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Research article

Linear superposition and interaction of Wronskian solutions to an extended (2+1)-dimensional KdV equation

Li Cheng[1,2,*], Yi Zhang[3] and Ying-Wu Hu[1]

[1] Normal School, Jinhua Polytechnic, Jinhua 321007, China
[2] Key Laboratory of Crop Harvesting Equipment Technology, Jinhua Polytechnic, Jinhua 321007, China
[3] Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

* **Correspondence:** Email: jhchengli@126.com; Tel: +8613757969689.

Abstract: The main purpose of this work is to discuss an extended KdV equation, which can provide some physically significant integrable evolution equations to model the propagation of two-dimensional nonlinear solitary waves in various science fields. Based on the bilinear Bäcklund transformation, a Lax system is constructed, which guarantees the integrability of the introduced equation. The linear superposition principle is applied to homogeneous linear differential equation systems, which plays a key role in presenting linear superposition solutions composed of exponential functions. Moreover, some special linear superposition solutions are also derived by extending the involved parameters to the complex field. Finally, a set of sufficient conditions on Wronskian solutions is given associated with the bilinear Bäcklund transformation. The Wronskian identities of the bilinear KP hierarchy provide a direct and concise way for proving the Wronskian determinant solution. The resulting Wronskian structure generates N-soliton solutions and a few of special Wronskian interaction solutions, which enrich the solution structure of the introduced equation.

Keywords: extended (2+1)-dimensional KdV equation; bilinear Bäcklund transformation; Lax pair; linear superposition solution; Wronskian interaction solution
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1. Introduction

Studying of exact solutions plays a vital role in revealing various mathematical and physical features of associated nonlinear evolution equations (NLEEs). There are diverse types of physically important exact solutions, such as soliton solutions, rational solutions, positons and complexiton solutions. Soliton solutions are a type of analytic solutions exponentially localized [1, 2]. Positon solutions contain only one class of transcendental functions-trigonometric functions, while complexiton solutions contain two classes of transcendental functions-exponential functions and trigonometric functions [3–5]. Such exact solutions have been explored owing to various specific mathematical techniques, including the Hirota bilinear approach [6–8], the extended transformed rational function technique [9], the symbolic computation method [10] and the Wronskian technique [11–14]. Among the existing methods, the Wronskian technique is widely used to construct exact solutions of bilinear differential equations, particularly rational solutions and complexiton solutions [3–5].

The standard Korteweg-de Vries (KdV) equation and the nonlinear Schrödinger (NLS) equation are two of the most famous integrable models in (1+1)-dimensions which possess very wide applications in a number of fields of nonlinear science. It is known that the KdV equation is applied to describing one-dimensional water waves of long wavelength and small amplitude on shallow-water waves with weakly non-linear restoring forces [15]. The NLS equation is also an interesting mathematical model that describes the motion of pulses in nonlinear optical fibers and of surface gravity waves in fluid dynamics. To explore abundant complex nonlinear phenomena in the real world, various (1+1)-dimensional extensions associated with the KdV and the NLS equations were constructed [16–21], including the time-fractional coupled Schrödinger-KdV equation [17] and the coupled nonlinear Schrödinger-KdV equation [19].

Just recently a novel integrable (2+1)-dimensional KdV equation

$$u_t = a(6uu_x + u_{xxx} - 3w_y) + b(2wu_x - z_y + u_{xxy} + 4uu_y),\ u_y = w_x,\ u_{yy} = z_{xx}, \tag{1.1}$$

has been systematically proposed as a two-dimensional extension of the standard KdV equation by Lou [22]. The Lax pair and dual Lax pair presented directly guarantee the integrability of Eq (1.1) [22]. More significantly, the missing D'Alembert type solutions and the soliton molecules have been obtained explicitly through the velocity resonant mechanism and the arbitrary traveling wave solutions, respectively.

The above-mentioned research inspires us to consider a new generalization of Eq (1.1), written as

$$a(6\chi uu_x + u_{xxx} + 3\delta v_{xy}) + b(u_{xxy} + 2\chi u_x v_x + 4\chi uu_y + \delta v_{yy}) + cu_t + du_x + hu_y = 0,\ v_{xx} = u_y, \tag{1.2}$$

where the constants a, b, c, χ and δ satisfy $\chi\delta c(a^2 + b^2) \neq 0$, but the constants d and h are arbitrary. Equation (1.2) can provide us with some integrable equations in (2+1)-dimensions to model the motion of two-dimensional nonlinear solitary waves in various science fields, such as plasma physics, fluid dynamics and nonlinear optics. Special physically important cases of Eq (1.2) have been investigated as follows:

• If we take $\chi = 1, \delta = c = -1, d = h = 0$ and $v_x = w$, Eq (1.2) reduces to the integrable (2+1)-dimensional KdV Eq (1.1). Equation (1.1) may be found potential applications in nonlinear science due to the existence of diverse solution structures [22].

• If we set $\chi = b = 1, \delta = -1, a = d = h = 0, c = -2$ and make use of the potential $u = \Psi_x$, Eq (1.2) is expressed as

$$\Psi_{xxxxy} + 4\Psi_{xxy}\Psi_x + 2\Psi_{xxx}\Psi_y + 6\Psi_{xy}\Psi_{xx} - \Psi_{yyy} - 2\Psi_{xxt} = 0, \tag{1.3}$$

which is the Date-Jimbo-Kashiwara-Miwa (DJKM) equation in (2+1)-dimensions [23,24]. The (2+1)-dimensional DJKM equation is one of the significant integrable models which possesses interesting properties, such as the bilinear Bäcklund transformation, Lax pairs and infinitely many conservation laws [23–25].

• Setting $\chi = -1, \delta = \alpha^2, a = h = 0, b = c = 1, d = \beta^2$ and using the potential $u = \Phi_x$ in Eq (1.2) yield the following extended Bogoyavlenskii's generalized breaking soliton equation introduced by Wazwaz [26]:

$$(\Phi_{xt} - 4\Phi_x\Phi_{xy} - 2\Phi_y\Phi_{xx} + \Phi_{xxxy})_x = -\alpha^2\Phi_{yyy} - \beta^2\Phi_{xxx}, \tag{1.4}$$

where the parameters α^2 and β^2 are real. Multiple soliton solutions have been derived for Eq (1.4) by applying the simplified Hereman's method.

• We take

$$b = h = 0,\ a = s_3,\ c = s_1,\ d = \alpha_1,\ \delta = \frac{s_4}{3s_3},\ \chi = \frac{s_2}{3s_3},$$

then Eq (1.2) becomes the extended Kadomtsev-Petviashvili (KP) equation discussed by Akinyemi [27] as follows:

$$s_1u_{tx} + \alpha_1u_{xx} + s_2(u^2)_{xx} + s_3u_{xxxx} + s_4u_{yy} = 0, \tag{1.5}$$

where the constants s_1–s_4 and α_1 satisfy $s_1s_2s_3s_4\alpha_1 \neq 0$. Abundant exact solutions including the shallow ocean wave solitons, Peregrine solitons, lumps and breathers solutions have been obtained in [27], which implicates Eq (1.5) might well be applied to explaining a variety of complex behaviors in ocean dynamics.

The main purpose of this paper is to construct abundant exact solutions including linear superposition solutions, soliton solutions and more importantly their interaction solutions for Eq (1.2). Note that Eq (1.2) has a trilinear form rather than the usual bilinear form [6]. Our work will provide a comprehensive way for building linear superposition solutions and Wronskian interaction solutions.

The following is the structure of the paper. In Section 2, on the basis of the bilinear Bäcklund transformation, we would like to derive a Lax pair and linear superposition solutions, which can contain some special linear superposition solutions. In Section 3, we will present a set of sufficient conditions which guarantees the Wronskian determinant is a solution of the trilinear form associated with Eq (1.2). And then we will construct a few special but meaningful Wronskian solutions, including N-soliton solutions and novel interaction solutions among different types Wronskian solutions. Our concluding remarks will be drawn in the last section.

2. Lax pair and linear superposition solutions

Under the logarithmic derivative transformation

$$u = \frac{2}{\chi}(\ln f)_{xx},\ v = \frac{2}{\chi}(\ln f)_y, \tag{2.1}$$

we transformed the extended Eq (1.2) into a trilinear form

$$\begin{aligned} &a(f^2f_{xxxxx} + 2ff_{xx}f_{xxx} - 5ff_xf_{xxxx} - 6f_{xx}^2f_x + 8f_{xxx}f_x^2 + 3\delta f^2f_{xyy} - 3\delta ff_xf_{yy} \\ &\quad -6\delta ff_yf_{xy} + 6\delta f_xf_y^2) + b(f^2f_{xxxxy} - ff_{xxxx}f_y + 2ff_{xx}f_{xxy} - 4ff_xf_{xxxy} \\ &\quad +4f_xf_{xxx}f_y - 2f_{xx}^2f_y - 4f_{xx}f_xf_{xy} + 4f_x^2f_{xxy} + \delta f^2f_{yyy} + 2\delta f_y^3 - 3\delta ff_yf_{yy}) \end{aligned}$$

$$+c(f^2 f_{xxt} - f f_{xx} f_t - 2 f f_x f_{xt} + 2 f_x^2 f_t) + d(f^2 f_{xxx} - 3 f f_x f_{xx} + 2 f_x^3)$$
$$+h(f^2 f_{xxy} - f f_{xx} f_y - 2 f f_x f_{xy} + 2 f_x^2 f_y) = 0. \tag{2.2}$$

It is known that Hirota bilinear derivatives with multiple variables read:

$$D_x^{n_1} D_t^{n_2} f \cdot g = (\partial_x - \partial_{x'})^{n_1} (\partial_t - \partial_{t'})^{n_2} f(x,t) g(x',t')|_{x'=x, t'=t}, \tag{2.3}$$

with n_1 and n_2 being arbitrary nonnegative integers [6]. By utilizing Hirota bilinear expressions, we can rewrite the trilinear form (2.2) as

$$D_x\Big[(3aD_x^4 + 9a\delta D_y^2 + 2bD_x^3 D_y + 3cD_x D_t + 3dD_x^2 + 3hD_x D_y) f \cdot f\Big] \cdot f^2$$
$$+D_y\Big[(bD_x^4 + 3b\delta D_y^2) f \cdot f\Big] \cdot f^2 = 0. \tag{2.4}$$

Theorem 2.1. *Assume that f and f' are two different solutions of Eq (2.2). Then we can give the following bilinear Bäcklund transformation of (2.2):*

$$(\tilde{\delta} D_y + D_x^2) f \cdot f' = 0, \tag{2.5a}$$

$$[aD_x^3 + bD_x^2 D_y - 3a\tilde{\delta} D_x D_y - b\tilde{\delta} D_y^2 + cD_t + dD_x + hD_y] f \cdot f' = 0, \tag{2.5b}$$

where $\tilde{\delta}^2 = \delta$.

By the same calculation introduced in [23, 24, 28], we would like to present a simple proof of Theorem 2.1 in Appendix A.

Setting

$$\psi = \frac{f}{f'}, \; u = \frac{2}{\chi} (\ln f')_{xx}, \tag{2.6}$$

and using the identities of double logarithmic transformations [6]

$$\begin{cases} (D_x f \cdot f')/f'^2 & = \psi_x, \\ (D_x^2 f \cdot f')/f'^2 & = \psi_{xx} + \chi u \psi, \\ (D_x^3 f \cdot f')/f'^2 & = \psi_{xxx} + 3\chi u \psi_x, \\ (D_x D_y f \cdot f')/f'^2 & = \psi_{xy} + \chi \partial_x^{-1} u_y \psi, \\ (D_x^2 D_y f \cdot f')/f'^2 & = \psi_{xxy} + 2\chi \partial_x^{-1} u_y \psi_x + \chi u \psi_y, \\ \cdots\cdots, \end{cases}$$

we can transform the bilinear Bäcklund transformation (2.5) into a Lax system of Eq (1.2) as follows:

$$L_1' \psi = \tilde{\delta} \psi_y + \psi_{xx} + \chi u \psi = 0, \tag{2.7a}$$

$$L_2' \psi = c\psi_t - \frac{2b}{\tilde{\delta}} \psi_{xxxx} + 4a\psi_{xxx} - \Big(\frac{4b\chi}{\tilde{\delta}} u + \frac{h}{\tilde{\delta}}\Big) \psi_{xx} + 2\Big(3a\chi u - \frac{2b\chi}{\tilde{\delta}} u_x + b\chi \partial_x^{-1} u_y + \frac{d}{2}\Big) \psi_x$$
$$-\Big(3a\chi\tilde{\delta} \partial_x^{-1} u_y + b\chi\tilde{\delta} \partial_x^{-2} u_{2y} + \frac{2b\chi^2}{\tilde{\delta}} u^2 - 3a\chi u_x + \frac{2b\chi}{\tilde{\delta}} u_{xx} - b\chi u_y + \frac{h}{\tilde{\delta}} \chi u\Big) \psi = 0, \tag{2.7b}$$

where

$$L_1' = \tilde{\delta} \partial_y + \partial_x^2 + \chi u, \tag{2.8a}$$

$$L_2' = c\partial_t - \frac{2b}{\tilde{\delta}}\partial_x^4 + 4a\partial_x^3 - \Big(\frac{4b\chi}{\tilde{\delta}}u + \frac{h}{\tilde{\delta}}\Big)\partial_x^2 + 2\Big(3a\chi u - \frac{2b\chi}{\tilde{\delta}}u_x + b\chi\partial_x^{-1}u_y + \frac{d}{2}\Big)\partial_x$$
$$-\Big(3a\chi\tilde{\delta}\partial_x^{-1}u_y + b\chi\tilde{\delta}\partial_x^{-2}u_{2y} + \frac{2b\chi^2}{\tilde{\delta}}u^2 - 3a\chi u_x + \frac{2b\chi}{\tilde{\delta}}u_{xx} - b\chi u_y + \frac{h}{\tilde{\delta}}\chi u\Big). \tag{2.8b}$$

Note that Eq (1.2) can be generated from the compatibility condition $[L_1', L_2'] = 0$ of the system (2.7), which represents the integrability of Eq (1.2).

In addition, taking $f' = 1$ as the seed solution in the system (2.5), we further obtain the following linear partial differential equations:

$$\tilde{\delta}f_y + f_{xx} = 0, \tag{2.9a}$$

$$af_{xxx} + bf_{xxy} - 3a\tilde{\delta}f_{xy} - b\tilde{\delta}f_{yy} + cf_t + df_x + hf_y = 0, \tag{2.9b}$$

which is rewritten as

$$\tilde{\delta}f_y + f_{xx} = 0, \tag{2.10a}$$

$$4af_{xxx} - \frac{2b}{\tilde{\delta}}f_{xxxx} + cf_t + df_x - \frac{h}{\tilde{\delta}}f_{xx} = 0. \tag{2.10b}$$

It is easy to see that the system (2.7) with the choice $u = 0$ is equivalent to the the pair of Eq (2.10). Generally, the linear superposition principle can not be directly applied to solutions of NLEEs. However, it is widely known that the linear superposition principle of solutions [29–32] can be applied to homogeneous linear differential equations. Therefore, solving the system (2.10) above, we can built a linear superposition solution formed by linear combinations of exponential functions as follows:

$$f = \sum_{i=1}^{N}\varepsilon_i e^{\theta_i+\theta_i^0}, \theta_i = k_i x - \frac{1}{\tilde{\delta}}k_i^2 y + \Big(\frac{2b}{\tilde{\delta}c}k_i^4 - \frac{4a}{c}k_i^3 + \frac{h}{\tilde{\delta}c}k_i^2 - \frac{d}{c}k_i\Big)t, \tag{2.11}$$

where the k_i's are arbitrary non-zero constants, the ε_i's, θ_i^0's are arbitrary constants and $\tilde{\delta}$ satisfies $\tilde{\delta}^2 = \delta$. The following are two cases which are composed of the product of two special functions for the trilinear form (2.2).

(a) The case of $\delta > 0$:

If we choose

$$k_i = r_i + l_i,\ \theta_i^0 = \theta_{i1}^0 + \theta_{i2}^0$$

and

$$k_i = r_i - l_i,\ \theta_i^0 = \theta_{i1}^0 - \theta_{i2}^0,\ r_i, l_i, \theta_{i1}^0, \theta_{i2}^0 \in \mathbb{R},\ 1 \le i \le N,$$

then the linear superposition solution (2.11) becomes

$$f = \sum_{i=1}^{N}\varepsilon_i e^{\theta_{i1}}e^{\theta_{i2}} \quad \text{and} \quad f = \sum_{i=1}^{N}\varepsilon_i e^{\theta_{i1}}e^{-\theta_{i2}},$$

with N-wave variables

$$\theta_{i1} = r_i x - \frac{1}{\tilde{\delta}}(r_i^2 + l_i^2)y + \Big[\frac{2b}{\tilde{\delta}c}(r_i^4 + 6r_i^2l_i^2 + l_i^4) - \frac{4a}{c}(r_i^3 + 3r_il_i^2) + \frac{h}{\tilde{\delta}c}(r_i^2 + l_i^2) - \frac{d}{c}r_i\Big]t + \theta_{i1}^0,$$

$$\theta_{i2} = l_i x - \frac{2}{\tilde{\delta}} r_i l_i y + \Big[\frac{8b}{\tilde{\delta}c}(r_i^3 l_i + r_i l_i^3) - \frac{4a}{c}(3r_i^2 l_i + l_i^3) + \frac{2h}{\tilde{\delta}c} r_i l_i - \frac{d}{c} l_i\Big] t + \theta_{i2}^0, \tag{2.12}$$

respectively. Therefore, two kinds of linear superposition formulas of Eq (2.2) composed of the product of exponential functions and hyperbolic functions can be expressed as

$$f = \sum_{i=1}^{N} \varepsilon_i e^{\theta_{i1}} \cosh(\theta_{i2}),\ f = \sum_{i=1}^{N} \varepsilon_i e^{\theta_{i1}} \sinh(\theta_{i2}), \tag{2.13}$$

where θ_{i1} and θ_{i2} are defined by (2.12).

By setting

$$k_i = r_i + \mathrm{I} l_i,\ \theta_i^0 = \theta_{i1}^0 + \mathrm{I}\theta_{i2}^0$$

and

$$k_i = r_i - \mathrm{I} l_i,\ \theta_i^0 = \theta_{i1}^0 - \mathrm{I}\theta_{i2}^0,\ r_i, l_i, \theta_{i1}^0, \theta_{i2}^0 \in \mathbb{R},\ \mathrm{I} = \sqrt{-1},\ l_i \neq 0,\ 1 \leq i \leq N,$$

respectively, we have

$$f = \sum_{i=1}^{N} \varepsilon_i e^{\mathrm{Re}(\theta_i)} e^{\mathrm{I}\times\mathrm{Im}(\theta_i)} \quad \text{and} \quad f = \sum_{i=1}^{N} \varepsilon_i e^{\mathrm{Re}(\theta_i)} e^{-\mathrm{I}\times\mathrm{Im}(\theta_i)},$$

where

$$\mathrm{Re}(\theta_i) = r_i x - \frac{1}{\tilde{\delta}}(r_i^2 - l_i^2) y + \Big[\frac{2b}{\tilde{\delta}c}(r_i^4 - 6r_i^2 l_i^2 + l_i^4) - \frac{4a}{c}(r_i^3 - 3r_i l_i^2) + \frac{h}{\tilde{\delta}c}(r_i^2 - l_i^2) - \frac{d}{c} r_i\Big] t + \theta_{i1}^0,$$

$$\mathrm{Im}(\theta_i) = l_i x - \frac{2}{\tilde{\delta}} r_i l_i y + \Big[\frac{8b}{\tilde{\delta}c}(r_i^3 l_i - r_i l_i^3) - \frac{4a}{c}(3r_i^2 l_i - l_i^3) + \frac{2h}{\tilde{\delta}c} r_i l_i - \frac{d}{c} l_i\Big] t + \theta_{i2}^0. \tag{2.14}$$

Thus two classes of solutions formed by the product of exponential functions and trigonometric functions appear as:

$$f = \sum_{i=1}^{N} \varepsilon_i e^{\mathrm{Re}(\theta_i)} \cos(\mathrm{Im}(\theta_i)),\ f = \sum_{i=1}^{N} \varepsilon_i e^{\mathrm{Re}(\theta_i)} \sin(\mathrm{Im}(\theta_i)), \tag{2.15}$$

with $\mathrm{Re}(\theta_i)$ and $\mathrm{Im}(\theta_i)$ being given by (2.14). Besides, applying the linear superposition principle, we may get the following mixed-type function solutions such as complexiton solutions:

$$f = \sum_{i=1}^{N} [\varepsilon_i e^{\theta_i} + \varrho_i e^{\mathrm{Re}(\theta_i)} \sin(\mathrm{Im}(\theta_i))], f = \sum_{i=1}^{N} [\varepsilon_i e^{\theta_i} + \varrho_i e^{\mathrm{Re}(\theta_i)} \cos(\mathrm{Im}(\theta_i))], \tag{2.16}$$

where the ε_i's, ϱ_i's are arbitrary constants, θ_i and $\mathrm{Re}(\theta_i)$, $\mathrm{Im}(\theta_i)$ are defined by (2.11) and (2.14), respectively.

(b) The case of $\delta < 0$:

Generally, the above solution (2.11) leads to complex-valued linear combinations if $\delta < 0$. However, we are more interested in real solutions. To this end, taking

$$k_i = \mathrm{I} r_i + l_i,\ \theta_i^0 = \mathrm{I}\theta_{i1}^0 + \theta_{i2}^0$$

and

$$k_i = \mathrm{I}r_i - l_i,\ \theta_i^0 = \mathrm{I}\theta_{i1}^0 - \theta_{i2}^0,\ r_i, l_i, \theta_{i1}^0, \theta_{i2}^0 \in \mathbb{R},\ \mathrm{I} = \sqrt{-1},\ l_i \neq 0,\ 1 \le i \le N,$$

in (2.11), respectively, we get the following two types of solutions of Eq (2.2):

$$f = \sum_{i=1}^{N} \varepsilon_i e^{\mathrm{I}\kappa_i} e^{\varsigma_i} \quad \text{and} \quad f = \sum_{i=1}^{N} \varepsilon_i e^{\mathrm{I}\kappa_i} e^{-\varsigma_i},$$

with N-wave variables

$$\kappa_i = r_i x - \frac{1}{\sqrt{-\delta}}(r_i^2 - l_i^2)y + \Big[\frac{2b}{\sqrt{-\delta}c}(6r_i^2 l_i^2 - r_i^4 - l_i^4) + \frac{4a}{c}(r_i^3 - 3r_i l_i^2) + \frac{h}{\sqrt{-\delta}c}(r_i^2 - l_i^2) - \frac{d}{c}r_i\Big]t + \theta_{i1}^0,$$

$$\varsigma_i = l_i x - \frac{2}{\sqrt{-\delta}}r_i l_i y + \Big[\frac{8b}{\sqrt{-\delta}c}(r_i l_i^3 - r_i^3 l_i) + \frac{4a}{c}(3r_i^2 l_i - l_i^3) + \frac{2h}{\sqrt{-\delta}c}r_i l_i - \frac{d}{c}l_i\Big]t + \theta_{i2}^0. \tag{2.17}$$

Similarly, the following linear superposition formulas:

$$f = \sum_{i=1}^{N} \varepsilon_i e^{\mathrm{I}\kappa_i} \cosh(\varsigma_i),\ f = \sum_{i=1}^{N} \varepsilon_i e^{\mathrm{I}\kappa_i} \sinh(\varsigma_i), \tag{2.18}$$

with κ_i and ς_i being defined by (2.17), solve the linear partial differential system (2.10). Let $N = 1$ in the solution (2.18). Through the transformation (2.1), a real solution of Eq (1.2) reads as

$$u = \frac{8l_1^2 e^{2\varsigma_1}}{\chi(1 + e^{2\varsigma_1})^2},\ v_x = \frac{-16l_1^2 r_1 e^{2\varsigma_1}}{\chi\sqrt{-\delta}(1 + e^{2\varsigma_1})^2}, \tag{2.19}$$

with ς_1 being defined by (2.17), which is nothing but the one-soliton solution of Eq (1.2).

Let us take the extended Bogoyavlenskii's generalized breaking soliton Eq (1.4) with $\alpha^2 > 0$ as an illustrative example to describe the propagations of linear superposition solutions. By the transformation (2.1), we have the following special N-order linear superposition solutions for Eq (1.4) with $\alpha^2 > 0$:

$$\Phi = -2(\ln f)_x,\ f = \sum_{i=1}^{N} \varepsilon_i e^{k_i x - \frac{1}{\alpha}k_i^2 y + \left(\frac{2}{\alpha}k_i^4 - \beta^2 k_i\right)t}, \tag{2.20}$$

$$\Phi = -2(\ln f)_x,$$

$$f = \sum_{i=1}^{N} \varepsilon_i e^{r_i x - \frac{1}{\alpha}(r_i^2 + l_i^2)y + \left[\frac{2}{\alpha}(r_i^4 + 6r_i^2 l_i^2 + l_i^4) - \beta^2 r_i\right]t} \times \cosh\Big(l_i x - \frac{2}{\alpha}r_i l_i y + [\frac{8}{\alpha}(r_i^3 l_i + r_i l_i^3) - \beta^2 l_i]t\Big), \tag{2.21}$$

and

$$\Phi = -2(\ln f)_x,$$

$$f = \sum_{i=1}^{N} \Big[\varepsilon_i e^{k_i x - \frac{1}{\alpha}k_i^2 y + \left(\frac{2}{\alpha}k_i^4 - \beta^2 k_i\right)t} + \varrho_i e^{r_i x - \frac{1}{\alpha}(r_i^2 - l_i^2)y + \left[\frac{2}{\alpha}(r_i^4 - 6r_i^2 l_i^2 + l_i^4) - \beta^2 r_i\right]t}$$

$$\times \cos\Big(l_i x - \frac{2}{\alpha}r_i l_i y + [\frac{8}{\alpha}(r_i^3 l_i - r_i l_i^3) - \beta^2 l_i]t\Big)\Big], \tag{2.22}$$

where the ε_i's, ϱ_i's, k_i's r_i's and l_i's are arbitrary real constants.

Three-dimensional plots of these linear superposition solutions are made with the aid of Maple plot tools, to depict the characteristics of linear superposition solutions. The solution (2.20) is a linear combination of N exponential wave solutions, also known as the N-wave solution [29]. Figure 1(a)–(c) displays that the three-dimensional plots of the linear superposition solution (2.20) for $N = 4, N = 5$ and $N = 6$ and correspondingly presents three-wave, four-wave and five-wave, respectively.

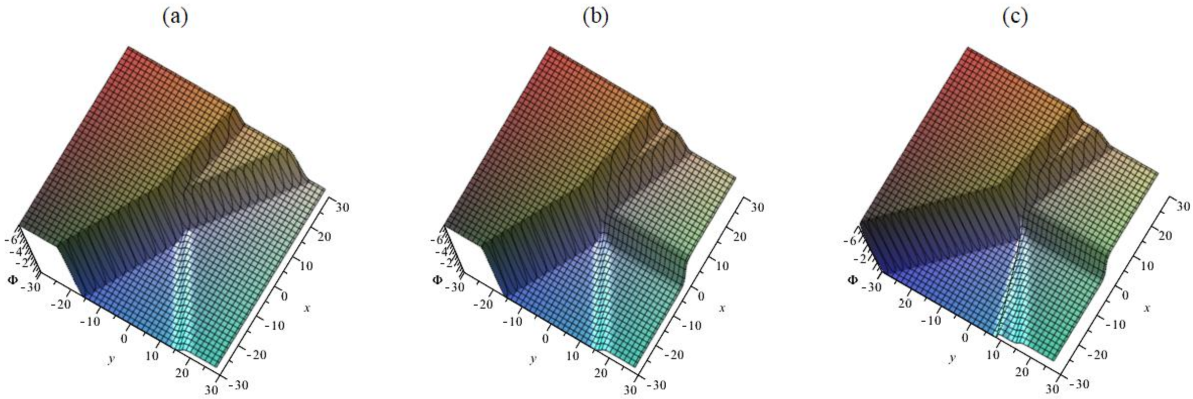

Figure 1. Three-dimensional plots of Φ determined by the linear superposition solution (2.20) with special parameters: (a) N=4, $\varepsilon_i = 1, 1 \leq i \leq 4, k_1 = -1.2, k_2 = -0.5, k_3 = 1.8, k_4 = 3, \alpha = 1, \beta = 2, t = 0$; (b) N=5, $\varepsilon_i = 1, 1 \leq i \leq 5, k_1 = -1.2, k_2 = -0.5, k_3 = 1.8, k_4 = 3, k_5 = 0.5, \alpha = 1, \beta = 2, t = 0$; (c) N=6, $\varepsilon_i = 1, 1 \leq i \leq 6, k_1 = -1.2, k_2 = -0.5, k_3 = 1.8, k_4 = 3, k_5 = 0.5, k_6 = -2, \alpha = 1, \beta = 2, t = 0$.

Figure 2(a)–(c) displays the two-dimensional density plots corresponding to Figure 1(a)–(c). We can see from Figures 1 and 2 that the solution (2.20) manifests a kink-shape traveling wave in the (x, y)-plane and the number of stripes increases with the increase of the positive integer N at $t = 0$. In addition, various phenomena of wave fusion or fission may occur when the traveling multi-kink waves propagate along different or same directions. If the coefficients ε_i's are all positive real constants, then the f in the solutions (2.20) and (2.21) is a positive function. But the coefficients ε_i's and ϱ_i's are all positive real constants can not guarantee that the function f is always positive in the complexiton solution (2.22).

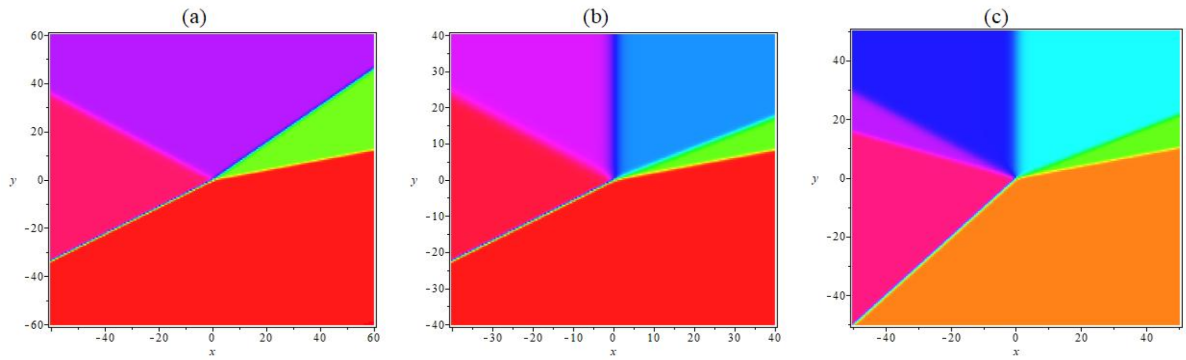

Figure 2. Two-dimensional density plots of Φ determined by the linear superposition solution (2.20). The related parameters are the same as in Figure 1.

Figure 3 shows three-dimensional graphics of the complexiton solutions, which possessing some singularities. It is obvious that the evolution of linear superposition solutions inclines to be more complicated with the increase of the positive integer N.

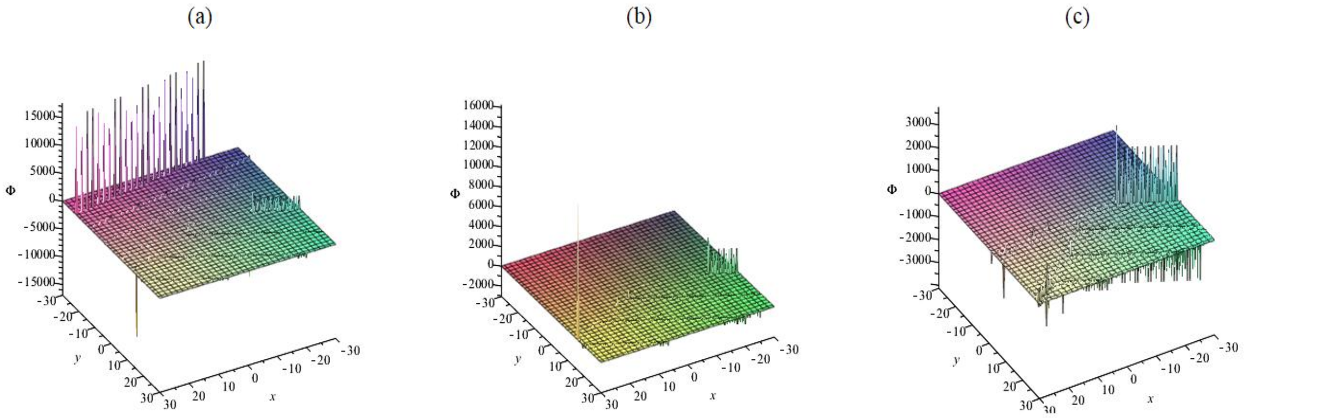

Figure 3. Three-dimensional plots of Φ determined by the linear superposition solution (2.22) with special parameters: (a) N=2, $\varepsilon_1 = \varepsilon_2 = 1, \varrho_1 = \varrho_2 = 1, k_1 = -1, k_2 = 2, r_1 = -1, l_1 = 2, r_2 = 3, l_2 = 1, \alpha = 1, \beta = 2, t = 0$; (b) N=3, $\varepsilon_i = 1, 1 \leq i \leq 3, \varrho_1 = 1, \varrho_2 = \varrho_3 = 0, k_1 = 1, k_2 = 1.5, k_3 = -2.5, r_1 = -1, l_1 = 2, \alpha = 1, \beta = 2, t = 0$; (c) N=3, $\varepsilon_i = 1, \varrho_i = 1, 1 \leq i \leq 3, k_1 = 1, k_2 = 3, k_3 = -0.5, r_1 = -1, l_1 = 2, r_2 = 3, l_2 = 1, r_3 = 1, l_3 = 1.5, \alpha = 1, \beta = 2, t = 0$.

3. Interaction of Wronskian solutions

Based on the pair of Eq (2.10), we now construct a set of sufficient conditions on Wronskian solutions for the extended (2+1)-dimensional KdV Eq (1.2). We first introduce the compact Freeman and Nimmo's notation [12, 13]:

$$W = W(\phi_1, \phi_2, \cdots, \phi_N) = \begin{vmatrix} \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)} \\ \phi_2 & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_N & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix} = |0, 1, \cdots, N-1| = |\widehat{N-1}|, \tag{3.1}$$

where $\phi_i^{(j)} = \dfrac{\partial^j \phi_i}{\partial x^j}, i, j \geq 1$. Then we present a set of sufficient conditions, which guarantees that the Wronskian determinant is a solution of the trilinear Eq (2.2).

Theorem 3.1. *Suppose that a set of functions $\phi_i = \phi_i(x, y, t), 1 \leq i \leq N$, meets the combined linear conditions as follows:*

$$\phi_{i,y} = -\frac{1}{\tilde{\delta}}\phi_{i,xx}, \tag{3.2a}$$

$$\phi_{i,t} = \frac{2b}{\tilde{\delta}c}\phi_{i,xxxx} - \frac{4a}{c}\phi_{i,xxx} + \frac{h}{\tilde{\delta}c}\phi_{i,xx} - \frac{d}{c}\phi_{i,x}, \tag{3.2b}$$

where $\tilde{\delta}^2 = \delta$. Then the Wronskian determinant $f = f_N = |\widehat{N-1}|$ defined by (3.1) solves the trilinear Eq (2.2).

The proof of Theorem 3.1 will be presented by using Wronskian identities of the bilinear KP hierarchy in Appendix B. We can check that the above sufficient conditions reduce to the ones introduced earlier studies for the (2+1)-dimensional DJKM Eq (1.3) [33,34].

It is known that Wronskian formulations provide us with a powerful technique to establish exact solutions including soliton solutions, rational solutions, positons, complexitons and their interaction solutions for NLEEs [3–5]. The interactions of mixed solutions for higher-dimensional NLEEs have become a more and more popular research topic recently [35–41]. The determination of mixed solutions is extremely useful to reveal specific physical characteristics modeled by these higher-dimensional equations. With the help of the homoclinic test method and the Hirota bilinear approach, different types of hybrid-type solutions were displayed, such as the breather-kink wave solutions [36], the mixed lump-kink solutions [37] and the periodic-kink wave solutions [38]. In this section, the major concern is to construct Wronskian interaction solutions among different types of Wronskian solutions determined by the set of Wronskian sufficient condition (3.2) to the trilinear Eq (2.2).

Let us consider the Wronskian sufficient condition (3.2). An N-soliton solution of Eq (2.2) associated with the Wronskian determinant can be expressed as

$$f = f_N = W(\phi_1, \phi_2, \cdots, \phi_N), \tag{3.3}$$

where

$$\begin{aligned} &\phi_i = e^{\xi_i} + e^{\widehat{\xi_i}},\ \xi_i = p_i x - \frac{1}{\tilde{\delta}} p_i^2 y + \Big(\frac{2b}{\tilde{\delta} c} p_i^4 - \frac{4a}{c} p_i^3 + \frac{h}{\tilde{\delta} c} p_i^2 - \frac{d}{c} p_i\Big) t + \text{constant}, \\ &\widehat{\xi_i} = q_i x - \frac{1}{\tilde{\delta}} q_i^2 y + \Big(\frac{2b}{\tilde{\delta} c} q_i^4 - \frac{4a}{c} q_i^3 + \frac{h}{\tilde{\delta} c} q_i^2 - \frac{d}{c} q_i\Big) t + \text{constant}, \end{aligned} \tag{3.4}$$

in which p_i's and q_i's are free parameters. Taking the variables transformations presented in [6] and introducing the following new parameters:

$$K_i = p_i - q_i,\ L_i = -\frac{1}{\tilde{\delta}}(p_i^2 - q_i^2),\ W_i = \frac{2b}{\tilde{\delta} c}(p_i^4 - q_i^4) - \frac{4a}{c}(p_i^3 - q_i^3) + \frac{h}{\tilde{\delta} c}(p_i^2 - q_i^2) - \frac{d}{c}(p_i - q_i), \tag{3.5}$$

the dispersion relation of Eq (2.2) can be written as

$$W_i = -\frac{a}{cK_i}(3\delta L_i^2 + K_i^4) - \frac{b}{cK_i^2}(\delta L_i^3 + K_i^4 L_i) - \frac{dK_i}{c} - \frac{hL_i}{c}. \tag{3.6}$$

Moreover, through the same calculation and variables transformations presented in [34], we obtain an equivalent representation of the N-soliton solution (3.3) as follows:

$$f = f_N = \sum_{\mu=0,1} e^{\sum\limits_{1\le i<j}^{N} \mu_i\mu_j \ln A_{ij} + \sum\limits_{i=1}^{N} \mu_i \eta_i}, \tag{3.7}$$

where

$$\eta_i = K_i x + L_i y + W_i t + \eta_i^0,\ A_{ij} = \frac{-K_i^2K_j^2(K_i - K_j)^2 + \delta(K_iL_j - K_jL_i)^2}{-K_i^2K_j^2(K_i + K_j)^2 + \delta(K_iL_j - K_jL_i)^2},\ 1 \le i < j \le N, \tag{3.8}$$

with W_i being defined by (3.6) and η_i^0's being arbitrary constants. Here $\mu = (\mu_1, \mu_2, \cdots, \mu_N)$, $\mu = 0, 1$ indicates that each μ_i takes 0 or 1. Also worth noting is that if we choose all the $A_{ij} = 0$, the N-soliton solution (3.7) can be reduced to the linear superposition solution (2.11).

Next we will show different types of Wronskian interaction solutions to Eq (1.2). In general, it will lead to complex-valued Wronskian interaction solutions when $\delta < 0$. For the convenience of the following discussion, we will only focus on the case of $\delta > 0$ in Eq (1.2).

Solving the following representative system

$$\phi_y = -\frac{1}{\tilde{\delta}}\phi_{xx}, \tag{3.9a}$$

$$\phi_t = \frac{2b}{\tilde{\delta}c}\phi_{xxxx} - \frac{4a}{c}\phi_{xxx} + \frac{h}{\tilde{\delta}c}\phi_{xx} - \frac{d}{c}\phi_x, \tag{3.9b}$$

with $\tilde{\delta}^2 = \delta > 0$, we get several special solutions for ϕ :

$$\phi_{\text{rational},1} = \alpha_1 x - \frac{d\alpha_1}{c}t + \alpha_2,\ \alpha_1, \alpha_2 = \text{constants}, \tag{3.10a}$$

$$\phi_{\text{rational},2} = \beta_1 x^2 + \beta_2 x - \frac{2\beta_1}{\tilde{\delta}}y + \frac{2\beta_1 h}{c\tilde{\delta}}t + \beta_3,\ d = 0,\ \beta_1, \beta_1, \beta_3 = \text{constants}, \tag{3.10b}$$

$$\begin{aligned} \phi_{\text{soliton}} &= e^{\xi} + e^{\widehat{\xi}},\ \xi = px - \frac{1}{\tilde{\delta}}p^2 y + \Big(\frac{2b}{\tilde{\delta}c}p^4 - \frac{4a}{c}p^3 + \frac{h}{\tilde{\delta}c}p^2 - \frac{d}{c}p\Big)t, \\ \widehat{\xi} &= qx - \frac{1}{\tilde{\delta}}q^2 y + \Big(\frac{2b}{\tilde{\delta}c}q^4 - \frac{4a}{c}q^3 + \frac{h}{\tilde{\delta}c}q^2 - \frac{d}{c}q\Big)t, \quad p, q = \text{constants}, \end{aligned} \tag{3.10c}$$

$$\begin{aligned} \phi_{\text{positon},1} &= e^{\widehat{\varphi}}\cos\varphi,\ \phi_{\text{positon},2} = e^{\widehat{\varphi}}\sin\varphi,\ \phi_{\text{complexiton}} = e^{\xi} + e^{\widehat{\varphi}}\cos\varphi, \\ \xi &= px - \frac{1}{\tilde{\delta}}p^2 y + \Big(\frac{2b}{\tilde{\delta}c}p^4 - \frac{4a}{c}p^3 + \frac{h}{\tilde{\delta}c}p^2 - \frac{d}{c}p\Big)t, \\ \widehat{\varphi} &= rx - \frac{1}{\tilde{\delta}}(r^2 - l^2)y + \Big[\frac{2b}{\tilde{\delta}c}(r^4 - 6r^2l^2 + l^4) - \frac{4a}{c}(r^3 - 3rl^2) + \frac{h}{\tilde{\delta}c}(r^2 - l^2) - \frac{d}{c}r\Big]t, \\ \varphi &= lx - \frac{2}{\tilde{\delta}}rly + \Big[\frac{8b}{\tilde{\delta}c}(r^3 l - rl^3) - \frac{4a}{c}(3r^2 l - l^3) + \frac{2h}{\tilde{\delta}c}rl - \frac{d}{c}l\Big]t,\ \ p, r, l = \text{constants}. \end{aligned} \tag{3.10d}$$

A few Wronskian interaction determinants between any two of a single soliton, a rational, a positon and a complexiton read as

$$f = W(\phi_{\text{rational},1}, \phi_{\text{soliton}}) = e^{\widehat{\xi}}\Big[\alpha_1 - q\Big(\alpha_1 x - \frac{d\alpha_1}{c}t + \alpha_2\Big) + \Big(\alpha_1 - p\Big(\alpha_1 x - \frac{d\alpha_1}{c}t + \alpha_2\Big)\Big)e^{\xi - \widehat{\xi}}\Big],$$

$$\begin{aligned} f = W(\phi_{\text{rational},2}, \phi_{\text{soliton}}) &= e^{\widehat{\xi}}\Big[2\beta_1 x + \beta_2 - q\Big(\beta_1 x^2 + \beta_2 x - \frac{2\beta_1}{\tilde{\delta}}y + \frac{2\beta_1 h}{c\tilde{\delta}}t + \beta_3\Big) \\ &+ \Big(2\beta_1 x + \beta_2 - p\Big(\beta_1 x^2 + \beta_2 x - \frac{2\beta_1}{\tilde{\delta}}y + \frac{2\beta_1 h}{c\tilde{\delta}}t + \beta_3\Big)\Big)e^{\xi - \widehat{\xi}}\Big],\ d = 0, \end{aligned}$$

$$f = W(\phi_{\text{rational},1}, \phi_{\text{positon},1}) = e^{\widehat{\varphi}}\Big[\Big(\alpha_1 x - \frac{d\alpha_1}{c}t + \alpha_2\Big)(r\cos\varphi - l\sin\varphi) - \alpha_1\cos\varphi\Big],$$

$$f = W(\phi_{\text{soliton}}, \phi_{\text{positon},2}) = e^{\widehat{\xi} + \widehat{\varphi}}\Big[(1 + e^{\xi - \widehat{\xi}})(r\sin\varphi + l\cos\varphi) - (pe^{\xi - \widehat{\xi}} + q)\sin\varphi\Big],$$

$$f=W(\phi_{\text{rational},1},\phi_{\text{complexiton}})=(\alpha_1 x-\frac{d\alpha_1}{c}t+\alpha_2)(pe^{\xi}+re^{\widehat{\varphi}}\cos\varphi-le^{\widehat{\varphi}}\sin\varphi)-\alpha_1(e^{\xi}+e^{\widehat{\varphi}}\cos\varphi),$$

where $\xi,\widehat{\xi}$ and $\varphi,\widehat{\varphi}$ are defined by (3.10c) and (3.10d), respectively. Moreover, by the transformation (2.1), the corresponding Wronskian interaction solutions of Eq (1.2) appear as

$$u=\frac{2(ff_{xx}-f_x^2)}{\chi f^2},\ v_x=\frac{2(ff_{xy}-f_xf_y)}{\chi f^2}. \tag{3.11}$$

For Eq (1.4) with $\alpha^2>0$, associated with (3.11), three special Wronskian interaction solutions are

$$\Phi_{rs}=-2\partial_x\ln W(\phi_{\text{rational},1},\phi_{\text{soliton}})=-2q-2\frac{\bar{f}_{rs,x}}{\bar{f}_{rs}}, \tag{3.12}$$

$$\bar{f}_{rs}=\alpha_1-q(\alpha_1x-\beta^2\alpha_1t+\alpha_2)+[\alpha_1-p(\alpha_1x-\beta^2\alpha_1t+\alpha_2)]e^{\xi'-\widehat{\xi}'},$$

$$\bar{f}_{rs,x}=-q\alpha_1+[-p\alpha_1+(\alpha_1-p(\alpha_1x-\beta^2\alpha_1t+\alpha_2))(p-q)]e^{\xi'-\widehat{\xi}'},$$

$$\Phi_{rp}=-2\partial_x\ln W(\phi_{\text{rational},1},\phi_{\text{positon},1})=-2r-2\frac{\bar{f}_{rp,x}}{\bar{f}_{rp}}, \tag{3.13}$$

$$\bar{f}_{rp}=(\alpha_1x-\beta^2\alpha_1t+\alpha_2)(r\cos\varphi'-l\sin\varphi')-\alpha_1\cos\varphi',$$

$$\bar{f}_{rp,x}=\alpha_1(r\cos\varphi'-l\sin\varphi'\Big)-(\alpha_1x-\beta^2\alpha_1t+\alpha_2)(rl\sin\varphi'+l^2\cos\varphi')+\alpha_1l\sin\varphi',$$

$$\Phi_{sp}=-2\partial_x\ln W(\phi_{\text{soliton}},\phi_{\text{positon},2})=-2(q+r)-2\frac{\bar{f}_{sp,x}}{\bar{f}_{sp}}, \tag{3.14}$$

$$\bar{f}_{sp}=(1+e^{\xi'-\widehat{\xi}'})(r\sin\varphi'+l\cos\varphi')-(pe^{\xi'-\widehat{\xi}'}+q)\sin\varphi',$$

$$\bar{f}_{sp,x}=e^{\xi'-\widehat{\xi}'}(p-q)(r\sin\varphi'+l\cos\varphi')+(1+e^{\xi'-\widehat{\xi}'})(rl\cos\varphi'-l^2\sin\varphi')$$

$$-e^{\xi'-\widehat{\xi}'}p(p-q)\sin\varphi'-(pe^{\xi'-\widehat{\xi}'}+q)l\cos\varphi',$$

where

$$\xi'-\widehat{\xi}'=(p-q)x-\frac{1}{\alpha}(p^2-q^2)y+\frac{2}{\alpha}(p^4-q^4)t-\beta^2(p-q)t,$$

$$\varphi'=lx-\frac{2}{\alpha}rly+\frac{8}{\alpha}(r^3l-rl^3)t-\beta^2lt,\ \ \alpha_1,\alpha_2,p,q,r,l=\text{constants}.$$

Figure 4 shows some singularities of these three Wronskian interaction solutions.

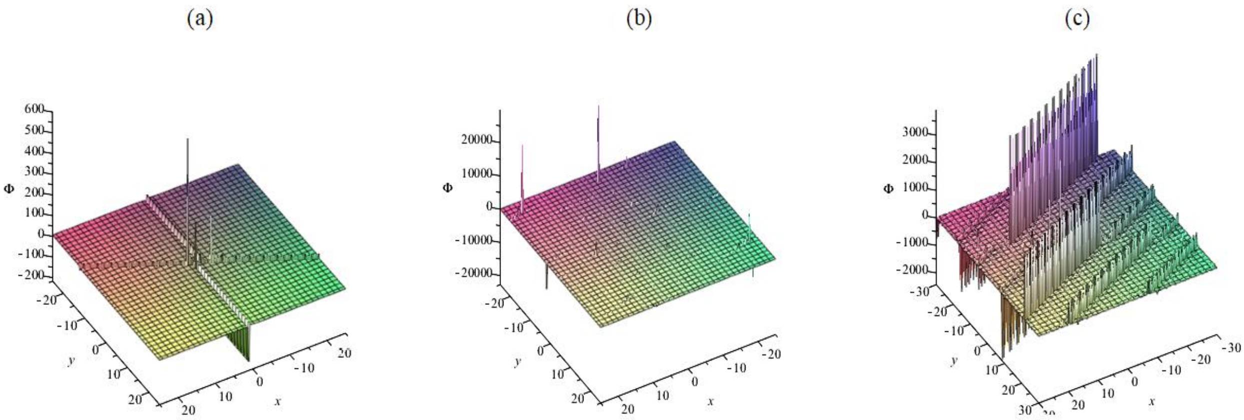

Figure 4. (a) The plot of the Wronskian interaction solution (3.12) with parameters: $p = 1, q = -3, \alpha_1 = 2, \alpha_2 = 0, \alpha = 1, \beta = 2, t = 0$; (b) The plot of the Wronskian interaction solution (3.13) with parameters: $r = 2, l = 1, \alpha_1 = 1, \alpha_2 = -2.1, \alpha = 1, \beta = 2, t = 0$; (c) The plot of the Wronskian interaction solution (3.14) with parameters: $p = 1, q = -3, r = 1, l = 2, \alpha = 1, \beta = 2, t = 0$.

4. Concluding remarks

In a word, on the basis of the bilinear Bäcklund transformation, we established a Lax system and linear superposition solutions composed of exponential functions for the trilinear Eq (2.2). Furthermore, by extending the involved parameters to the complex field, we obtained some special linear superposition solutions, such as the linear superposition formula of the product of exponential functions and trigonometric functions, the linear superposition formula of the product of exponential functions and hyperbolic functions, and the mixed-type function solutions. Finally, a set of sufficient conditions, which guarantees that the Wronskian determinant is a solution of Eq (2.2), was given associated with the bilinear Bäcklund transformation. The resulting Wronskian structure generated the N-soliton solution and a few special Wronskian interaction solutions for Eq (1.2).

In a sense, the presented results in this paper extend existing studies, because many soliton equations can be used as special cases of Eq (1.2). The bilinear Bäcklund transformation (2.5) and Lax system (2.7) indicate the integrability of Eq (1.2). By employing Wronskian identities of the bilinear KP hierarchy and properties of Hirota operators, we provide a direct and simple verification of the Wronskian determinant solution, which avoids a lengthy and complex proof process. Our studies also demonstrate the diversity and richness of solution structures of the introduced equation.

We remark that Eq (1.2) possesses a kind of traveling wave solutions in the form:

$$u = \frac{2}{\chi}(\ln g(\tau))_{xx},\ v = \frac{2}{\chi}(\ln g(\tau))_{y},\ \tau = x - \frac{a}{b}y - \Big(\frac{2\delta a^3}{cb^2} + \frac{d}{c} - \frac{ah}{bc}\Big)t + \tau^0,$$

where the coefficients a and b are two nonzero constants, g is an arbitrary function and τ^0 is an arbitrary constant.

Therefore Eq (1.2) has a large number of solution formulas including soliton molecules [22, 42]. Furthermore, in our future work, lump solutions [10, 34, 43–45] and nonsingular complexiton solutions [10] are expected to be studied for Eq (1.2) with $\delta < 0$. These wonderful solutions will be useful for analyzing nonlinear phenomena in realistic applications.

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Conflict of interest

The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

Appendix A

We provide a simple proof of Theorem 2.1.

Proof. Let us start from a key function

$$\begin{aligned} P = {} & f'^4\Big\{D_x[(3aD_x^4 + 9a\delta D_y^2 + 2bD_x^3D_y + 3cD_xD_t)f\cdot f]\cdot f^2 \\ & + D_y[(bD_x^4 + 3b\delta D_y^2)f\cdot f]\cdot f^2\Big\} - f^4\Big\{D_x[(3aD_x^4 + 9a\delta D_y^2 \\ & + 2bD_x^3D_y + 3cD_xD_t)f'\cdot f']\cdot f'^2 + D_y[(bD_x^4 + 3b\delta D_y^2)f'\cdot f']\cdot f'^2\Big\}. \qquad (A.1) \end{aligned}$$

By applying (2.5a) and Theorem 2.1 obtained in [28], we have

$$P = 6D_x\Big\{D_x[(aD_x^3 + bD_x^2D_y - 3a\tilde{\delta}D_xD_y - b\tilde{\delta}D_y^2 + cD_t)f\cdot f']\cdot ff'\Big\}\cdot f^2f'^2. \qquad (A.2)$$

Let us introduce another key function

$$Q = f'^4\Big\{D_x[(3dD_x^2 + 3hD_xD_y)f \cdot f] \cdot f^2\Big\} - f^4\Big\{D_x[(3dD_x^2 + 3hD_xDy)f' \cdot f'] \cdot f'^2\Big\}. \qquad (A.3)$$

By employing the folllowing exchange identities for Hirota's bilinear operators [6]:

$$(D_xF \cdot G)H^2 - G^2(D_xT \cdot H) = D_x(FH - GT) \cdot GH, \qquad (A.4a)$$

$$(D_yD_xF \cdot F)G^2 - F^2(D_yD_xG \cdot G) = 2D_x(D_yF \cdot G) \cdot FG = 2D_y(D_xF \cdot G) \cdot FG, \qquad (A.4b)$$

a direct computation shows

$$\begin{aligned} Q &= D_x\Big\{f'^2(3dD_x^2 + 3hD_xD_y)f \cdot f - f^2(3dD_x^2 + 3hD_xD_y)f' \cdot f'\Big\} \cdot f^2f'^2 \\ &= 6D_x\Big\{D_x[(dD_x + hD_y)f \cdot f'] \cdot ff'\Big\} \cdot f^2f'^2. \qquad (A.5) \end{aligned}$$

It further follows that

$$\begin{aligned} P + Q = {} & f'^4\Big\{D_x[(3aD_x^4 + 9a\delta D_y^2 + 2bD_x^3D_y + 3cD_xD_t + 3dD_x^2 + 3hD_xD_y)f \cdot f] \cdot f^2 \\ & + D_y[(bD_x^4 + 3b\delta D_y^2)f \cdot f] \cdot f^2\Big\} - f^4\Big\{D_x[(3aD_x^4 + 9a\delta D_y^2 + 2bD_x^3D_y \\ & + 3cD_xD_t + 3dD_x^2 + 3hD_xD_y)f' \cdot f'] \cdot f'^2 + D_y[(bD_x^4 + 3b\delta D_y^2)f' \cdot f'] \cdot f'^2\Big\} \\ = {} & 6D_x\Big\{D_x[(aD_x^3 + bD_x^2D_y - 3a\tilde{\delta}D_xD_y - b\tilde{\delta}D_y^2 + cD_t + dD_x + hD_y)f \cdot f'] \cdot ff'\Big\} \cdot f^2f'^2. \qquad (A.6) \end{aligned}$$

Thus, the system of bilinear Eq (2.5) guarantees $P + Q = 0$, which indicates that the system (2.5) yields a Bäcklund transformation for Eq (2.2).

Appendix B

It is widely known that the first two equations of the KP hierarchy [46] may be expressed in Hirota bilinear form as

$$(D_1^4 - 4D_1D_3 + 3D_2^2)f \cdot f = 0, \qquad (B.1a)$$

$$[(D_1^3 + 2D_3)D_2 - 3D_1D_4]f \cdot f = 0, \qquad (B.1b)$$

where f denotes a function related to variables $x_j,\ j = 1, 2, 3, \ldots$, and $D_j \equiv D_{x_j}$. To prove Theorem 3.1, we first give two helpful lemmas in terms of Hirota differential operators.

Lemma B.1. *Suppose that a group of functions $\phi_i = \phi_i(x_1, x_2, x_3, \ldots), (1 \le i \le N)$, satisfies that*

$$\partial_{x_j}\phi_i = \frac{\partial^j\phi_i}{\partial x^j},\ j = 1, 2, 3, \cdots. \qquad (B.2)$$

Then the Wronskian determinant $f = f_N = |\widehat{N-1}|$ defined by (3.1) solves the bilinear Eqs (B.1a) and (B.1b).

Lemma B.1 has been proved in [6, 47]. This lemma demonstrates that the bilinear forms $(B.1a)$ and $(B.1b)$ become the Plücker relations for determinants if the function f is written as the Wronskian determinant [48]. That is, $(B.1a)$ and $(B.1b)$ can be transformed into the following Wronskian identities:

$$(D_1^4 - 4D_1D_3 + 3D_2^2)|\widehat{N-1}| \cdot |\widehat{N-1}| = 24(|\widehat{N-1}||\widehat{N-3}, N, N+1| \\ -|\widehat{N-2}, N||\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+1||\widehat{N-3}, N-1, N|) \equiv 0, \qquad (B.3a)$$

and

$$[(D_1^3 + 2D_3)D_2 - 3D_1D_4]|\widehat{N-1}| \cdot |\widehat{N-1}| = 12(|\widehat{N-1}||\widehat{N-3}, N, N+2| \\ -|\widehat{N-2}, N||\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+2||\widehat{N-3}, N-1, N|) \\ -12(|\widehat{N-1}||\widehat{N-4}, N-2, N, N+1| - |\widehat{N-2}, N||\widehat{N-4}, N-2, N-1, N+1| \\ +|\widehat{N-2}, N+1||\widehat{N-4}, N-2, N-1, N|) \equiv 0, \qquad (B.3b)$$

respectively. The first identity $(B.3a)$ is nothing but a Plücker relation, and the second identity $(B.3b)$ is a combination of two Plücker relations.

Lemma B.2. *Let Wronskian entries $\phi_i = \phi_i(x, y, t)$, $1 \leq i \leq N$, in the Wronskian determinant (3.1) satisfy (B.2) and*

$$\partial_y \phi_i = (a_1\partial_x + a_2\partial_x^2 + \cdots + a_m\partial_x^m)\phi_i \equiv (a_1\partial_{x_1} + a_2\partial_{x_2} + \cdots + a_m\partial_{x_m})\phi_i, \qquad (B.4a)$$

$$\partial_t \phi_i = (b_1\partial_x + b_2\partial_x^2 + \cdots + b_n\partial_x^n)\phi_i \equiv (b_1\partial_{x_1} + b_2\partial_{x_2} + \cdots + b_n\partial_{x_n})\phi_i, \qquad (B.4b)$$

where m, n are nonnegative integers, and $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n$ are arbitrary constants. Then

$$f = f_N = |\widehat{N-1}|$$

defined by (3.1) yields

$$D_y^{n_1} D_t^{n_2} f \cdot f = (a_1D_1 + a_2D_2 + \cdots + a_mD_m)^{n_1}(b_1D_1 + b_2D_2 + \cdots + b_nD_n)^{n_2} f \cdot f, \qquad (B.5)$$

with n_1, n_2 are nonnegative integers and $D_j \equiv D_{x_j}$. The associated Hirota's bilinear operators D_{x_j}, $j = 1, 2, 3, \ldots,$ are defined by the expression (2.3).

Lemma B.2 has been proved in [49]. Next, we present the proof of Theorem 3.1.

Proof. Introducing an auxiliary variable z, we may rewrite Eq (2.4) as

$$D_x\Big[(3aD_x^4 + 9a\delta D_y^2 + 2bD_x^3D_y + 3cD_xD_t + 3dD_x^2 + 3hD_xD_y - D_yD_z)f \cdot f\Big] \cdot f^2 \\ +D_y\Big[(bD_x^4 + 3b\delta D_y^2 + D_xD_z)f \cdot f\Big] \cdot f^2 = 0. \qquad (B.6)$$

Let the Wronskian entries ϕ_i, $1 \leq i \leq N$, meets

$$\phi_{i,z} = -4b\phi_{i,xxx}.$$

Applying the conditions (3.2) and Lemma B.2, we have

$$D_x^4 f \cdot f = D_1^4 f \cdot f, \;\; D_y^2 f \cdot f = \frac{1}{\delta} D_2^2 f \cdot f, \;\; D_x^3 D_y f \cdot f = -\frac{1}{\tilde{\delta}} D_1^3 D_2 f \cdot f,$$

$$D_xD_t f \cdot f = D_1\Big(\frac{2b}{\tilde{\delta}c}D_4 - \frac{4a}{c}D_3 + \frac{h}{\tilde{\delta}c}D_2 - \frac{d}{c}D_1\Big) f \cdot f,$$

$$D_yD_zf\cdot f=\frac{4b}{\tilde{\delta}}D_2D_3f\cdot f,\quad D_xD_yf\cdot f=-\frac{1}{\tilde{\delta}}D_1D_2f\cdot f,$$
$$D_xD_zf\cdot f=-4bD_1D_3f\cdot f,\cdots.$$

Substituting the above derivatives into $(B.6)$ and employing Lemma B.2, a direct calculation yields

$$\begin{aligned}
&(3aD_x^4+9a\delta D_y^2+2bD_x^3D_y+3cD_xD_t+3dD_x^2+3hD_xD_y-D_yD_z)f\cdot f\\
&=(3aD_1^4+9aD_2^2-\frac{2b}{\tilde{\delta}}D_1^3D_2+\frac{6b}{\tilde{\delta}}D_1D_4-12aD_1D_3+\frac{3h}{\tilde{\delta}}D_1D_2-3dD_1^2\\
&+3dD_1^2-\frac{3h}{\tilde{\delta}}D_1D_2-\frac{4b}{\tilde{\delta}}D_2D_3)f\cdot f\\
&=3a(D_1^4+3D_2^2-4D_1D_3)f\cdot f-\frac{2b}{\tilde{\delta}}(D_1^3D_2+2D_2D_3-3D_1D_4)f\cdot f\\
&=0,
\end{aligned}\qquad (B.7)$$

and

$$(bD_x^4+3b\delta D_y^2+D_xD_z)f\cdot f=b(D_1^4+3D_2^2-4D_1D_3)f\cdot f=0,\qquad (B.8)$$

where the Wronskian identities $(B.3a)$ and $(B.3b)$ have been applied. Furthermore, we have

$$D_x\Big[(3aD_x^4+9a\delta D_y^2+2bD_x^3D_y+3cD_xD_t+3dD_x^2+3hD_xD_y)|\widehat{N-1}|\cdot|\widehat{N-1}|\Big]\cdot|\widehat{N-1}|^2$$
$$+D_y\Big[(bD_x^4+3b\delta D_y^2)|\widehat{N-1}|\cdot|\widehat{N-1}|\Big]\cdot|\widehat{N-1}|^2=0.$$

Thus, the Wronskian determinant $f=f_N=|\widehat{N-1}|$ solves Eq (2.2).