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Research article

Note on fuzzifying probability density function and its properties

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Abstract: This paper applies the concepts of fuzzifying functions to the probability density function of a random variable and introduce a fuzzifying probability to better understand the probability arising from the uncertainties of the probability density function. Using the fuzzifying probability, we derive the fuzzifying expected value and the fuzzifying variance of a random variable with the fuzzifying probability density function. Additionally, we provide examples of a fuzzifying probability density function to validate that the proposed fuzzy concepts generalize crisp expected value and variance in probability theory.

Keywords: fuzzifying function; fuzzifying probability density function; fuzzifying probability; fuzzifying expected value
Mathematics Subject Classification: 28E10, 46S40

1. Introduction

In 1965, Zadeh [18] introduced the concept of fuzzy theory, which has since undergone extensive research and various applications, including Choquet integrals of set-valued functions [5, 6, 20–22], fuzzy set-valued measures [9, 10, 16], fuzzy random variable applications [1, 4, 17], theory for general quantum systems interacting with linear dissipative systems [3], and more. The relationship between fuzzy theory and probability theory has been a subject of much discussion [1, 12, 14], as both frameworks aim to capture the concept of uncertainty using membership functions and probability density functions (PDFs) whose values lie within the interval [0, 1].

Fuzzy theory and probability theory are two distinct mathematical frameworks, each with their own approach to modeling uncertainty. Fuzzy theory represents imprecision and vagueness in human reasoning using fuzzy sets, which assign degrees of membership to elements of a universe of discourse.

Probability theory deals with randomness and uncertainty using probability distributions, which assign probabilities to outcomes of a random event. Despite their differences, both frameworks allow the expression of uncertainty using values that lie within the range of [0,1], and they provide a means for decision-making under uncertainty, incorporating expert knowledge and data.

Research has explored the relationships between fuzzy theory and probability theory, revealing similarities in terms of mathematical structure and some analytical tools. For instance, fuzzy measures can be viewed as a generalization of probability measures, and Choquet integrals of set-valued functions are analogous to probability integrals. While the majority of fuzzy probability measure theories [7, 13, 15, 19] have traditionally considered probability as the expected value of the membership function of fuzzy events, however, by using fuzzifying a PDF we define the fuzzifying probability of crisp events.

In this study, we propose the concept of fuzzifying probability for continuous random variables in the context of crisp events, along with its properties and associations with conventional probability theories. Therefore, fuzzy theory can be seen as an extension or generalization of probability theory. The main objective of this study is to introduce fuzzifying probability density functions and to investigate related properties by applying the concepts of fuzzifying probability of crisp events. This approach enables investigation of the ambiguities of the PDF and their impact on probability theories. Relevant definitions in probability theory will be briefly recalled to facilitate this investigation.

Definition 1.1. *[8] Let S be a sample space and X be a real-valued continuous random variable on S. Then, a function $f_x : S \to \mathbb{R}^+$ is a PDF of X if it satisfies the following criteria:*

(i) $f_X(x)$ *is positive everywhere in the support* S*, i.e.,* $f_X(x) > 0$ *for all* $x \in S$*, and*
(ii) $\int_S f_X(x)dx = 1$.

If $f_X(x)$ is a PDF of the random variable X, then the probability P that X belongs to an event E is defined as

$$P(X \in E) = \int_E f_X(x)dx.$$

Definition 1.2. *[8] Let $\mathcal{A}$ be the σ-algebra of a sample space S. A real-valued function P on $\mathcal{A}$ is a probability if P satisfies the following properties:*

(i) $P(E) \geq 0$ *for all* $E \in \mathcal{A}$,
(ii) $P(S) = 1$ *and* $P(\emptyset) = 0$*, and*
(iii) For any sequence of events $\{E_1, E_2, \cdots\}$ *with* $E_i \cap E_j = \emptyset$ $(i \neq j)$*, it holds*

$$P(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n).$$

We next recall some basic fuzzy theory notions and definitions. Let U and V be two universal sets and $g : U \to V$ be a crisp function between these sets. Then the fuzzifying function $\widetilde{g} : U \to \mathcal{F}(V)$ is a mapping from the same domain to a new range $\mathcal{F}(V)$ comprising the family of all fuzzy sets on V. The fuzzy set $\widetilde{A} \in \mathcal{F}(V)$ of V can be expressed as

$$\widetilde{A} = \{(v, \mathfrak{m}_{\widetilde{A}}(v)) \big| \, v \in V\},$$

where $\mathfrak{m}_{\widetilde{A}} : V \to [0,1]$ is a membership function of $\widetilde{A}$ (For more details see [2,11]). Recall that a fuzzy set $\widetilde{A}$ is said to be normal if there exists $v_0 \in V$ such that $\mathfrak{m}_{\widetilde{A}}(v_0) = 1$.

Let $I([0,1])$ be the set of all intervals in $[0,1]$ whose elements are described as

$$I([0,1]) := \left\{[a^-, a^+] \,\middle|\, 0 \le a^- \le a^+ \le 1\right\}.$$

In particular, we consider $a = [a,a]$ for any $a \in [0,1]$. Then the interval operators in $I([0,1])$ are defined as follows.

Definition 1.3. *[5, 6, 12] For each $\bar{a} = [a^-, a^+], \bar{b} = [b^-, b^+] \in I([0,1])$, the arithmetic, comparison, and inclusion operators can be expressed as follows.*

(i) $\bar{a} + \bar{b} = [a^- + b^-, a^+ + b^+]$,
(ii) $k\bar{a} = [ka^-, ka^+]$ *for all* $k \in [0,1]$,
(iii) $\bar{a}\bar{b} = [a^-b^-, a^+b^+]$,
(iv) $\bar{a} \wedge \bar{b} = [a^- \wedge b^-, a^+ \wedge b^+]$,
(v) $\bar{a} \vee \bar{b} = [a^- \vee b^-, a^+ \vee b^+]$,
(vi) $\bar{a} \le \bar{b}$ *if and only if* $a^- \le b^-$ *and* $a^+ \le b^+$,
(vii) $\bar{a} < \bar{b}$ *if and only if* $\bar{a} \le \bar{b}$ *and* $\bar{a} \ne \bar{b}$, *and*
(viii) $\bar{a} \subseteq \bar{b}$ *if and only if* $b^- \le a^-$ *and* $a^+ \le b^+$.

Also, algebraic operations of fuzzy sets are defined as follows.

Definition 1.4. *[11] Let X be a nonempty set and $\widetilde{A}$ and $\widetilde{B}$ be fuzzy sets of X.*

(i) The α-cut $\widetilde{A}_\alpha$ of a fuzzy set $\widetilde{A}$ is defined as

$$\widetilde{A}_\alpha = \{x \in X | \mathfrak{m}_{\widetilde{A}}(x) \ge \alpha\}.$$

(ii) The algebraic sum $\widetilde{A} + \widetilde{B}$ of two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ of X is defined as

$$(\widetilde{A} + \widetilde{B})_\alpha = \widetilde{A}_\alpha + \widetilde{B}_\alpha \text{ for all } \alpha \in [0,1],$$

provided $\widetilde{A}_\alpha + \widetilde{B}_\alpha \subseteq [0,1]$.

(iii) The algebraic product $\widetilde{A}\widetilde{B}$ of two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ of X is defined as

$$(\widetilde{A}\widetilde{B})_\alpha = \widetilde{A}_\alpha \widetilde{B}_\alpha \text{ for all } \alpha \in [0,1].$$

Let A be a measurable subset of U and f be an integrable function on U. If $\widetilde{f}$ is a fuzzifying function, then the fuzzifying integral [11] of $\widetilde{f}$ over A is defined as

$$(\mathcal{F}) \int_A \widetilde{f}(x)dx := \left\{ \left(\left[\int_A f_\alpha^-(x)dx, \int_A f_\alpha^+(x)dx \right], \alpha \right) \middle| \alpha \in [0,1] \right\}, \tag{1.1}$$

where f_α^- and f_α^+ are α-cut functions of $\widetilde{f}(x)$, i.e.,

$$\left(\widetilde{f}(x)\right)_\alpha = [f_\alpha^-(x),\ f_\alpha^+(x)] \text{ for all } x \in A.$$

2. Fuzzifying PDF and fuzzifying probability

Let S be a sample space with continuous random variable $X : S \to \mathbb{R}$ and $\mathcal{F}(\mathbb{R}^+)$ be the family of all fuzzy sets on $[0, \infty)$. Using the concepts [2] of fuzzifying functions to a PDF $f_X : S \to [0, \infty)$, we define a fuzzifying PDF $\widetilde{f_X}$ as follows. In order to facilitate theoretical development throughout the remainder of the paper, it is assumed that the fuzzifying PDF $\widetilde{f_X}$ is integrable for all α-cuts.

Definition 2.1. *Let X be a continuous random variable and f_X be a PDF of X. Then we define the fuzzifying PDF $\widetilde{f_X} : S \to \mathcal{F}(\mathbb{R}^+)$ by fuzzifying f_X that satisfies the following conditions:*

(i) $\widetilde{f_X}(x) > 0$ *for all $x \in S$, i.e.,*

$$\mathfrak{m}_{\widetilde{f_X(x)}} > 0 \text{ for all } x \in S,$$

where $\mathfrak{m}_{\widetilde{f_X(x)}} > 0$ means that there exists $u \in \mathbb{R}^+$ such that $\mathfrak{m}_{\widetilde{f_X(x)}}(u) > 0$.

(ii) The fuzzifying integration (1.1) *of $\widetilde{f_X}$ satisfies*

$$(\mathcal{F}) \int_S \widetilde{f_X}(x)dx = \widetilde{1},$$

where $\widetilde{1}$ is a convex fuzzy set [11] of 1 with $\mathfrak{m}_{\widetilde{1}}(1) = 1$.

Note that from (1.1)*, the fuzzy set $\widetilde{1}$ in Definition 2.1 (ii) has its α-cuts*

$$(\widetilde{1})_\alpha = \begin{cases} \left[\int_S f^-_{X_\alpha}(x)dx, \int_S f^+_{X_\alpha}(x)dx\right] & \text{if } 0 \le \alpha < 1, \\ \int_S f_X(x)dx = 1 & \text{if } \alpha = 1. \end{cases}$$

If $\widetilde{f_X}$ is a fuzzifying PDF of X, then the fuzzifying probability $\widetilde{P}$ that X belongs to some event E is given by the fuzzifying integral of $\widetilde{f_X}$ over E, i.e.,

$$\widetilde{P}(X \in E) = (\mathcal{F}) \int_E \widetilde{f_X}(x)dx. \tag{2.1}$$

We consider the fuzzifying probability using the concept of fuzzifying functions in a similar way.

Definition 2.2. *Let $\mathcal{A}$ be a σ-algebra of a sample space S and $P : \mathcal{A} \to [0, 1]$ be a probability. Then the fuzzifying function $\widetilde{P} : \mathcal{A} \to F([0, 1])$ is called the fuzzifying probability if the following conditions are satisfied:*

(i) $0 \le \widetilde{P}(E) \le 1$ *for each event E of S.*

(ii) $\widetilde{P}(S) = \widetilde{1}$*, where $\widetilde{1}$ is a convex fuzzy set satisfying $\mathfrak{m}_{\widetilde{1}}(1) = 1$.*

(iii) For any sequence of events $\{E_1, E_2, \cdots\}$ with $E_i \cap E_j = \emptyset$ $(i \ne j)$, it holds

$$\widetilde{P}\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \widetilde{P}(E_n).$$

The following theorem follows from Definitions 2.1 and 2.2.

Theorem 2.3. *Let $\widetilde{f_X}$ be a fuzzifying PDF for a continuous random variable X and $\widetilde{P}$ be the fuzzifying probability with the density function $\widetilde{f_X}$ given by* (2.1)*. Then $\widetilde{P}$ is a fuzzifying probability.*

Proof. We need only show that $\widetilde{P}$ satisfies the three conditions in Definition 2.2.

(i) Let E be an element of S. Then from (1.1) and (2.1),

$$\begin{aligned}\widetilde{P}(X \in E) &= (\mathcal{F}) \int_E \widetilde{f}_X(x)dx \\ &= \left\{ \left(\left[\int_E f^-_{X_\alpha}(x)dx, \int_E f^+_{X_\alpha}(x)dx \right], \alpha \right) \middle| \alpha \in [0,1] \right\}.\end{aligned}$$

Since $0 \leq \int_E f^-_{X_\alpha}(x)dx \leq \int_E f^+_{X_\alpha}(x)dx \leq 1$ for all $\alpha \in [0,1]$, it implies $0 \leq \widetilde{P}(X \in E) \leq 1$. Thus the first condition holds.

(ii) Since $\widetilde{f}_{X_1}(x) = f_X(x)$ for all $x \in S$, the α-cut of $\widetilde{P}(X \in S)$ at $\alpha = 1$ can be expressed as

$$\begin{aligned}\left(\widetilde{P}(X \in S)\right)_1 &= \left[\int_S f^-_{X_1}(x)dx, \int_S f^+_{X_1}(x)dx \right] \\ &= \left[\int_S f_X(x)dx, \int_S f_X(x)dx \right] \\ &= 1.\end{aligned}$$

Hence the second condition is satisfied.

(iii) Let $\{E_1, E_2, \cdots\}$ be a sequence of disjoint events. Then

$$\begin{aligned}\widetilde{P}(X \in \bigcup_{n=1}^{\infty} E_n) &= \left\{ \left(\left[\int_{\bigcup_{n=1}^{\infty} E_n} f^-_{X_\alpha}(x)dx, \int_{\bigcup_{n=1}^{\infty} E_n} f^+_{X_\alpha}(x)dx \right], \alpha \right) \middle| \alpha \in [0,1] \right\} \\ &= \left\{ \left(\left[\sum_{n=1}^{\infty} \left(\int_{E_n} f^-_{X_\alpha}(x)dx, \int_{E_n} f^+_{X_\alpha}(x)dx \right] \right), \alpha \right) \middle| \alpha \in [0,1] \right\} \\ &= \sum_{n=1}^{\infty} \left\{ \left(\left[\int_{E_n} f^-_{X_\alpha}(x)dx, \int_{E_n} f^+_{X_\alpha}(x)dx \right], \alpha \right) \middle| \alpha \in [0,1] \right\} \\ &= \bigcup_{n=1}^{\infty} \widetilde{P}(X \in E_n).\end{aligned}$$

Thus, third condition is satisfied, which completes the proof.

□

Remark 2.4. *Theorem 2.3 confirms the fuzzifying probability is a fuzzifying probability. Thus, we consider the fuzzifying probability to be* $\widetilde{P}(E) = \widetilde{P}(X \in E)$.

Recall the negative-scalar product [11]: for $k \in \mathbb{R}^- = (-\infty, 0)$ and some interval $[a,b]$ in $\mathbb{R} = (-\infty, \infty)$ with $a \leq b$, the product $[a,b]$ by k can be expressed as

$$k[a,b] = [kb, ka]. \tag{2.2}$$

Consider a fuzzy set $\widetilde{P}^*(E)$ for $E \subseteq S$ whose α-cuts are defined by

$$\left(\widetilde{P}^*(E)\right)_\alpha = \begin{cases} \left[\int_E f^+_{X_\alpha}(x)dx, \int_E f^-_{X_\alpha}(x)dx\right] & \text{if } 0 \leq \alpha < 1, \\ \int_E f_X(x)dx = P(E) & \text{if } \alpha = 1. \end{cases} \tag{2.3}$$

Then the fuzzifying probability establishes the following property.

Theorem 2.5. *Let X be a continuous random variable on a sample space S and $\widetilde{P}$ be a fuzzifying probability. Then*

(i) $\widetilde{P}(E^c) = \widetilde{1} - \widetilde{P}^*(E)$ *for* $E \subseteq S$*,*
(ii) $\widetilde{P}(\emptyset) = 0$*,*
(iii) If $E_1 \subseteq E_2$ *in* S*, then* $\widetilde{P}(E_1) \leq \widetilde{P}(E_2)$*.*

Proof. We need only show that $\widetilde{P}$ satisfies the conditions.

(i) From (2.2) with $E^c = S - E$,

$$\begin{aligned}\widetilde{P}(E^c) &= (\mathcal{F})\int_{E^c} \widetilde{f}_X(x)dx \\ &= \left\{\left(\left[\int_{E^c} f^-_{X_\alpha}(x)dx, \int_{E^c} f^+_{X_\alpha}(x)dx\right], \alpha\right) \middle| \alpha \in [0,1]\right\} \\ &= \left\{\left(\left[\int_S f^-_{X_\alpha}(x)dx - \int_E f^-_{X_\alpha}(x)dx, \int_S f^+_{X_\alpha}(x)dx - \int_E f^+_{X_\alpha}(x)dx\right], \alpha\right) \middle| \alpha \in [0,1]\right\} \\ &= \left\{\left(\left[\int_S f^-_{X_\alpha}(x)dx, \int_S f^+_{X_\alpha}(x)dx\right] - \left[\int_E f^+_{X_\alpha}(x)dx, \int_E f^-_{X_\alpha}(x)dx\right], \alpha\right) \middle| \alpha \in [0,1]\right\} \\ &= \widetilde{1} - \widetilde{P}^*(E),\end{aligned}$$

where the fuzzy set $\widetilde{P}^*(E)$ is given by (2.3).

(ii) The second condition is trivially satisfied by the definition

$$\widetilde{P}(\emptyset) = \left\{\left(\left[\int_\emptyset f^-_{X_\alpha}(x)dx, \int_\emptyset f^+_{X_\alpha}(x)dx\right], \alpha\right) \middle| \alpha \in [0,1]\right\} = 0.$$

(iii) Since $\int_{E_2} f^-_{X_\alpha}(x)dx \leq \int_{E_1} f^-_{X_\alpha}(x)dx$ and $\int_{E_1} f^+_{X_\alpha}(x)dx \leq \int_{E_2} f^+_{X_\alpha}(x)dx$ for all $\alpha \in [0,1]$,

$$\begin{aligned}\widetilde{P}(E_1) &= \left\{\left(\left[\int_{E_1} f^-_{X_\alpha}(x)dx, \int_{E_1} f^+_{X_\alpha}(x)dx\right], \alpha\right) \middle| \alpha \in [0,1]\right\} \\ &\leq \left\{\left(\left[\int_{E_2} f^-_{X_\alpha}(x)dx, \int_{E_2} f^+_{X_\alpha}(x)dx\right], \alpha\right) \middle| \alpha \in [0,1]\right\} \\ &= \widetilde{P}(E_2).\end{aligned}$$

□

We present an example of the fuzzifying probability obtained from a fuzzifying PDF.

Example 2.6. *Let X be a continuous random variable with PDF $f_X(x) = 3x^2,\ 0 \leq x \leq 1$. Then, we consider the fuzzifying PDF $\widetilde{f}_X(x) = \widetilde{3}x^2,\ 0 \leq x \leq 1$ of f_X, where a fuzzy set $\widetilde{3}$ of the constant 3 is given by*

$$\mathfrak{m}_{\widetilde{3}}(u) = \begin{cases} u - 2 & \text{if } 2 \leq u \leq 3, \\ -\frac{1}{2}u + \frac{5}{2} & \text{if } 3 < u \leq 5. \end{cases}$$

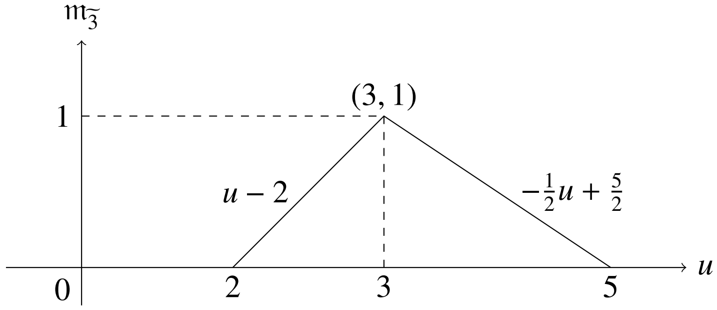

Figure 1. Membership function of a fuzzy set $\widetilde{3}$.

Note that the membership function of the fuzzifying function is given by

$$\mathfrak{m}_{\widetilde{f_X(x)}}(u) = \mathfrak{m}_{\widetilde{3}x^2}(u) = \begin{cases} (u-2)x^2 & \text{if } 2 \le u \le 3, \\ (-\frac{1}{2}u + \frac{5}{2})x^2 & \text{if } 3 < u \le 5. \end{cases}$$

From Definition 2.1 (iii), the corresponding fuzzifying probability can be expressed as

$$\begin{aligned} \widetilde{P}(0 < X < \frac{1}{3}) =& (\mathcal{F}) \int_0^{\frac{1}{3}} \widetilde{f_X}(x)dx \\ =& \left\{ \left(\left[\int_0^{\frac{1}{3}} f^-_{X_\alpha}(x)dx, \int_0^{\frac{1}{3}} f^+_{X_\alpha}(x)dx \right], \alpha \right) \middle| \alpha \in [0,1] \right\}, \end{aligned} \tag{2.4}$$

where

$$\widetilde{f}_{X_\alpha}(x) = \left[f^-_{X_\alpha}(x), f^+_{X_\alpha}(x) \right] := \left[(\alpha+2)x^2, (5-2\alpha)x^2 \right] \text{ for all } \alpha \in [0,1].$$

Thus, from (2.4)*,*

$$\begin{aligned} \widetilde{P}(0 < X < \frac{1}{3}) &= \left\{ \left(\left[\int_0^{\frac{1}{3}} (\alpha+2)x^2dx, \int_0^{\frac{1}{3}} (5-2\alpha)x^2dx \right], \alpha \right) \middle| \alpha \in [0,1] \right\} \\ &= \left\{ \left(\left[\frac{\alpha+2}{3^4}, \frac{5-2\alpha}{3^4} \right], \alpha \right) \middle| \alpha \in [0,1] \right\}, \end{aligned} \tag{2.5}$$

and hence,

$$\left(\widetilde{P}(0 < X < \frac{1}{3}) \right)_\alpha = \left[\frac{\alpha+2}{3^4}, \frac{5-2\alpha}{3^4} \right].$$

Therefore, the membership of the fuzzifying probability $\widetilde{P}$ *for* $0 < X < \frac{1}{3}$ *is given by*

$$\mathfrak{m}_{\widetilde{P}(0<X<\frac{1}{3})}(u) = \begin{cases} 3^4u - 2 & \text{if } \frac{2}{3^4} \le u \le \frac{1}{3^3}, \\ \frac{5-3^4u}{2} & \text{if } \frac{1}{3^3} \le u \le \frac{5}{3^4}. \end{cases}$$

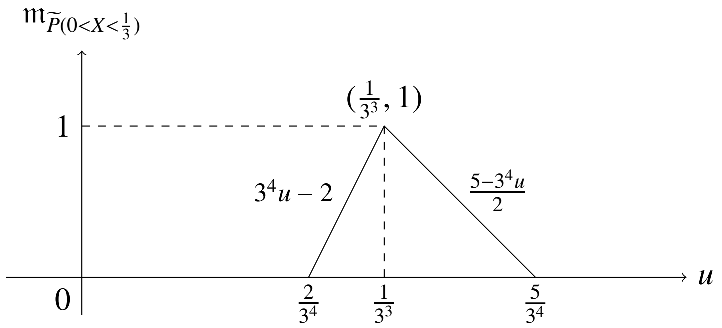

Figure 2. Membership function of fuzzifying probability $\widetilde{P}(0 < X < \frac{1}{3})$.

Note that the probability P over $0 < X < \frac{1}{3}$ *is given by*

$$P(0 < X < \frac{1}{3}) = \int_0^{\frac{1}{3}} 3x^2 dx = \frac{1}{3^3}.$$

Therefore, as observed in the graph of the membership function $\mathfrak{m}_{\widetilde{P}(0<X<\frac{1}{3})}$ *in Figure 2, we see that* $\widetilde{P}(0 < X < \frac{1}{3})$ *establishes a normal fuzzy set of* $\frac{1}{3^3}$ *since* $\mathfrak{m}_{\widetilde{P}(0<X<\frac{1}{3})}(\frac{1}{3^3}) = 1$.

3. Fuzzifying expected values

We define the fuzzifying expected value of a random variable X with the fuzzifying PDF $\widetilde{f_X}$ as

$$\widetilde{E}(X) = (\mathcal{F}) \int x\widetilde{f_X}(x)dx = \left\{ \left(\left[\int x f^-_{X_\alpha}(x)dx, \int x f^+_{X_\alpha}(x)dx \right], \alpha \right) \middle| \, \alpha \in [0, 1] \right\}$$

and the fuzzifying expected value for a measurable function $g(X)$ of X for $\widetilde{f_X}$ as

$$\widetilde{E}(g(X)) = (\mathcal{F}) \int g(x)\widetilde{f_X}(x)dx.$$

Thus, we can derive the fuzzifying n-th moment of a random variable as follows.

Theorem 3.1. *Let X be a continuous random variable with PDF* f_X *and* $\mu_n = E(X^n)$ *be the n-th moment about the origin for X. If* $\widetilde{f_X}$ *is a fuzzifying PDF, then* $\widetilde{E}(X^n) = \widetilde{\mu}_n$ *is a fuzzy set of* μ_n *and* $(\widetilde{\mu}_n)_1 = \mu_n$ *for each* $n \in \mathbb{N}$.

Proof. The definition of $\widetilde{E}$ directly provides that

$$\begin{aligned} \widetilde{E}(X^n) =& (\mathcal{F}) \int x^n \widetilde{f_X}(x)dx \\ =& \left\{ \left(\left[\int x^n f^-_{X_\alpha}(x)dx, \int x^n f^+_{X_\alpha}(x)dx \right], \alpha \right) \middle| \, \alpha \in [0, 1] \right\}, \end{aligned} \tag{3.1}$$

hence $\widetilde{E}(X^n) = \widetilde{\mu}_n$ is a fuzzy set of μ_n. The α-cut of $\widetilde{E}(X^n)$ at $\alpha = 1$ in (3.1) can be expressed as

$$\begin{aligned} \left(\widetilde{E}(X^n)\right)_1 =& \left[\int x^n f^-_{X_1}(x)dx, \int x^n f^+_{X_1}(x)dx \right] \\ =& \left[\int x^n f_X(x)dx, \int x^n f_X(x)dx \right] \\ =& E(X^n), \end{aligned} \tag{3.2}$$

thus $(\widetilde{\mu_n})_1 = \mu_n$. □

We now proceed to introduce the concept of the fuzzifying variance of a random variable with a fuzzifying PDF, expressed in terms of the fuzzifying expected value.

Theorem 3.2. *If X is a random variable with a fuzzifying PDF f_X and $\mu = E(X)$ is the expected value of X, then fuzzifying variance $\widetilde{Var}(X)$ of X can be expressed as*

$$\widetilde{Var}(X) = \widetilde{E}((X-\mu)^2) = \widetilde{E}(X^2) - 2\mu\widetilde{\mu} + \widetilde{1}\mu^2,$$

where $\widetilde{\mu} = \widetilde{E}(X)$ and $\widetilde{1} = \left\{\left(\left[\int f^-_{X_\alpha}(x)dx, \int f^+_{X_\alpha}(x)dx\right], \alpha\right)\middle|\, \alpha \in [0,1]\right\}$.

Proof. From the definition of the fuzzifying variance,

$$\begin{aligned}\widetilde{Var}(X) =& \widetilde{E}((X-\mu)^2)\\ =& \left\{\left(\left[\int x^2 f^-_{X_\alpha}(x)dx, \int x^2 f^+_{X_\alpha}(x)dx\right], \alpha\right)\middle|\, \alpha \in [0,1]\right\}\\ &- 2\mu\left\{\left(\left[\int x f^-_{X_\alpha}(x)dx, \int x f^+_{X_\alpha}(x)dx\right], \alpha\right)\middle|\, \alpha \in [0,1]\right\}\\ &+ \mu^2\left\{\left(\left[\int f^-_{X_\alpha}(x)dx, \int f^+_{X_\alpha}(x)dx\right], \alpha\right)\middle|\, \alpha \in [0,1]\right\}\\ =& \widetilde{E}(X^2) - 2\mu\widetilde{\mu} + \widetilde{1}\mu^2.\end{aligned}$$

□

Remark 3.3. *Theorem 3.2 shows the fuzzy set*

$$\widetilde{1} = \left\{\left(\left[\int f^-_{X_\alpha}(x)dx, \int f^+_{X_\alpha}(x)dx\right], \alpha\right)\middle|\, \alpha \in [0,1]\right\}$$

is a generalization of the constant 1. *Since $f^-_{X_1} = f^+_{X_1} = f_X$ when $\alpha = 1$, $(\widetilde{1})_1$ is a PDF of X, hence*

$$(\widetilde{1})_1 = \left[\int f^-_{X_1}(x)dx, \int f^+_{X_1}(x)dx\right] = \left[\int f_X(x)dx, \int f_X(x)dx\right] = \int f_X(x)dx = 1.$$

We extend Example 2.6 to introduce the concept of fuzzifying expected value and fuzzifying variance, and establish their relationship with the corresponding crisp measures.

Example 3.4. *Consider $\widetilde{f_X}(x) = \widetilde{3}x^2$ in Example 2.6. Then, the fuzzifying expected value of X when $n = 1$ in Theorem 3.1 is*

$$\widetilde{E}(X) = \left\{\left(\left[\int x f^-_{X_\alpha}(x)dx, \int x f^+_{X_\alpha}(x)dx\right], \alpha\right)\middle|\, \alpha \in [0,1]\right\}, \tag{3.3}$$

where $\widetilde{f^-_{X_\alpha}}(x) = (\alpha+2)x^2$ and $\widetilde{f^+_{X_\alpha}}(x) = (5-2\alpha)x^2$ for all $\alpha \in [0,1]$.

Therefore,

$$\begin{aligned}\widetilde{E}(X) &= \left\{\left(\left[\int x^3(\alpha+2)dx, \int x^3(5-2\alpha)dx\right], \alpha\right)\middle|\, \alpha \in [0,1]\right\} \\ &= \left\{\left(\left[\frac{\alpha+2}{4}, \frac{5-2\alpha}{4}\right], \alpha\right)\middle|\, \alpha \in [0,1]\right\}.\end{aligned} \tag{3.4}$$

Thus, $\left(\widetilde{E}(X)\right)_\alpha = \left[\frac{\alpha+2}{4}, \frac{5-2\alpha}{4}\right]$, *and hence*

$$\mathfrak{m}_{\widetilde{E}(X)}(u) = \begin{cases} 4u-2 & \text{if } \frac{1}{2} \le u \le \frac{3}{4}, \\ \frac{5-4u}{2} & \text{if } \frac{3}{4} \le u \le \frac{5}{4}. \end{cases} \tag{3.5}$$

Since $E(X) = \int_0^1 x^3 dx = \frac{3}{4}$, $\widetilde{E}(X)$ *can be represented by a fuzzy set* $\widetilde{\frac{3}{4}}$ *(see Figure 3).*

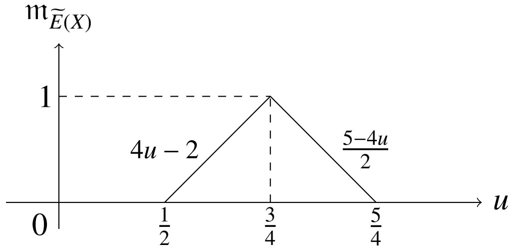

Figure 3. Membership function of fuzzifying expected value $\widetilde{E}(X)$.

We can express $\widetilde{E}(X^2)$ *and* $E(X^2)$ *as*

$$\begin{aligned}\widetilde{E}(X^2) &= \left\{\left(\left[\int x^4(\alpha+2)dx, \int x^4(5-2\alpha)dx\right], \alpha\right)\middle|\, \alpha \in [0,1]\right\} \\ &= \left\{\left(\left[\frac{\alpha+2}{5}, \frac{5-2\alpha}{5}\right], \alpha\right)\middle|\, \alpha \in [0,1]\right\},\end{aligned} \tag{3.6}$$

hence $\left(\widetilde{E}(X)\right)_\alpha = \left[\frac{\alpha+2}{5}, \frac{5-2\alpha}{5}\right]$ *for all* $\alpha \in [0,1]$ *and* $E(X^2) = \int_0^1 3x^4 dx = \frac{3}{5}$. *Thus* $\widetilde{E}(X^2)$ *comprises a fuzzy set* $\widetilde{\frac{3}{5}}$ *(Figure 3). From Theorem 3.2,*

$$\widetilde{Var}(X) = \widetilde{E}(X^2) - 2\widetilde{\mu} + \widetilde{1}\mu^2,$$

where $\widetilde{1}$ *satisfies*

$$\begin{aligned}\widetilde{1} &= \left\{\left(\left[\int_0^1 x^2(\alpha+2)dx, \int_0^1 x^2(5-2\alpha)dx\right], \alpha\right)\middle|\, \alpha \in [0,1]\right\} \\ &= \left\{\left(\left[\frac{\alpha+2}{3}, \frac{5-2\alpha}{3}\right], \alpha\right)\middle|\, \alpha \in [0,1]\right\},\end{aligned}$$

and hence the membership function of $\widetilde{1}$ *is*

$$\mathfrak{m}_{\widetilde{1}}(u) = \begin{cases} 3u-2 & \text{if } \frac{2}{3} \le u \le 1, \\ \frac{5-3u}{2} & \text{if } 1 \le u \le \frac{5}{3}. \end{cases}$$

From (2.2),

$$\left(\widetilde{Var}(X)\right)_\alpha = \left(\widetilde{E}(X^2)\right)_\alpha - 2\mu(\widetilde{\mu})_\alpha + \mu^2(\widetilde{1})_\alpha = \left[\frac{\alpha+2}{80}, \frac{5-2\alpha}{80}\right]$$

for each $\alpha \in [0,1]$. *Thus, the fuzzifying variance* $\widetilde{Var}(X)$ *is*

$$\widetilde{Var}(X) = \left\{ \left(\left[\frac{\alpha+2}{80}, \frac{5-2\alpha}{80}\right], \alpha \right) \middle| \alpha \in [0,1] \right\},$$

and the membership function of the fuzzifying variance is

$$\mathfrak{m}_{\widetilde{Var}(X)}(u) = \begin{cases} 80u - 2 & \text{if } \frac{1}{40} \le u \le \frac{3}{80}, \\ \frac{5-80u}{2} & \text{if } \frac{3}{80} \le u \le \frac{5}{80}. \end{cases}$$

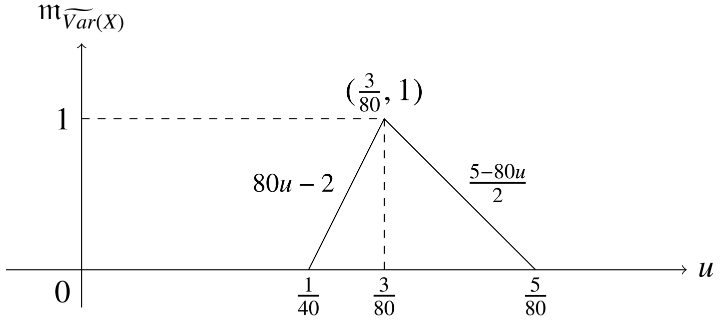

Figure 4. Membership function of fuzzifying variance $\widetilde{Var}(X)$.

Since $Var(X) = E(X^2) - (E(X))^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{80}$, $\widetilde{Var}(X)$ *is a fuzzy set of* $\frac{3}{80}$ *(see Figure 4).*

In conclusion, the expression for the linearity of expectations for a random variable with a fuzzifying PDF is as follows.

Theorem 3.5. *Let* g_j *be integrable functions of a random variable* X *and* k_j *be positive integers for* $j = 1, 2, \cdots, m$. *Then,*

$$\widetilde{E}\left(\sum_{j=1}^{m} k_j g_j(X)\right) = \sum_{j=1}^{m} k_j \widetilde{E}\left(g_j(X)\right).$$

Proof. Since $g_j(X)\widetilde{f_X}(x) = g_j(X)[f^-_{X_\alpha}(x), f^+_{X_\alpha}(x)] = [g_j(X)f^-_{X_\alpha}(x), g_j(X)f^+_{X_\alpha}(x)]$ for all x,

$$\begin{aligned}
\widetilde{E}\left(\sum_{j=1}^{m} k_j g_j(X)\right) &= (\mathcal{F}) \int \sum_{j=1}^{m} k_j g_j(X) \widetilde{f_X}(x) dx \\
&= \left\{ \left(\left[\int \sum_{j=1}^{m} k_j g_j(X) f^-_{X_\alpha}(x) dx, \int \sum_{j=1}^{m} k_j g_j(X) f^+_{X_\alpha}(x) dx \right], \alpha \right) \middle| \alpha \in [0,1] \right\} \\
&= \left\{ \left(\left[\sum_{j=1}^{m} k_j \int g_j(X) f^-_{X_\alpha}(x) dx, \sum_{j=1}^{m} k_j \int g_j(X) f^+_{X_\alpha}(x) dx \right], \alpha \right) \middle| \alpha \in [0,1] \right\}
\end{aligned}$$

$$= \left\{ \sum_{j=1}^{m} k_j \left(\left[\int g_j(X) f^{-}_{X_\alpha}(x)dx, \int g_j(X) f^{+}_{X_\alpha}(x)dx \right], \alpha \right) \middle| \alpha \in [0,1] \right\}$$

$$= \sum_{j=1}^{m} k_j \left((\mathcal{F}) \int g_j(X) \widetilde{f}_X(x)dx \right),$$

which confirms linearity of fuzzifying expectations $\widetilde{E}\left(g_j(X)\right)$. □

4. Conclusions

In this study, the concept of fuzzifying functions has been introduced to probability theory as a means of developing a fuzzifying PDF and a fuzzifying probability. Through this approach, we aim to investigate the ambiguities inherent in probability theories that are affected by uncertainties in the PDF. The validity of the fuzzifying probability was established through Theorem 2.3, while Theorems 3.1 and 3.2 provided the fuzzifying n-th moment about the origin of a random variable and the fuzzifying variance, respectively. To demonstrate the utility of our approach, we presented modeled examples in which the fuzzifying functions were shown to generalize crisp functions in probability theory. Examples 2.6 and 3.4 illustrated the fuzzifying probability and the fuzzifying expected value, respectively. Furthermore, we extended the concept of fuzzifying functions to Bernoulli, Poisson, and geometric random variables, among others, thus enabling us to investigate the uncertainties in probability theories arising from the ambiguities in PDFs. In summary, our approach of employing fuzzifying functions allows for the investigation of the impact of uncertainties in PDFs on probability theories, and our findings suggest that the concept of fuzzifying functions has the potential to enhance our understanding of probability theory.

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Conflict of interest

The authors declare no conflict of interest.

References

1. S. B. Boswell, M. S. Taylor, An application of a fuzzy random variable to vulnerability modeling, *Fuzzy Sets Syst.*, **33** (1989), 19–28. http://doi.org/10.1016/0165-0114(89)90213-3
2. V. Dimiev, Fuzzifying functions, *Fuzzy Sets Syst.*, **33** (1989), 47–58. http://doi.org/10.1016/0165-0114(89)90216-9
3. R. Feynman, F. Vernon Jr., The theory of a general quantum system interacting with a linear dissipative system, *Ann. Phy.*, **24** (1963), 118–173. http://doi.org/10.1016/0003-4916(63)90068-X
4. G. Hesamian, M. Akbari, J. Zendehdel, Location and scale fuzzy random variables, *Int. J. Syst. Sci.*, **51** (2019), 229–241 . https://doi.org/10.1080/00207721.2019.1701131

5. L.-C. Jang, A note on convergence properties of interval-valued capacity functionals and Choquet integrals, *Inform. Sci.*, **183** (2012), 151–158. https://doi.org/10.1016/j.ins.2011.09.011
6. L.-C. Jang, B. Kil, Y. Kim, J. Kwon, Some properties of Choquet integrals of set-valued functions, *Fuzzy Sets Syst.*, **91** (1997), 95–98. https://doi.org/10.1016/S0165-0114(96)00124-8
7. E. P. Klement, W. Schwyhla, R. Lowen, Fuzzy probability measures, *Fuzzy Sets Syst.*, **5** (1981), 21–30. http://doi:10.1016/0165-0114(81)90031-2
8. A. Leon-Garcia, *Probability, statistics, and random processes for electrical engineering*, Upper Saddle River: Pearson/Prentice Hall, 2008. https://books.google.co.kr/books?id=GUJosCkbBywC
9. L. S. Li, Z. H. Sheng, The fuzzy set-valued measures generated by fuzzy random variables, *Fuzzy Sets Syst.*, **97** (1998), 203–209. https://doi.org/10.1016/S0165-0114(96)00344-2
10. Z. Li, Z. Wang, Q. Li, P. Wang, C.-F. Wen, Uncertainty measurement for a fuzzy set-valued information system, *Int. J. Mach. Learn. Cyb.*, **12** (2021), 1769–1787. https://doi.org/10.1007/s13042-020-01273-6
11. K. Lee, *First Course on Fuzzy Theory and Applications*, Berlin, Heidelberg: Springer, 2005. https://doi.org/10.1007/3-540-32366-X
12. L. Li, Z. Sheng, The fuzzy set-valued measures generated by fuzzy random variables, *Fuzzy Sets Syst.* **97** (1998), 203–209. https://doi.org/10.1016/S0165-0114(96)00344-2
13. O. A. Provotar, O. O. Provotar, Fuzzy Probabilities of Fuzzy Events, *Cyb. Syst. Anal.*, **56** (2020), 171–180. https://doi.org/10.1007/s10559-020-00232-x
14. O. Rebrov, G. Kuleshova, Fuzzy robust estimates of location and scale parameters of a fuzzy random variable, *ENVIRONMENT. TECHNOLOGIES. RESOURCES., Proceedings of the International Scientific and Practical Conference*, **2** (2021), 181–186. https://doi.org/10.17770/etr2021vol2.6566
15. J. Talasová, O. Pavlacka, Fuzzy probability spaces and their applications in decision making, *Aust. J. Stat.*, **35** (2006), 347–356.
16. J. Wu, H. Liu, Autocontinuity of set-valued fuzzy measures and applications, *Fuzzy Sets Syst.*, **175** (2011), 57–64. https://doi.org/10.1016/j.fss.2010.12.003
17. S. Wang, Y.-K. Liu, J. Watada, Fuzzy random renewal process with queueing applications, *Comput. Math. Appl.*, **57** (2009), 1232–1248. http://doi.org/10.1016/j.camwa.2009.01.030
18. L. Zadeh, Fuzzy sets, *Inform. Control*, **8** (1965), 338–353. http://doi.org/10.1016/S0019-9958(65)90241-X
19. L. Zadeh, Probability measures of fuzzy events, *J. Math. Anal. Appl.*, **23** (1968), 421–427. http://doi.org/10.1016/0022-247X(68)90078-4
20. D. Zhang, C. Guo, Choquet integrals of set-valued functions with respect to set-valued fuzzy measures, *Fuzzy Sets Syst.*, **457** (2023), 80–104. http://doi.org/10.1016/j.fss.2022.08.025
21. D. Zhang, C. Guo, D. Liu, Set-valued Choquet integrals revisited, *Fuzzy Sets Syst.*, **147** (2004), 475–485. http://doi.org/10.1016/j.fss.2004.04.005
22. D. Zhang, R. Mesiar, E. Pap, Jensen's inequality for choquet integral revisited and a note on jensen's inequality for generalized choquet integral, *Fuzzy Sets Syst.*, **430** (2022), 79–87. https://doi.org/10.1016/j.fss.2021.09.004

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