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Research article

Real positive solutions of operator equations $AX = C$ and $XB = D$

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Abstract: In this paper, we mainly consider operator equations $AX = C$ and $XB = D$ in the framework of Hilbert space. A new representation of the reduced solution of $AX = C$ is given by a convergent operator sequence. The common solutions and common real positive solutions of the system of two operator equations $AX = C$ and $XB = D$ are studied. The detailed representations of these solutions are provided which extend the classical closed range case with a short proof.

Keywords: reduced solution; real positive solution; operator equation
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1. Introduction

The study of operator equations has a long history. Khatri and Mitra [10], Wu and Cain [15], Xiong and Qin [16] and Yuan et al. [18] studied matrix equations $AX = C$ and $XB = C$ in matrix algebra. Dajić and Koliha [1], Douglas [4] and Liang and Deng [12] investigated these equations in operator space. Xu et al. [5, 13, 14, 17], Fang and Yu [6] studied these equations of adjoint operator space on Hilbert C^*-modules. The famous Douglas' range inclusion theorem played a key role in the existence of the solutions of equation $AX = C$. Many scholars discussed the existence and the general formulae of self-adjoint solutions (resp. positive or real positive solutions) of one equation or common solutions of two equations. In finite dimensional case, Groβ gave the necessary and sufficient conditions for the matrix equation $AX = C$ to have a real positive solution in [7]. However, the detailed formula for the real positive solution of this equation is not fully provided. Dajíc and Koliha [2] first provided a general form of real positive solutions of equation $AX = C$ in Hilbert space under certain conditions. Recently, the real positive solutions of $AX = C$ were considered with corresponding operator A not necessarily having closed range in [6, 12] for adjoint operators. However, these formulae of real positive solutions still have some additional restrictions. In [16], the authors considered the equivalent conditions for the existence of common real positive solutions to the pair of the matrix equations $AX = C$, $XB = D$ in matrix algebra and offered partial common real positive solutions.

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces. We denote the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ and by $\mathcal{B}(\mathcal{H})$ when $\mathcal{H} = \mathcal{K}$. For an operator A, we shall denote by A^*, $R(A)$, $\overline{R(A)}$ and $N(A)$ the adjoint, the range, the closure of the range and the null space of A, respectively. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive ($A \geq 0$), if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. Note that a positive operator has unique square root and

$$|A| := (A^*A)^{\frac{1}{2}}$$

is the absolute value of A. Let $A^\dagger$ be the Moore-Penrose inverse of A, which is bounded if and only if $R(A)$ is closed [9]. An operator A is called real positive if

$$Re(A) := \frac{1}{2}(A + A^*) \geq 0.$$

The set of all real positive operators in $\mathcal{B}(\mathcal{H})$ is denoted by $Re\mathcal{B}_+(\mathcal{H})$. An operator sequence $\{T_n\}$ convergents to $T \in \mathcal{B}(\mathcal{H})$ in strong operator topology if $\|T_n x - Tx\| \to 0$ as $n \to \infty$ for any $x \in \mathcal{H}$. Denote by

$$T = s.o. - lim_{n\to\infty} T_n.$$

Let P_M denote the orthogonal projection on the closed subspace M. The identity onto a closed subspace M is denoted by I_M or I, if there is no confusion.

In this paper, we focus our work on the problem of characterizing the real positive solutions of operator equations $AX = C$ and $XB = D$ with corresponding operators A and B not necessarily having closed ranges in infinite dimensional Hilbert space. Our current goal is three-fold:

Firstly, by using polar decomposition and the strong operator convergence, a completely new representation of the reduced solution F of operator equation $AX = C$ is given by

$$F = s.o. - lim_{n\to\infty}(|A| + \frac{1}{n}I)^{-1} U_A^* C,$$

where $A = U_A|A|$ is the polar decomposition of A. This solution has property $F = A^\dagger C$ when $R(A)$ is closed. Furthermore, the necessary and sufficient conditions for the existence of real positive solutions and the detailed formulae for the real positive solutions of equation $AX = C$ are obtained, which improves the related results in [2, 7, 12]. Some comments on the reduced solution and real positive solutions are given.

Next, we discuss common solutions of the system of two operator equations $AX = C$ and $XB = D$. The necessary and sufficient conditions for the existence common solutions and the detailed representations of general common solutions are provided, which extends the classical closed range case with a short proof. Furthermore, we consider the problem of finding the sufficient conditions which ensure this system has a real positive solution as well as the formula of these real positive solutions.

Finally, two examples are provided. As shown by Example 4.1, the system of equations $AX = C$ and $XB = D$ has common real positive solutions under above given sufficient conditions. It is shown by Example 4.2 that a gap is unfortunately contained in the original paper [16], where the authors gave two equivalent conditions for the existence of common real positive solutions for this system in matrix algebra. Here, it is still an open question to give an equivalent condition for the existence of common real positive solutions of $AX = C$ and $XB = D$ in infinite dimensional Hilbert space.

2. The reduced solution of $AX = C$

In general, $A^{\dagger}C$ is the reduced solution of equation $AX = C$ if $R(A)$ is closed. Liang and Deng gave an expression of the reduced solution denoted $A^{\ddagger}C$ through the spectral measure of positive operators where operator A may not be closed, but sometimes $A^{\ddagger}$ is only a symbol since the spectral integral $\int_0^{\infty} \lambda^{\ddagger} dE_{\lambda}$ (see [12]) may be divergent. In this section, we will give a new representation of reduced solution of $AX = C$ by operator sequence. we begin with some lemmas.

Lemma 2.1. *([8]) Suppose that $\{T_n\}_{n=1}^{\infty}$ is an invertible positive operator sequence in $\mathcal{B}(\mathcal{H})$ and*

$$T = s.o. - lim_{n\to\infty} T_n.$$

If T is also invertible and positive, then

$$T^{-1} = s.o. - lim_{n\to\infty} T_n^{-1}.$$

For an operator $T \in \mathcal{B}(\mathcal{H})$, then T has a unique polar decomposition $T = U_T|T|$, where U_T is a partial isometry and $P_{\overline{R(T^*)}} = U_T^* U_T$([8]). Denote

$$T_n = (\frac{1}{n} I_{\mathcal{H}} + |T|)^{-1},$$

for each positive integer n. In [11], the authors paid attention to the operator sequence $T_n|T|$ over Hilbert C^*-module. It was given that $T_n|T|$ converges to $P_{\overline{R(T^*)}}$ in strong operator topology for T is a positive operator. Here we have some relative results as follows,

Lemma 2.2. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ and $T_n = (\frac{1}{n} I_{\mathcal{H}} + |T|)^{-1}$. Then the following statements hold:*

(i) *([8, 11]) $P_{\overline{R(T^*)}} = s.o. - lim_{n\to\infty} T_n|T|$.*

(ii) *$T^{\dagger} = s.o. - lim_{n\to\infty} T_n U_T^*$ if $R(T)$ is closed.*

Proof. We only prove the statement (ii). If $R(T)$ is closed, then $R(T^*)$ is also closed. It is natural that $\mathcal{H} = R(T^*) \oplus N(T) = R(T) \oplus N(T^*)$. Thus T and $|T|$ have the following matrix forms, respectively,

$$T = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix} \to \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix} \tag{2.1}$$

and

$$|T| = (T^*T)^{\frac{1}{2}} = \begin{pmatrix} (T_{11}^* T_{11})^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix} \to \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix}, \tag{2.2}$$

where $T_{11} \in \mathcal{B}(R(T^*), R(T))$ is invertible. Then $T^{\dagger}$ has the matrix form

$$T^{\dagger} = \begin{pmatrix} T_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix} \to \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix}. \tag{2.3}$$

And from (2.2), it is easy to get that

$$\frac{1}{n} I_{\mathcal{H}} + |T| = \begin{pmatrix} \frac{1}{n} I_{R(T^*)} + |T_{11}| & 0 \\ 0 & \frac{1}{n} I_{N(A)} \end{pmatrix} : \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix} \to \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix}. \tag{2.4}$$

By the invertiblity of diagonal operator matrix and the above matrix form (2.4), we have

$$T_n = (\frac{1}{n}I_{\mathcal{H}} + |T|)^{-1} = \begin{pmatrix} (\frac{1}{n}I_{R(T^*)} + |T_{11}|)^{-1} & 0 \\ 0 & nI_{N(A)} \end{pmatrix} : \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix} \to \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix}. \tag{2.5}$$

Because the operator sequence

$$\{\frac{1}{n}I_{R(T^*)} + |T_{11}|\} \subset \mathcal{B}(R(T^*))$$

converges to $|T_{11}|$ and

$$\frac{1}{n}I_{R(T^*)} + |T_{11}|$$

is invertible in $\mathcal{B}(R(T^*))$ for each n, then the operator sequence

$$\{(\frac{1}{n}I_{R(T^*)} + |T_{11}|)^{-1}\} \subset \mathcal{B}(R(T^*))$$

converges to $|T_{11}|^{-1}$ by Lemma 2.1. Moreover, partial isometry U_T has the matrix form

$$U_T = \begin{pmatrix} U_{11} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix} \to \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix} \tag{2.6}$$

with respect to the space decomposition $\mathcal{H} = R(T^*) \oplus N(T)$, where U_{11} is a unitary from $R(T^*)$ onto $R(T)$. Then $T_{11} = U_{11}|T_{11}|$ and $T_{11}^{-1} = |T_{11}|^{-1}U_{11}^*$ by $T = U_T|T|$ and the matrix forms (2.1), (2.2), and (2.6). From matrix forms (2.5) and (2.6), we have

$$T_nU_T^* = \begin{pmatrix} (\frac{1}{n}I_{R(T^*)} + |T_{11}|)^{-1}U_{11}^* & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix} \to \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix}.$$

Since the operator sequence $(\frac{1}{n}I_{R(T^*)} + |T_{11}|)^{-1}U_{11}^*$ converges to

$$|T_{11}|^{-1}U_{11}^* = T_{11}^{-1},$$

we obtain that the operator sequence $\{T_nU_T^*\}$ converges to $\begin{pmatrix} T_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}$, which is equivalent to $T^\dagger$ by the matrix form (2.3). □

In Lemma 2.2, the statement (ii) is not necessarily true if $R(T)$ is not closed. The following is an example.

Example 2.3. *Let* $\{e_1, e_2, \cdots\}$ *be a basis of an infinite Hilbert space* $\mathcal{H}$. *Define an operator* T *as follows,*

$$Te_k = \frac{1}{k}, k = 1, 2, \cdots, n, \cdots.$$

Then

$$T_n = (\frac{1}{n}I_{\mathcal{H}} + |T|)^{-1} = (\frac{1}{n}I_{\mathcal{H}} + T)^{-1},$$

since T is a positive operator. It is easy to get that

$$T_n e_k = \frac{kn}{n+k} e_k$$

for any k. Suppose

$$x = \sum_{k=1}^{\infty} \frac{1}{k} e_k.$$

Then $x \in \mathcal{H}$ with

$$\|x\|^2 = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

By direct computing, we have

$$\|T_n x\| \cdot \|x\| \geq | < T_n x, x > | = \sum_{k=1}^{\infty} \frac{n}{k(n+k)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{n+k} \right) > \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{n+k} > \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2},$$

that is, $\|T_n x\| > \frac{1}{\|x\|} \left(\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2} \right)$. It is shown that the number sequence $\{\|T_n x\|\}$ is divergent since the harmonic progression $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent. This implies that the operator sequence $\{T_n\}$ is divergent in strong operator topology.

Lemma 2.4. *([4]) Let $A, C \in \mathcal{B}(\mathcal{H})$. The following three conditions are equivalent:*

(1) $\mathcal{R}(C) \subseteq \mathcal{R}(A)$.

(2) *There exists $\lambda > 0$ such that $CC^* \leq \lambda AA^*$.*

(3) *There exists $D \in \mathcal{B}(\mathcal{H})$ such that $AD = C$.*

If one of these conditions holds then there exists a unique solution $F \in \mathcal{B}(\mathcal{H})$ of the equation $AX = C$ such that $R(F) \subseteq R(A^*)$ and $N(F) = N(C)$. This solution is called the reduced solution.

Theorem 2.5. *Let $A, C \in \mathcal{B}(\mathcal{H})$ be such that the equation $AX = C$ has a solution. Then the unique reduced solution F can be formulated by*

$$F = s.o. - lim_{n\to\infty}(|A| + \frac{1}{n}I)^{-1} U_A^* C,$$

where $U_A \in \mathcal{B}(H)$ is the partial isometry associated to the polar decomposition $A = U_A|A|$. Particularly, if $R(A)$ is closed, the reduced solution F is $A^\dagger C$.

Proof. Since $A = U_A|A|$, we have $U_A^* A = |A|$. Assuming that X is a solution of $AX = C$, then $|A|X = U_A^* C$. By multiplication $(|A| + \frac{1}{n}I)^{-1}$ on the left, it follows that

$$(|A| + \frac{1}{n}I)^{-1}|A|X = (|A| + \frac{1}{n}I)^{-1} U_A^* C.$$

From Lemma 2.2 (i), $\{(|A| + \frac{1}{n})^{-1}|A|X\}$ is convergent to $P_{\overline{R(A^*)}}X$ in strong operator topology. So

$$P_{\overline{R(A^*)}}X = s.o. - lim_{n\to\infty}(|A| + \frac{1}{n}I)^{-1} U_A^* C \triangleq F.$$

This shows that

$$AF = AP_{\overline{R(A^*)}}X = AX = C,$$

$R(F) \subseteq \overline{R(A^*)}$ and $N(F) \subseteq N(C)$. By the definition of F, we know $N(C) \subseteq N(F)$. So $N(C) = N(F)$. Suppose that F' is another reduced solution, then $A(F - F') = 0$ and so

$$R(F - F') \subseteq N(A) \cap \overline{R(A^*)} = \{0\}.$$

It is shown that $F - F' = 0$ and then $F = F'$. That is to say, F is the unique reduced solution of $AX = C$. If $R(A)$ is closed, $F = A^{\dagger}C$ by Lemma 2.2 (ii). □

Remark 2.6. *Although* $(|A| + \frac{1}{n}I)^{-1}U_A^*$ *does not necessarily converge in strong operator topology by Example 2.3,* $(|A| + \frac{1}{n}I)^{-1}U_A^*C$ *is convergent from the proof of the Theorem 2.5 if* $R(C) \subseteq R(A)$.

As a consequence of the above Theorem 2.5, Lemma 2.4 and Theorem 3.5 in [12], the following result holds.

Theorem 2.7. *([12]) Let* $A, C \in \mathcal{B}(\mathcal{H})$. *Then* $AX = C$ *has a solution if and only if* $R(C) \subseteq R(A)$. *In this case, the general solutions can be represented by*

$$X = F + (I - P)Y, \ \forall Y \in \mathcal{B}(\mathcal{H}),$$

where F *is the reduced solution of* $AX = C$ *and* $P = P_{\overline{R(A^*)}}$. *Particularly, if* $R(A)$ *is closed, then the general solutions can be represented by*

$$X = A^{\dagger}C + (I - P)Y, \ \ \forall Y \in \mathcal{B}(\mathcal{H}).$$

3. Real positive solutions of $AX = C, XB = D$

In this section, we mainly study the real positive solutions of equation

$$AX = C \tag{3.1}$$

and the system of equations

$$AX = C, XB = D. \tag{3.2}$$

Firstly, some preliminaries are given. For a given operator $A \in \mathcal{B}(\mathcal{H})$, denote $P = P_{\overline{R(A^*)}}$ in the sequel.

Lemma 3.1. *Let* $A, C \in \mathcal{B}(\mathcal{H})$ *be such that Eq* (3.1) *has a solution and* F *is the reduced solution. Then the following statements hold:*

(i) $CA^* + AC^* \geq 0$ *if and only if* $FP + PF^* \geq 0$.

(ii) $R(FP + PF^*)$ *is closed if* $R(CA^* + AC^*)$ *is closed.*

Proof. (i) Assume that $CA^* + AC^* \geq 0$. For any $x \in \mathcal{H}$, there exist $y \in \overline{R(A^*)}$ and $z \in N(A)$ such that $x = y + z$. And there exists $\{A^*y_n\} \subset R(A^*)$ such that $lim_{n\to\infty}A^*y_n = y$. According to $R(F) \subset \overline{R(A^*)}$, by direct computing,

$$\begin{aligned}
\langle (FP_{\overline{R(A^*)}} + P_{\overline{R(A^*)}}F^*)x, x\rangle &= \langle (FP + PF^*)y, y\rangle \\
&= lim_{n\to\infty}\langle (F + F^*)A^*y_n, A^*y_n\rangle \\
&= lim_{n\to\infty}\langle A(F + F^*)A^*y_n, y_n\rangle \\
&= \langle (CA^* + AC^*)y, y\rangle \geq 0.
\end{aligned}$$

This shows that $FP + PF^* \geq 0$.

Contrarily, if $FP + PF^* \geq 0$, it is elementary to get that

$$\begin{aligned}\langle (CA^* + AC^*)x, x\rangle &= \langle (AFA^* + AF^*A^*)x, x\rangle = \langle (F + F^*)A^*x, A^*x\rangle \\ &= \langle (F + F^*)PA^*x, PA^*x\rangle = \langle (FP + PF^*)A^*x, A^*x\rangle \geq 0,\end{aligned}$$

for any $x \in \mathcal{H}$ by $R(F) \subset \overline{R(A^*)}$. This implies that $CA^* + AC^* \geq 0$.

(ii) Since $R(CA^* + AC^*) = R(A(F + F^*)A^*)$ is closed, we have $R(A(F + F^*)P)$ is closed. It follows that $R(P(F + F^*)A^*)$ is closed and so $R(P(F + F^*)P)$ is closed. □

Lemma 3.2. *([3]) Let*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}).$$

The following two statements hold:

(i) *There exists an operator A_{22} such that $A \geq 0$ if and only if $A_{11} \geq 0$ and $R(A_{12}) \subseteq R(A_{11}^{\frac{1}{2}})$.*

(ii) *Suppose that A_{11} is an invertible and positive operator. Then $A \geq 0$ if and only if $A_{22} \geq A_{12}^*A_{11}^{-1}A_{12}$.*

Lemma 3.3. *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K}).$$

Then $A \geq 0$ if and only if

(i) *$A_{11} \geq 0$.*

(ii) *$A_{22} \geq X^*P_{\overline{R(A_{11})}}X$, where X is a solution of $A_{11}^{\frac{1}{2}}X = A_{12}$.*

Proof. Assume that the statements (i) and (ii) hold. Then

$$\begin{aligned} A &= \begin{pmatrix} A_{11} & A_{11}^{\frac{1}{2}}X \\ X^*A_{11}^{\frac{1}{2}} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{11}^{\frac{1}{2}}P_{\overline{R(A_{11})}}X \\ X^*P_{\overline{R(A_{11})}}A_{11}^{\frac{1}{2}} & A_{22} \end{pmatrix} \\ &= \begin{pmatrix} A_{11}^{\frac{1}{2}} & P_{\overline{R(A_{11})}}X \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} A_{11}^{\frac{1}{2}} & P_{\overline{R(A_{11})}}X \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A_{22} - X^*P_{\overline{R(A_{11})}}X \end{pmatrix} \\ &\geq 0. \end{aligned}$$

Conversely, if $A \geq 0$, it follows from Lemma 3.2 that $A_{11} \geq 0$. And also we get that

$$A_{11}^{\frac{1}{2}}X = A_{12}$$

has a solution X since $R(A_{12}) \subset R(A_{11}^{\frac{1}{2}})$. Moreover, $A \geq 0$ shows that for any positive integer n,

$$A + \frac{1}{n}P_{\mathcal{H}} \geq 0,$$

that is,

$$\begin{pmatrix} A_{11} + \frac{1}{n}I_{\mathcal{H}} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \geq 0.$$

By Lemma 3.2 again, we have

$$A_{22} \geq A_{12}^*(A_{11} + \frac{1}{n}I_{\mathcal{H}_1})^{-1}A_{12} = X^*A_{11}^{\frac{1}{2}}(A_{11} + \frac{1}{n}I_{\mathcal{H}_1})^{-1}A_{11}^{\frac{1}{2}}X.$$

From Lemma 2.2 (i), it is immediate that

$$A_{11}^{\frac{1}{2}}(A_{11} + \frac{1}{n}I_{\mathcal{H}_1})^{-1}A_{11}^{\frac{1}{2}} = (A_{11} + \frac{1}{n}I_{\mathcal{H}_1})^{-1}A_{11} \to P_{\overline{R(A_{11})}}(n \to \infty)$$

in strong operator topology. Thus

$$X^*P_{\overline{R(A_{11})}}X = s.o. - lim_{n\to\infty}X^*A_{11}^{\frac{1}{2}}(A_{11} + \frac{1}{n}I_{\mathcal{H}_1})^{-1}A_{11}^{\frac{1}{2}}X$$

holds. Therefore $A_{22} \geq X^*P_{\overline{R(A_{11})}}X$. □

Corollary 3.4. *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$$

be such that A_{11} has closed range. Then $A \geq 0$ if and only if

(i) $A_{11} \geq 0$.

(ii) $A_{22} \geq X^*A_{11}X$, *where X is a solution of $A_{11}X = A_{12}$.*

Proof. Since A_{11} is an operator with closed range, then $R(A_{11}^{\frac{1}{2}}) = R(A_{11})$. Thus the solvability of the two equations $A_{11}X = A_{12}$ and $A_{11}^{\frac{1}{2}}X = A_{12}$ are equivalent. It is easy to check that $A_{11}^{\frac{1}{2}}X$ is a solution of $A_{11}^{\frac{1}{2}}X = A_{12}$ if X is a solution of $A_{11}X = A_{12}$. From the Lemma 3.3, the result holds. □

In [12], the author gave a formula of real positive solutions for Eq (3.1) which is only a restriction on the general solutions. Here we obtain a specific expression of the real positive solutions.

Theorem 3.5. *Let $A, C \in \mathcal{B}(\mathcal{H})$. The Eq* (3.1) *has real positive solutions if and only if $R(C) \subseteq R(A)$ and $CA^* + AC^* \geq 0$. In this case, the real positive solutions can be represented by*

$$X = F - (I-P)F^* + (I-P)Y^*(PF^* + FP)^{\frac{1}{2}} + \frac{1}{2}(I-P)Y^*P_{\overline{R(F_0)}}Y(I-P) + (I-P)Z(I-P), \quad (3.3)$$

for any $Y \in \mathcal{B}(\mathcal{H})$, $Z \in Re\mathcal{B}_+(\mathcal{H})$, where F is the reduced solution of $AX = C$ and $F_0 = PF^* + FP$. Particularly, if $R(CA^* + AC^*)$ is closed, then the real positive solutions can be represented by

$$X = F - (I-P)F^* + (I-P)Y^*(PF^* + FP) + (I-P)Y^*FPY(I-P) + (I-P)Z(I-P). \quad (3.4)$$

Proof. From Theorem 2.7, $AX = C$ has a solution if and only if $R(C) \subseteq R(A)$. Suppose that X is a solution of $AX = C$. There exists an operator $\overline{Y} \in \mathcal{B}(\mathcal{H})$ such that

$$X = F + (I-P)\overline{Y}.$$

Set $\mathcal{H} = \overline{R(A^*)} \oplus N(A)$. Then F has the following matrix form

$$F = \begin{pmatrix} F_{11} & F_{12} \\ 0 & 0 \end{pmatrix}, \quad (3.5)$$

since $R(F) \subset \overline{R(A^*)}$. The operator $\overline{Y}$ can be expressed as

$$\overline{Y} = \begin{pmatrix} \overline{Y}_{11} & \overline{Y}_{12} \\ \overline{Y}_{21} & \overline{Y}_{22} \end{pmatrix}. \tag{3.6}$$

This follows that

$$X + X^* = \begin{pmatrix} F_{11} + F_{11}^* & F_{12} + \overline{Y}_{21}^* \\ \overline{Y}_{21} + F_{12}^* & \overline{Y}_{22} + \overline{Y}_{22}^* \end{pmatrix}. \tag{3.7}$$

"⇒": If X is a real positive solution, then $F_{11} + F_{11}^* \geq 0$ from Lemma 3.3 and the matrix form (3.7) of $X + X^*$. And so $PF^* + FP \geq 0$. According to Lemma 3.1, $CA^* + AC^* \geq 0$.

"⇐": Assume that $CA^* + AC^* \geq 0$, then $PF^* + FP \geq 0$ and so $F_{11} + {F_{11}}^* \geq 0$. Denote $X_0 = F - (I - P)F^*$. Then $AX_0 = C$ and

$$X_0 + X_0^* = \begin{pmatrix} F_{11} + F_{11}^* & 0 \\ 0 & 0 \end{pmatrix} \geq 0.$$

That is, X_0 is a real positive solution of $AX = C$.

Next, we analyse the general form of real positive solutions. Suppose that X is a real positive solution of $AX = C$. Then $X + X^*$ has the matrix form (3.7). From Lemma 3.3, there exists an operator Y_{12} such that

$$\overline{Y}_{21}^* = (F_{11} + F_{11}^*)^{\frac{1}{2}} Y_{12} - F_{12}$$

and

$$\overline{Y}_{22} + \overline{Y}_{22}^* \geq Y_{12}^* P_{\overline{R(F_{11}+F_{11}^*)}} Y_{12}.$$

Let

$$Z_{22} = \overline{Y}_{22} - \frac{1}{2} Y_{12}^* P_{\overline{R(F_{11}+F_{11}^*)}} Y_{12}.$$

In this case, X can be represented by the following form

$$\begin{aligned} X &= \begin{pmatrix} F_{11} & F_{12} \\ {Y_{12}}^*(F_{11} + {F_{11}}^*)^{\frac{1}{2}} - {F_{12}}^* & \frac{1}{2}{Y_{12}}^* Y_{12} + Z_{22} \end{pmatrix} \\ &= F - (I - P)F^* + (I - P)Y^*(PF^* + FP)^{\frac{1}{2}} + \tfrac{1}{2}(I - P)Y^* P_{\overline{R(F_0)}} Y(I - P) + (I - P)Z(I - P), \end{aligned}$$

for some $Y \in \mathcal{B}(\mathcal{H})$, $Z \in Re\mathcal{B}_+(\mathcal{H})$, where $F_0 = PF^* + FP$.

On the contrary, for any operator $Y \in \mathcal{B}(\mathcal{H})$ and $Z \in Re\mathcal{B}_+(\mathcal{H})$ such that X has the form (3.3), it is clear that $AX = C$. Let $Y = (Y_{ij})_{2\times 2}$ and $Z = (Z_{ij})_{2\times 2}$ with respect to the space decomposition $\mathcal{H} = \overline{R(A^*)} \oplus N(A)$. Then

$$\begin{aligned} X + X^* &= \begin{pmatrix} F_{11} + F_{11}^* & (F_{11} + F_{11}^*)^{\frac{1}{2}} Y_{12} \\ {Y_{12}}^*(F_{11} + F_{11}^*)^{\frac{1}{2}} & \frac{1}{2}{Y_{12}}^* Y_{12} + Z_{22} + {Z_{22}}^* \end{pmatrix} \\ &= \begin{pmatrix} (F_{11} + F_{11}^*)^{\frac{1}{2}} & Y_{12} \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} (F_{11} + F_{11}^*)^{\frac{1}{2}} & Y_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & Z_{22} + {Z_{22}}^* \end{pmatrix} \\ &\geq 0. \end{aligned}$$

Particularly, if $R(CA^* + AC^*)$ is closed, then $R(F_{11} + F_{11}^*)$ is closed by Lemma 3.1. Suppose that X is a real positive solution of $AX = C$, then $X + X^*$ has the matrix form (3.7). From Corollary 3.4, there exists an operator $Y_{12} \in \mathcal{B}(N(A), \overline{R(A^*)})$ such that

$$\overline{Y}_{21}^* = (F_{11} + F_{11}^*)Y_{12} - F_{12}$$

and

$$\overline{Y}_{22} + \overline{Y}_{22}^* \geq Y_{12}^*(F_{11} + F_{11}^*)Y_{12}.$$

Let

$$Z_{22} = \overline{Y}_{22} - Y_{12}^* F_{11} Y_{12}.$$

Consequently,

$$\begin{aligned} X &= \begin{pmatrix} F_{11} & F_{12} \\ Y_{12}^*(F_{11} + F_{11}^*) - F_{12}^* & Y_{12}^* F_{11} Y_{12} + Z_{22} \end{pmatrix} \\ &= F - (I - P)F^* + (I - P)Y^*(PF^* + FP) + (I - P)Y^* FPY(I - P) + (I - P)Z(I - P), \end{aligned}$$

for some $Y \in \mathcal{B}(\mathcal{H})$, $Z \in Re\mathcal{B}_+(\mathcal{H})$.

On the contrary, for any $Y \in \mathcal{B}(\mathcal{H})$ and $Z \in Re\mathcal{B}_+(\mathcal{H})$ such that X has a form (3.2), it is clear that $AX = C$ and also we have

$$\begin{aligned} X + X^* &= \begin{pmatrix} F_{11} + F_{11}^* & (F_{11} + F_{11}^*)Y_{12} \\ {Y_{12}}^* F_{11} + F_{11}^* & {Y_{12}}^* F_{11} + F_{11}^* Y_{12} + Z_{22} + {Z_{22}}^* \end{pmatrix} \\ &= \begin{pmatrix} (F_{11} + F_{11}^*)^{\frac{1}{2}} & 0 \\ Y_{12}^*(F_{11} + F_{11}^*)^{\frac{1}{2}} & 0 \end{pmatrix} \begin{pmatrix} (F_{11} + F_{11}^*)^{\frac{1}{2}} & (F_{11} + F_{11}^*)^{\frac{1}{2}} Y_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & Z_{22} + Z_{22}^* \end{pmatrix} \\ &\geq 0. \end{aligned}$$

□

Next, we consider the solutions of $AX = C, XB = D$.

Theorem 3.6. *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$. Then the system of Eq* (3.2) *has a solution if and only if $R(C) \subseteq R(A)$, $R(D^*) \subseteq R(B^*)$ and $AD = CB$. In this case, the general common solution can be represented by*

$$X = F - (I - P)H^* + (I - P)Z(I - P_{\overline{R(B)}}),$$

for any $Z \in \mathcal{B}(\mathcal{H})$, where F is the reduced solution of (3.1) *and H is the reduced solution of $B^*X = D^*$. Particularly, if $R(A)$ and $R(B)$ are closed, the general solution can be represented by*

$$X = A^\dagger C + DB^\dagger - A^\dagger ADB^\dagger + (I - A^\dagger A)Z(I - BB^\dagger).$$

Proof. The necessity is clear. We only need to prove the sufficient condition. From $R(C) \subseteq R(A)$ and Theorem 2.7, a solution X of Eq (3.1) can be represented by

$$X = F + (I - P)Y,$$

for some $Y \in \mathcal{B}(\mathcal{H})$, where F is the reduced solution. Substitute above X into equation $XB = D$, we have

$$(I - P)YB = D - FB.$$

This shows that

$$(I - P)YB = (I - P)(D - FB)$$

and then

$$(I - P)YB = (I - P)D,$$

since $R(F) \subseteq \overline{R(A^*)}$ and $(I - P)F = 0$. Moreover, $YB = D$ has a solution since $R(D^*) \subseteq R(B^*)$. Denote H is the reduced solution of $B^*\hat{Y} = D^*$. Then

$$R(H(I - P)) \subseteq \overline{R(B)}$$

and

$$N(H(I - P)) = N(D^*(I - P)),$$

since $R(H) \subseteq \overline{R(B)}$ and $N(H) = N(D^*)$. So $H(I - P)$ is the reduced solution of $B^*\hat{Y} = D^*(I - P)$ and then there exists $Z \in \mathcal{B}(\mathcal{H})$ such that

$$Y^*(I - P) = H(I - P) + (I - P_{\overline{R(B)}})Z.$$

That is to say, the system of equations $AX = C$, $XB = D$ has common solutions. The general solution can be represented by

$$X = F - (I - P)H^* + (I - P)Z(I - P_{\overline{R(B)}}), \text{ for any } Z \in \mathcal{B}(\mathcal{H}).$$

Especially, if $R(A)$ and $R(B)$ are closed, then $F = A^\dagger C$ and $H = B^{*\dagger}D^*$. It follows that the general common solution is

$$X = A^\dagger C + (I - A^\dagger A)DB^\dagger + (I - A^\dagger A)Z(I - BB^\dagger),$$

for any $Z \in \mathcal{B}(\mathcal{L}, \mathcal{H})$. □

Theorem 3.7. *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$ and $Q = P_{\overline{R((I-P)B)}}$. The system of Eq* (3.2) *has a real positive solution if the following statements hold:*

(i) *$R(C) \subseteq R(A)$, $R(D^*) \subseteq R(B^*)$, $AD = CB$,*

(ii) *CA^*, $(D^* + B^*F)(I - P)B$ are real positive operators,*

(iii) *$R((D^* + B^*F)(I - P)) \subseteq R(B^*(I - P))$,*

where F is the reduced solution of $AX = C$. In this case, one of common real positive solutions can be represented by

$$X = F - (I - P)F^* + (I - P)H^*(I - P) - (I - P)H(I - P - Q) + (I - P - Q)Z(I - P - Q), \quad (3.8)$$

for any $Z \in Re\mathcal{B}_+(\mathcal{H})$, where H is the reduced solution of

$$B^*(I - P)X = (D^* + B^*F)(I - P).$$

Proof. Combined Theorems 3.5 and 3.6 with statements (i) and (ii), the system of equations $AX = C$, $XB = D$ has common solutions. For any $Y \in Re\mathcal{B}_+(\mathcal{H})$,

$$X = F - (I - P)F^* + (I - P)Y(I - P)$$

is a real positive solution of $AX = C$. We only need to prove that there exists $Y \in Re\mathcal{B}_+(\mathcal{H})$ such that above X is also a solution of $XB = D$. Since

$$\mathcal{H} = \overline{R(A^*)} \oplus N(A) = \overline{R(A)} \oplus N(A^*)$$

and also

$$\mathcal{H} = \overline{R(B^*)} \oplus N(B),$$

then A, F and B have the following operator matrix forms,

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \overline{R(A^*)} \\ N(A) \end{pmatrix} \to \begin{pmatrix} \overline{R(A)} \\ N(A^*) \end{pmatrix}, \tag{3.9}$$

$$F = \begin{pmatrix} F_{11} & F_{12} \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \overline{R(A^*)} \\ N(A) \end{pmatrix} \to \begin{pmatrix} \overline{R(A^*)} \\ N(A) \end{pmatrix}, \tag{3.10}$$

$$B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & 0 \end{pmatrix} : \begin{pmatrix} \overline{R(B^*)} \\ N(B) \end{pmatrix} \to \begin{pmatrix} \overline{R(A^*)} \\ N(A) \end{pmatrix}, \tag{3.11}$$

respectively, where A_{11} is an injective operator from $\overline{R(A^*)}$ into $\overline{R(A)}$ with dense range. The operator D has the matrix form

$$D = \begin{pmatrix} D_{11} & 0 \\ D_{21} & 0 \end{pmatrix} : \begin{pmatrix} \overline{R(B^*)} \\ N(B) \end{pmatrix} \to \begin{pmatrix} \overline{R(A^*)} \\ N(A) \end{pmatrix}, \tag{3.12}$$

since $R(D^*) \subseteq R(B^*)$. Let

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} : \begin{pmatrix} \overline{R(A^*)} \\ N(A) \end{pmatrix} \to \begin{pmatrix} \overline{R(A^*)} \\ N(A) \end{pmatrix}. \tag{3.13}$$

Then X has a matrix form as follows:

$$X = \begin{pmatrix} F_{11} & F_{12} \\ -F_{12}^* & Y_{22} \end{pmatrix} : \begin{pmatrix} \overline{R(A^*)} \\ N(A) \end{pmatrix} \to \begin{pmatrix} \overline{R(A^*)} \\ N(A) \end{pmatrix}. \tag{3.14}$$

By the matrix forms (3.11) and (3.14) of B and X, respectively, it is easy to get that

$$B^*X^* = \begin{pmatrix} B_{11}^* & B_{21}^* \\ 0 & 0 \end{pmatrix} \begin{pmatrix} F_{11}^* & -F_{12} \\ F_{12}^* & Y_{22}^* \end{pmatrix} = \begin{pmatrix} B_{11}^*F_{11}^* + B_{21}^*F_{12}^* & -B_{11}^*F_{12} + B_{21}^*Y_{22}^* \\ 0 & 0 \end{pmatrix}. \tag{3.15}$$

Combining $AD = CB = AFB$ with the matrix forms (3.9)–(3.12), we have

$$AD = \begin{pmatrix} A_{11}D_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_{11}(F_{11}B_{11} + F_{12}B_{21}) & 0 \\ 0 & 0 \end{pmatrix} = AFB.$$

It is immediate that

$$A_{11}(F_{11}B_{11} + F_{12}B_{21}) = A_{11}D_{11}.$$

And so

$$(B_{11}^{*}F_{11}^{*} + B_{21}^{*}F_{12}^{*})A_{11}^{*} = D_{11}^{*}A_{11}^{*}.$$

Therefore,

$$B_{11}^{*}F_{11}^{*} + B_{21}^{*}F_{12}^{*} = D_{11}^{*}, \tag{3.16}$$

since $R(A_{11}^{*})$ has dense range in $\overline{R(A_{11}^{*})}$. From the statement (iii), $R(D_{21}^{*} + B_{11}^{*}F_{12}) \subseteq R(B_{21}^{*})$ holds. Moreover, by the statement (ii) $(D^{*} + B^{*}F)(I - P)B$ being real positive, we have $(D_{21}^{*} + B_{11}^{*}F_{12})$ is real positive. These infer that the equation

$$B_{21}^{*}\widehat{Y_{22}} = D_{21}^{*} + B_{11}^{*}F_{12} \tag{3.17}$$

has real positive solutions by Theorem 3.5. Suppose that H_{22} is the reduced solution of Eq (3.17), we can deduce that

$$H = \begin{pmatrix} 0 & 0 \\ 0 & H_{22} \end{pmatrix}$$

is the reduced solution of the following equation

$$B^{*}(I - P)\widehat{Y} = (D^{*} + B^{*}F)(I - P), \tag{3.18}$$

since $R(H) \subseteq (I - P)B$ and

$$N(H) = N((D^{*} + B^{*}F)(I - P)).$$

Using Theorem 3.5 again, $H - (I - Q)H^{*} \geq 0$ is a real positive solution of (3.18). Then $(I - P)(H - (I - Q)H^{*})(I - P)$ is also a real positive solution of (3.18). Because of

$$I - P \geq P_{\overline{R((I-P)B)}} = Q,$$

we have

$$(I - P)(I - Q) = (I - Q)(I - P) = (I - P - Q).$$

Let

$$X_0 = F - (I - P)F^{*} + (I - P)H^{*}(I - P) - (I - P)H(I - P - Q).$$

That is,

$$X_0 = \begin{pmatrix} F_{11} & F_{12} \\ -F_{12}^{*} & H_{22}^{*} - H_{22}(I_{N(A)} - Q) \end{pmatrix} : \begin{pmatrix} \overline{R(A^{*})} \\ N(A) \end{pmatrix} \to \begin{pmatrix} \overline{R(A^{*})} \\ N(A) \end{pmatrix}.$$

Put $X = X_0$ into formula (3.15), combining Eqs (3.16) and (3.17) with the matrix form (3.12) and (3.15), we can get $B^{*}X_0^{*} = D^{*}$. So X_0 is a common real positive solution of Eq (3.2). Furthermore, it is easy to verify that

$$A(I - P - Q)Z(I - P - Q) = 0$$

and

$$(I - P - Q)Z(I - P - Q)B = 0,$$

for any $Z \in \mathcal{B}(\mathcal{H})$. Therefore

$$X = X_0 + (I - P - Q)Z(I - P - Q), \text{ for any } Z \in Re\mathcal{B}_{+}(\mathcal{H})$$

is a real positive solution of system equations $AX = C, XB = D$. □

4. Examples

Some examples are given in this section to demonstrate our results are valid. And also it is shown that a gap is unfortunately contained in the original paper [16] about existence of real positive solutions.

Example 4.1. *Let $\mathcal{H}$ be the infinite Hilbert space as in Example 2.3 with a basis $\{e_1, e_2, \cdots\}$ and $Te_k = \frac{1}{k}e_k, \forall k \geq 1$. As well known the range of $T \in \mathcal{B}(\mathcal{H})$ is not closed and $\overline{R(T)} = \mathcal{H}$. Define operators $A, C, B, D \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}$ as follows:*

$$A = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} T & T \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & T \\ 0 & -T \end{pmatrix}.$$

Then

$$A \geq 0,\ U_A = \begin{pmatrix} I_{\mathcal{H}} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$(|A| + \frac{1}{n}I)^{-1}U_A^*C = \begin{pmatrix} (T + \frac{1}{n}I_{\mathcal{H}})^{-1}T & (T + \frac{1}{n}I_{\mathcal{H}})^{-1}T \\ 0 & 0 \end{pmatrix}.$$

By Theorem 2.5 and Lemma 2.2 (i), the reduced solution F of the operator equation $AX = C$ has the following form

$$F = s.o. - lim_{n\to\infty}(|A| + \frac{1}{n}I)^{-1}U_A^*C = \begin{pmatrix} I_{\mathcal{H}} & I_{\mathcal{H}} \\ 0 & 0 \end{pmatrix}.$$

Denote

$$P = P_{\overline{R(A^*)}} = \begin{pmatrix} I_{\mathcal{H}} & 0 \\ 0 & 0 \end{pmatrix}.$$

From Theorem 3.5, we know that the formula of real positive solutions of $AX = C$ is

$$X = \begin{pmatrix} I_{\mathcal{H}} & I_{\mathcal{H}} \\ -I_{\mathcal{H}} + \sqrt{2}Y_{12}^* & \frac{1}{2}Y_{12}^*Y_{12} + Z_{22} \end{pmatrix}, \text{ for any } Y_{12} \in \mathcal{B}(\mathcal{H}), Z_{22} \in Re\mathcal{B}_+(\mathcal{H}).$$

Furthermore, it is easy to check that

(i) $AD = CB, R(C) = R(A), R(D^*) = R(B^*)$.

(ii) $CA^* = A$ *and* $(D^* + B^*F)(I - P)B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *are real positive operators.*

(iii) $(D^* + B^*F)(I - P) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $B^*(I - P) = \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}$.

By Theorem 3.7, we have the system of equations

$$AX = C, XB = D$$

has common real positive solutions. Moreover,

$$Q = P_{\overline{R((I-P)B)}} = 0.$$

So one of real positive solutions has the following form

$$X = F - (I-P)F^* + (I-P-Q)Z(I-P-Q) = \begin{pmatrix} I_{\mathcal{H}} & I_{\mathcal{H}} \\ -I_{\mathcal{H}} & Z_{22} \end{pmatrix}, \text{ for any } Z_{22} \in Re\mathcal{B}_+(\mathcal{H}).$$

Whereas the statements (i)–(iii) in Theorem 3.7 are only sufficient conditions for the existence of common real positive solutions of system (3.2)*. The following is an example.*

Example 4.2. *Let A, B, C, D be 2×2 complex matrices. Denote*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & \sqrt{2} - \frac{1}{2} \end{pmatrix}, X_0 = \begin{pmatrix} 1 & 1 \\ \sqrt{2} - 1 & \frac{1}{2} \end{pmatrix}.$$

By direct computing, $AX_0 = C$, $X_0B = D$ and

$$X_0 + X_0^* = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 1 & 0 \end{pmatrix}^* \geq 0.$$

So X_0 is a common real positive solution of system equations $AX = C, XB = D$. But in this case, Theorem 3.7 does not work. In fact,

$$F = A^\dagger C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \ P = P_{R(A^*)} = A.$$

Hence,

$$(D^* + B^*F)(I-P) = \left(\begin{pmatrix} 1 & \frac{1}{2} \\ 2 & \sqrt{2} - \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \sqrt{2} + \frac{1}{2} \end{pmatrix},$$

and

$$B^*(I-P) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

This shows that the statement (iii)

$$R((D^* + B^*F)(I-P)) \subseteq R(B^*(I-P))$$

in Theorem 3.7 does not hold.

Xiong and Qin gave two equivalent conditions (Theorems 2.1 and 2.2 in [16]) for the existence of common real positive solutions for $AX = C, XB = D$ in matrix algebra. But unfortunately, there is a gap in these results. Here, Example 4.2 is a counterexample. In fact, $r(A) = 1 \neq 2$, where $r(A)$ stands for the rank of matrix A and $A = A^* = CA^*$. Simplifying by elementary block matrix operations, we have

$$r\begin{pmatrix} AA^* & 0 & AC^* + CA^* \\ A^* & D & C^* \\ 0 & B & -A^* \end{pmatrix} = r\begin{pmatrix} A & 0 & 2A \\ A & D & C^* \\ 0 & B & -A \end{pmatrix} = r\begin{pmatrix} A & 0 & 0 \\ 0 & D & C^* - 2A \\ 0 & B & -A \end{pmatrix}. \tag{4.1}$$

Moreover,

$$
\begin{aligned}
r\begin{pmatrix} D & C^* - 2A \\ B & -A \end{pmatrix} &= r\begin{pmatrix} 1 & 2 & -1 & 0 \\ \frac{1}{2} & \sqrt{2}-\frac{1}{2} & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & 1 & -1 & 0 \\ \frac{1}{2} & \sqrt{2}-\frac{1}{2} & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \\
&= r\begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \sqrt{2}-\frac{1}{2} & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \sqrt{2}+\frac{1}{2} & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \\
&= r\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 3.
\end{aligned}
$$

Therefore, combining the above result with formula (4.1)*, we obtain that*

$$
r\begin{pmatrix} AA^* & 0 & AC^* + CA^* \\ A^* & D & C^* \\ 0 & B & -A^* \end{pmatrix} = r(A) + r\begin{pmatrix} D & C^* - 2A \\ B & -A \end{pmatrix} = 4. \tag{4.2}
$$

Then

$$
r(A) \neq 2 \text{ and } r\begin{pmatrix} AA^* & 0 & AC^* + CA^* \\ A^* & D & C^* \\ 0 & B & -A^* \end{pmatrix} \neq 2r(A). \tag{4.3}
$$

Noting that

$$
AD = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ \frac{1}{2} & \sqrt{2}-\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}.
$$

Using elementary row-column transformation again, we have

$$
\begin{aligned}
r\begin{pmatrix} B & A^* \\ AD & CA^* \end{pmatrix} &= r\begin{pmatrix} B & A \\ AD & A \end{pmatrix} = r\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= r\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 2.
\end{aligned} \tag{4.4}
$$

From formulaes (4.1)*,* (4.2) *and* (4.4)*, it is natural to get that*

$$
r(AD - CB) + r\begin{pmatrix} AA^* & 0 & AC^* + CA^* \\ A^* & D & C^* \\ 0 & B & -A^* \end{pmatrix} \neq r(A) + r\begin{pmatrix} B & A^* \\ AD & CA^* \end{pmatrix}, \tag{4.5}
$$

since $AD = CB$. But $AX = C, XB = D$ have a common real positive solution. The statements (4.3) *and* (4.5) *shows that Theorems 2.1 and 2.2 in [16] do not work, respectively. Actually, the conditions in [16] are also only sufficient conditions for the existence of common real positive solutions of Eq* (3.2). *So here is still an open question.*

Question 4.3. *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$. Give an equivalent condition for the existence of common real positive solutions of $AX = C, XB = D$.*

5. Conclusions

In this work, a new representation of reduced solution of $AX = C$ is given by a strong operator convergent sequence. This result provides us a method to discuss the general solutions of Eq (3.2). By making full use of block operator matrix methods, the formula of real positive solutions of $AX = C$ is obtained in Theorem 3.5, which is the basis of finding common real positive solutions of Eq (3.2). Through Example 4.1, it is demonstrated that Theorem 3.7 is useful to find some common real positive solutions. But unfortunately, it is complicated to consider all the common real positive solutions by using of the method in Theorem 3.7. Maybe, we need some other techniques. This will be our next problem to solve.

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Conflict of interest

The authors declare no conflicts of interest.

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