Then there exists a unique $P \in \pi(\alpha)$ such that $x \in P$. Let $P = y\alpha^{-1}$ for some $y \in X\alpha \cap Y$. Thus $P \in R_A$ for some $A \in X/E$. If $P \in R_B$ for some $B \in X/E$, then $y \in B \cap Y$. Hence, $B = A$ and so γ is well-defined. We claim $\gamma \in \overline{S}_E(X, Y)$. Let $x, y \in Y$ and $(x, y) \in E$. Then $(x\alpha, y\alpha) \in E$ and $x\alpha, y\alpha \in Y$. There exists a unique $A \in X/E$ such that $x\alpha, y\alpha \in A$. Thus $P = (x\alpha)\alpha^{-1}$, $Q = (y\alpha)\alpha^{-1}$ and so $P, Q \in R_A$. We may assume $P \neq Q$. Note that $x \in P$, $y \in Q$ and $(x, y) \in E$. Since $|R_A| = 2$, we get $R_A \subseteq \pi_{A'\cap Y}(\alpha)$. This implies that $x_P, x_Q \in A' \cap Y$. Hence $(x\gamma, y\gamma) \in E$ and $x\gamma, y\gamma \in Y$. Therefore $\gamma \in \overline{S}_E(X, Y)$. For each $P \in \pi(\alpha)$ and $x \in P$, we have $x\gamma = x_P$. Clearly, $\pi(\gamma) = \pi(\alpha)$. We have $(\gamma, \alpha) \in \mathcal{R}^*$. Consider $x\gamma^2 = x_p\gamma = x_p = x\gamma$. Thus γ is an idempotent. Hence $\overline{S}_E(X, Y)$ is right abundant. This means that $\overline{S}_E(X, Y)$ is abundant.

Case 2. $|\{A \in X/E : |A \cap Y| \geq 2\}| \leq 1$. Let $A \in X/E$ be such that $|A \cap Y| \geq 2$. Then, for all $B \in X/E$, $B \neq A$ implies $|B \cap Y| \leq 1$. For each $P \in \pi(\alpha)$, if $P \cap A \cap Y \neq \emptyset$, then we choose $x_P \in P \cap A \cap Y$. If $P \cap Y \neq \emptyset$, then choose $x_P \in P \cap Y$. If $P \cap Y = \emptyset$, then choose $x_P \in P$. Define $\gamma : X \to X$ by $x\gamma = x_P$ where $P \in \pi(\alpha)$ and $x \in P$. Let $x, y \in Y$ and $(x, y) \in E$. If $x, y \in A$ for some $A \in X/E$, then $x \in P \cap A \cap Y$ and $y \in Q \cap A \cap Y$. Thus $x_P, x_Q \in A \cap Y$. Therefore $(x\gamma, y\gamma) \in E$ and $x\gamma, y\gamma \in Y$. Hence $\gamma \in \overline{S}_E(X, Y)$. If $x, y \notin A$, then $x, y \in B \cap Y$ for some $B \in X/E$. Thus $x = y$ and so $x\gamma = y\gamma$. Therefore $\gamma \in \overline{S}_E(X, Y)$. For each $P \in \pi(\alpha)$ and $x \in P$, we have $x\gamma = x_P$. Clearly, $\pi(\gamma) = \pi(\alpha)$. We obtain that $(\gamma, \alpha) \in \mathcal{R}^*$. From $x_P \in P$, we have $x_P\gamma = x_P$. This implies that $x\gamma^2 = x_P\gamma = x_P = x\gamma$. Hence γ is an idempotent. This means that $\overline{S}_E(X, Y)$ is right abundant. Therefore, $\overline{S}_E(X, Y)$ is abundant. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

References

1. R. Chinram, S. Baupradist, Magnifying elements in semigroups of transformations with invariant set, *Asian-Eur. J. Math.*, **12** (2019), 1950056. http://dx.doi.org/10.1142/S1793557119500566
2. W. Choomanee, P. Honyam, J. Sanwong, Regularity in semigroups of transformations with invariant sets, *International Journal of Pure and Applied Mathematics*, **87** (2013), 151–164. http://dx.doi.org/10.12732/ijpam.v87i1.9
3. J. Howie, *Fundamentals of semigroup theory*, Oxford: Clarendon Press, 1995.
4. P. Honyam, J. Sanwong, Semigroups of transformations with invariant set, *J. Korean Math. Soc.*, **48** (2011), 289–300. http://dx.doi.org/10.4134/JKMS.2011.48.2.289
5. T. Kaewnoi, M. Petapirak, R. Chinram, Magnifying elements in a semigroup of transformations preserving equivalence relation, *Korean J. Math.*, **27** (2019), 269–277. http://dx.doi.org/10.11568/kjm.2019.27.2.269
6. E. Lyapin, *Semigroups*, Providence: American Mathematical Society, 1963.
7. S. Nenthein, P. Youngkhong, Y. Kemprasit, Regular elements of some transformation semigroups, *Pure Mathematics and Applications*, **16** (2005), 307–314.
8. H. Pei, Regularity and Green's relations for semigroups of transformations that preserve an equivalence, *Commun. Algebra*, **33** (2005), 109–118. http://dx.doi.org/10.1081/AGB-200040921
9. H. Pei, Equivalences, α-semigroups and α-congruences, *Semigroup Forum*, **49** (1994), 49–58. http://dx.doi.org/10.1007/BF02573470
10. H. Pei, A regular α-semigroup inducing a certain lattice, *Semigroup Forum*, **53** (1996), 98–113. http://dx.doi.org/10.1007/BF02574125
11. H. Pei, On the rank of the semigroup $T_E(X)$, *Semigroup Forum*, **70** (2005), 107–117. http://dx.doi.org/10.1007/s00233-004-0135-z
12. H. Pei, H. Zhou, Abundant semigroups of transformations preserving an equivalence relation, *Algebr. Colloq.*, **18** (2011), 77–82. http://dx.doi.org/10.1142/S1005386711000034
13. L. Sun, J. Sun, A note on naturally ordered semigroups of transformations with invariant set, *Bull. Aust. Math. Soc.*, **91** (2015), 264–267. http://dx.doi.org/10.1017/S0004972714000860
14. L. Sun, L. Wang, Natural partial order in semigroups of transformations with invariant set, *Bull. Aust. Math. Soc.*, **87** (2013), 94–107. http://dx.doi.org/10.1017/S0004972712000287
15. L. Sun, H. Pei, Z. Cheng, Naturally ordered transformation semigroups preserving an equivalence, *Bull. Austral. Math. Soc.*, **78** (2008), 117–128. http://dx.doi.org/10.1017/S0004972708000543

16. L. Sun, Compatibility on naturally ordered transformation semigroups preserving an equivalence, *Semigroup Forum*, **98** (2019), 75–82. http://dx.doi.org/10.1007/s00233-018-9965-y

17. J. Symons, Some result concerning a transformation semigroup, *J. Aust. Math. Soc.*, **19** (1975), 413–425. http://dx.doi.org/10.1017/S1446788700034455

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