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Research article

On fixed point results in $\mathcal{F}$-metric spaces with applications

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Abstract: The aim of this research article is to define locally rational contractions concerning control functions of one variable in the background of $\mathcal{F}$-metric spaces and establish common fixed point results. We also introduce (α^*-ψ)-contractions and generalized (α^*,ψ, $\delta_{\mathcal{F}}$)-contractions in $\mathcal{F}$-metric spaces and obtain fixed points of multifunctions. A non trivial example is also furnished to manifest the originality of the fundamental result. As application, we investigate the solution of nonlinear neutral differential equation.

Keywords: $\mathcal{F}$-metric space; fixed point; locally rational contractions; generalized (α^*,ψ, $\delta_{\mathcal{F}}$)-contractive multifunctions; differential equation
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1. Introduction

The theory of fixed points is investigated to be the utmost fascinating and dynamic field of research in the progress of mathematical analysis. In this way, the conception of metric space [1] is one of the basic parts of mathematical sciences. Because of its outstanding and extraordinary improvement in various fields, it has been extended and generalized in different directions.

Recently, many compulsive generalizations (or extensions) of the concept of metric space came into sight. The famous extensions of the concept of metric spaces have been done by Bakhtin [2] which was formally defined by Czerwik [3] in 1993. Czerwik [3] gave the idea of b-metric space which broadens the notion of metric space by improving the triangle equality metric axiom by putting a constant $s \geq 1$ multiplied to the right-hand side of the inequality, and is one of the enormously applied generalizations for metric spaces. Berinde et al. [4] gave a brief survey on the development of fixed point theory, specially on b-metric spaces and discussed some important related aspects of it. Brzdek [5] proved and discussed some fixed point results for nonlinear operators, acting on some classes of functions with values in a b-metric space. Paluszyński et al. [6] discussed quasi-metric space as an extension of classical metric space and its involvement in a part of harmonic analysis related to the theory of spaces of homogeneous type. Khamsi et al. [7] reintroduced the notion of b-metric space with the name metric-type and proved some theorems in this recently introduced space. In [8], Branciari gave the concept of rectangular metric space and generalized the classical metric space by replacing the triangle inequality with more general inequality that is called rectangular inequality. This rectangular inequality consists of a distance between four points. In 2018, Jleli et al. [9] introduced a fascinating generalization of classical metric space, b-metric space and rectangular metric space that is famous as an $\mathcal{F}$-metric space. Subsequently, Al-Mazrooei et al. [10] used the notion of $\mathcal{F}$-metric space and proved some results for rational inequality that includes some non-negative constants.

On the other hand, Samet et al. [11] defined the notion of α-admissibility and (α-ψ)-contraction in the background of metric spaces and proved some results for these contractions. Subsequently, Asl et al. [12] generalized the above concept of α-admissibility and gave the notion of α^*-admissible mappings and established fixed point results for multivalued mappings in 2013. Recently, Hussain et al. [13] defined the notion of Ćirić type (α-ψ)-contraction in the framework of $\mathcal{F}$-metric space and proved fixed point theorems.

In this article, we establish common fixed point results for locally rational contractions concerning control functions of one variable in the background of $\mathcal{F}$-metric spaces. We also establish fixed points of (α^*-ψ)-contractive and generalized (α^*,ψ, $\delta_{\mathcal{F}}$)-contractive multifunctions. An important example is also included to display the originality of our principal result.

2. Preliminaries

Czerwik [3] gave the concept of b-metric space in this manner.

Definition 1. ([3]) Let $\Theta \neq \emptyset$ and $s \geq 1$ be a constant. A function $\kappa : \Theta \times \Theta \to [0, \infty)$ is called a b-metric if the following assertions hold:

($b1$) $\kappa(\rho, \hbar) \geq 0$ and $\kappa(\rho, \hbar) = 0$ if and only if $\rho = \hbar$,
($b2$) $\kappa(\rho, \hbar) = \kappa(\hbar, \rho)$,
($b3$) $\kappa(\rho, \varphi) \leq s[\kappa(\rho, \hbar) + \kappa(\hbar, \varphi)]$,
for all $\rho, \hbar, \varphi \in \Theta$.
Then the pair (Θ, κ) is known as a b-metric space.
Jleli et al. [9] gave the following notion of $\mathcal{F}$-metric space in this way.
Let $\mathcal{F}$ be the family of continuous functions $f : (0, +\infty) \to \mathbb{R}$ satisfying

($\mathcal{F}_1$) f is non-decreasing,
($\mathcal{F}_2$) for each $\{\rho_\jmath\} \subseteq \mathbb{R}^+$, $\lim_{\jmath\to\infty} \rho_\jmath = 0$ if and only if $\lim_{\jmath\to\infty} f(\rho_\jmath) = -\infty$.

Definition 2. ([9]) Let $\Theta \neq \emptyset$ and $\kappa : \Theta\times\Theta \to [0, +\infty)$ be a function satisfying the following conditions

(D_1) $(\rho, \hbar) \in \Theta \times \Theta$, $\kappa(\rho, \hbar) = 0$ if and only if $\rho = \hbar$,

(D$_2$) $\kappa(\rho,\hbar) = \kappa(\hbar,\rho)$, for all $(\rho,\hbar) \in \Theta \times \Theta$,
(D$_3$) for every $(\rho,\hbar) \in \Theta \times \Theta$ and $(\rho_i)_{i=1}^N \subset \Theta$ with

$$(\rho_1, \rho_N) = (\rho,\hbar),$$

there exists $(f, \mathfrak{h}) \in \mathcal{F} \times [0,+\infty)$ such that

$$\kappa(\rho,\hbar) > 0 \quad \text{implies} \quad f(\kappa(\rho,\hbar)) \leq f(\sum_{i=1}^{N-1} \kappa(\rho_i,\rho_{i+1})) + \mathfrak{h},$$

for all $N \in \mathbb{N}$ and $N \geq 2$.

Then (Θ,κ) is called an $\mathcal{F}$-metric space.

Example 1. Let $\Theta = \mathbb{R}$ and $\kappa : \Theta \times \Theta \to [0,+\infty)$ be defined by

$$\kappa(\rho,\hbar) = \begin{cases} (\rho-\hbar)^2 & \text{if } (\rho,\hbar) \in [0,2] \times [0,2] \\ |\rho-\hbar| & \text{if } (\rho,\hbar) \notin [0,2] \times [0,2] \end{cases}$$

with $f(\iota) = \ln(\iota)$ and $\mathfrak{h} = \ln(2)$, then (Θ,κ) is $\mathcal{F}$-metric space.

Definition 3. ([9]) Let (Θ,κ) be $\mathcal{F}$-metric space,

(i) a sequence $\{\rho_\jmath\}$ in Θ is said to be $\mathcal{F}$-convergent to $\rho \in \Theta$ if $\{\rho_\jmath\}$ is convergent to ρ with regard to the $\mathcal{F}$-metric κ;

(ii) a sequence $\{\rho_\jmath\}$ is $\mathcal{F}$-Cauchy, if

$$\lim_{\jmath,m\to\infty} \kappa(\rho_\jmath,\rho_m) = 0;$$

(iii) if every $\mathcal{F}$-Cauchy sequence in $\mathcal{F}$-metric space (Θ,κ) is $\mathcal{F}$-convergent to an element of Θ, then (Θ,κ) is said to be $\mathcal{F}$-complete.

Theorem 1. *([9]) Let (Θ,κ) be $\mathcal{F}$-complete $\mathcal{F}$-metric space and $\mathfrak{L} : \Theta \to \Theta$. Assume that there exists $\alpha \in [0,1)$ such that*

$$\kappa(\mathfrak{L}(\rho), \mathfrak{L}(\hbar)) \leq \alpha\kappa(\rho,\hbar)$$

for all $\rho,\hbar \in \Theta$, then $\mathfrak{L}$ has a unique fixed point $\rho^* \in \Theta$. Moreover, for any $\rho_0 \in \Theta$, the sequence $\{\rho_\jmath\} \subset \Theta$ defined by

$$\rho_{\jmath+1} = \mathfrak{L}(\rho_\jmath), \quad \jmath \in \mathbb{N},$$

is $\mathcal{F}$-convergent to ρ^*.

Subsequently, Hussain et al. [13] defined α-ψ-contraction in the background of $\mathcal{F}$-metric spaces and generalized the main result of Jleli et al. [9]. Later on, Ahmad et al. [10] defined a rational contraction in $\mathcal{F}$-metric space and proved the following result as generalization of main theorem of Jleli et al. [9].

Theorem 2. *([10]) Let (Θ,κ) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and $\mathfrak{L} : \Theta \to \Theta$. Suppose that there exists $\alpha,\beta \in [0,1)$ such that*

$$\kappa(\mathfrak{L}(\rho), \mathfrak{L}(\hbar)) \leq \alpha\kappa(\rho,\hbar) + \beta\frac{\kappa(\rho,\mathfrak{L}\rho)\kappa(\hbar,\mathfrak{L}\hbar)}{1+\kappa(\rho,\hbar)}$$

for all $\rho,\hbar \in \Theta$, then $\mathfrak{L}$ has a unique fixed point.

For more details in the direction of metric space, b-metric space and $\mathcal{F}$-metric space, we refer the researchers [14–28].

3. Main results

We state our main result in this way.

Theorem 3. *Let* (Θ, κ) *be an* $\mathcal{F}$*-complete* $\mathcal{F}$*-metric space and* $\mathfrak{L}_1, \mathfrak{L}_2 : \Theta \to \Theta$*. If there exist* $r > 0$ *and mappings* $\alpha, \beta : \Theta \to [0, 1)$ *such that*

(a) $\alpha(\mathfrak{L}_1\rho) \leq \alpha(\rho)$ and $\alpha(\mathfrak{L}_2\rho) \leq \alpha(\rho)$
$\beta(\mathfrak{L}_1\rho) \leq \beta(\rho)$ and $\beta(\mathfrak{L}_2\rho) \leq \beta(\rho)$,
(b) $\alpha(\rho) + \beta(\rho) < 1$,
(c)

$$\kappa(\mathfrak{L}_1\rho, \mathfrak{L}_2\hbar) \leq \alpha(\rho)\kappa(\rho, \hbar) + \beta(\rho)\frac{\kappa(\rho, \mathfrak{L}_1\rho)\kappa(\hbar, \mathfrak{L}_2\hbar)}{1 + \kappa(\rho, \hbar)}, \tag{3.1}$$

for all $\rho_0, \rho, \hbar \in \overline{B(\rho_0, r)}$ and

$$\kappa(\rho_0, \mathfrak{L}_1\rho_0) \leq (1 - \lambda)r \tag{3.2}$$

where $\lambda = \frac{\alpha(\rho_0)}{1-\beta(\rho_0)} < 1$, then there exists a unique point $\rho^* \in \overline{B(\rho_0, r)}$ such that $\mathfrak{L}_1\rho^* = \mathfrak{L}_2\rho^* = \rho^*$.

Proof. For $\rho_0 \in \overline{B(\rho_0, r)}$, define the sequence $\{\rho_\jmath\}$ by

$$\rho_{2\jmath+1} = \mathfrak{L}_1\rho_{2\jmath} \text{ and } \rho_{2\jmath+2} = \mathfrak{L}_2\rho_{2\jmath+1}$$

for all $\jmath = 0, 1, 2, ...$ By inequality (3.2), we have

$$\kappa(\rho_0, \rho_1) = \kappa(\rho_0, \mathfrak{L}_1\rho_0) \leq (1 - \lambda)r \leq r$$

that is, $\rho_1 \in \overline{B(\rho_0, r)}$. Assume that $\rho_2, \rho_3, ...\rho_\jmath \in \overline{B(\rho_0, r)}$ for some $j \in \mathbb{N}$. Now if, $2k + 1 \leq \jmath$, then by inequality (3.1), we have

$$\begin{aligned}
\kappa(\rho_{2k+1}, \rho_{2k+2}) &= \kappa(\mathfrak{L}_1\rho_{2k}, \mathfrak{L}_2\rho_{2k+1}) \leq \alpha(\rho_{2k})\kappa(\rho_{2k}, \rho_{2k+1}) \\
&\quad +\beta(\rho_{2k})\frac{\kappa(\rho_{2k}, \mathfrak{L}_1\rho_{2k})\kappa(\rho_{2k+1}, \mathfrak{L}_2\rho_{2k+1})}{1 + \kappa(\rho_{2k}, \rho_{2k+1})} \\
&= \alpha(\rho_{2k})\kappa(\rho_{2k}, \rho_{2k+1}) \\
&\quad +\beta(\rho_{2k})\frac{\kappa(\rho_{2k}, \rho_{2k+1})\kappa(\rho_{2k+1}, \rho_{2k+2})}{1 + \kappa(\rho_{2k}, \rho_{2k+1})} \\
&\leq \alpha(\rho_{2k})\kappa(\rho_{2k}, \rho_{2k+1}) \\
&\quad +\beta(\rho_{2k})\kappa(\rho_{2k+1}, \rho_{2k+2}).
\end{aligned}$$

By the sequence

$$\begin{aligned}
\kappa(\rho_{2k+1}, \rho_{2k+2}) &\leq \alpha\left(\mathfrak{L}_2\rho_{2\jmath-1}\right)\kappa(\rho_{2k}, \rho_{2k+1}) \\
&\quad +\beta\left(\mathfrak{L}_2\rho_{2\jmath-1}\right)\kappa(\rho_{2k+1}, \rho_{2k+2}) \\
&\leq \alpha\left(\rho_{2\jmath-1}\right)\kappa(\rho_{2k}, \rho_{2k+1}) \\
&\quad +\beta\left(\rho_{2\jmath-1}\right)\kappa(\rho_{2k+1}, \rho_{2k+2}) \\
&= \alpha\left(\mathfrak{L}_1\rho_{2\jmath-2}\right)\kappa(\rho_{2k}, \rho_{2k+1})
\end{aligned}$$

$$
\begin{aligned}
& +\beta\left(\mathfrak{L}_1\rho_{2\jmath-2}\right)\kappa\left(\rho_{2k+1},\rho_{2k+2}\right) \\
\leq\ & \alpha\left(\rho_{2\jmath-2}\right)\kappa\left(\rho_{2k},\rho_{2k+1}\right) \\
& +\beta\left(\rho_{2\jmath-2}\right)\kappa\left(\rho_{2k+1},\rho_{2k+2}\right) \\
\leq\ & \ldots \leq \alpha\left(\rho_0\right)\kappa\left(\rho_{2k},\rho_{2k+1}\right) \\
& +\beta\left(\rho_0\right)\kappa\left(\rho_{2k+1},\rho_{2k+2}\right)
\end{aligned}
$$

and hence

$$
\begin{aligned}
\kappa\left(\rho_{2k+1},\rho_{2k+2}\right) &\leq \frac{\alpha\left(\rho_0\right)}{1-\beta\left(\rho_0\right)}\kappa\left(\rho_{2k},\rho_{2k+1}\right) \\
&= \lambda\kappa\left(\rho_{2k},\rho_{2k+1}\right).
\end{aligned} \tag{3.3}
$$

Similarly, if $2k \leq \jmath$, we deduce

$$
\begin{aligned}
\kappa\left(\rho_{2k},\rho_{2k+1}\right) &\leq \frac{\alpha\left(\rho_0\right)}{1-\beta\left(\rho_0\right)}\kappa\left(\rho_{2k-1},\rho_{2k}\right) \\
&= \lambda\kappa\left(\rho_{2k-1},\rho_{2k}\right).
\end{aligned} \tag{3.4}
$$

Thus by inequalities (3.3) and (3.4), we have

$$
\kappa\left(\rho_{2k+1},\rho_{2k+2}\right) \leq \lambda\kappa\left(\rho_{2k},\rho_{2k+1}\right) \leq \ldots \leq \lambda^{2k+1}\kappa\left(\rho_0,\rho_1\right) \tag{3.5}
$$

$$
\kappa\left(\rho_{2k},\rho_{2k+1}\right) \leq \lambda\kappa\left(\rho_{2k-1},\rho_{2k}\right) \leq \ldots \leq \lambda^{2k}\kappa\left(\rho_0,\rho_1\right). \tag{3.6}
$$

Thus by inequalities (3.5) and (3.6), we have

$$
\kappa\left(\rho_\jmath,\rho_{\jmath+1}\right) \leq \lambda^\jmath\kappa\left(\rho_0,\rho_1\right)
$$

for some $\jmath \in \mathbb{N}$. Now

$$
\begin{aligned}
f\left(\kappa\left(\rho_0,\rho_{\jmath+1}\right)\right) &\leq f\left(\kappa\left(\rho_0,\rho_{\jmath+1}\right)+\kappa\left(\rho_1,\rho_2\right)+\ldots+\kappa\left(\rho_j,\rho_{\jmath+1}\right)\right) \\
&\leq f\left(\left(1+\ldots+\lambda^{\jmath-1}+\lambda^\jmath\right)\kappa\left(\rho_0,\rho_1\right)\right) \\
&\leq f\left(\frac{(1-\lambda^\jmath)}{1-\lambda}\kappa\left(\rho_0,\rho_1\right)\right) \\
&\leq f\left(\frac{(1-\lambda^\jmath)}{1-\lambda}(1-\lambda)r\right) \\
&< f\left(r\right).
\end{aligned}
$$

By $(\mathcal{F}_1)$, we get $\rho_{\jmath+1} \in \overline{B(\rho_0, r)}$. Thus $\rho_\jmath \in \overline{B(\rho_0, r)}$, for all $\jmath \in \mathbb{N}$. Then it follows that

$$
\kappa\left(\rho_\jmath,\rho_{\jmath+1}\right) \leq \lambda^\jmath\kappa\left(\rho_0,\rho_1\right)
$$

for all $\jmath \in \mathbb{N}$. Let $(f, \mathfrak{h}) \in \mathcal{F} \times [0, +\infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$
0 < \iota < \delta \Longrightarrow f(\iota) < f(\delta) - \mathfrak{h}. \tag{3.7}
$$

Hence, by (3.7), $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$, we have

$$f(\sum_{i=\jmath}^{m-1}\kappa(\rho_i,\rho_{i+1}))\leq f(\sum_{i=\jmath}^{m-1}\lambda^{\jmath}(\kappa(\rho_0,\rho_1)))\leq f(\sum_{\jmath\geq\jmath(\epsilon)}\lambda^{\jmath}\kappa(\rho_0,\rho_1))<f(\epsilon)-h \tag{3.8}$$

for $m>\jmath\geq\jmath(\epsilon)$. Using (D_3) and (3.8), we obtain $\kappa(\rho_\jmath,\rho_m)>0$, $m>\jmath\geq\jmath(\epsilon)$ implies

$$f(\kappa(\rho_\jmath,\rho_m))\leq f(\sum_{i=\jmath}^{m-1}\kappa(\rho_i,\rho_{i+1}))+\mathfrak{h}<f(\epsilon)$$

which yields by $(\mathcal{F}_1)$ that $\kappa(\rho_\jmath,\rho_m)<\epsilon$, $m>\jmath\geq\jmath(\epsilon)$. It shows that $\{\rho_\jmath\}$ is $\mathcal{F}$-Cauchy sequence in $\overline{B(\rho_0,r)}$. Now, $\overline{B(\rho_0,r)}$ is $\mathcal{F}$-complete since $\overline{B(\rho_0,r)}$ is $\mathcal{F}$-closed in Θ, so there exists $\rho^*\in\overline{B(\rho_0,r)}$ such that the sequence $\{\rho_\jmath\}$ is $\mathcal{F}$-convergent to ρ^*, i.e.,

$$\lim_{\jmath\to\infty}\kappa(\rho_\jmath,\rho^*)=0. \tag{3.9}$$

Now, we show that ρ^* is fixed point of $\mathfrak{L}_1$. We contrary suppose that $\kappa(\rho^*,\mathfrak{L}_1\rho^*)>0$. Then from (3.1), $(\mathcal{F}_1)$ and (D_3), we have

$$\begin{aligned}
f(\kappa(\rho^*,\mathfrak{L}_1\rho^*)) &\leq f(\kappa\left(\rho^*,\mathfrak{L}_2\rho_{2\jmath+1}\right)+\kappa\left(\mathfrak{L}_2\rho_{2\jmath+1},\mathfrak{L}_1\rho^*\right))+\mathfrak{h}\\
&\leq f\left(\kappa\left(\rho^*,\mathfrak{L}_2\rho_{2\jmath+1}\right)+\kappa\left(\mathfrak{L}_1\rho^*,\mathfrak{L}_2\rho_{2\jmath+1}\right)\right)+\mathfrak{h}\\
&\leq f\left(\kappa\left(\rho^*,\rho_{2\jmath+2}\right)+\left(\begin{array}{c}\alpha(\rho^*)\,\kappa\left(\rho^*,\rho_{2\jmath+1}\right)\\ +\beta(\rho^*)\,\frac{\kappa(\rho^*,\mathfrak{L}_1\rho^*)\kappa\left(\rho_{2\jmath+1},\mathfrak{L}_2\rho_{2\jmath+1}\right)}{1+\kappa\left(\rho^*,\rho_{2\jmath+1}\right)}\end{array}\right)\right)+\mathfrak{h}\\
&\leq f\left(\kappa\left(\rho^*,\rho_{2\jmath+2}\right)+\left(\begin{array}{c}\alpha(\rho^*)\,\kappa\left(\rho^*,\rho_{2\jmath+1}\right)\\ +\beta(\rho^*)\,\frac{\kappa(\rho^*,\mathfrak{L}_1\rho^*)\kappa\left(\rho_{2\jmath+1},\rho_{2\jmath+2}\right)}{1+\kappa\left(\rho^*,\rho_{2\jmath+1}\right)}\end{array}\right)\right)+\mathfrak{h}.
\end{aligned}$$

Taking the limit as $\jmath\to\infty$ and using $(\mathcal{F}_2)$ and (8), we have

$$\lim_{\jmath\to\infty}f(\kappa(\rho^*,\mathfrak{L}_1\rho^*))\leq\lim_{\jmath\to\infty}f\left(\kappa\left(\rho^*,\rho_{2\jmath+2}\right)+\left(\begin{array}{c}\alpha(\rho^*)\,\kappa\left(\rho^*,\rho_{2\jmath+1}\right)\\ +\beta(\rho^*)\,\frac{\kappa(\rho^*,\mathfrak{L}_1\rho^*)\kappa\left(\rho_{2\jmath+1},\rho_{2\jmath+2}\right)}{1+\kappa\left(\rho^*,\rho_{2\jmath+1}\right)}\end{array}\right)\right)+\mathfrak{h}=-\infty,$$

which implies that $\kappa(\rho^*,\mathfrak{L}_1\rho^*)=0$, a contradiction. Thus $\rho^*=\mathfrak{L}_1\rho^*$. Now we prove that ρ^* is fixed point of $\mathfrak{L}_2$. Then from (3.1), $(\mathcal{F}_1)$ and (D_3), we have

$$\begin{aligned}
f(\kappa(\rho^*,\mathfrak{L}_2\rho^*)) &\leq f(\kappa\left(\rho^*,\mathfrak{L}_1\rho_{2\jmath}\right)+\kappa\left(\mathfrak{L}_1\rho_{2\jmath},\mathfrak{L}_2\rho^*\right))+\mathfrak{h}\\
&\leq f\left(\kappa\left(\rho^*,\rho_{2\jmath+1}\right)+\left(\begin{array}{c}\alpha\left(\rho_{2\jmath}\right)\kappa\left(\rho_{2\jmath},\rho^*\right)\\ +\beta\left(\rho_{2\jmath}\right)\frac{\kappa\left(\rho_{2\jmath},\mathfrak{L}_1\rho_{2\jmath}\right)\kappa(\rho^*,\mathfrak{L}_2\rho^*)}{1+\kappa\left(\rho_{2\jmath},\rho^*\right)}\end{array}\right)\right)+\mathfrak{h}\\
&\leq f\left(\kappa\left(\rho^*,\rho_{2\jmath+1}\right)+\left(\begin{array}{c}\alpha\left(\rho_{2\jmath}\right)\kappa\left(\rho_{2\jmath},\rho^*\right)\\ +\beta\left(\rho_{2\jmath}\right)\frac{\kappa\left(\rho_{2\jmath},\rho_{2\jmath+1}\right)\kappa(\rho^*,\mathfrak{L}_2\rho^*)}{1+\kappa\left(\rho_{2\jmath},\rho^*\right)}\end{array}\right)\right)+\mathfrak{h}.
\end{aligned}$$

Taking the limit as $\jmath \to \infty$ and using $(\mathcal{F}_2)$ and (8), we have

$$\lim_{\jmath\to\infty} f(\kappa(\rho^*, \mathfrak{L}_2\rho^*)) \leq \lim_{\jmath\to\infty} f\left(\kappa\left(\rho^*, \rho_{2\jmath+1}\right) + \left(\begin{array}{c} \alpha\left(\rho_{2\jmath}\right)\kappa\left(\rho_{2\jmath}, \rho^*\right) \\ +\beta\left(\rho_{2\jmath}\right)\frac{\kappa\left(\rho_{2\jmath}, \rho_{2\jmath+1}\right)\kappa(\rho^*, \mathfrak{L}_2\rho^*)}{1+\kappa\left(\rho_{2\jmath}, \rho^*\right)} \end{array}\right)\right) + \mathfrak{h} = -\infty,$$

which implies that $\kappa(\rho^*, \mathfrak{L}_1\rho^*) = 0$, a contradiction. Thus $\rho^* = \mathfrak{L}_2\rho^*$.Thus ρ^* is a common fixed point of $\mathfrak{L}_1$ and $\mathfrak{L}_2$. Now we prove that ρ^* is unique. We suppose that

$$\rho^/ = \mathfrak{L}_1\rho^/ = \mathfrak{L}_2\rho^/$$

but $\rho^* \neq \rho^/$. Now from (3.1), we have

$$\begin{aligned} \kappa\left(\rho^*, \rho^/\right) &= \kappa\left(\mathfrak{L}_1\rho^*, \mathfrak{L}_2\rho^/\right) \\ &\leq \alpha(\rho^*)\kappa\left(\rho^*, \rho^/\right) + \beta(\rho^*)\frac{\kappa(\rho^*, \mathfrak{L}\rho^*)\kappa\left(\rho^/, \mathfrak{L}_2\rho^/\right)}{1+\kappa(\rho^*, \rho^/)} \\ &= \alpha(\rho^*)\kappa\left(\rho^*, \rho^/\right) + \beta(\rho^*)\frac{\kappa(\rho^*, \rho^*)\kappa\left(\rho^/, \rho^/\right)}{1+\kappa(\rho^*, \rho^/)}. \end{aligned}$$

This implies that, we have

$$\kappa\left(\rho^*, \rho^/\right) \leq \alpha(\rho^*)\kappa\left(\rho^*, \rho^/\right).$$

As $\alpha(\rho^*) < 1$, we have

$$\kappa\left(\rho^*, \rho^/\right) = 0.$$

Thus $\rho^* = \rho^/$. □

Example 2. Let Θ=$\left\{S_\jmath = 2\jmath + 1 : \jmath \in \mathbb{N}\right\}$ be endowed with the $\mathcal{F}$-metric

$$\kappa(\rho, \hbar) = \begin{cases} 0, \text{ if } \rho = \hbar \\ 2^{|\rho-\hbar|}, \text{ if } \rho \neq \hbar \end{cases}$$

for all $\rho, \hbar \in \Theta$ and $f(\iota) = \ln\iota$. Then (Θ, κ) is an $\mathcal{F}$-complete $\mathcal{F}$-metric space. Define the mapping $\mathfrak{L}_1, \mathfrak{L}_2 : \Theta \to \Theta$ by

$$\mathfrak{L}_1\left(S_\jmath\right) = \begin{cases} S_1, & \text{if } \jmath = 1, \\ S_2, & \text{if } \jmath = 2, \\ S_{\jmath-2}, & \text{if } \jmath \geq 3 \end{cases}$$

and

$$\mathfrak{L}_2\left(S_\jmath\right) = \begin{cases} S_1, & \text{if } \jmath = 1, 2 \\ S_{\jmath-1}, & \text{if } \jmath \geq 3 \end{cases}.$$

Suppose that $m \neq \jmath$, then

$$\begin{aligned} \kappa\left(\mathfrak{L}_1\left(S_\jmath\right), \mathfrak{L}_2(S_m)\right) &= 2^{\left|S_{\jmath-2}-S_{m-1}\right|} \\ &= 2^{|2(\jmath-m)-2|} \\ &< 2^{-1}\cdot 2^{|2(\jmath-m)|} \\ &\leq \alpha\left(S_\jmath\right)\kappa\left(S_\jmath, S_m\right) + \beta\left(S_\jmath\right)\frac{\kappa\left(S_\jmath, \mathfrak{L}_1S_\jmath\right)\kappa(S_m, \mathfrak{L}_2S_m)}{1+\kappa\left(S_\jmath, S_m\right)}. \end{aligned}$$

Thus all the assertions of Theorem 3 hold with $\alpha : \Theta\times\Theta\to[0,1)$ defined by $\alpha\left(S_J\right) = 2^{-1}$ and any $\beta : \Theta\to[0,1)$. Hence S_1 is a unique common fixed point of $\mathfrak{L}_1$ and $\mathfrak{L}_2$.

Corollary 1. *Let (Θ,κ) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and $\mathfrak{L} : \Theta \to \Theta$. If there exist $r > 0$ and mappings $\alpha,\beta : \Theta\to[0,1)$ such that*

(a) $\alpha(\mathfrak{L}\rho) \leq \alpha(\rho)$ and $\beta(\mathfrak{L}\rho) \leq \beta(\rho)$,
(b) $\alpha(\rho) + \beta(\rho) < 1$,
(c)
$$\kappa(\mathfrak{L}\rho,\mathfrak{L}\hbar) \leq \alpha(\rho)\kappa(\rho,\hbar) + \beta(\rho)\frac{\kappa(\rho,\mathfrak{L}\rho)\kappa(\hbar,\mathfrak{L}\hbar)}{1+\kappa(\rho,\hbar)},$$

for all $\rho_0,\rho,\hbar \in \overline{B(\rho_0,r)}$ and
$$\kappa(\rho_0,\mathfrak{L}\rho_0) \leq (1-\lambda)r$$
where $\lambda = \frac{\alpha(\rho_0)}{1-\beta(\rho_0)} < 1$, then there exists a unique point $\rho^* \in \overline{B(\rho_0,r)}$ such that $\mathfrak{L}\rho^* = \rho^*$.

Corollary 2. *Let (Θ,κ) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and $\mathfrak{L}_1,\mathfrak{L}_2 : \Theta \to \Theta$. If there exist $r > 0$ and mappings $\alpha : \Theta\to[0,1)$ such that*

(a) $\alpha(\mathfrak{L}_1\rho) \leq \alpha(\rho)$ and $\alpha(\mathfrak{L}_2\rho) \leq \alpha(\rho)$
(b) $\alpha(\rho) < 1$,
(c)
$$\kappa(\mathfrak{L}_1\rho,\mathfrak{L}_2\hbar) \leq \alpha(\rho)\kappa(\rho,\hbar),$$

for all $\rho_0,\rho,\hbar \in \overline{B(\rho_0,r)}$ and
$$\kappa(\rho_0,\mathfrak{L}_1\rho_0) \leq (1-\lambda)r$$
where $\lambda = \alpha(\rho_0) < 1$, then there exists a unique point $\rho^* \in \overline{B(\rho_0,r)}$ such that $\mathfrak{L}_1\rho^* = \mathfrak{L}_2\rho^* = \rho^*$.

Corollary 3. *Let (Θ,κ) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and $\mathfrak{L}_1,\mathfrak{L}_2 : \Theta \to \Theta$. If there exist $r > 0$ and mapping $\beta : \Theta\to[0,1)$ such that*

(a) $\beta(\mathfrak{L}_1\rho) \leq \beta(\rho)$ and $\beta(\mathfrak{L}_2\rho) \leq \beta(\rho)$,
(b) $\alpha(\rho) + \beta(\rho) < 1$,
(c)
$$\kappa(\mathfrak{L}_1\rho,\mathfrak{L}_2\hbar) \leq \beta(\rho)\frac{\kappa(\rho,\mathfrak{L}_1\rho)\kappa(\hbar,\mathfrak{L}_2\hbar)}{1+\kappa(\rho,\hbar)},$$

for all $\rho_0,\rho,\hbar \in \overline{B(\rho_0,r)}$ and
$$\kappa(\rho_0,\mathfrak{L}_1\rho_0) \leq (1-\lambda)r$$
where $\lambda = \frac{1}{1-\beta(\rho_0)} < 1$, then there exists a unique point $\rho^* \in \overline{B(\rho_0,r)}$ such that $\mathfrak{L}_1\rho^* = \mathfrak{L}_2\rho^* = \rho^*$.

Corollary 4. *Let (Θ,κ) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and $\mathfrak{L}_1,\mathfrak{L}_2 : \Theta \to \Theta$. If there exist $r > 0$ and $\alpha,\beta \in [0,1)$ such that*

(a) $\alpha + \beta < 1$,
(b)
$$\kappa(\mathfrak{L}_1\rho,\mathfrak{L}_2\hbar) \leq \alpha\kappa(\rho,\hbar) + \beta\frac{\kappa(\rho,\mathfrak{L}_1\rho)\kappa(\hbar,\mathfrak{L}_2\hbar)}{1+\kappa(\rho,\hbar)},$$

for all $\rho_0, \rho, \hbar \in \overline{B(\rho_0, r)}$ and

$$\kappa(\rho_0, \mathfrak{L}_1\rho_0) \leq (1-\lambda)r$$

where $\lambda = \frac{\alpha}{1-\beta} < 1$, then there exists a unique point $\rho^* \in \overline{B(\rho_0, r)}$ such that $\mathfrak{L}_1\rho^* = \mathfrak{L}_2\rho^* = \rho^*$.

Corollary 5. *Let (Θ, κ) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and $\mathfrak{L}_1, \mathfrak{L}_2 : \Theta \to \Theta$. If there exist $r > 0$ and $\alpha \in [0, 1)$ such that*

$$\kappa(\mathfrak{L}_1\rho, \mathfrak{L}_2\hbar) \leq \alpha\kappa(\rho, \hbar),$$

for all $\rho_0, \rho, \hbar \in \overline{B(\rho_0, r)}$ and

$$\kappa(\rho_0, \mathfrak{L}_1\rho_0) \leq (1-\lambda)r$$

where $\lambda = \alpha < 1$, then there exists a unique point $\rho^* \in \overline{B(\rho_0, r)}$ such that $\mathfrak{L}_1\rho^* = \mathfrak{L}_2\rho^* = \rho^*$.

Corollary 6. *Let (Θ, κ) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and $\mathfrak{L}_1, \mathfrak{L}_2 : \Theta \to \Theta$. If there exist $r > 0$ and $\beta \in [0, 1)$ such that*

$$\kappa(\mathfrak{L}_1\rho, \mathfrak{L}_2\hbar) \leq \beta\frac{\kappa(\rho, \mathfrak{L}_1\rho)\kappa(\hbar, \mathfrak{L}_2\hbar)}{1 + \kappa(\rho, \hbar)},$$

for all $\rho_0, \rho, \hbar \in \overline{B(\rho_0, r)}$ and

$$\kappa(\rho_0, \mathfrak{L}_1\rho_0) \leq (1-\lambda)r$$

where $\lambda = \frac{1}{1-\beta} < 1$, then there exists a unique point $\rho^* \in \overline{B(\rho_0, r)}$ such that $\mathfrak{L}_1\rho^* = \mathfrak{L}_2\rho^* = \rho^*$.

Remark 1. If we set $\mathfrak{L}_1 = \mathfrak{L}_2 = \mathfrak{L}$ in the Corollary 5, the we get the main result of Samet et al. [9].

4. α^*-ψ-contractive multivalued results

Let Ψ represents the set of all nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{\jmath=1}^{\infty} \psi^{\jmath}(\iota) < +\infty$, $\forall\ \iota > 0$, where $\psi^{\jmath}$ is the $\jmath$-th iterate of these nondecreasing functions ψ.

Now we state a lemma which is useful in the sequel.

Lemma 1. *For $\psi \in \Psi$, these conditions hold:*

(i) $(\psi^{\jmath}(\iota))_{\jmath\in\mathbb{N}}$ converges to 0 as $\jmath \to \infty$, $\forall \iota \in (0, +\infty)$,
(ii) $\psi(\iota) < \iota$, $\forall\ \iota > 0$,
(iii) $\psi(\iota) = 0$ iff $\iota = 0$.

Samet et al. [11] gave the theory of α-admissibility and proved the following result.

Definition 4. ([11]) A mapping $\mathfrak{L} : \Theta \to \Theta$ is called an α-admissible if there exists a mapping $\alpha : \Theta \times \Theta \to [0, +\infty)$ satisfying

$$\rho, \hbar \in \Theta, \quad \alpha(\rho, \hbar) \geq 1 \quad \Longrightarrow \quad \alpha(\mathfrak{L}\rho, \mathfrak{L}\hbar) \geq 1.$$

Theorem 4. *([11]) Let (Θ, κ) be a complete metric space and $\mathfrak{L}$ be α-admissible mapping. Assume that*

$$\alpha(\rho, \hbar)\kappa(\mathfrak{L}\rho, \mathfrak{L}\hbar) \leq \psi(\kappa(\rho, \hbar))$$

for all $\rho, \hbar \in \Theta$, where $\psi \in \Psi$, also

(i) there exists $\rho_0 \in \Theta$ such that $\alpha(\rho_0, \mathfrak{L}\rho_0) \geq 1$;

(ii) either $\mathfrak{L}$ is continuous or for any sequence $\{\rho_\jmath\}$ in Θ with $\alpha(\rho_\jmath, \rho_{\jmath+1}) \geq 1$ for all $\jmath \in \mathbb{N}$ and $\rho_\jmath \to \rho$ as $\jmath \to +\infty$, we have $\alpha(\rho_\jmath, \rho) \geq 1$ $\forall \jmath \in \mathbb{N}$.

Then $\mathfrak{L}$ has a fixed point.

In 2013, Asl et al. [12] gave the notion of α^*-admissible mappings in this way.

Definition 5. ([12]) Let $\alpha : \Theta \times \Theta \to [0, +\infty)$ be a function and $\mathfrak{L} : \Theta \to CL(\Theta)$ be multivalued mapping. Then $\mathfrak{L}$ is said to be α^*-admissible mapping if

$$\forall \rho, \hbar \in \Theta, \quad \alpha^*(\rho, \hbar) \geq 1 \quad \Longrightarrow \quad \alpha^*(\mathfrak{L}\rho, \mathfrak{L}\hbar) \geq 1$$

where $\alpha^*(\mathfrak{L}(\rho), \mathfrak{L}(\hbar)) = \inf\{\alpha(a, b) : a \in \mathfrak{L}(\rho), b \in \mathfrak{L}(\hbar)\}$.

Lemma 2. *Let (Θ, κ) be an $\mathcal{F}$-metric space and let $\mathfrak{R} \in CL(\Theta)$. Then, for each $\rho \in \Theta$ with $\kappa(\rho, \mathfrak{R}) > 0$ and $q > 1$, there exists an member $\hbar \in \mathfrak{R}$ such that*

$$\kappa(\rho, \hbar) \leq q\kappa(\rho, \mathfrak{R}).$$

Let (Θ, κ) be an $\mathcal{F}$-metric space. We represent by $N(\Theta)$ by set of non empty subsets of Θ, by $CL(\Theta)$ the set of all nonempty closed subsets of Θ and $B(\Theta)$ the set of all nonempty bounded subsets of Θ. Now for $\mathfrak{R} \in N(\Theta)$ and $\rho \in \Theta$, $\kappa(\rho, \mathfrak{R}) = \inf\{\kappa(\rho, \hbar) : \hbar \in \mathfrak{R}\}$. Also for $\mathfrak{R}_1, \mathfrak{R}_2 \in B(\Theta)$, $\delta_{\mathcal{F}}(\mathfrak{R}_1, \mathfrak{R}_2) = \sup\{\kappa(\rho, \hbar) : \rho \in \mathfrak{R}_1, \hbar \in \mathfrak{R}_2\}$. Whenever $\mathfrak{R}_1 = \{\rho\}$, we represent $\delta_{\mathcal{F}}(\mathfrak{R}_1, \mathfrak{R}_2)$ by $\delta_{\mathcal{F}}(\rho, \mathfrak{R}_2)$. Let $(\Theta, \preceq, \kappa)$ be an ordered $\mathcal{F}$-metric space and $\mathfrak{R}_1, \mathfrak{R}_2 \subseteq \Theta$. We say that $\mathfrak{R}_1 \prec_r \mathfrak{R}_2$, if for every $\rho \in \mathfrak{R}_1, \hbar \in \mathfrak{R}_2$, we have $\rho \preceq \hbar$.

Definition 6. Let (Θ, κ) be an $\mathcal{F}$-metric space. A closed-valued multifunction $\mathfrak{L} : \Theta \to CL(\Theta)$ is said to be $(\alpha^*$-$\psi)$- contractive multifunction if there exists two functions $\alpha : \Theta \times \Theta \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha^*(\mathfrak{L}(\rho), \mathfrak{L}(\hbar))\kappa(\hbar, \mathfrak{L}(\hbar)) \leq \psi(\kappa(\rho, \hbar)) \tag{4.1}$$

for each $\rho \in \Theta$ and $\hbar \in \mathfrak{L}(\rho)$.

Theorem 5. *Let (Θ, κ) be an $\mathcal{F}$-metric space and $\mathfrak{L} : \Theta \to CL(\Theta)$ be an α^*-admissible and $(\alpha^*$-$\psi)$-contractive multifunction. Also suppose that the following assertions holds:*

(i) (Θ, κ) is $\mathcal{F}$-complete;
(ii) there exists $\rho_0 \in \Theta$ and $\rho_1 \in \mathfrak{L}(\rho_0)$ such that $\alpha(\rho_0, \rho_1) \geq 1$.
Then ρ is a fixed point of $\mathfrak{L}$ iff $g(\xi) = \kappa(\xi, \mathfrak{L}\xi)$ is lower semi-continuous at ρ.

Proof. Let $\rho_0 \in \Theta$ be an arbitrary element. Since $\mathfrak{L}(\rho_0) \neq \emptyset$, so there exists $\rho_1 \in \mathfrak{L}(\rho_0)$. If $\rho_0 = \rho_1$, then ρ_0 is a fixed point of $\mathfrak{L}$ and we have nothing to prove. As $\mathfrak{L}(\rho_1) \neq \emptyset$. So if $\rho_1 \in \mathfrak{L}(\rho_1)$, then ρ_1 is a fixed point of $\mathfrak{L}$. Let $\rho_1 \notin \mathfrak{L}(\rho_1)$. Since $\mathfrak{L}$ is α^*-admissible, so $\alpha^*(\mathfrak{L}(\rho_0), \mathfrak{L}(\rho_1)) \geq 1$. Thus by (4.1), we have

$$\begin{aligned} 0 &< \kappa(\rho_1, \mathfrak{L}(\rho_1)) \leq \alpha^*(\mathfrak{L}(\rho_0), \mathfrak{L}(\rho_1))\kappa(\rho_1, \mathfrak{L}(\rho_1)) \\ &\leq \psi(\kappa(\rho_0, \rho_1)). \end{aligned} \tag{4.2}$$

For given $q > 1$ and by Lemma 2, $\exists \rho_2 \in \mathfrak{L}(\rho_1)$ such that

$$0 < \kappa(\rho_1, \rho_2) < q\kappa(\rho_1, \mathfrak{L}(\rho_1)). \tag{4.3}$$

Thus by (4.2) and (4.3), we have

$$0 < \kappa(\rho_1, \rho_2) \leq q\psi(\kappa(\rho_0, \rho_1)). \tag{4.4}$$

It is clear that $\rho_2 \neq \rho_1$. As $\kappa(\rho_1, \rho_2) < q\psi(\kappa(\rho_0, \rho_1))$. Since ψ is strictly increasing, so $\psi(\kappa(\rho_1, \rho_2)) < \psi(q\psi(\kappa(\rho_0, \rho_1)))$. Put $q_1 = \frac{\psi(q\psi(\kappa(\rho_0,\rho_1)))}{\psi(\kappa(\rho_1,\rho_2))}$. Then $q_1 > 1$. If $\rho_2 \in \mathfrak{L}(\rho_2)$, then ρ_2 is fixed point of $\mathfrak{L}$. Assume that $\rho_2 \notin \mathfrak{L}(\rho_2)$. As $\alpha^*(\rho_1, \rho_2) \geq 1$ and $\mathfrak{L}$ is α^*-admissible, so $\alpha^*(\mathfrak{L}(\rho_1), \mathfrak{L}(\rho_2)) \geq 1$. Then from (4.1), we get

$$\begin{aligned} 0 &< \kappa(\rho_2, \mathfrak{L}(\rho_2)) \leq \alpha^*(\mathfrak{L}(\rho_1), \mathfrak{L}(\rho_2))\kappa(\rho_2, \mathfrak{L}(\rho_2)) \\ &\leq \psi(\kappa(\rho_1, \rho_2)). \end{aligned} \tag{4.5}$$

For given $q_1 > 1$ and by Lemma 2, $\exists \rho_3 \in \mathfrak{L}(\rho_2)$ such that

$$0 < \kappa(\rho_2, \rho_3) < q\kappa(\rho_2, \mathfrak{L}(\rho_2)). \tag{4.6}$$

Thus by (4.5) and (4.6), we have

$$\begin{aligned} 0 &< \kappa(\rho_2, \rho_3) \leq q_1\psi(\kappa(\rho_1, \rho_2)) \\ &= \psi(q\psi(\kappa(\rho_0, \rho_1))). \end{aligned}$$

It is clear that $\rho_3 \neq \rho_2$. As $\kappa(\rho_2, \rho_3) < \psi(q\psi(\kappa(\rho_0, \rho_1)))$. Since ψ is strictly increasing, so $\psi(\kappa(\rho_2, \rho_3)) < \psi^2(q\psi(\kappa(\rho_0, \rho_1)))$. Put $q_2 = \frac{\psi^2(q\psi(\kappa(\rho_0,\rho_1)))}{\psi(\kappa(\rho_2,\rho_3))}$. Then $q_2 > 1$. If $\rho_3 \in \mathfrak{L}(\rho_3)$, then ρ_3 is fixed point of $\mathfrak{L}$. Assume that $\rho_3 \notin \mathfrak{L}\rho_3$. As $\alpha^*(\rho_2, \rho_3) \geq 1$ and $\mathfrak{L}$ is α^*-admissible, so $\alpha^*(\mathfrak{L}(\rho_2), \mathfrak{L}(\rho_3)) \geq 1$. Then from (4.1), we get

$$\begin{aligned} 0 &< \kappa(\rho_3, \mathfrak{L}(\rho_3)) \leq \alpha^*(\mathfrak{L}(\rho_2), \mathfrak{L}(\rho_3))\kappa(\rho_3, \mathfrak{L}(\rho_3)) \\ &\leq \psi^2(q\psi(\kappa(\rho_0, \rho_1))). \end{aligned} \tag{4.7}$$

For given $q_2 > 1$ and by Lemma 2, $\exists \rho_4 \in \mathfrak{L}(\rho_3)$ such that

$$0 < \kappa(\rho_3, \rho_4) < q_2\kappa(\rho_3, \mathfrak{L}(\rho_3)). \tag{4.8}$$

Thus by (4.7) and (4.8), we have

$$0 < \kappa(\rho_3, \rho_4) \leq \psi^2(q\psi(\kappa(\rho_0, \rho_1))).$$

Pursuing the same, we obtain $\{\rho_\jmath\}$ in Θ such that $\rho_\jmath \in \mathfrak{L}\rho_{\jmath-1}$ and $\rho_\jmath \neq \rho_{\jmath-1}$, and

$$\kappa(\rho_\jmath, \rho_{\jmath+1}) \leq \psi^{\jmath-1}(q\psi(\kappa(\rho_0, \rho_1))) \tag{4.9}$$

for all $\jmath$, which yields that

$$\sum_{i=\jmath}^{m-1} \kappa(\rho_i, \rho_{i+1}) \leq \sum_{i=\jmath}^{m-1} \psi^{i-1}(q\psi(\kappa(\rho_0, \rho_1))). \tag{4.10}$$

Now for $m > \jmath$. Fix $\epsilon > 0$ and let $\jmath(\epsilon) \in \mathbb{N}$ such that $\sum_{\jmath \geq \jmath(\delta)} \psi^{i-1}(q\psi(\kappa(\rho_0, \rho_1))) < \epsilon$. Now suppose that $(f, \mathfrak{h}) \in \mathcal{F} \times [0, +\infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathcal{F}_2)$, $\exists\, \delta > 0$ such that

$$0 < \iota < \delta \Longrightarrow f(\iota) < f(\delta) - \mathfrak{h}. \tag{4.11}$$

Hence, by (4.10), (4.11) and $(\mathcal{F}_1)$, we have

$$f(\sum_{i=\jmath}^{m-1} \kappa(\rho_i, \rho_{i+1})) \leq f(\sum_{i=\jmath}^{m-1} \psi^{i-1}(q\psi(\kappa(\rho_0, \rho_1))) \leq f(\sum_{\jmath \geq \jmath(\delta)} \psi^{i-1}(q\psi(\kappa(\rho_0, \rho_1))) < f(\epsilon) - \mathfrak{h} \tag{4.12}$$

for $m > \jmath \geq \jmath(\epsilon)$. Using (D_3) and (4.12), we obtain $\kappa(\rho_\jmath, \rho_m) > 0$, $m > \jmath \geq \jmath(\epsilon)$ implies

$$f(\kappa(\rho_\jmath, \rho_m)) \leq f(\sum_{i=\jmath}^{m-1} \kappa(\rho_i, \rho_{i+1})) + \mathfrak{h} < f(\epsilon)$$

which implies by $(\mathcal{F}_1)$ that $\kappa(\rho_\jmath, \rho_m) < \epsilon$, $m > \jmath \geq \jmath(\epsilon)$. This proves that $\{\rho_\jmath\}$ is $\mathcal{F}$-Cauchy. Since (Θ, κ) is $\mathcal{F}$-complete, there exists $\rho^* \in \Theta$ such that $\{\rho_\jmath\}$ is $\mathcal{F}$-convergent to ρ^*, i.e.,

$$\lim_{\jmath \to \infty} \kappa(\rho_\jmath, \rho^*) = 0. \tag{4.13}$$

Suppose $g(\xi) = \kappa(\xi, \mathfrak{L}\xi)$ is lower semi-continuous at ρ, then

$$f\left(\kappa(\rho^*, \mathfrak{L}(\rho^*))\right) \leq f\left(\liminf_{\jmath} g(\rho_\jmath)\right) = f\left(\liminf_{\jmath} \kappa(\rho_\jmath, \mathfrak{L}(\rho_\jmath))\right) = -\infty,$$

which implies that $\kappa(\rho^*, \mathfrak{L}(\rho^*)) = 0$. Therefore $\rho^* \in \mathfrak{L}(\rho^*)$. By the closedness of $\mathfrak{L}$ it yields that $\rho^* \in \mathfrak{L}(\rho^*)$. Conversely, assume that ρ^* is a fixed point of $\mathfrak{L}$, then $\kappa(\rho^*, \mathfrak{L}(\rho^*)) = 0$, which implies that

$$g(\rho^*) = 0 \leq \liminf_{\jmath} g(\rho_\jmath).$$

□

Corollary 7. *Let $(\Theta, \preceq, \kappa)$ be an ordered $\mathcal{F}$-metric space, $\psi \in \Psi$ be a strictly increasing mapping and $\mathfrak{L} : \Theta \to CL(\Theta)$ be a mapping such that for each $\rho \in \Theta$ and $\hbar \in \mathfrak{L}(\rho)$ with $\rho \preceq \hbar$, we have*

$$\kappa(\hbar, \mathfrak{L}(\hbar)) \leq \psi(\kappa(\rho, \hbar)).$$

Also, assume that

(i) (Θ, κ) is $\mathcal{F}$-complete;
(ii) there exists $\rho_0 \in \Theta$ and $\rho_1 \in \mathfrak{L}(\rho_0)$ such that $\rho_0 \preceq \rho_1$;
(iii) if $\rho \preceq \hbar$, then $\mathfrak{L}\rho \prec_r \mathfrak{L}\hbar$.
Then ρ is a fixed point of $\mathfrak{L}$ iff $g(\xi) = \kappa(\xi, \mathfrak{L}\xi)$ is lower semi-continuous at ρ.

Proof. Define $\alpha : \Theta \times \Theta \to [0, \infty)$ by

$$\alpha(\rho, \hbar) = \begin{cases} 1, & \text{if } \rho \preceq \hbar, \\ 0, & \text{otherwise.} \end{cases}$$

By using condition (i) and the definition of α, we have $\alpha(\rho_0, \rho_1) = 1$. Also, from condition (iii), we have $\rho \preceq \hbar$, then $\mathfrak{L}\rho \prec_r \mathfrak{L}\hbar$, by using the definitions of α and $\prec_r$,wehave $\alpha(\rho, \hbar) = 1$ implies $\alpha^*(\mathfrak{L}\rho, \mathfrak{L}\hbar) = 1$. Furthermore, it is simple to check that $\mathfrak{L}$ is a strictly generalized (α^*, ψ)-contractive mapping. Therefore, by Theorem 5, ρ is a fixed point of $\mathfrak{L}$ if and only if $g(\xi) = \kappa(\xi, \mathfrak{L}\xi)$ is lower semi-continuous at ρ. □

Definition 7. Let (Θ, κ) be an $\mathcal{F}$-metric space and $\mathfrak{L} : \Theta \to B(\Theta)$ be a mapping. We say that $\mathfrak{L} : \Theta \to B(\Theta)$ is said to be generalized $(\alpha^*,\psi, \delta_{\mathcal{F}})$-contractive mapping if there exists two functions $\alpha : \Theta\times\Theta \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha^*(\mathfrak{L}(\rho), \mathfrak{L}(\hbar))\delta_{\mathcal{F}}(\hbar, \mathfrak{L}(\hbar)) \leq \psi(\kappa(\rho, \hbar)) \tag{4.14}$$

for each $\rho \in \Theta$ and $\hbar \in \mathfrak{L}(\rho)$.

Theorem 6. *Let (Θ, κ) be an $\mathcal{F}$-metric space and $\mathfrak{L} : \Theta \to B(\Theta)$ be an α^*-admissible and generalized $(\alpha^*,\psi, \delta_{\mathcal{F}})$-contractive mapping. Also suppose that the following assertions holds:*

(i) (Θ, κ) is $\mathcal{F}$-complete;
(ii) there exists $\rho_0 \in \Theta$ and $\rho_1 \in \mathfrak{L}(\rho_0)$ such that $\alpha(\rho_0, \rho_1) \geq 1$.
Then there exists $\rho \in \Theta$ such that $\{\rho\} = \mathfrak{L}(\rho)$ iff $g(\xi) = \kappa(\xi, \mathfrak{L}(\xi))$ is lower semi-continuous at ρ.

Proof. By the hypothesis of the theorem, there exist $\rho_0 \in \Theta$ and $\rho_0 \in \mathfrak{L}(\rho_0)$ such that $\alpha(\rho_0, \rho_1) \geq 1$. Assume that $\rho_0 \neq \rho_1$, for otherwise, ρ_0 is a fixed point. Let $\rho_1 \notin \mathfrak{L}(\rho_1)$. As $\mathfrak{L}$ is α^*-admissible, we have $\alpha^*(\mathfrak{L}(\rho_0), \mathfrak{L}(\rho_1)) \geq 1$. Then

$$\begin{aligned} \delta_{\mathcal{F}}(\rho_1, \mathfrak{L}(\rho_1)) &\leq \alpha^*(\mathfrak{L}(\rho_0), \mathfrak{L}(\rho_1))\delta_{\mathcal{F}}(\rho_1, \mathfrak{L}(\rho_1)) \\ &\leq \psi(\kappa(\rho_0, \rho_1)). \end{aligned} \tag{4.15}$$

Since $\mathfrak{L}(\rho_1) \neq \emptyset$, there is $\rho_2 \in \mathfrak{L}(\rho_1)$. Then

$$0 < \kappa(\rho_1, \rho_2) \leq \delta_{\mathcal{F}}(\rho_1, \mathfrak{L}(\rho_1)). \tag{4.16}$$

From (4.15) and (4.16), we have

$$0 < \kappa(\rho_1, \rho_2) \leq \psi(\kappa(\rho_0, \rho_1)).$$

Since ψ is nondecreasing, we have

$$\psi(\kappa(\rho_1, \rho_2)) \leq \psi^2(\kappa(\rho_0, \rho_1)).$$

As $\rho_2 \in \mathfrak{L}\rho_1$, we have $\alpha(\rho_1, \rho_2) \geq 1$. Since $\mathfrak{L}(\rho_2) \neq \emptyset$, there is $\rho_3 \in \mathfrak{L}(\rho_2)$. Assume that $\rho_2 \neq \rho_3$, for otherwise, ρ_2 is a fixed point of $\mathfrak{L}$. Then

$$\begin{aligned} \delta_{\mathcal{F}}(\rho_2, \mathfrak{L}(\rho_2)) &\leq \alpha^*(\mathfrak{L}(\rho_1), \mathfrak{L}(\rho_2))\delta_{\mathcal{F}}(\rho_2, \mathfrak{L}(\rho_2)) \\ &\leq \psi(\kappa(\rho_1, \rho_2)). \end{aligned} \tag{4.17}$$

Since $\mathfrak{L}(\rho_2) \neq \emptyset$, there is $\rho_3 \in \mathfrak{L}(\rho_2)$. Then

$$0 < \kappa(\rho_2, \rho_3) \leq \delta_{\mathcal{F}}(\rho_2, \mathfrak{L}(\rho_2)). \tag{4.18}$$

From (4.17) and (4.18), we have

$$0 < \kappa(\rho_2, \rho_3) \leq \psi(\kappa(\rho_1, \rho_2)).$$

By (4.16), we have

$$\begin{aligned} 0 &< \kappa(\rho_2, \rho_3) \leq \psi(\kappa(\rho_1, \rho_2)) \\ &\leq \psi^2(\kappa(\rho_0, \rho_1)). \end{aligned}$$

Since ψ is nondecreasing, we have

$$\psi(\kappa(\rho_2, \rho_3)) \leq \psi^3(\kappa(\rho_0, \rho_1)).$$

By continuing in this way, we get a sequence $\{\rho_\jmath\}$ in Θ such that $\rho_{\jmath+1} \in \mathfrak{L}(\rho_\jmath)$ and $\rho_\jmath \neq \rho_{\jmath+1}$ for $\jmath = 0, 1, 2, ...$ Further we have

$$0 < \kappa(\rho_\jmath, \rho_{\jmath+1}) \leq \delta_{\mathcal{F}}(\rho_\jmath, \mathfrak{L}(\rho_\jmath)) \leq \psi^\jmath(\kappa(\rho_0, \rho_1)). \tag{4.19}$$

This yields that

$$\sum_{i=\jmath}^{m-1} \kappa(\rho_i, \rho_{i+1}) \leq \sum_{i=\jmath}^{m-1} \psi^i(\kappa(\rho_0, \rho_1)). \tag{4.20}$$

Now for $m > \jmath$. Fix $\epsilon > 0$ and let $\jmath(\epsilon) \in \mathbb{N}$ such that $\sum_{\jmath \geq \jmath(\delta)} \psi^i(\kappa(\rho_0, \rho_1)) < \epsilon$. Now assume that $(f, \mathfrak{h}) \in \mathcal{F} \times [0, +\infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathcal{F}_2)$, $\exists\, \delta > 0$ such that

$$0 < \iota < \delta \Longrightarrow f(\iota) < f(\delta) - \mathfrak{h}. \tag{4.21}$$

Hence, by (4.20), (4.21) and $(\mathcal{F}_1)$, we have

$$f\left(\sum_{i=\jmath}^{m-1} \kappa(\rho_i, \rho_{i+1})\right) \leq f\left(\sum_{i=\jmath}^{m-1} \psi^i(\kappa(\rho_0, \rho_1))\right) \leq f\left(\sum_{\jmath \geq \jmath(\delta)} \psi^i(\kappa(\rho_0, \rho_1))\right) < f(\epsilon) - \mathfrak{h} \tag{4.22}$$

for $m > \jmath \geq \jmath(\epsilon)$. Using (D_3) and (4.22), we obtain $\kappa(\rho_\jmath, \rho_m) > 0$, $m > \jmath \geq \jmath(\epsilon)$ implies

$$f\left(\kappa(\rho_\jmath, \rho_m)\right) \leq f\left(\sum_{i=\jmath}^{m-1} \kappa(\rho_i, \rho_{i+1})\right) + \mathfrak{h} < f(\epsilon)$$

which implies by $(\mathcal{F}_1)$ that $\kappa(\rho_\jmath, \rho_m) < \epsilon$, $m > \jmath \geq \jmath(\epsilon)$. This proves that $\{\rho_\jmath\}$ is $\mathcal{F}$-Cauchy. Since (Θ, κ) is $\mathcal{F}$-complete, there exists $\rho^* \in \Theta$ such that $\{\rho_\jmath\}$ is $\mathcal{F}$-convergent to ρ^*. Letting $\jmath \to \infty$ in (4.19), we have

$$\lim_{\jmath \to \infty} \delta_{\mathcal{F}}(\rho_\jmath, \mathfrak{L}(\rho_\jmath)) = 0.$$

Suppose $g(\xi) = \delta_{\mathcal{F}}(\xi, \mathfrak{L}\xi)$ is lower semi-continuous at ρ, then by $(\mathcal{F}_1)$, we have

$$f\left(\delta_{\mathcal{F}}(\rho^*, \mathfrak{L}(\rho^*))\right) \leq f\left(\liminf_{\jmath} g(\rho_\jmath)\right) = f\left(\liminf_{\jmath} \delta_{\mathcal{F}}(\rho_\jmath, \mathfrak{L}(\rho_\jmath))\right) = -\infty.$$

Therefore $\{\rho^*\} \in \mathfrak{L}(\rho^*)$, because $\delta_{\mathcal{F}}(\mathfrak{R}_1, \mathfrak{R}_2) = 0$ implies $\mathfrak{R}_1 = \mathfrak{R}_2 = \{a\}$. Conversely, suppose that $\{\rho^*\} \in \mathfrak{L}(\rho^*)$, then

$$g(\rho^*) = 0 \leq \liminf_{\jmath} g(\rho_\jmath).$$

□

Corollary 8. *Let $(\Theta, \preceq, \kappa)$ be an ordered $\mathcal{F}$-metric space, $\psi \in \Psi$ be a strictly increasing mapping and $\mathfrak{L} : \Theta \to B(\Theta)$ be a mapping such that for each $\rho \in \Theta$ and $\hbar \in \mathfrak{L}(\rho)$ with $\rho \preceq \hbar$, we have*

$$\delta_{\mathcal{F}}(\hbar, \mathfrak{L}(\hbar)) \leq \psi(\kappa(\rho, \hbar)).$$

Also, assume that

(i) (Θ, κ) is $\mathcal{F}$-complete,
(ii) there exists $\rho_0 \in \Theta$ and $\{\rho_0\} \in \mathfrak{L}(\rho_0)$ i.e., there exists $\rho_1 \in \mathfrak{L}(\rho_0)$ such that $\rho_0 \preceq \rho_1$,
(iii) if $\rho \preceq \hbar$, then $\mathfrak{L}\rho \prec_r \mathfrak{L}\hbar$.
Then there exists $\rho \in \Theta$ such that $\{\rho\} = \mathfrak{L}(\rho)$ iff $g(\xi) = \kappa(\xi, \mathfrak{L}(\xi))$ is lower semi-continuous at ρ.

Proof. Define $\alpha : \Theta \times \Theta \to [0, \infty)$ by

$$\alpha(\rho, \hbar) = \begin{cases} 1, & \text{if } \rho \preceq \hbar, \\ 0, & \text{otherwise.} \end{cases}$$

By using condition (i) and the definition of α, we have $\alpha(\rho_0, \rho_1) = 1$. Also, from condition (iii), we have $\rho \preceq \hbar$, then $\mathfrak{L}\rho \prec_r \mathfrak{L}\hbar$, by using the definitions of α and $\prec_r$,wehave $\alpha(\rho, \hbar) = 1$ implies $\alpha^*(\mathfrak{L}\rho, \mathfrak{L}\hbar) = 1$. Furthermore, it is simple to check that $\mathfrak{L}$ is a strictly generalized $(\alpha^*, \psi, \delta_{\mathcal{F}})$-contractive mapping. Therefore, by Theorem 5, there exists $\rho \in \Theta$ such that $\{\rho\} = \mathfrak{L}(\rho)$ if and only if $g(\xi) = \delta_{\mathcal{F}}(\xi, \mathfrak{L}\xi)$ is lower semi-continuous at ρ. □

5. Applications

A representative stability result based on fixed point theory arguments follows a number of basic arguments adapted to the special structure of the equation under consideration. It leads to large number of results in the literature for different classes of equations, see [29, 30]. In the present section, we investigate the existence of solution of differential equation

$$\rho^{/}(\iota) = -a(\iota)\rho(\iota) + b(\iota)g(\rho(\iota - r(\iota))) + c(\iota)\rho^{/}(\iota - r(\iota)). \tag{5.1}$$

We state a lemma of Djoudi et al. [31] which will be used in proving of our theorem.

Lemma 3. *([31]) Assume that $r^{/}(\iota) \neq 1$ $\forall \iota \in \mathbb{R}$. Then $\rho(\iota)$ is a solution of (5.1) if and only if*

$$\begin{aligned} \rho(\iota) \quad &= \quad \left(\rho(0) - \frac{c(0)}{1 - r^{/}(0)}\rho(-r(0))\right) e^{-\int_0^\iota a(s)ds} + \frac{c(\iota)}{1 - r^{/}(\iota)}\rho(\iota - r(\iota)) \\ &\quad - \int_0^\iota \left(h(\upsilon)\right)\rho(\upsilon - r(\upsilon))) - b(\upsilon)g\left(\rho(\upsilon - r(\upsilon))\right)) e^{-\int_\upsilon^\iota a(s)ds} d\upsilon \end{aligned} \tag{5.2}$$

where

$$h(\upsilon) = \frac{r^{//}(\upsilon)c(\upsilon) + \left(c^{/}(\upsilon) + c(\upsilon)a(\upsilon)\right)(1 - r^{/}(\upsilon))}{(1 - r^{/}(\upsilon))^2}. \tag{5.3}$$

Now suppose that $\vartheta : (-\infty, 0] \to \mathbb{R}$ is a bounded and continuous function, then $\rho(\iota) = \rho(\iota, 0, \vartheta)$ is a solution of (5.1) if $\rho(\iota) = \vartheta(\iota)$ for $\iota \leq 0$ and satisfies (5.1) for $\iota \geq 0$. Assume that $\mathfrak{C}$ is the collection of $\rho : \mathbb{R} \to \mathbb{R}$ which are continuous. Define $\aleph_\vartheta$ by

$$\aleph_\vartheta = \{\rho : \mathbb{R} \to \mathbb{R} \text{ such that } \vartheta(\iota) = \rho(\iota) \text{ if } t \leq 0,\ \rho(\iota) \to 0 \text{ as } \iota \to \infty,\ \rho \in \mathfrak{C}\}.$$

Then $\aleph_\vartheta$ is a Banach space endowed with $\|\cdot\|$.

Lemma 4. *([13]) The space $(\aleph_\vartheta, \|\cdot\|)$ with the $\mathcal{F}$-metric d defined by*

$$d(\iota, \iota^*) = \|\iota - \iota^*\| = \sup_{\rho \in I} |\iota(\rho) - \iota^*(\rho)|$$

for all $\iota, \iota^* \in \aleph_\vartheta$, is $\mathcal{F}$-metric space.

Theorem 7. *Let $\mathfrak{L} : \aleph_\vartheta \to \aleph_\vartheta$ be a mapping defined by*

$$\begin{aligned}(\mathfrak{L}\rho)(\iota) &= \left(\rho(0) - \frac{c(0)}{1 - r^/(0)}\rho(-r(0))\right) e^{-\int_0^\iota a(s)ds} + \frac{c(\iota)}{1 - r^/(\iota)}\tau(\iota - r(\iota)) \\ &\quad - \int_0^\iota (h(\upsilon)\rho(\upsilon - r(\upsilon)) - b(\upsilon)g\left(\rho(\upsilon - r(\upsilon))\right)) e^{-\int_\upsilon^\iota a(s)ds} d\upsilon, \ \iota \geq 0 \end{aligned} \tag{5.4}$$

for all $\rho \in \aleph_\vartheta$. Assume that there exists $\alpha : \aleph_\vartheta \times \aleph_\vartheta \to [0, 1)$ such that

$$\alpha\left(\rho(\iota), \hbar(\iota)\right) = \left\{ \left| \frac{c(\iota)}{1 - r^/(\iota)} \right| + \int_0^\iota \left(|h(\upsilon)| + |b(\upsilon)|\right) e^{-\int_\upsilon^\iota a(s)ds} \right\} < 1.$$

Then $\mathfrak{L}$ has a fixed point.

Proof. It follows from (5.3) that $\mathfrak{L}(\rho), \mathfrak{L}(\hbar) \in \aleph_\vartheta$. Now from (5.4), we have

$$\begin{aligned}|(\mathfrak{L}\rho)(\iota) - (\mathfrak{L}\hbar)(\iota)| &\leq \left| \frac{c(\iota)}{1 - r^/(\iota)} \right| \|\rho - \hbar\| \\ &\quad + \int_0^\iota |h(\upsilon)(\rho(\upsilon - r(\upsilon))) - \hbar(\upsilon - r(\upsilon))| e^{-\int_\upsilon^\iota a(s)ds} \\ + &\quad \int_0^\iota |(b(\upsilon))g(\rho(\upsilon - r(\upsilon))) - g(\hbar(\upsilon - r(\upsilon)))| e^{-\int_\upsilon^\iota a(s)ds} \\ &\leq \left\{ \left| \frac{c(\iota)}{1 - r^/(\iota)} \right| + \int_0^\iota \left(|h(\upsilon)| + |b(\upsilon)|\right) e^{-\int_\upsilon^\iota a(s)ds} \right\} \|\rho - \hbar\| \\ &\leq \alpha(\rho) \|\rho - \hbar\| \\ &\leq \alpha(\rho) \|\rho - \hbar\| + \beta(\rho) \frac{\|\rho - \mathfrak{L}\rho\| \|\hbar - \mathfrak{L}\hbar\|}{1 + \|\rho - \hbar\|}.\end{aligned}$$

Hence,

$$\kappa(\mathfrak{L}\rho, \mathfrak{L}\hbar) \leq \alpha(\rho) \kappa(\rho, \hbar) + \beta(\rho) \frac{\kappa(\rho, \mathfrak{L}\rho)\kappa(\hbar, \mathfrak{L}\hbar)}{1 + \kappa(\rho, \hbar)}$$

for any $\beta : \Theta \to [0, 1)$. Thus all the assumptions of Corollary 1 are satisfied and $\mathfrak{L}$ has a unique fixed point in $\aleph_\vartheta$ which solves (5.1). □

6. Conclusions

This article is precised on the notion of $\mathcal{F}$-metric space to prove common fixed points of six mappings for generalized rational contractions involving control functions of one variable. A non-trivial example is also provided to show the validity of obtained results. We also established fixed points of (α^*-ψ)-contractive and generalized (α^*,ψ, $\delta_{\mathcal{F}}$)-contractive multifunctions. As application, we discussed the solution of nonlinear neutral differential equation.

Common fixed points of fuzzy mappings in the background of $\mathcal{F}$-metric space can be interesting outline for the future work in this direction. Differential and integral inclusions can be investigated as applications of these results.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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