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Research article

The connection between ordinary and soft σ-algebras with applications to information structures

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Abstract: The paper presents a novel analysis of interrelations between ordinary (crisp) σ-algebras and soft σ-algebras. It is known that each soft σ-algebra produces a system of crisp (parameterized) σ-algebras. The other way round is also possible. That is to say, one can generate a soft σ-algebra from a system of crisp σ-algebras. Different methods of producing soft σ-algebras are discussed by implementing two formulas. It is demonstrated how these formulas can be used in practice with the aid of some examples. Furthermore, we study the fundamental properties of soft σ-algebras. Lastly, we show that elements of a soft σ-algebra contain information about a specific event.

Keywords: soft set; soft σ-algebra; random variable; probability
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1. Introduction

In today's world, the mathematical modeling and manipulation of various kinds of uncertainties has turned into an increasingly important issue in solving complex problems in a variety of fields such as engineering, economics, environmental science, medicine, and social sciences. Although probability theory, fuzzy set theory [1], rough set theory [2], and interval mathematics [3], and so on are well-known and effective tools for dealing with ambiguity and uncertainty, each has its own set of limitations; one of a common major weakness among these mathematical techniques is the limitation of parametrization tools.

Molodtsov [4], in 1999, originated the soft set theory as is a mathematical tool for dealing with uncertainty which is free of the challenges related with the earlier mentioned theories. Soft sets were presented as a collection of parameterized possibilities of a universe. The principle of soft set theory is based on the idea of parameterization, which suggests that complex objects should be perceived from myriad aspects and each solo facet only provide a partial and approximate description of the whole entity (see, [5]). This leads to the rapid growth of soft set theory and its relevant areas in a short amount of time (see, [6–8]). This development of soft sets is not free from real life applications (see, [9–12]).

Multiple researchers have applied soft set theory to various mathematical structures like soft group theory [13], soft ring theory [14], soft category theory [15], soft topology [16, 17], infra-soft topology [18], supra-soft topology [19, 20], etc. In the frame of soft topologies, it has been studied the concepts and properties related to soft separation axioms [21, 22] and extensions of soft open sets [23–25] amply. Hybridized (mixed) methods were developed to address more complex real-world problems in various disciplines. Dubois and Prade [26] were the first to mix fuzzy and rough set theories. In this fashion many generalizations of soft sets appeared in the literature (see, [27–31]).

The concept information structure is useful for stochastic control with partial observations and necessary for control of networked stochastic systems. The term information structure was first appeared by Witsenhausen in [32] and Marschak and Radner [33], pages 47–49. The information structure based on partitions offers a fairly simple illustration of how information spreads through a series of increasingly finer partitions. However, in a general probabilistic context, this structure is insufficient to explain the spread of information. The set of probable events is actually much richer than the set of partitions. Determining a structure of events as well as partitions is therefore essential. Filtration describes the pattern of events that defines the spread of knowledge. But in the discrete case, particularly finite, information structure and filtration have the same meaning.

The idea behind filtration is to identify all known events at any particular time. It is assumed that each dealing instant t can be associated with a σ-algebra of events $\Sigma_t \subseteq \Sigma$ composed of all events known previous to or at time t. It is believed that events are never "forgotten", i.e. $\Sigma_t \subseteq \Sigma_s$, if $t < s$. As a result, a time ordering is formed. This ordering is created by an increasing chain of σ-algebras, each of which corresponds to the time when all of its events are known. This is a filtration process. A filtration, denoted as $\{\Sigma_t\}$, is thus an increasing series of all σ-algebras. Each is linked with an instant t. In some other resources, an information structure on a simple space is considered a σ-algebra of events (see, [34])

A σ-algebra on a universal (crisp) set is crucial for the growth of several disciplines, including mathematical analysis, probability theory, dynamical systems and statistical information theory. The concept of σ-algebras in the context of soft set theory was introduced by Khameneh and Kilicman [35]. They studied some basic properties of soft σ-algebras. Some other basic properties of this concept were mentioned in [36]. Various properties and relations of soft σ-algebras remain untouched. Thence, we further study this concept and establish a general construction for producing soft σ-algebras. The construction helps us to study the properties of soft σ-algebras by means of properties of crisp σ-algebras.

The content of the paper is arranged as follows: We present an overview of the literature on soft set theory and probability theory in Section 2. Section 3 focuses on the study of soft σ-algebras, along with some operations. Section 4 establishes two formulas to find the relationships between crisp and soft σ-algebras. Section 5 provides some applications of soft σ-algebras in information structures. Section 6 ends the work with a brief conclusion.

2. Preliminaries

Let X be an initial universe, A be a set of parameters, and $\mathcal{P}(X)$ denote the set of all subsets of X.

Definition 2.1. *[4] Let $F : \mathbb{W} \rightarrow \mathcal{P}(X)$ be a set-valued mapping and let $\mathbb{W} \subseteq A$. An order pair $(F, \mathbb{W}) = \{(w, F(w)) : w \in \mathbb{W}\}$ is said to be the soft set over X.*

A soft set over X is considered as a parameterized class of subsets of X. The family of all soft subsets of X with the parametric set A (resp. $\mathbb{W}$) is denoted by $S_A(X)$ (resp. $S_{\mathbb{W}}(X)$).

Remark 2.1. *The soft set $(F, \mathbb{W})$ can be easily extended to the soft set (F, A) by giving $F(w) = \emptyset$ for each $w \in A - \mathbb{W}$.*

Definition 2.2. *[7] The soft complement $(F, \mathbb{W})^c$ of a soft set $(F, \mathbb{W})$ is a soft set $(F^c, \mathbb{W})$, where $F^c : \mathbb{W} \rightarrow \mathcal{P}(X)$ is a mapping having the property that $F^c(w) = X - F(w)$ for all $w \in \mathbb{W}$.*

Notice that $((F, \mathbb{W})^c)^c = (F, \mathbb{W})$.

Definition 2.3. *[37] A soft set $(F, \mathbb{W})$ over X is called null with respect to $\mathbb{W}$, $\Phi_{\mathbb{W}}$, if $F(w) = \emptyset$ for all $w \in \mathbb{W}$ and is called absolute with respect to $\mathbb{W}$, $X_{\mathbb{W}}$, if $F(w) = X$ for all $w \in \mathbb{W}$. The null and absolute soft sets are respectively denoted by Φ_A and X_A.*

Evidently, $\Phi_A^c = X_A$ and $X_A^c = \Phi_A$.

Definition 2.4. *[38] A soft set $(F, \mathbb{W})$ is called finite (resp. countable) [38] if $F(w)$ is finite (resp. countable) for each $w \in \mathbb{W}$. Otherwise, it is called infinite (resp. uncountable).*

Definition 2.5. *[16] A soft set $(F, \mathbb{W})$ over X is called a (an ordinary) soft point if $F(w) = \{x\}$ for all $w \in \mathbb{W}$, where $x \in X$. It is denoted by $(\{x\}, \mathbb{W})$ (or shortly x). It is said $x \tilde{\in} (F, \mathbb{W})$ if $x \in F(w)$ for all $w \in \mathbb{W}$.*

Definition 2.6. *[7, 39] Let $\mathbb{W}_1, \mathbb{W}_2 \subseteq A$. It is said that $(F_1, \mathbb{W}_1)$ is a soft subset of $(F_2, \mathbb{W}_2)$ (written by $(F_1, \mathbb{W}_1) \tilde{\subseteq} (F_2, \mathbb{W}_2)$) if $\mathbb{W}_1 \subseteq \mathbb{W}_2$ and $F_1(w) \subseteq F_2(w)$ for all $w \in \mathbb{W}_1$. And $(F_1, \mathbb{W}_1)$ is soft equal to $(F_2, \mathbb{W}_1)$ if $(F_1, \mathbb{W}_1) \tilde{\subseteq} (F_2, \mathbb{W}_2)$ and $(F_2, \mathbb{W}_2) \tilde{\subseteq} (F_1, \mathbb{W}_1)$.*

Definition 2.7. *[37, 40] Let $\{(F_i, \mathbb{W}) : i \in I\}$ be an indexed family of soft sets over X.*

(1) The soft intersection of $(F_i, \mathbb{W})$, for $i \in I$, is a soft set $(F, \mathbb{W})$ such that $F(w) = \bigcap_{i \in I} F_i(w)$ for all $w \in \mathbb{W}$ and is denoted by $(F, \mathbb{W}) = \tilde{\bigcap}_{i \in I} (F_i, \mathbb{W})$.

(2) The soft union of $(F_i, \mathbb{W})$, for $i \in I$, is a soft set $(F, \mathbb{W})$ such that $F(w) = \bigcup_{i \in I} F_i(w)$ for all $w \in \mathbb{W}$ and is denoted by $(F, \mathbb{W}) = \tilde{\bigcup}_{i \in I} (F_i, \mathbb{W})$.

Definition 2.8. *[41] Let P be a probability function defined on the collection of all possible events $\mathcal{A}$ from a sample space X. Then P is called distribution if $\sum_{x \in X} P(x) = 1$. The set of all elements with non-zero probability is called support of P.*

Definition 2.9. *[41] Let $(X, \mathcal{E}, P)$ be a probability space. A random variable R is a mapping defined on X, which to an outcome $x \in X$ associates the value $R(x)$ in a suitable set S of possible values of some quantity of interest. The function f has to be sufficiently well-behaved so that we can discuss the probabilities that the quantity will take on particular values. The idea of measurable function expresses the condition of well-behavedness of R. For $F \subseteq S$, the set of all the outcomes $x \in X$ such that $R(x) \in F$ for an event $R^{-1}(F) = \{x \in X : R(x) \in F\} \in \mathcal{E}$.*

Definition 2.10. *[34] Let X be a sample space or a state space. An event of the state space X is a collection of outcomes, which is mathematically a subset of X. An information structure is a collection of specific events. Those events represent only those that may be observed.*

Remark 2.2. *An information structure $\mathcal{E}$ in the language of this work is a soft σ-algebra on the sample space X (c.f., Note 1 in [34]). Each element of $\mathcal{E}$ contain certain information.*

3. Soft σ-algebras

Definition 3.1. *[35] A family $\Sigma\tilde{\subseteq}S_{\mathbb{W}}(X)$ is called a soft σ-algebra on X if*

(1) $\Phi_{\mathbb{W}}$ is in Σ,

(2) if $(F,\mathbb{W})$ is in Σ, then $(F,\mathbb{W})^c$ is in Σ, and,

(3) if $(F_n,\mathbb{W})$ is in Σ, for all $n\in\mathbb{N}$, then $\tilde{\bigcup}_{n\in\mathbb{N}}(F_n,\mathbb{W})$ is in Σ.

If (3) in the above definition holds true for finitely many members of Σ, then we call Σ a soft algebra on X.

Example 3.1. *The collections $\{\Phi_{\mathbb{W}}, X_{\mathbb{W}}\}$ and $S_{\mathbb{W}}(X)$ are trivially soft σ-algebras. They are respectively the smallest and the largest soft algebras.*

Example 3.2. *For any $(F,\mathbb{W})\tilde{\in}S_{\mathbb{W}}(X)$, $\{\Phi_{\mathbb{W}}, (F,\mathbb{W}), (F,\mathbb{W})^c, X_{\mathbb{W}}\}$ is a soft σ-algebra.*

Readily, each soft σ-algebra is a soft algebra but not the converse.

Example 3.3. *Let X be an infinite set and let $\mathbb{W}$ be a set of parameters. The collection*

$$\Sigma = \{(F,\mathbb{W})\tilde{\in}S_{\mathbb{W}}(X) : \textit{either } (F,\mathbb{W}) \textit{ or } (F,\mathbb{W})^c \textit{ is finite}\}$$

is a soft algebra but not a soft σ-algebra.

Lemma 3.1. *Let $\{\Sigma_i : i\in I\}$ be an indexed family of soft σ-algebras on X. Then $\tilde{\bigcap}_{i\in I}\Sigma_i$ is a soft σ-algebra.*

Proof. Since each soft σ-algebra Σ_i contains $\Phi_{\mathbb{W}}$, so $\tilde{\bigcap}_{i\in I}\Sigma_i$ is not null and it contains $\Phi_{\mathbb{W}}$. Let $(F,\mathbb{W})\tilde{\in}\tilde{\bigcap}_{i\in I}\Sigma_i$. Then $(F,\mathbb{W})\tilde{\in}\Sigma_i$ for each $i\in I$ and therefore $(F,\mathbb{W})^c\tilde{\in}\Sigma_i$ for each $i\in I$. Hence, $(F,\mathbb{W})^c\tilde{\in}\tilde{\bigcap}_{i\in I}\Sigma_i$. For the same reason, $\tilde{\bigcap}_{i\in I}\Sigma_i$ is closed under the countable soft union. □

On the other hand, the union of two soft σ-algebras need not be a soft σ-algebra.

Example 3.4. *Let $X=\{1,2,3\}$ and let $\mathbb{W}=\{w_1,w_2\}$. Given the following two soft σ-algebras: $\Sigma_1 = \{\Phi_{\mathbb{W}}, (F_1,\mathbb{W}), (F_2,\mathbb{W}), X_{\mathbb{W}}\}$ and $\Sigma_2=\{\Phi_{\mathbb{W}}, (F_3,\mathbb{W}), (F_4,\mathbb{W}), X_{\mathbb{W}}\}$, where $(F_1,\mathbb{W})=\{(w_1,\{1\}), (w_2,\emptyset)\}$, $(F_1,\mathbb{W})=\{(w_1,\{2,3\}), (w_2,X)\}$, $(F_3,\mathbb{W})=\{(w_1,\emptyset), (w_2,\{2,3\})\}$, and $(F_4,\mathbb{W})=\{(w_1,X), (w_2,\{1\})\}$. Then $\Sigma_1\tilde{\cup}\Sigma_2$ is not a soft σ-algebra.*

However, the next result demonstrates that the union of soft σ-algebras is a σ-soft algebra under certain conditions.

Definition 3.2. *A subfamily $\mathcal{F}$ of $S_{\mathbb{W}}(X)$ is called a partition of $X_{\mathbb{W}}$ (or soft partition of X) if*

(1) $\Phi_{\mathbb{W}} \tilde{\notin} \mathcal{F}$,

(2) $\tilde{\bigcup}_{(F,\mathbb{W})\tilde{\in}\mathcal{F}}(F,\mathbb{W}) = X_{\mathbb{W}}$,

(3) for all $(F,\mathbb{W}),(G,\mathbb{W})\tilde{\in}\mathcal{F}$ *with* $(F,\mathbb{W}) \neq (G,\mathbb{W})$ *implies* $(F,\mathbb{W})\tilde{\cap}(G,\mathbb{W}) = \Phi_{\mathbb{W}}$.

Theorem 3.1. *Let* $\{\Sigma_i : i \in I\}$ *be an indexed family of soft* σ*-algebras on* $(X_i,\mathbb{W})$ *and let* $\{(X_i,\mathbb{W}) : i \in I\}$ *be a partition of* $X_{\mathbb{W}}$. *Then* $\Sigma = \{\tilde{\bigcup}_{i\in I}(F_i,\mathbb{W}) : (F_i,\mathbb{W})\tilde{\in}\Sigma_i\}$ *is a soft* σ*-algebra on* $X_{\mathbb{W}}$.

Proof. Clearly $\Phi_{\mathbb{W}}\tilde{\in}\Sigma_i$ for each i. Then $\tilde{\bigcup}_{i\in I}\Phi_{\mathbb{W}} = \Phi_{\mathbb{W}}\tilde{\in}\Sigma$. If $(F,\mathbb{W})\tilde{\in}\Sigma$, then $(F,\mathbb{W}) = \tilde{\bigcup}_{i\in I}(F_i,\mathbb{W})$, where $(F_i,\mathbb{W})\tilde{\in}\Sigma_i$. Therefore, $(F,\mathbb{W})^c = X_{\mathbb{W}} - \tilde{\bigcup}_{i\in I}(F_i,\mathbb{W}) = \tilde{\bigcup}_{i\in I}[(X_i,\mathbb{W}) - (F_i,\mathbb{W})]$. Since each Σ_i is a soft σ-algebra on $(X_i,\mathbb{W})$, so $(X_i,\mathbb{W}) - (F_i,\mathbb{W})\tilde{\in}\Sigma_i$ for all $i \in I$. Consequently, $(F,\mathbb{W})^c\tilde{\in}\Sigma$. Let $(G_n,\mathbb{W})\tilde{\in}\Sigma$ for $n = 1,2,\ldots$. Then, for each n, $(G_n,\mathbb{W}) = \tilde{\bigcup}_{i\in I}(F_{n,i},\mathbb{W})$, where $(F_{n,i},\mathbb{W})\tilde{\in}\Sigma_i$. Therefore,

$$\tilde{\bigcup}_{n=1}^{\infty}(G_n,\mathbb{W}) = \tilde{\bigcup}_{n=1}^{\infty}\tilde{\bigcup}_{i\in I}(F_{n,i},\mathbb{W}) = \tilde{\bigcup}_{i\in I}\tilde{\bigcup}_{n=1}^{\infty}(F_{n,i},\mathbb{W}).$$

Since $\tilde{\bigcup}_{n=1}^{\infty}(F_{n,i},\mathbb{W})\tilde{\in}\Sigma_i$ which implies that $\tilde{\bigcup}_{n=1}^{\infty}(G_n,\mathbb{W})\tilde{\in}\Sigma$. Hence, Σ is a soft σ-algebra. □

Lemma 3.2. *Let* C *be a subcollection of* $S_{\mathbb{W}}(X)$. *Then there exists a unique* σ*-algebra* Σ *on* X *containing* C *in the sense that if* Σ^* *is any other algebra containing* C, *then* $\Sigma\tilde{\subseteq}\Sigma^*$.

Proof. Since $S_{\mathbb{W}}(X)$ is a soft σ-algebra containing C, so such a soft σ-algebra always exists. Therefore, the soft σ-algebra Σ obtained in Lemma 3.1 is the required soft σ-algebra. □

We refer to the above soft σ-algebra as the soft σ-algebra on X generated by C and denote it by $\sigma(C)$. If C is a countable collection, then $\sigma(C)$ is called the countably generated soft σ-algebra. The soft σ-algebra considered in Example 3.2 is the soft σ-algebra generated by $\{(F,\mathbb{W})\}$.

Theorem 3.2. *Let* $C, C_i\tilde{\subseteq}S_{\mathbb{W}}(X)$, *for* $i = 0,1,2$, *and let* Σ *be any soft* σ*-algebra on* X. *The soft* σ*-algebra* $\sigma(C)$ *generated by* C *has the following properties:*

(1) $C\tilde{\subseteq}\sigma(C)\tilde{\subseteq}\Sigma$.

(2) $\sigma(\sigma(C)) = \sigma(C)$.

(3) C *is a soft* σ*-algebra iff* $C = \sigma(C)$.

(4) $C_1\tilde{\subseteq}C_2$ *implies* $\sigma(C_1)\tilde{\subseteq}\sigma(C_2)$.

(5) $\sigma(C_1),\sigma(C_2)\tilde{\subseteq}\sigma(C_1)\tilde{\cup}\sigma(C_2)\tilde{\subseteq}\sigma(C_1\tilde{\cup}C_2) = \sigma(\sigma(C_1)\tilde{\cup}\sigma(C_2))$.

(6) $C\tilde{\subseteq}C_0\tilde{\subseteq}\sigma(C)$ *implies* $\sigma(C_0) = \sigma(C)$.

Proof. (1) It follows from the definition of $\sigma(C)$.

(2) The first direction, $\sigma(C)\tilde{\subseteq}\sigma(\sigma(C))$, follows from (1). Now, we always have $\sigma(C)\tilde{\subseteq}\sigma(C)$, so $\sigma(C)$ is a soft σ-algebra including $\sigma(C)$. Since $\sigma(\sigma(C))$ is the smallest soft σ-algebra including $\sigma(C)$. Thus, $\sigma(\sigma(C))\tilde{\subseteq}\sigma(C)$. Hence, $\sigma(\sigma(C)) = \sigma(C)$.

(3) Straightforward.

(4) By (1), $C_2\tilde{\subseteq}\sigma(C_2)$. Since $C_1\tilde{\subseteq}C_2$, then $C_1\tilde{\subseteq}C_2\tilde{\subseteq}\sigma(C_2)$ and so $\tilde{\subseteq}\sigma(C_2)$ is a soft σ-algebra containing C_1. But $\sigma(C_1)$ is the smallest soft σ-algebra containing C_1. Thus, $\sigma(C_1)\tilde{\subseteq}\sigma(C_2)$.

(5) We only prove the last equality, other inclusions can be concluded easily. Since $C_1\tilde{\subseteq}C_1\tilde{\cup}C_2$, by (4), $\sigma(C_1)\tilde{\subseteq}\sigma(C_1\tilde{\cup}C_2)$. Similarly, $\sigma(C_2)\tilde{\subseteq}\sigma(C_1\tilde{\cup}C_2)$. Therefore, $\sigma(C_1)\tilde{\cup}\sigma(C_1)\tilde{\subseteq}\sigma(C_1\tilde{\cup}C_2)$. By (4), $\sigma[\sigma(C_1)\tilde{\cup}\sigma(C_1)]\tilde{\subseteq}\sigma(C_1\tilde{\cup}C_2)$.

On the other hand, since $C_1\tilde{\subseteq}\sigma(C_1)$ and $C_2\tilde{\subseteq}\sigma(C_2)$, then $C_1\tilde{\cup}C_2\tilde{\subseteq}\sigma(C_1)\tilde{\cup}\sigma(C_2)$. By (4), $\sigma(C_1\tilde{\cup}C_2)\tilde{\subseteq}\sigma[\sigma(C_1)\tilde{\cup}\sigma(C_2)]$. Hence, $\sigma(C_1\tilde{\cup}C_2) = \sigma(\sigma(C_1)\tilde{\cup}\sigma(C_2))$.

(6) It follows from (2) and (4). □

Proposition 3.1. *Let* $(Y,\mathbb{W})\tilde{\in}S_{\mathbb{W}}(X) - \Phi_{\mathbb{W}}$ *and let* Σ *be a soft* σ*-algebra on* X*. Then*

$$\Sigma\tilde{\cap}(Y,\mathbb{W}) = \{(F,\mathbb{W})\tilde{\cap}(Y,\mathbb{W}) : (F,\mathbb{W})\tilde{\in}\Sigma\}$$

is a soft σ*-algebra on* $(Y,\mathbb{W})$ *and is denoted by* $\Sigma|_{(Y,\mathbb{W})}$ *or simply* $\Sigma|_Y$*.*

Proof. Since $\Phi_{\mathbb{W}}\tilde{\cap}(Y,\mathbb{W}) = \Phi_{\mathbb{W}}$ and $\Phi_{\mathbb{W}}\tilde{\in}\Sigma$, so $\Phi_{\mathbb{W}}\tilde{\in}\Sigma|_{(Y,\mathbb{W})}$. If $(G,\mathbb{W})\tilde{\in}\Sigma|_{(Y,\mathbb{W})}$, then $(G,\mathbb{W}) = (F,\mathbb{W})\tilde{\cap}(Y,\mathbb{W})$ for some $(F,\mathbb{W})\tilde{\in}\Sigma$. Since $(F,\mathbb{W})^c\tilde{\in}\Sigma$, we have $(G,\mathbb{W})^c = (F,\mathbb{W})^c\tilde{\cap}(Y,\mathbb{W})$. Thus, $(G,\mathbb{W})^c\tilde{\in}\Sigma|_{(Y,\mathbb{W})}$. Suppose that $(G_1,\mathbb{W}),(G_2,\mathbb{W}),\cdots\tilde{\in}\Sigma|_{(Y,\mathbb{W})}$. Then $(G_n,\mathbb{W}) = (F_n,\mathbb{W})\tilde{\cap}(Y,\mathbb{W})$ for some $(F_n,\mathbb{W})\tilde{\in}\Sigma$. Since $\tilde{\cup}_{n=1}^{\infty}(F_n,\mathbb{W})\tilde{\in}\Sigma$, so

$$\tilde{\bigcup}_{n=1}^{\infty}(G_n,\mathbb{W}) = \tilde{\bigcup}_{n=1}^{\infty}[(F_n,\mathbb{W})\tilde{\cap}(Y,\mathbb{W})] = [\tilde{\bigcup}_{n=1}^{\infty}(F_n,\mathbb{W})]\tilde{\cap}(Y,\mathbb{W}).$$

This implies that $\tilde{\cup}_{n=1}^{\infty}(G_n,\mathbb{W})\tilde{\in}\Sigma|_{(Y,\mathbb{W})}$. □

Theorem 3.3. *Let* C *be a subcollection of* $S_{\mathbb{W}}(X)$ *and let* $(Y,\mathbb{W})\tilde{\in}S_{\mathbb{W}}(X) - \Phi_{\mathbb{W}}$*. Then*

$$\sigma(C\tilde{\cap}(Y,\mathbb{W})) = \sigma(C)\tilde{\cap}(Y,\mathbb{W}),$$

where $C\tilde{\cap}(Y,\mathbb{W}) = \{(F,\mathbb{W})\tilde{\cap}(Y,\mathbb{W}) : (F,\mathbb{W})\tilde{\in}C\}$*.*

Proof. Let $(G,\mathbb{W})\tilde{\in}C\tilde{\cap}(Y,\mathbb{W})$. Then $(G,\mathbb{W}) = (F,\mathbb{W})\tilde{\cap}(Y,\mathbb{W})$ for some $(F,\mathbb{W})\tilde{\in}C\tilde{\subseteq}\sigma(C)$. Therefore, $(G,\mathbb{W})\tilde{\in}\sigma(C)\tilde{\cap}(Y,\mathbb{W})$ and, hence, $C\tilde{\cap}(Y,\mathbb{W})\tilde{\subseteq}\sigma(C)\tilde{\cap}(Y,\mathbb{W})$. Since, by Proposition 3.1, $\sigma(C)\ \tilde{\cap}(Y,\mathbb{W})$ is a soft σ-algebra on Y. Thus, Lemma 3.2 guarantees that $\sigma(C\tilde{\cap}(Y,\mathbb{W}))\tilde{\subseteq}\sigma(C)\tilde{\cap}(Y,\mathbb{W})$. To prove the other way of the inclusion, we first need to check that

$$\Sigma = \{(H,\mathbb{W})\tilde{\in}S_{\mathbb{W}}(X) : (H,\mathbb{W})\tilde{\cap}(Y,\mathbb{W})\tilde{\in}\sigma(C\tilde{\cap}(Y,\mathbb{W}))\}$$

is a soft σ-algebra. Evidently, $\Phi_{\mathbb{W}}\tilde{\in}\Sigma$ because $\Phi_{\mathbb{W}}\tilde{\cap}(Y,\mathbb{W}) = \Phi_{\mathbb{W}}\tilde{\in}\sigma(C\tilde{\cap}(Y,\mathbb{W}))$. If $(H,\mathbb{W})\tilde{\in}\Sigma$, then $(H,\mathbb{W})\tilde{\cap}(Y,\mathbb{W})\tilde{\in}\ \sigma(C\tilde{\cap}(Y,\mathbb{W}))$. Since $(H,\mathbb{W})^c\tilde{\cap}(Y,\mathbb{W}) = (Y,\mathbb{W}) - [(H,\mathbb{W})\tilde{\cap}(Y,\mathbb{W})]$, this means that $(H,\mathbb{W})^c\tilde{\in}\Sigma$. If $(H_1,\mathbb{W}),(H_2,\mathbb{W}),\cdots\tilde{\in}\Sigma$, then $(H_1,\mathbb{W})\tilde{\cap}(Y,\mathbb{W}),(H_2,\mathbb{W})\tilde{\cap}(Y,\mathbb{W}),\cdots\tilde{\in}\sigma(C\tilde{\cap}(Y,\mathbb{W}))$. Therefore,

$$[\tilde{\bigcup}_{n=1}^{\infty}(H_n,\mathbb{W})]\tilde{\cap}(Y,\mathbb{W}) = \tilde{\bigcup}_{n=1}^{\infty}[(H_n,\mathbb{W})\tilde{\cap}(Y,\mathbb{W})]\tilde{\in}\sigma(C\tilde{\cap}(Y,\mathbb{W})).$$

Hence, $\tilde{\cup}_{n=1}^{\infty}(H_n,\mathbb{W})\tilde{\in}\Sigma$. This proves that Σ is a soft σ-algebra on X. Suppose that $(F,\mathbb{W})\tilde{\in}C$. Then $(F,\mathbb{W})\tilde{\cap}(Y,\mathbb{W})\tilde{\in}C\tilde{\cap}(Y,\mathbb{W})\tilde{\subseteq}\sigma(C\tilde{\cap}(Y,\mathbb{W}))$ and, so, $(F,\mathbb{W})\tilde{\in}\Sigma$. Thus, $C\tilde{\subseteq}\Sigma$ and, by Lemma 3.2, $\sigma(C)\tilde{\subseteq}\Sigma$. Now, if $(G,\mathbb{W})\tilde{\in}\sigma(C)\tilde{\cap}(Y,\mathbb{W})$, then $(G,\mathbb{W}) = (F,\mathbb{W})\tilde{\cap}(Y,\mathbb{W})$ for some $(F,\mathbb{W})\tilde{\in}\sigma(C)\ \tilde{\subseteq}\Sigma$ and, hence, $(G,\mathbb{W})\tilde{\in}\sigma(C\tilde{\cap}(Y,\mathbb{W}))$. This concludes that $\sigma(C)\tilde{\cap}(Y,\mathbb{W})\tilde{\subseteq}\sigma(C\tilde{\cap}(Y,\mathbb{W}))$. Thus, $\sigma(C\tilde{\cap}(Y,\mathbb{W})) = \sigma(C)\tilde{\cap}(Y,\mathbb{W})$. □

Definition 3.3. *[42,43] Let X, Y be two different universes parameterized by $\mathbb{W}, \mathbb{W}'$, respectively, and let $p : X \to Y, q : \mathbb{W} \to \mathbb{W}'$ be mappings. The image of a soft set $(F, \mathbb{W})\widetilde{\subseteq}(X, \mathbb{W})$ under $f_{p,q}$ or simply $f : (X, \mathbb{W}) \to (Y, \mathbb{W}')$ is a soft subset $f(F, \mathbb{W}) = (f(F), q(\mathbb{W}))$ of $(Y, \mathbb{W}')$ which is given by*

$$f(F)(w') = \begin{cases} \bigcup\limits_{w \in q^{-1}(w') \cap \mathbb{W}} p\left(F(w)\right), & q^{-1}(w') \cap \mathbb{W} \neq \emptyset, \\ \emptyset, & \text{otherwise}, \end{cases}$$

for each $w' \in \mathbb{W}'$.

The inverse image of a soft set $(G, \mathbb{W}')\widetilde{\subseteq}(Y, \mathbb{W}')$ under f is a soft subset $f^{-1}(G, \mathbb{W}') = (f^{-1}(G), q^{-1}(\mathbb{W}'))$ such that

$$(f^{-1}(G)(w) = \begin{cases} p^{-1}\left(G(q(w))\right), & q(w) \in \mathbb{W}', \\ \emptyset, & \text{otherwise}, \end{cases}$$

for each $w \in \mathbb{W}$.

The soft mapping f is bijective if both p and q are bijective.

Lemma 3.3. *Let $f : (X, \mathbb{W}) \longrightarrow (Y, \mathbb{W}')$ be a soft mapping. Then*

(1) if Σ is a soft σ-algebra on X, then $\{(G, \mathbb{W}')\tilde{\subseteq}(Y, \mathbb{W}') : f^{-1}(G, \mathbb{W}')\tilde{\in}\Sigma\}$ is a soft σ-algebra on Y.

(2) if Σ' is a soft σ-algebra on Y, the set $f^{-1}(\Sigma')$ is a soft σ-algebra on X.

Proof. Both (1) and (2) follow from the fact that $f^{-1}(\Phi_{\mathbb{W}}) = \Phi_{\mathbb{W}}, f^{-1}(Y_{\mathbb{W}'} - (G, \mathbb{W}')) = X_{\mathbb{W}} - f^{-1}(G, \mathbb{W}')$ for each soft set $(G, \mathbb{W}')$ over Y and $f^{-1}(\bigcup_{n \geq 1}(G_n, \mathbb{W}')) = \bigcup_{n \geq 1} f^{-1}(G_n, \mathbb{W}')$ for each collection $\{(G_n, \mathbb{W}') : n \in \mathbb{N}\}$ of soft sets over Y (see Theorem 14 in [42]). □

We shall remark that the direct image of a soft σ-algebra under a soft mapping need not be a soft σ-algebra.

Example 3.5. *Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$, and $\mathbb{W} = \mathbb{W}' = \{w_1, w_2\}$. Define the mapping $f_{p,q}$ by*

$$p(x) = \begin{cases} y_1, \text{ if } x \neq x_3; \\ y_2, \text{ if } x = x_3, \end{cases}$$

and $q(w) = w$, for all $w \in \mathbb{W}$. Let $\Sigma = \{\Phi_{\mathbb{W}}, (F_1, \mathbb{W}), (F_2, \mathbb{W}), X_{\mathbb{W}}\}$, where $(F_1, \mathbb{W}) = \{(w_1, \{x_1\}), (w_2, \emptyset)\}$ and $(F_2, \mathbb{W}) = \{(w_1, \{x_2, x_3\}), (w_2, X)\}$. The image of the soft σ-algebra Σ is $f_{p,q}(\Sigma) = \{\Phi_{\mathbb{W}}, (G_1, \mathbb{W}), \tilde{Y}\}$, where $(G_1, \mathbb{W}) = \{(w_1, \{y_1\}), (w_2, \emptyset)\}$, which is not a soft σ-algebra on Y.

Theorem 3.4. *Let $f : (X, \mathbb{W}) \longrightarrow (Y, \mathbb{W}')$ be a soft mapping. Then*

$$f^{-1}(\sigma(C)) = \sigma(f^{-1}(C)),$$

for each collection C of soft sets over Y.

Proof. By Lemma 3.3 (2) $f^{-1}(\sigma(C))$ is a soft σ-algebra and $f^{-1}(C)\tilde{\subseteq}f^{-1}(\sigma(C))$, and this implies that $\sigma(f^{-1}(C))\tilde{\subseteq} f^{-1}(\sigma(C))$. But, by Lemma 3.3 (1), the set $\Sigma = \{(G, \mathbb{W}')\tilde{\subseteq}(Y, \mathbb{W}') : f^{-1}(G, \mathbb{W}')\tilde{\in}\sigma(f^{-1}(C))\}$ is a soft σ-algebra, and it contains C; hence $\sigma(C)\tilde{\subseteq}\Sigma$. Therefore, $f^{-1}(\sigma(C))\tilde{\subseteq}f^{-1}(\Sigma) = \{(F, \mathbb{W})\tilde{\subseteq}(X, \mathbb{W}) : (F, \mathbb{W}) = f^{-1}(G, \mathbb{W}')$ for some $(G, \mathbb{W}')\tilde{\in}\Sigma\}\tilde{\subseteq}\sigma(f^{-1}(C))$. This shows that $f^{-1}(\sigma(C)) = \sigma(f^{-1}(C))$. □

4. Relationships between crisp and soft σ-algebras

Proposition 4.1. *Let Σ be a soft σ-algebra on X parameterized by $\mathbb{W}$. Then $\Sigma_w = \{F(w) : (F, \mathbb{W})\tilde{\in}\Sigma\}$ is a σ-algebra on X for each $w \in \mathbb{W}$.*

Proof. Since $\Phi_{\mathbb{W}}\tilde{\in}\Sigma$, then $\emptyset \in \Sigma_w$ for each $w \in \mathbb{W}$. Let $F(w) \in \Sigma_w$. Then $(F, \mathbb{W})\tilde{\in}\Sigma$ and, since, Σ is a soft σ-algebra, so $(F, \mathbb{W})^c\tilde{\in}\Sigma$. But, $(F, \mathbb{W})^c = \{(w, X - F(w)) : w \in \mathbb{W}\}$ and, so, $F^c(w) = X - F(w) \in \Sigma_w$. Let $\{F_n(w) : n \in \mathbb{N}\}$ be an indexed family of sets in Σ_w. Then, for each n, $(F_n, \mathbb{W})\tilde{\in}\Sigma$ and, thus, $\tilde{\cup}_{n\in\mathbb{N}}(F_n, \mathbb{W})\tilde{\in}\Sigma$. Hence, $\cup_{n\in\mathbb{N}}F_n(w) \in \Sigma_w$. □

We declare that the above result proved in Theorem 3 in [35], but we give a clearer proof. The converse of this lemma is not always true (see Example 5 in [35]), particularly when $|\mathbb{W}| > 1$. The scenario is different when $|\mathbb{W}| = 1$, where $|\mathbb{W}|$ is the cardinality of $\mathbb{W}$.

Theorem 4.1. *Let $\mathbb{W} = \{e\}$ and let $\mathcal{F}$ be a family of subsets of X. Then $\Sigma = \{(w, F(w)) : F(w) \in \mathcal{F}\}$ is a soft σ-algebra iff $\Sigma_w = \{F(w) : (w, F(w))\tilde{\in}\Sigma\}$ is a (crisp) σ-algebra on X.*

Proof. The first direction follows from Proposition 4.1.

Conversely, Suppose Σ_w is a σ-algebra. Clearly, $\emptyset \in \Sigma_w$ and, so, $(w, \emptyset) = \Phi_{\mathbb{W}}\tilde{\in}\Sigma$. Let $(F, \mathbb{W})\tilde{\in}\Sigma$. Then $(F, \mathbb{W}) = (w, F(w))$ and $F(w) \in \Sigma_w$. Since Σ_w is a σ-algebra, so $F^c(w) \in \Sigma_w$. Therefore, $(F, \mathbb{W})^c = (w, F^c(w))\tilde{\in}\Sigma$. Let $\{(F_n, \mathbb{W}) : n \in \mathbb{N}\}$ be a collection of soft sets in Σ. This implies that $(F_n, \mathbb{W}) = (w, F_n(w))$ for each $n \in \mathbb{N}$ and, thus, $\cup_{n\in\mathbb{N}}F_n(w) \in \Sigma_w$ as Σ_w is a σ-algebra on X. Therefore,

$$\tilde{\bigcup}_{n\in\mathbb{N}}(F_n, \mathbb{W}) = \tilde{\bigcup}_{n\in\mathbb{N}}(w, F_n(w)) = (w, \bigcup_{n\in\mathbb{N}} F_i(w))\tilde{\in}\Sigma.$$

This proves that Σ is a soft σ-algebra on X. □

Theorem 4.2. *Let $\mathcal{E}$ be a (crisp) σ-algebra on X and $\mathbb{W}$ be a set of parameters. Then*

(1) the family $\Sigma(\mathcal{E})$ of all soft sets $(F, \mathbb{W})$ forms a soft σ-algebra on X, where $(F, \mathbb{W}) = \{(w, F(w)) : F(w) \in \mathcal{E}$ for each $w \in \mathbb{W}\}$.

(2) the family $\widehat{\Sigma}(\mathcal{E})$ of all soft sets $(F, \mathbb{W})$ forms a soft σ-algebra on X, where $(F, \mathbb{W}) = \{(w, F(w)) : F(w) = F(w') \in \mathcal{E}$ for each $w, w' \in \mathbb{W}\}$.

Proof. The proof is quite comparable to the second part of the proof for the preceding result. □

Remark 4.1. *The above formula holds true for any family of subsets of a given universe. Namely, we generate a collection of soft sets from a crisp family of sets.*

The soft σ-algebra $\Sigma(\mathcal{E})$ is called a soft σ-algebra on X generated by $\mathcal{E}$ with respect to $\mathbb{W}$. And the soft σ-algebra $\widehat{\Sigma}(\mathcal{E})$ is called an extended soft σ-algebra on X via $\mathcal{E}$.

Observe that $\Sigma_w = \widehat{\Sigma}_w = \mathcal{E}$ for each $w \in \mathbb{W}$.

The following examples show how the process in Theorem 4.2 can be used:

Example 4.1. *Let $X = \{x_1, x_2, x_3, x_4\}$ and $\mathbb{W} = \{w_1, w_2\}$. Consider the following crisp σ-algebra on X: $\mathcal{E} = \{\emptyset, \{x_1, x_2\}, \{x_3, x_4\}, X\}$.*

Applying the first formula, we conclude the following soft σ-algebra on X:

$$\Sigma(\mathcal{E}) = \Big\{\Phi_{\mathbb{W}}, (F_1, \mathbb{W}), (F_2, \mathbb{W}), \cdots, (F_{14}, \mathbb{W}), X_{\mathbb{W}}\Big\},$$

where

$(F_1, \mathbb{W}) = \{(w_1, \emptyset), (w_2, \{x_1, x_2\})\}$,
$(F_2, \mathbb{W}) = \{(w_1, \emptyset), (w_2, \{x_3, x_4\})\}$,
$(F_3, \mathbb{W}) = \{(w_1, \emptyset), (w_2, X)\}$,
$(F_4, \mathbb{W}) = \{(w_1, \{x_1, x_2\}), (w_2, \emptyset)\}$,
$(F_5, \mathbb{W}) = \{(w_1, \{x_1, x_2\}), (w_2, \{x_1, x_2\})\}$,
$(F_6, \mathbb{W}) = \{(w_1, \{x_1, x_2\}), (w_2, \{x_3, x_4\})\}$,
$(F_7, \mathbb{W}) = \{(w_1, \{x_1, x_2\}), (w_2, X)\}$,
$(F_8, \mathbb{W}) = \{(w_1, \{x_3, x_4\}), (w_2, \emptyset)\}$,
$(F_9, \mathbb{W}) = \{(w_1, \{x_3, x_4\}), (w_2, \{x_1, x_2\})\}$,
$(F_{10}, \mathbb{W}) = \{(w_1, \{x_3, x_4\}), (w_2, \{x_3, x_4\})\}$,
$(F_{11}, \mathbb{W}) = \{(w_1, \{x_3, x_4\}), (w_2, X)\}$,
$(F_{12}, \mathbb{W}) = \{(w_1, X), (w_2, \emptyset)\}$,
$(F_{13}, \mathbb{W}) = \{(w_1, X), (w_2, \{x_1, x_2\})\}$, *and*
$(F_{14}, \mathbb{W}) = \{(w_1, X), (w_2, \{x_3, x_4\})\}$.

Applying the second formula, we conclude the following soft σ-algebra on X:

$$\widehat{\Sigma}(\mathcal{E}) = \Big\{\Phi_{\mathbb{W}}, (G_1, \mathbb{W}), (G_2, \mathbb{W}), X_{\mathbb{W}}\Big\},$$

where

$(G_1, \mathbb{W}) = \{(w_1, \{x_1, x_2\}), (w_2, \{x_1, x_2\})\}$ *and*
$(G_2, \mathbb{W}) = \{(w_1, \{x_3, x_4\}), (w_2, \{x_3, x_4\})\}$.

A less trivial example is

Example 4.2. *Let $\mathcal{E}_{cc}$ be the countable-cocountable σ-algebra on X, where $\mathcal{E}_{cc} = \{F \subseteq X : F \text{ is countable or } F^c \text{ is countable}\}$. If X is finite, $\mathcal{E}_{cc} = \mathcal{P}(X)$. The soft σ-algebra $\Sigma(\mathcal{E}_{cc})$ on X generated by $\mathcal{E}_{cc}$ is given by the following family:*

$$\Sigma(\mathcal{E}_{cc}) = \Big\{\{(w, F(w)) : w \in \mathbb{W}\} \tilde{\in} S_{\mathbb{W}}(X) : F(w) \text{ or } F(w)^c \text{ is countable, for each } w \in \mathbb{W}\Big\}.$$

When we need to define the countable-cocountable σ-algebra on X in soft settings, we normally do so as follows:

$$\Sigma_{cc} = \Big\{(F, \mathbb{W}) \tilde{\in} S_{\mathbb{W}}(X) : (F, \mathbb{W}) \text{ or } (F, \mathbb{W})^c \text{ is countable}\Big\}.$$

However, both $\Sigma_{cc} = \Sigma(\mathcal{E}_{cc})$ are equivalent. This example demonstrates how effectively our formulas produce soft σ-algebras. We shall call Σ_{cc} the countable-cocountable soft σ-algebra on X.

It is worth noting that the general formula for constructing soft σ-algebra provided by Theorem 4.2 can be improved by the use of several crisp σ-algebras.

Corollary 4.1. *Let $\mathscr{E} = \{\mathcal{E}_w : w \in \mathbb{W}\}$ be a collection of (crisp) σ-algebras on X indexed by a set of parameters $\mathbb{W}$. The family $\Sigma(\mathscr{E})$ of all soft sets $(F, \mathbb{W})$ forms a soft σ-algebra on X, where $(F, \mathbb{W}) = \{(w, F(w)) : F(w) \in \mathcal{E}_w$ for each $w \in \mathbb{W}\}$.*

Notice that if, for each $w, w' \in \mathbb{W}$, $\mathcal{E}_w = \mathcal{E}_{w'} = \mathcal{E}$, then $\Sigma(\mathscr{E}) = \Sigma(\mathcal{E})$.

We have seen that each soft σ-algebra produces a family of (crisp) σ-algebras of size $\leq |\mathbb{W}|$, so we obtain the following:

Lemma 4.1. *Let $\mathscr{E} = \{\mathcal{E}_w : w \in \mathbb{W}\}$ be the family of all crisp σ-algebras on X from a soft σ-algebra Σ. Then*

$$\Sigma \tilde{\subseteq} \Sigma(\mathscr{E}).$$

Proof. It can be concluded from the definition of soft sets and the soft σ-algebra generated by $\mathscr{E}$. □

Theorem 4.3. *Let $\{\mathcal{E}_w : w \in \mathbb{W}\}$ be a family of crisp σ-algebras on X indexed by $\mathbb{W}$. Then*

$$\Sigma(\tilde{\bigcap} \mathcal{E}_w) = \tilde{\bigcap} \Sigma(\mathcal{E}_w).$$

Proof. Let $(F, \mathbb{W}) \tilde{\in} \Sigma(\tilde{\bigcap} \mathcal{E}_w)$. Then

$$\begin{aligned}
&\Longrightarrow \quad F(w) \in \bigcap \mathcal{E}_w,\\
&\Longrightarrow \quad F(w) \in \mathcal{E}_w, \text{ for each } w \in \mathbb{W},\\
&\Longrightarrow \quad \{(w, F(w)) : w \in \mathbb{W}\} \in \Sigma(w_w), \text{ for each } w \in \mathbb{W},\\
&\Longrightarrow \quad (F, \mathbb{W}) \tilde{\in} \tilde{\bigcap} \Sigma(w_w).
\end{aligned}$$

Hence, $\Sigma(\tilde{\bigcap} \mathcal{E}_w) \tilde{\subseteq} \tilde{\bigcap} \Sigma(\mathcal{E}_w)$. The converse can be followed by reversing the above steps. □

Consequently, we have the following corollary:

Corollary 4.2. *Let $\mathcal{E}$ be a subcollection of $S_{\mathbb{W}}(X)$ such that $\Phi_{\mathbb{W}}, X_{\mathbb{W}} \tilde{\in} \mathcal{E}$. Then*

$$\Sigma(\sigma(\mathcal{E})) = \sigma(\Sigma(\mathcal{E})),$$

where $\Sigma(\mathcal{E})$ is the family of all soft sets $\{(w, F(w)) : w \in \mathbb{W}\}$ such that $F(w) \in \mathcal{E}$ for each $w \in \mathbb{W}$.

5. Applications

The concept of σ-algebras plays a crucial role not only in mathematics but real life problems in economics and finance. σ-algebra has an intuitive interpretation in probability theory. In this direction, the concept of soft σ-algebra is used here to characterize information that can be observed, implying that the soft σ-algebra contains information. This concept is demonstrated via basic examples.

Example 5.1. *Suppose that two players 1 and 2 are gambling on the outcomes of a coin tosses. Here, our set of parameters is $\mathbb{W} = \{\text{player 1}, \text{player 2}\} = \{w_1, w_2\}$ and the sample space for each player is $X_k = \{H, T\}$ for $k = 1, 2$. Assuming that they gamble on the following results obtained by the random variable R_k and the coin.*

$$R_k(x) = \begin{cases} 1, & x = H, \text{means the tossing is } H; \\ 0, & x = T, \text{means the tossing is } T. \end{cases}$$

From Formula (1) in Theorem 4.2 and Remark 4.1, the sample space for this game in soft settings will be $X_{\mathbb{W}} = \Sigma(\{X_1, X_2\}) = \Big\{\{(w_1, H), (w_2, H)\}, \{(w_1, H), (w_2, T)\}, \{(w_1, T), (w_2, H)\}, \{(w_1, T), (w_2, T)\}\Big\}$.

When discussing information in soft σ*-algebra, the notion of time is involved.*

At time zero, player 1 and player 2 are unaware of the outcomes other than one of the events in $X_{\mathbb{W}}$*. As a result, the information at time zero that they can both discuss is the soft* σ*-algebra generated by* $C_0 = \{X_{\mathbb{W}}\}$ *which is* $\Sigma_0 = \{\Phi_{\mathbb{W}}, X_{\mathbb{W}}\}$.

At time one, the coin had been tossed once by each player, only. They are aware of the events in the collection

$$C_1 = \{\{(w_1, HH), (w_2, HT)\}, \{(w_1, TT), (w_2, TH)\}\}.$$

Therefore, the information at this time that they both can discuss is the soft σ*-algebra* Σ_1 *generated* C_1*. Therefore,* $\Sigma_1 = \Big\{\Phi_{\mathbb{W}}, \{(w_1, HH), (w_2, HT)\}, \{(w_1, TT), (w_2, TH)\}, X_{\mathbb{W}}\Big\}$.

At time two, the coin had been tossed twice. They are aware of the events in the collection

$$C_2 = \{\{(w_1, H), (w_2, H)\}, \{(w_1, H), (w_2, T)\}, \{(w_1, T), (w_2, H)\}, \{(w_1, T), (w_2, T)\}\}$$

which means they know everything about the gambling results. Hence, the information at this stage that they both can discuss is the soft σ*-algebra* Σ_2 *generated by* C_2 *which is the family of all soft subsets of* $X_{\mathbb{W}}$*. That is,* $\Sigma_2 = \Big\{\Phi_{\mathbb{W}}, (F_1, \mathbb{W}), (F_2, \mathbb{W}), \ldots, (F_{14}, \mathbb{W}), X_{\mathbb{W}}\Big\}$*, where*

$(F_1, \mathbb{W}) = \{(w_1, H), (w_2, H)\}$,
$(F_2, \mathbb{W}) = \{(w_1, H), (w_2, T)\}$,
$(F_3, \mathbb{W}) = \{(w_1, T), (w_2, H)\}$,
$(F_4, \mathbb{W}) = \{(w_1, T), (w_2, T)\}$,
$(F_5, \mathbb{W}) = \{(w_1, HH), (w_2, TH)\}$,
$(F_6, \mathbb{W}) = \{(w_1, HT), (w_2, HH)\}$,
$(F_7, \mathbb{W}) = \{(w_1, HT), (w_2, TH)\}$,
$(F_8, \mathbb{W}) = \{(w_1, HT), (w_2, TH)\}$,
$(F_9, \mathbb{W}) = \{(w_1, TH), (w_2, TT)\}$,
$(F_{10}, \mathbb{W}) = \{(w_1, TT), (w_2, HT)\}$,
$(F_{11}, \mathbb{W}) = \{(w_1, HHT), (w_2, HTH)\}$,
$(F_{12}, \mathbb{W}) = \{(w_1, HHT), (w_2, HTT)\}$,
$(F_{13}, \mathbb{W}) = \{(w_1, HTH), (w_2, HHH)\}$*, and*
$(F_{14}, \mathbb{W}) = \{(w_1, HTT), (w_2, TTT)\}$.

At each time $t > 0$*, the generating soft* σ*-algebra gives us more information about the happening events; and evidently,* $\Sigma_0 \tilde{\subseteq} \Sigma_1 \tilde{\subseteq} \Sigma_2$.

Each event can be assigned to a probability by the following formula:

$$\tilde{P}(F, \mathbb{W}) = P_2(\psi(F, \mathbb{W})), \tag{5.1}$$

where ψ *is a mapping from* Σ_2 *into* $\mathcal{E} = \sigma(\{HH\}, \{HT\}, \{TH\}, \{TT\})$ *defined by*

$$\psi(\{(w, F(w)) : w \in \mathbb{W}\}) = \{\prod_{i=1}^{m} f_j(w_i) : j = 1, \ldots, N\} \tag{5.2}$$

such that P_2 *is the (natural) probability of tossing a coin twice,* $m = |\mathbb{W}|$*,* $N = |F(w)|$ *for any* $w \in \mathbb{W}$*, and* $f_j(w) \in F(w)$*. To apply the above formulas, we need to take some soft sets in* Σ_2*. Let us examine* $(F_2, \mathbb{W})$*,* $(F_{10}, \mathbb{W})$*, and* $(F_{11}, \mathbb{W})$*. Now,*

$$\begin{aligned}\tilde{P}(F_1, \mathbb{W}) &= \tilde{P}(\{(w_1, H), (w_2, T)\})\\ &= P_2(\psi(\{(w_1, H), (w_2, T)\}))\\ &= P_2(\{H, T\}) = 1/4,\end{aligned}$$

$$\begin{aligned}\tilde{P}(F_{10}, \mathbb{W}) &= \tilde{P}(\{(w_1, TT), (w_2, HT)\})\\ &= P_2(\psi(\{(w_1, TT), (w_2, HT)\}))\\ &= P_2(\{TH, TT\}) = 1/2, \text{ and}\end{aligned}$$

$$\begin{aligned}\tilde{P}(F_{11}, \mathbb{W}) &= \tilde{P}(\{(w_1, HHT), (w_2, HTH)\})\\ &= P_2(\psi(\{(w_1, HHT), (w_2, HTH)\}))\\ &= P_2(\{HH, HT, TH\}) = 3/4.\end{aligned}$$

Remark 5.1. *The construction given in the above example is true for any (finite) numbers n of players and the results are consistent with the natural probability of tossing a coin n times.*

Definition 5.1. *Let* $(X, \mathcal{E}, P)$ *be a probability space and let* $(F, \mathbb{W})$ *be an element in the soft* σ*-algebra* Σ *generated by* $\mathcal{E}$ *with respect to* $\mathbb{W}$*. A parameter* w_1 *is said to be more informative than* w_2 *if* $P(F(w_1)) > P(F(w_2))$*.*

In the earlier example, we have shown how explicitly the concept of soft σ-algebra involved in applications. In the next illustrations, we may implicitly mention this concept.

Example 5.2. *Suppose that our game is rolling two dices. The possible outcomes are* $X = \{(i, j) : i,\ j = 1, 2, \ldots, 6\}$*. Let* $\mathcal{E}$ *be the* σ*-algebra generated by singletons in* X *and let* $\Sigma(\mathcal{E})$ *be the soft* σ*-algebra generated by* $\mathcal{E}$ *with respect to an appropriate set of parameters* $A = \{w_1, w_2, \ldots, w_{12}\}$*. The probability* P_{2d} *of each* $F(w)$*,* $w \in S \cup M$ *is presented in Tables 1 and 2, where* $S = \{sum(i, j) : i, j = 1, 2, \ldots, 6\}$ *and* $M = \{max(i, j) : i, j = 1, 2, \ldots, 6\}$*.*

Table 1. The probability of $F(w)$, $w \in S$.

sum(i,j)	2	3	4	5	6	7	8	9	10	11	12
probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Table 2. The probability of $F(w)$, $w \in M$.

max(i,j)	1	2	3	4	5	6
probability	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Take $\mathbb{W} = \{w_1, w_2\}$*, where* $w_1 := max(i, j) = 2$ *and* $w_2 := sum(i, j) = 10$*. The probability* $\tilde{Q}$ *of the soft set* $(F, \mathbb{W}) = \{(w_1, \{(1, 2), (2, 2), (2, 1)\}), (w_2, \{(4, 6), (5, 5), (6, 4)\})\}$ *is* $1/6$*. From Example 3.4,* *one can verify that the soft set* $(F, \mathbb{W})$ *exists in* $\Sigma(\mathcal{E})$ *and each missing parameter in the preceding computation is assigned to the empty set, which has a probability of zero.*

According to term given in Definition 5.1, a parameter e that correspond to $\max = 6$ *is more informative as it has the highest probability. Thus, the player shall receive enough information on the game and gamble on this parameter.*

Example 5.3. *Let* $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ *be our universe, where* x_1 *= less than 9th grade,* x_2 *= 9th to 12th grade (no diploma),* x_3 *= high school graduate or equivalent,* x_4 *= some college (no degree),* x_5 *= associate's degree,* x_6 *= bachelor's degree, and* x_7 *= graduate or professional degree. Suppose the set of parameters* $A = \{w_1, w_2\}$*, where* w_1 *= male and* w_2 *= female. Let* $\mathcal{E}$ *be the* σ*-algebra generated by singletons in X and let* $\Sigma(\mathcal{E})$ *be the soft* σ*-algebra generated by* $\mathcal{E}$ *with respect to A. Suppose we choose a random US residents over the age of 25. Based on data from the* 2010 *American Community Survey, Table 3 shows the distribution of education levels attained by gender:*

Table 3. The probability of the soft sample space $X_{\mathbb{W}}$.

$X_{\mathbb{W}}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
w_1	0.07	0.10	0.30	0.22	0.06	0.16	0.09
w_2	0.13	0.09	0.20	0.24	0.08	0.17	0.09

We now demonstrate how the elements of $\Sigma(\mathcal{E})$ *provide information about specific events. Assume that* $(F, A) = \{(w_1, \{x_1, x_3\}), (w_2, \{x_2\})\}$*,* $(G, A) = \{(w_1, \{x_4\}), (w_2, \{x_4\})\}$*, and* $(H, A) = \{(w_1, \emptyset), (w_2, \{x_6, x_7\})\}$*. Then* (F, A) *represents a man who has not accomplished a 9th grade certificate, a man who has graduated from high school, or a woman who has achieved a 9th to 12th grade certificate. The soft set represents a man who has not accomplished a 9th grade certificate, a man who has graduated from high school, or a woman who has achieved a 9th to 12th grade certificate. The soft set* (G, A) *represents a resident who graduated from a college with no degree. And* (H, A) *represents a man with at least a bachelor's degree. The probability of them are as follows:*

$$\begin{aligned}\tilde{P}(F, \mathbb{W}) &= \tilde{P}(\{(w_1, \{x_1\})\}) + \tilde{P}(\{(w_1, \{x_3\})\}) + \tilde{P}(\{(w_2, \{x_2\})\}) \\ &= 0.07 + 0.30 + 0.09 = 0.46.\end{aligned}$$

$$\begin{aligned}\tilde{P}(G, \mathbb{W}) &= (\tilde{P}(\{(w_1, \{x_4\})\}) + \tilde{P}(\{(w_2, \{x_4\})\}))/2 \\ &= (0.22 + 0.24)/2 = 0.23.\end{aligned}$$

$$\begin{aligned}\tilde{P}(H, \mathbb{W}) &= \tilde{P}(\{(w_1, \emptyset)\}) + \tilde{P}(\{(w_2, \{x_6, x_7\})\}) \\ &= \tilde{P}(\{(w_1, \emptyset)\}) + \tilde{P}(\{(w_2, \{x_7\})\}) + \tilde{P}(\{(w_2, \{x_7\})\}) \\ &= 0 + 0.17 + 0.09 = 0.26.\end{aligned}$$

We finish this part by stating that the soft σ-algebras mentioned in Examples 5.1, 5.2, and 5.3 represent information structures on finite sets of states according to Remark 2.2.

6. Conclusions and future work

Molodtsov presented several potential applications of soft set theory, and many researchers followed his direction and used soft sets in various fields. This paper contributes to the field of soft set theory by investigating the concept of soft σ-algebras. We have discussed some operations on soft σ-algebras. We have seen that a soft σ-algebra can produce a family of crisp σ-algebras. Then, we have established two remarkable formulas by which one can construct a soft σ-algebra from a family of crisp σ-algebras. This construction provides a general framework for studying soft σ-algebras and investigating their properties in relation to their counterparts in crisp σ-algebras. In particular, one can study probability theory, measure theory, etc. in the context of soft set theory without defining all the related terminologies. As applications to this construction, we have offered two examples and shown that one can compute the probability of a soft set in the soft σ algebra generated by a crisp σ-algebra with respect to the given set of parameters. The stated examples represent the information structures of two players. We draw the readers' attention to the obtained results are consistent with crisp σ-algebras when the set of parameters contains a single point.

The results obtained in this paper are preliminary and many extensions to this work are needed. A deep study is needed to determine the accurate relationship between soft σ-algebra and information. The reason is that a decision-maker may strictly prefer the soft σ-algebra generated by the full information over the soft σ-algebra generated by the incomplete information. The soft version of Doob-Dynkin lemma will help us determining which σ-algebra represents information. This could be the best suggested directions for future research.

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Conflict of interest

The authors declare no conflict of interest.

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