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Research article

Controllability of fractional differential evolution equation of order $\gamma \in (1, 2)$ with nonlocal conditions

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Abstract: This paper investigates the existence of positive mild solutions and controllability for fractional differential evolution equations of order $\gamma \in (1, 2)$ with nonlocal conditions in Banach spaces. Our approach is based on Schauder's fixed point theorem, Krasnoselskii's fixed point theorem, and the Arzelà-Ascoli theorem. Finally, we include an example to verify our theoretical results.

Keywords: controllability; fractional differential evolution equations; positive mild solution; fixed points
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1. Introduction

Fractional calculus is a rapidly growing area with numerous applications in various fields, ranging from engineering and technology to finance and natural phenomena. Many physical processes in real-world events exhibit fractional-order behavior, such as biological systems, earthquake vibrations, electrical circuits, viscoelasticity, natural phenomena, and heat flow in materials, among others. Fractional calculus is a useful tool for studying these phenomena, and as a result, many researchers have investigated various fractional differential equations. This topic continues to be a popular area for research. For further study on fractional calculus, one can refer to the references [1, 2] and monographs by Kilbas et al. [3], Zhou et al. [4], Oldham and Spanier [5] and Podlubny [6].

In recent years, many researchers have investigated the existence of solutions to various classes of fractional differential equations, obtaining excellent results in this field through the use of fixed point theorems, evolution families, and semigroup techniques. In particular, the existence of mild solutions for various fractional differential equations in Banach and Hilbert spaces has received significant attention. Byszewski [7] studied the existence of solutions with nonlocal conditions for evolution equations in Banach spaces, and subsequently, due to their many applications, various researchers have obtained results using nonlocal conditions. For example, Mophou and N'Guerekata [8] investigated the existence of mild solutions for some fractional differential equations, Shu and Wang [9] studied the existence of mild solutions for fractional differential equations of order $1 < \gamma < 2$, Wang and Shu [10] discussed the existence of positive mild solutions for fractional differential evolution equations of order $1 < \alpha < 2$, and Balachandran and Sakthivel [11] proved the existence of solutions of the neutral functional integrodifferential equation in Banach spaces.

Controllability is a fundamental concept in control theory, which refers to the ability to steer a system from any initial state to any desired final state using an appropriate control input. The study of controllability began with the work of Kalman in the 1960s, who developed the necessary and sufficient conditions for controllability of finite-dimensional linear systems. Since then, many researchers have worked on controllability problems in both finite-dimensional and infinite-dimensional systems, developing various techniques and tools for analysis and control design. In the last few decades, many researchers have worked on the controllability of different fractional differential equations with nonlocal conditions. In this direction Arthi et al. [12] studied the controllability of damped second-order neutral integrodifferential systems with nonlocal conditions, Mohan Raja et al. [13] studied results on the existence and controllability of fractional integrodifferential system of order $1 < r < 2$,. Wang and Zhou [14] investigated the existence and controllability of semilinear fractional differential inclusions while Liu and Zeng [15] studied the existence and controllability of fractional evolution inclusions of Clarke's subdifferential type. Ji et al. [16] established controllability results for impulsive differential systems with nonlocal conditions, Madmudov et al. [17] verified the approximate controllability of fractional integrodifferential equations involving nonlocal initial conditions. In addition, Gorniewics et al. [18] worked on the existence and controllability of functional differential inclusions with nonlocal conditions while Guo et al. [19] studied the optimal control problem of random impulsive differential equations. Most recently, Chendrayan et al. [20] established results on controllability of Sobolev-type fractional stochastic hemivariational inequalitities of order $r \in (1, 2)$, Mohan Raja et al. [21] proved some beautiful results related to the controllability of fractional integrodifferential inclusions of order $r \in (1, 2)$, Chendrayan et al. [22] worked on the controllability of fractional stochastic Volterra-Fredholm integro-differential systems of order $1 < r < 2$, Arora et al. [23] discussed the controllability with monotonic nonlinearity of nonlocal fractional semilinear equations of order $1 < r < 2$, Mohan Raja et al. [24] established results on the approximate controllability for fractional integrodifferential systems of order $1 < r < 2$, Ma et al. [25] proved new results related to the controllability of Sobolev-type fractional differential equations of order $1 < r < 2$ with finite delay and Shu et al. [26] discussed the existence of mild solutions and approximate controllability for Riemann-Liouville fractional stochastic evolution equations of order $1 < \alpha < 2$ with nonlocal conditions. Here, we reference some work recently published on the aforesaid area, where authors used the tools of fixed point theory and applied analysis like [27, 28].

Motivated by the work as mentioned in the above discussion, by using semigroup theory, mild solutions, sectorial operator, control theory, fractional calculus, Arzelà-Ascoli theorem, Krasnoselskii's and Schauder's fixed point theorems, in this manuscript, we study the existence and controllability of fractional differential evolution equation of order $1 < \gamma < 2$ with nonlocal conditions in Banach space

Y of the form

$$\begin{cases} {}^{c}D^{\gamma} r(\xi) = \mathcal{A}r(\xi) + \mathcal{B}v(\xi) + g(\xi, r(\xi)) + \int_0^{\xi} p(\xi-\zeta) f(\zeta, r(\zeta)) d\xi, \xi \in J = [0, K], \\ r(0) + n(r) = r_0 \in Y, \ r'(0) + m(r) = r_1 \ \in Y, \end{cases} \tag{1.1}$$

where ${}^{c}D^{\gamma}$ is the Caputo fractional derivative of order $1 < \gamma < 2$, $\mathcal{A} : D(\mathcal{A}) \subset Y \to Y$ is a sectorial operator of type $(W, \vartheta, \gamma, \delta)$, $\mathcal{B} : Y \to Y$ is a bounded linear operator, $v(\cdot) \in L^2(J, Y)$ is control function. The nonlinear functions $g, f : J \times Y \to P$ are continuous functions, where P is a positive cone, $p : [0, K] \to R^{+}$ is an integrable function, and $p = \max_{\xi \in [o,K]} \int_0^{\xi} |p(\xi - \zeta)| d\zeta$, $n, m : Y \to Y$ are continuous functions.

The organization of this manuscript is as follows: Section 2 includes some definitions, lemmas, remarks, and theorems that will be used to prove our main results. Section 3 presents the existence and controllability results of a fractional differential evolution equation with nonlocal conditions. In the last section, Section 4, an example is included to illustrate the applicability of our main results.

2. Preliminaries

Now, we present some definitions, remarks, and theorems which will be used in the proof of the main results. Let $(Y, \|\cdot\|)$ be an ordered Banach space and P be a cone in which a partial ordering in Y is defined by $h \le k$ if and only if $k - h \in P$. If there exist a positive constant N such that $\vartheta \le h \le k$ implies $\|h\| \le N\|k\|$ then P is said to be a normal, where ϑ represents the zero element of Y, and the smallest N is called the normal constant of P. If the interior of P is nonempty then P is called solid cone. whenever $h - k \in P$ and P is solid cone then we write $h \ll k$. For further study on cone theory, see [29].

Throughout this manuscript, we assume that P is a positive cone of ordered Banach space Y, then $T = \{h \in C(J, Y) : h(\xi) \ge \vartheta$, for all $\xi \in J\}$ is also the positive cone of $C(J, Y)$.

Definition 2.1. *The fractional integral of order γ of a function g is denoted by $I^{\gamma} g$ and defined as*

$$I_0^{\gamma} g(\xi) = \frac{1}{\Gamma(\gamma)} \int_0^{\xi} (\xi - \zeta)^{\gamma - 1} g(\zeta) d\zeta, \ \xi > 0, \ \gamma > 0.$$

Definition 2.2. *The Caputo derivative of order γ of a function g is denoted by ${}^{c}D^{\gamma} g$ and defined as*

$${}^{c}D_0^{\gamma} g(\xi) = \frac{1}{\Gamma(m - \gamma)} \int_0^{\xi} (\xi - \zeta)^{m - \gamma - 1} g^{m}(\zeta) d\zeta, \ \xi > 0, \ m - 1 \le \gamma \le m.$$

Definition 2.3. *[6] The Mittag-Leffler function is denoted by $E_{\alpha,\beta}(s)$ and defined as*

$$E_{\alpha,\beta}(s) = \sum_{h=0}^{\infty} \frac{s^h}{\Gamma(\alpha h + \beta)} = \frac{1}{2\pi i} \int_{\mathcal{H}_\alpha} e^{\nu} \frac{\nu^{\alpha - \beta}}{\nu^{\alpha} - s} d\nu, \ \ \alpha, \beta > 0, s \in C,$$

here $\mathcal{H}_\alpha$ denotes Hankle path, a contour starting and ending at $-\infty$, and encircles the disc $|\nu| \le |s|^{\frac{1}{\alpha}}$ counterclockwise.

Now, we present well-known results which will be used in the proof of main results.

Theorem 2.4. *(Schauder's fixed point theorem) Let Y be a Banach space and X be a nonempty, bounded, convex and closed subset of Y, and let $K : X \to X$ is a compact operator. Then the operator K has at least one fixed point in X.*

Theorem 2.5. *(Krasnoselskii's fixed point theorem) Let Y be a Banach space and $W \neq \emptyset$ be a closed, convex subset of Y. And let $\mathcal{A}$ and $\mathcal{B}$ be operators satisfies the following three conditions*

(1) $\mathcal{A}h + \mathcal{B}k \in W, forh, k \in W$;
(2) $\mathcal{A}$ continuous and compact;
(3) $\mathcal{B}$ is a contraction mapping.

Then there exists $z \in W \ni z = \mathcal{A}z + \mathcal{B}z$

Theorem 2.6. *(Arzelà-Ascoli Theorem) Let $\mathcal{D}$ be compact and $\{g_m(\xi)\}_{m=1}^{\infty}$ be a sequence of continuous functions defined on $\mathcal{D}$. If $\mathcal{G} = \{g_m : m \in N\}$ is uniformly bounded and equicontinuous on $\mathcal{D}$. Then there exists a subsequence $\{g_{mk}\}_{k=1}^{\infty}$ that converges uniformly to a function $g \in C(\mathcal{D}, R)$.*

Theorem 2.7. *[30] Let $\mathcal{A}$ be a densely defined operator in Y satisfies*

(1) *For some $0 < \vartheta < \frac{\pi}{2}, \nu + \mathcal{T}_\vartheta = \{\nu + \eta : \eta \in C, |Arg(-\eta)| < \vartheta\}$.*

(2) *There exists a constant W, such that*

$$\|(\eta I - \mathcal{A})^{-1}\| \le \frac{W}{|\eta - \nu|}, \quad \notin \nu + \mathcal{T}_\vartheta.$$

Then (A) is an infinitesimal generator of a semigroup $\mathcal{S}(\xi)$ satisfies $\|\mathcal{S}(\xi)\| \le C$. Also, $\mathcal{S}(\xi) = \frac{1}{2\pi i}\int_C e^{\eta\xi} R(\eta, \mathcal{A}) d\eta$, for $\eta \in c$, $\eta \notin \nu + \mathcal{T}_\vartheta$, where C is a suitable path.

Definition 2.8. *[9] Let $\mathcal{A}$ be a closed linear operator from $D(Y) \subseteq Y$ into Y. $\mathcal{A}$ is said to be a sectorial operator of type $(W, \vartheta, \gamma, \nu)$ if there exist $W > 0, 0 < \vartheta \le \frac{\pi}{2}$ and $\nu \in R$ such that out side of the following sector $\nu + \mathcal{T}_\vartheta$, $\gamma-$resolvent of $\mathcal{A}$ exists,*

$$\nu + \mathcal{T}_\vartheta = \{\nu + \eta^\gamma : \gamma \in C, |Arg(-\eta^\gamma)| < \vartheta\}$$

and

$$\|(\eta I - \mathcal{A})^{-1}\| \le \frac{W}{|\eta - \nu|}, \quad \notin \nu + \mathcal{T}_\vartheta.$$

Remark 2.9. [9] If $\mathcal{A}$ is sectorial operator of type $(W, \vartheta, \gamma, \nu)$, then the operator $\mathcal{A}$ is the infinitesimal generator of a $\gamma-$resolvent family $\{\mathcal{S}_\gamma(\xi)\}_{\xi \ge 0}$ in a Banach space, where $\mathcal{S}_\gamma(\xi) = \frac{1}{2\pi i}\int_c e^{\eta\xi} R(\eta^\gamma, \mathcal{A}) d\eta$.

Definition 2.10. *Let $R(\xi)_{(\xi \ge 0)}$ be a $\gamma-$resolvent solution operator in Y. When $R(\xi)h \ge \vartheta$ for all $h \ge \vartheta$, $h \in Y$, and $\xi \ge 0$, then $R(\xi)_{\xi \ge 0}$ is called to be positive.*

Definition 2.11. *[31] The linear operator $\mathcal{A} : D(\mathcal{A}) \subseteq Y \to Y$ is said to be non-negative if $\mathcal{A}$ satisfies the following conditions:*

(1) For every value $\eta > 0$ and every $q \in D(\mathcal{A})$, we have

$$\eta\|q\| \le T\|\eta q + \mathcal{A}\eta\|.$$

(2) $R(\eta I + \mathcal{A}) = Y$ for all values of $\eta > 0$.

Definition 2.12. *[31] Let $\mathcal{A}$ be a linear operator and if for $K = 1$ the linear operator $\mathcal{A}$ satisfy condition (1) in Definition 2.11, then $\mathcal{A}$ is said to be accretive. And if $\mathcal{A}$ also satisfy condition (2) in Definition 2.11, then $\mathcal{A}$ is said to be $m-$accretive.*

Definition 2.13. *$\mathcal{A}$ is said to be a sectorial accretive operator of type $(W, \vartheta, \gamma, \nu)$ if and only if $\mathcal{A}$ is accretive and $\mathcal{A}$ is a sectorial operator of type $(W, \vartheta, \gamma, \nu)$.*

Remark 2.14. [31] Let $(\mathcal{Y}, \langle\cdot,\cdot\rangle)$ is a Hilbert space. Then $\mathcal{A}$ is to be accretive if and only if $Re(\mathcal{A}q, q) \geq 0$ for all $q \in D(\mathcal{A})$. In particular, if Y is a Hilbert space with field of real numbers and $\mathcal{A}$ is positive, then $(\mathcal{A}q, q) \geq 0$ for every $q \in D(\mathcal{A})$. And note that if an order Banach space is a real space, implying that if $\mathcal{A}$ is accretive and Y is an ordered Banach space, then $(\mathcal{A}q, q) \geq 0$ for all $q \in D(\mathcal{A})$.

Definition 2.15. *[9] A function $r \in C([0, K], Y)$ is said to be a mild solution of (1.1) if it satisfies*

$$\begin{aligned} r(\xi) &= \mathcal{T}_\gamma(\xi)[r_0 - n(r)] + Q_\gamma(\xi)[q_1 - m(q)] \\ &\quad + \int_o^\xi \mathcal{S}_\gamma(\xi - \zeta)[g(\zeta, r(\zeta)) + \int_0^\zeta p(\zeta - \lambda) f(\lambda, q(\lambda)) d\lambda] d\zeta. \end{aligned}$$

Lemma 2.16. *[10] If $\mathcal{A}$ is a sectorial operator of type $(W, \vartheta, \gamma, \nu)$, then we have*

$$\mathcal{T}_\gamma(\xi) = \tfrac{1}{2\pi i} \int_c e^{\eta\xi} \eta^{\gamma-1} R(\eta^\gamma, \mathcal{A}) d\eta = E_{\gamma,1}(\mathcal{A}\xi^\gamma) = \textstyle\sum_{h=0}^{\infty} \frac{(\mathcal{A}\xi^\gamma)^h}{\Gamma(1+\gamma h)},$$

$$\mathcal{S}_\gamma(\xi) = \tfrac{1}{2\pi i} \int_c e^{\eta\xi} R(\eta^\gamma, \mathcal{A}) d\eta = \xi^{\gamma-1} E_{\gamma,\gamma}(\mathcal{A}\xi^\gamma) = \xi^{\gamma-1} \textstyle\sum_{h=0}^{\infty} \frac{(\mathcal{A}\xi^\gamma)^h}{\Gamma(\gamma+\gamma h)}$$

and

$$Q_\gamma(\xi) = \tfrac{1}{2\pi i} \int_c e^{\eta\xi} \eta^{\gamma-2} R(\eta^\gamma, \mathcal{A}) d\eta = \xi E_{\gamma,2}(\mathcal{A}\xi^\gamma) = \xi \textstyle\sum_{h=0}^{\infty} \frac{(\mathcal{A}\xi^\gamma)^h}{\Gamma(2+\gamma h)}.$$

Remark 2.17. In the range $1 < \gamma < 2$, the Mittag-Leffler function $E_{\gamma,1}(h)$ is well known to have a finite number of real zeros. [32] deduces that the operator $\mathcal{T}_\gamma(\xi)$ is non-positive by Lemma 2.16.

Remark 2.18. It follows from Remark 2.14 and Definition 2.13 that if $\mathcal{A}$ is a sectorial operator of type $(W, \vartheta, \gamma, \nu)$ and Y is an ordered Banach space, then the $\gamma-$resolvent families $\{\mathcal{S}_\gamma(\xi)\}_{\xi\geq 0}$, $\{\mathcal{T}_\gamma(\xi)\}_{\xi\geq 0}$, and $\{Q_\gamma(\xi)\}_{\xi\geq 0}$, are positive.

Theorem 2.19. *[9] Let $\mathcal{A}$ be a sectorial operator of type $(W, \vartheta, \gamma, \nu)$. Then the following two estimates are hold for $\|\mathcal{T}_\gamma(\xi)\|$.*

(1) Let $\nu \geq 0$. Given $\varphi \in (0, \pi)$, we have

$$\begin{aligned} \|\mathcal{T}_\gamma(\xi)\| &\leq \frac{T_1(\vartheta,\varphi) W e^{[T_1(\vartheta,\varphi)(1+\nu\xi^\gamma)]}[(1 + \frac{\sin\varphi}{\sin(\varphi-\vartheta)^{\frac{1}{\gamma}}}) - 1]}{\pi \sin^{1+\frac{1}{\gamma}} \vartheta} (1 + \nu\xi^\gamma) \\ &\quad + \frac{\Gamma(\gamma) W}{\pi(1 + \nu\xi^\gamma)|\cos\frac{\pi-\varphi|^\gamma}{\gamma}} \sin\vartheta \sin\varphi, \end{aligned}$$

for $\xi > 0$, where $T_1(\vartheta, \varphi) = \max\{1, \frac{\sin\vartheta}{\sin(\vartheta-\varphi)}\}$

(2) Assume $\nu < 0$. Given $T_1(\vartheta, \pi)$, we have

$$\|\mathcal{T}_\gamma(\xi)\| \leq \Big(\frac{eW[(1+\sin\varphi)^{\frac{1}{\gamma}}-1]}{\pi|\cos\varphi|^{1+\frac{1}{\gamma}}} + \frac{\Gamma(\gamma)W}{\pi|\cos\varphi||\cos\frac{\pi-\varphi}{\gamma}|^{\gamma}}\Big)\frac{1}{1+|\nu|\xi^\gamma},$$

for $\xi > 0$.

Theorem 2.20. *[9] Let $\mathcal{A}$ be a sectorial operator of type $(W, \vartheta, \gamma, \nu)$. Then the following two estimates are hold for $\|\mathcal{S}_\gamma(\xi)\|$ and $\|\mathcal{Q}_\gamma(\xi)\|$.*

(1) Let $\nu \geq 0$. Given $\varphi \in (0, \pi)$, we have

$$\begin{aligned}\|\mathcal{S}_\gamma(\xi)\| &\leq \frac{We^{[T_1(\vartheta,\varphi)(1+\nu\xi^\gamma)]}[(1+\frac{\sin\varphi}{\sin(\varphi-\vartheta)^{\frac{1}{\gamma}}})-1]}{\pi\sin^{1+\frac{1}{\gamma}}\vartheta}(1+\nu\xi^\gamma)^{\frac{1}{\gamma}}\xi^{\gamma-1}\\ &\quad+\frac{W\xi^{\gamma-1}}{\pi(1+\nu\xi^\gamma)|\cos\frac{\pi-\varphi|^\gamma}{\gamma}}\sin\vartheta\sin\varphi,\\ \|\mathcal{Q}_\gamma(\xi)\| &\leq \frac{WT_1(\vartheta,\varphi)e^{[T_1(\vartheta,\varphi)(1+\nu\xi^\gamma)]}[(1+\frac{\sin\varphi}{\sin(\varphi-\vartheta)^{\frac{1}{\gamma}}})-1]}{\pi\sin^{\frac{\gamma+2}{\gamma}}\vartheta}(1+\nu\xi^\gamma)^{\frac{\gamma-1}{\gamma}}\xi^{\gamma-1}\\ &\quad+\frac{W\gamma\Gamma(\gamma)}{\pi(1+\nu\xi^\gamma)|\cos\frac{\pi-\varphi|^\gamma}{\gamma}}\sin\vartheta\sin\varphi,\end{aligned}$$

for $\xi > 0$, where $T_1(\vartheta, \varphi) = \max\{1, \frac{\sin\vartheta}{\sin(\vartheta-\varphi)}\}$

(2) Assume $\nu < 0$. Given $T_1(\vartheta, \pi)$, we have

$$\|\mathcal{S}_\gamma(\xi)\| \leq \Big(\frac{eW[(1+\sin\varphi)^{\frac{1}{\gamma}}-1]}{\pi|\cos\varphi|} + \frac{W}{\pi|\cos\varphi||\cos\frac{\pi-\varphi}{\gamma}|}\Big)\frac{\xi^{\gamma-1}}{1+|\nu|\xi^\gamma},$$

$$\|\mathcal{Q}_\gamma(\xi)\| \leq \Big(\frac{eW[(1+\sin\varphi)^{\frac{1}{\gamma}}-1]}{\pi|\cos\varphi|^{1+\frac{2}{\gamma}}} + \frac{\gamma\Gamma(\gamma)W}{\pi|\cos\varphi||\cos\frac{\pi-\varphi}{\gamma}|}\Big)\frac{1}{1+|\nu|\xi^\gamma},$$

for $\xi > 0$.

3. Mild solutions and controllability

From Theorems 2.19 and 2.20, it is obvious because of the estimation on $\mathcal{T}_\gamma(\xi)$, $\mathcal{Q}_\gamma(\xi)$ and $\mathcal{S}_\gamma(\xi)$, that $\mathcal{T}_\gamma(\xi)$, $\mathcal{Q}_\gamma(\xi)$ and $\mathcal{S}_\gamma(\xi)$ are bounded and $\mathcal{A}$ is a linear operator. Now, we make the following (H1)–(H4) assumptions:

H1. $\mathcal{A}$ is a sectorial operator of type $(W, \vartheta, \gamma, \nu)$ and generates compact $\gamma-$resolvent families $\{\mathcal{S}_\gamma(\xi)\}_{\xi\geq 0}$, $\{\mathcal{T}_\gamma(\xi)\}_{\xi\geq 0}$ and $\{\mathcal{Q}_\gamma(\xi)\}_{\xi\geq 0}$.

H2. There exists $\widetilde{W} > 0$, such that for any $\xi \in J$, we have $\sup_{\xi\in J}\|\mathcal{T}_\gamma(\xi)| \leq \widetilde{W}$, $\sup_{\xi\in J}\|\mathcal{Q}_\gamma(\xi)| \leq \widetilde{W}$ and $\sup_{\xi\in J}\|\mathcal{S}_\gamma(\xi)| \leq \widetilde{W}$.

H3. The functions $g, f : J \times Y \to P$ are jointly continuous and for each $h > 0$ there exist $\nu_h, \mu_h \in L([0, K], R^+)$ such that $\sup_{\|r\| \le h} \|g(\xi, r)\| \le \nu_h(\xi)$, $\sup_{\|r\| \le h} \|f(\xi, r)\| \le \mu_h(\xi)$

H4. $r_0 - n(r)$, $r_1 - m(r) \in C(Y, P)$ and there exist $a, b, c, d > 0$ such that $\|n(r)\| \le c\|r\| + d$, $\|m(r)\| \le a\|r\| + b$, for all $r \in Y$.

Theorem 3.1. *Let (H1)–(H4) hold and* $\widetilde{W} < 1$, *then problem (1.1) has at least one positive mild solution on J.*

Proof.

$$\begin{aligned} r(\xi) &= \mathcal{T}_\gamma(\xi)[r_0 - n(r)] + Q_\gamma(\xi)[r_1 - m(r)] \\ &\quad + \int_0^\xi \mathcal{S}_\gamma(\xi - \zeta)\Big[g(\zeta, r(\zeta)) + \int_0^\zeta p(\zeta - \lambda) f(\lambda, r(\lambda)) d\lambda\Big] d\zeta. \end{aligned}$$

Choose

$$h \ge \frac{\widetilde{W}(K\nu_h(\xi)) + Kp\mu_h(\xi)) + \|r_0\| + \|r_1\| + d + b}{1 - \widetilde{W}(c + a)}$$

and suppose $\Theta = \{r \in T : \|r\| \le h\}$. Define the operator $\Upsilon : \Theta \to C(J, Y)$ by

$$\begin{aligned} (\Upsilon r)(\xi) &= \mathcal{T}_\gamma(\xi)[r_0 - n(r)] + Q_\gamma(\xi)[r_1 - m(r)] \\ &\quad + \int_0^\xi \mathcal{S}_\gamma(\xi - \zeta)\Big[g(\zeta, r(\zeta)) + \int_0^\zeta p(\zeta - \lambda) f(\lambda, r(\lambda)) d\lambda\Big] d\zeta. \end{aligned}$$

Step 1. First we show that $\Upsilon\Theta \subseteq \Theta$. For any $r \in \Theta$, based on assumption (H3) and (H4), for $\xi \in J$, we have

$$r_0 - n(r) \ge \vartheta, r_1 - m(r) \ge \vartheta, \tag{3.1}$$

$$g(\xi, r(\xi)) \ge \vartheta, \int_0^\xi p(\xi - \zeta) f(\zeta, r(\zeta)) d\zeta \ge \vartheta. \tag{3.2}$$

Because of (H2), we know that the operator $\mathcal{A}$ is a sectorial accretive of type $(W, \vartheta, \gamma, \nu)$ and generates compact and positive γ-resolvent families $\{\mathcal{T}_\gamma(\xi)\}_{\xi \ge 0}$, $\{Q_\gamma(\xi)\}_{\xi \ge 0}$ and $\{\mathcal{S}_\gamma(\xi)\}_{\xi \ge 0}$. Then we have

$$\mathcal{T}_\gamma(\xi)[r_0 - n(r)] \ge \vartheta, Q_\gamma(\xi)[r_1 - n(r)] \ge \vartheta, \tag{3.3}$$

$$\int_0^\xi \mathcal{S}_\gamma(\xi - \zeta)\Big[g(\zeta, r(\zeta)) + \int_0^\zeta p(\zeta - \lambda) f(\lambda, r(\lambda)) d\lambda\Big] d\zeta \ge \vartheta. \tag{3.4}$$

Consequently, we obtain

$$(\Upsilon r) \ge \vartheta, for\, r \in \Theta. \tag{3.5}$$

Also we have

$$
\begin{aligned}
\|(\Upsilon r)(\xi)\| \leq & \|\mathcal{T}_{\gamma}(\xi)\| \cdot \|r_0 - n(r)\| + \|\mathcal{Q}_{\gamma}(\xi)\| \cdot \|r_1 - n(r)\| \\
& + \int_0^{\xi} \|\mathcal{S}_{\gamma}(\xi-\zeta)\| \left\| g(\zeta, r(\zeta)) + \int_0^{\zeta} p(\zeta-\lambda) f(\lambda, r(\lambda)) d\lambda \right\| d\zeta \\
\leq & \widetilde{W}\Big(\|r_0\| + \|n(r)\| + \|r_1\| + \|m(r)\| + K\|g(\zeta, r(\zeta))\| \\
& + K \int_0^{\zeta} |p(\zeta-\lambda)| \cdot \|f(\lambda, r(\lambda))\| d\zeta \Big) \\
\leq & \widetilde{W}(\|r_0\| + c\|r\| + d + \|r_1\| + a\|r\| + b \\
& + K\nu_h(\xi) + Kp\mu_h(\xi)) \\
\leq & h.
\end{aligned} \tag{3.6}
$$

From (3.6), we obtain $\Upsilon\Theta \subseteq \Theta$, for all $r \in \Theta$.

Step 2. Now, we prove the continuity of Υ.

Suppose $\{r_i\}$ be a sequence in Θ such that $\|r_i - r\| \to 0$. Noting that g, f, n, m are continuous, as $i \to \infty$ we have

$$n(r_i) \to n(r), m(r_i) \to m(r), \tag{3.7}$$

$$g(\xi, r_i(\xi)) \to g(\xi, r(\xi)), f(\xi, r_i(\xi)) \to f(\xi, r(\xi)). \tag{3.8}$$

For all $\xi \in J$, we get

$$
\begin{aligned}
\|(\Upsilon r_i)(\xi) - (\Upsilon r)(\xi)\| \leq & \|\mathcal{T}_{\gamma}(\xi)\| \cdot \|n(r_i) - n(r)\| + \|\mathcal{Q}_{\gamma}(\xi)\| \cdot \|m(r_i) - m(r)\| \\
& + \int_0^{\xi} \|\mathcal{S}_{\gamma}(\xi-\zeta)\| \cdot \Big[\|g(\zeta, r_i(\zeta)) - g(\zeta, r(\zeta))\| \\
& + \int_0^{\zeta} |p(\zeta-\lambda)| \cdot \|f(\lambda, r_i(\lambda)) - f(\lambda, r(\lambda))\| d\lambda \Big] d\zeta \\
\leq & \widetilde{W}\|n(r_i) - n(r)\| + \widetilde{W}\|m(r_i) - m(r)\| \\
& + \widetilde{W}K\|g(\zeta, r_i(\zeta)) - g(\zeta, r(\zeta))\| \\
& + \widetilde{W}Kp\|f(\zeta, r_i(\zeta)) - f(\zeta, r(\zeta))\|.
\end{aligned}
$$

Combining (3.7) and (3.8), we obtain $\lim_{m\to\infty}(\Upsilon r_i)(\xi) = (\Upsilon r)(\xi)$, that is, Υ is continuous.

Step 3. In this step, we show that $\{(\Upsilon r)(\xi) : r \in \Theta\}$ is uniformly bounded. We have

$$
\begin{aligned}
\|(\Upsilon r)(\xi)\| \leq & \|\mathcal{T}_{\gamma}(\xi)\| \cdot \|r_0 - n(r)\| + \|\mathcal{Q}_{\gamma}(\xi)\| \cdot \|r_1 - n(r)\| \\
& + \int_0^{\xi} \|\mathcal{S}_{\gamma}(\xi-\zeta)\| \Big[\|g(\zeta, r(\zeta))\| + \int_0^{\zeta} |p(\zeta-\lambda)| \cdot \|f(\lambda, r(\lambda))\| d\lambda \Big] d\zeta \\
\leq & \widetilde{W}\Big(\|r_0\| + \|n(r)\| + \|r_1\| + \|m(r)\| + K\|g(\zeta, r(\zeta))\| \\
& + K \int_0^{\zeta} |p(\zeta-\lambda)| \|f(\lambda, r(\lambda))\| d\zeta \Big)
\end{aligned}
$$

$$\begin{aligned}&\leq \widetilde{W}(\|r_0\| + c\|r\| + d + \|r_1\| + a\|r\| + b\\&\quad +K\nu_h(\xi) + Kp\mu_h(\xi))\\&\leq h\\&< \infty.\end{aligned}$$

So, $\{(\Upsilon r)(\xi) : r \in \Theta\}$ is uniformly bounded.

Step 4. Now, we prove that $\Upsilon(\Theta)$ is equicontinuous. The function $\{(\Upsilon r)(\xi) : r \in \Theta\}$ are equicontinuous at $\xi = 0$. For $0 < \xi_1 < \xi_2 \leq K$ and $r \in \Theta$, we have

$$\begin{aligned}\|(\Upsilon r)(\xi_2) - (\Upsilon r)(\xi_1)\| &\leq \|\mathcal{T}_\gamma(\xi_2) - \mathcal{T}_\gamma(\xi_1)\| \cdot \|r_0 - n(r)\|\\&\quad +\|\mathcal{Q}_\gamma(\xi_2) - \mathcal{Q}_\gamma(\xi_1)\| \cdot \|r_1 - m(r)\|\\&\quad + \int_0^{\xi_1} \|\mathcal{S}_\gamma(\xi_2 - \zeta) - \mathcal{S}_\gamma(\xi_1 - \zeta)\| \Big[\|g(\zeta, r(\zeta))\|\\&\quad + \int_0^{\zeta} |p(\lambda - \zeta)| \cdot \|f(\lambda, r(\lambda))\| d\lambda\Big] d\zeta\\&\quad + \int_{\xi_1}^{\xi_2} \|\mathcal{S}_\gamma(\xi_2 - \zeta)\| \Big[\|g(\zeta, r(\zeta))\|\\&\quad + \int_0^{\zeta} |p(\lambda - \zeta)| \cdot \|f(\lambda, r(\lambda))\| d\lambda\Big] d\zeta\\&\leq I_1 + I_2 + I_3,\end{aligned}$$

where

$$\begin{aligned}I_1 &= \|\mathcal{T}_\gamma(\xi_2) - \mathcal{T}_\gamma(\xi_1)\| \cdot \|r_0) - n(r)\| + \|\mathcal{Q}_\gamma(\xi_2) - \mathcal{Q}_\gamma(\xi_1)\| \cdot \|r_1 - m(r)\|,\\I_2 &= \int_0^{\xi_1} \|\mathcal{S}_\gamma(\xi_2 - \zeta) - \mathcal{S}_\gamma(\xi_1 - \zeta)\| \cdot \Big[\|g(\zeta, r(\zeta))\| + \int_0^{\zeta} |p(\lambda - \zeta)| \|f(\lambda, r(\lambda))\| d\lambda\Big] d\zeta,\\I_3 &= \int_{\xi_1}^{\xi_2} \|\mathcal{S}_\gamma(\xi_2 - \zeta)\| \Big[\|g(\zeta, r(\zeta))\| + \int_0^{\zeta} |p(\lambda - \zeta)| \|f(\lambda, r(\lambda))\| d\lambda\Big] d\zeta.\end{aligned}$$

The continuity of functions $\xi \to \|\mathcal{T}_\gamma(\xi)\|$, $\xi \to \|\mathcal{Q}_\gamma(\xi)\|$ for $\xi \in (0, K]$, gives us to get that $\lim_{\xi_1 \to \xi_2} I_1 = 0$. In fact, we have

$$I_2 = \int_0^{\xi_1} \|\mathcal{S}_\gamma(\xi_2 - \zeta) - \mathcal{S}_\gamma(\xi_1 - \zeta)\| \cdot [\nu_h(\xi) + Kp\mu_h(\xi)] d\zeta.$$

So, the continuity of functions $\xi \to \|\mathcal{S}_\gamma(\xi)\|$, $\xi \to \|\mathcal{S}_\gamma(\xi)\|$ for $\xi \in (0, K]$, gives us to get that $\lim_{\xi_1 \to \xi_2} I_2 = 0$. We have

$$\begin{aligned}I_3 &= \int_{\xi_1}^{\xi_2} \|\mathcal{S}_\gamma(\xi_2 - \zeta)\| (\nu_h(\xi) + p\mu_h(\xi)) d\zeta\\&\leq \widetilde{W}(\nu_h(\xi) + p\mu_h(\xi)) |\xi_2 - \xi_1|.\end{aligned}$$

Consequently, $\lim_{\xi_1 \to \xi_2} I_3 = 0$. Thus, for $\xi \in J$, $\{(\Upsilon r)(\xi) : r \in \Theta\}$ is a family of equicontinuous functions. As a consequence of steps 1–4 with the Arzelà-Ascoli theorem, we have proved that

$\{(\Upsilon r)(\xi) : r \in \Theta\}$ is relative compact, therefore Υ is compact. So, by using Theorem 2.4, Υ has at least one fixed point on J. As $(\Upsilon r)(\xi) \geq \vartheta$ when $r \in \Theta$, problem (1.1) has at least one positive mild solution on J.

□

Next, if (H4) does not satisfied then we prove an existence result, for that result we need the following two assumptions.

H5. $r_0 - n(r),\ r_1 - m(r) : Y \to P$ are bounded and continuous on Y.

H6. There exist $L_1, L_2 > 0$ such that for every $\xi \in J$, $r, u \in Y$ we have
$\|g(\xi, r) - g(\xi, u)\| \leq L_1\|r - u\|$, $\|f(\xi, r) - f(\xi, u)\| \leq L_2\|r - u\|$.

Theorem 3.2. *Let (H1)–(H3), (H5) and (H6) hold. If* $\widetilde{W}K(L_1 + pL_2) < 1$, *then (1.1) has at least one positive mild solution on J.*

Proof. Select

$$h \geq \widetilde{W}(\|r_0 - n(r)\| + \|r_1 - m(r)\| + K\nu_h(\xi) + Kp\mu_h(\xi)),$$

and consider $\Omega = \{r \in T : \|r\| \leq h\}$. Define operators $\mathcal{T}, \mathcal{S}$ on Ω by

$$(\mathcal{T}r)(\xi) = \mathcal{T}_\gamma(\xi)[r_0 - n(r)] + Q_\gamma(r_1 - m(r))],$$

$$(\mathcal{S}r)(\xi) = \int_0^\xi \mathcal{S}_\gamma(\xi - \zeta)\Big[g(\zeta, r(\zeta)) + \int_0^\zeta p(\zeta - \lambda)f(\lambda, r(\lambda))d\lambda\Big]d\zeta.$$

Step 1. First we show that when $r, u \in \Omega$, then $\mathcal{T}r + \mathcal{S}u \in \Omega$. Similar to (3.3) and (3.4), for $r, u \in \Omega$, we obtain

$$\mathcal{T}_\gamma(\xi)[r_0 - n(r)] \geq \vartheta, Q_\gamma(\xi)[r_1 - m(r)] \geq \vartheta,$$

$$\int_0^\xi \mathcal{S}_\gamma(\xi - \zeta)\Big[g(\zeta, r(\zeta)) + \int_0^\zeta p(\zeta - \lambda)f(\lambda, r(\lambda))d\lambda\Big]d\zeta \geq \vartheta.$$

Consequently, we obtain

$$(\mathcal{T}r)(\xi) + (\mathcal{S}u)(\xi) \geq \vartheta, for\, r, u \in \Omega. \tag{3.9}$$

Following from (H1), (H3) and (H5), we have

$$\begin{aligned}\|(\mathcal{T}r)(\xi) - (\mathcal{S}u)(\xi)\| \leq& \|\mathcal{T}_\gamma(\xi)\| \cdot \|r_0 - n(r)\| + \|Q_\gamma(\xi)\| \cdot \|r_1 - m(r)\| \\ &+ \int_0^\xi \|\mathcal{S}_\gamma(\xi - \zeta)\| \cdot \Big\|g(\zeta, r(\zeta)) \int_0^\zeta p(\zeta - \lambda)f(\lambda, r(\lambda))d\lambda\Big\| \\ \leq& \widetilde{W}(\|r_0 - n(r)\| + \|r_1 - m(r)\| + K\nu_h(\xi) + Kp\mu_h(\xi)) \\ \leq& h.\end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we have $\mathcal{T}r + \mathcal{S}u \in \Omega$, for $r, u \in \Omega$.

Step 2. Now, we show that $\mathcal{S}$ is contraction. For any $r, u \in \Omega$, we get

$$\begin{aligned}\|(\mathcal{S}r)(\xi) - (\mathcal{S}u)(\xi)\| &\leq \int_0^{\xi} \|\mathcal{S}_\gamma(\xi - \zeta)\| \cdot \|g(\zeta, r(\zeta)) - g(\zeta, r(\zeta))\| d\zeta \\ &\quad + \int_0^{\zeta} \|\mathcal{S}_\gamma(xi - \zeta)\| \cdot \Big(\int_0^{\zeta} |p(\zeta - \lambda)| \cdot \|g(\lambda, r(\lambda)) - g(\lambda, u(\lambda))\| \Big) d\zeta \\ &\leq \widetilde{W}\mathcal{S}L_1 \|r(\xi) - u(\xi)\| + \widetilde{W}\mathcal{S}L_2 p \|r(\xi) - u(\xi)\| \\ &\leq \widetilde{W}\mathcal{S}(L_1 + pL_2)\|r - u\|.\end{aligned}$$

As $\widetilde{W}\mathcal{S}(L_1 + pL_2) < 1$, operator $\mathcal{S}$ is a contraction.

Step 3. In this step, we show that $\mathcal{T}$ is continuous. Let $r_i, r \in \Omega$, $\|r_i(\xi) - r(\xi)\| \to 0$ as $i \to 0$. As we know n, m are continuous, we have

$$n(r_i) \to n(r), \quad m(r_i) \to m(r), \; as\, m \to \infty \tag{3.11}$$

Then

$$\begin{aligned}\|(\mathcal{T}r_i)(\xi) - (\mathcal{T}r)(\xi)\| &\leq \|\mathcal{T}_\gamma(\xi)\| \cdot \|n(r_i) - n(r)\| + \|\mathcal{Q}_\gamma(\xi)\| \cdot \|m(r_i) - m(r)\| \\ &\leq \widetilde{W}(\|n(r_i) - n(r)\| + \|m(r_i) - m(r)\|).\end{aligned}$$

From (3.11), we have $\lim_{m\to\infty}(\mathcal{T}r_i)(\xi) = (\mathcal{T}r)(\xi)$. That is, operator $\mathcal{T}$ is continuous.

Step 4. In this step, we show that $\mathcal{T}$ is uniformly bounded. For $r \in \Omega$, we have

$$\begin{aligned}\|(\mathcal{T}r)(\xi)\| &\leq \|\mathcal{T}_\gamma(\xi)\| \cdot \|r_0 - n(r)\| + \|\mathcal{Q}_\gamma(\xi)\| \cdot \|r_1 - m(r)\| \\ &\leq \widetilde{W}(\|r_0 - n(r)\| + \|r_1 - m(r)\|) \\ &< \infty.\end{aligned}$$

This completes that $\mathcal{T}$ is uniformly bounded.

Step 5. In this step, we show that $\mathcal{T}(\Omega)$ is equicontinuous. Evidently, $(\mathcal{T}r)(\xi)$ is equicontinuous at $\xi = 0$. For $0 < \xi_1 < \xi_2 \leq K$, $r \in \Omega$, we have

$$\begin{aligned}\|(\mathcal{T}r)(\xi_2) - (\mathcal{T}r)(\xi_1)\| &\leq \|\mathcal{T}_\gamma(\xi_2) - \mathcal{T}_\gamma(\xi_1)\| \cdot \|r_0 - n(r)\| \\ &\quad + \|\mathcal{Q}_\gamma(\xi_2) - \mathcal{Q}_\gamma(\xi_1)\| \cdot \|r_1 - m(r)\|.\end{aligned}$$

In view of (H5), $\|r_0 - n(r)\|$, $\|r_1 - m(r)\|$ are bounded, so the continuity of function $\xi \to \|\mathcal{T}_\gamma(\xi)\|$, $\xi \to \|\mathcal{Q}_\gamma(\xi)\|$ for $\xi \in (0, K]$, enable us to obtain that

$$\lim_{\xi_1 \to \xi_2} (\mathcal{T}r)(\xi_1) = (\mathcal{S}r)(\xi_2).$$

This completes that $\mathcal{T}(\Omega)$ is equicontinuous. As a consequence of the above steps with Arzelà-Ascoli theorem, $\mathcal{T}$ is compact and also all the conditions of Theorem 2.5 are satisfied, so we get that (1.1) has at least one mild solution on J. Given that $\mathcal{T}r + \mathcal{S}r \geq \vartheta$ for $r \in \Omega$, we get that (1.1) has at least one positive mild solution on J.

□

Here we list definition of controllability and some reasonable hypotheses which will help us in the proof of controllability result.

Definition 3.3. *System (1.1) is said to be controllable on* $[0, d]$, *if, for any* $r_0, s_0 \in \overline{D(\mathcal{A})}$ *there exists a control* $v \in L^2(J, Y)$ *such that a mild solution* r *of (1.1) satisfy* $r(d) + n(r) = s_0$.

H7. $n, m : Y \to \overline{D(\mathcal{A})}$ are continuous functions. For any $v_1, v_2 \in Y$, there exist $l_1, l_2 \geq 0$, such that $\|n(v_1) - n(v_2)\| \leq l_1\|v_1 - v_2\|$ and $\|m(v_1) - m(v_2)\| \leq l_2\|v_1 - v_2\|$.

H8. The linear operator $W : L^2(J, Y) \to Y$ defined by $Wv = \int_0^d \mathcal{S}_\gamma(d-\zeta)\mathcal{B}v(\zeta)d\zeta$ induces an invertible operator W^- defined on $L^2(J, Y)/KerW$, and there exists $\mathcal{M} > 0$, such that $\|\mathcal{B}W^-\| \leq \mathcal{M}$.

H9. For any $h > 0$, there exists a function $v_h \in L(J, R^+)$ such that $\sup_{\|r\|\leq h} \|g(\xi, r(\xi))\| \leq v_h(\xi)$.

Theorem 3.4. *Let (H7)–(H9) hold and suppose that*

$$\widetilde{W}(l_1 + l_2) < 1, \tag{3.12}$$

$$\|g(\zeta, r(\zeta)) + \int_0^\zeta p(\zeta - \lambda) f(\lambda, r(\lambda))d\lambda\| \leq v_z(\zeta), \tag{3.13}$$

and

$$\begin{aligned}\widetilde{W} \quad &\Big(\|r_0\| + \|r_1\| + l_1 z + \|n(0)\| + l_2 z + \|m(0)\|\Big) \\ &+ d\widetilde{W}\mathcal{M}\Big(\|s_0\| + l_1 z + \|n(0)\| + \widetilde{W}\|u_0\| + \widetilde{W}l_1 z + \widetilde{W}\|n(0)\| + \widetilde{W}\|u_1\| \\ &+ \widetilde{W}l_2 z + \widetilde{W}\|m(0)\| + \widetilde{W}\int_0^d v_z(\zeta)d\zeta\Big) < z,\end{aligned} \tag{3.14}$$

for some $z > 0$. *Then system (1.1) is controllable on J.*

Proof. Set $\mathcal{B}_z = \{r \in Y : \|r\| \leq z\}$. For $r \in \mathcal{B}_z$, define the operator $\Lambda = \Lambda_1 + \Lambda_2$, where

$$(\Lambda_1 r)(\xi) = \mathcal{T}_\gamma(\xi)(r_0 - n(r)) + \mathcal{Q}_\gamma(\xi)(r_1 - m(r))$$

and

$$(\Lambda_2 r)(\xi) = \int_0^\xi \mathcal{S}_\gamma(\xi - \zeta)[g(\zeta, r(\zeta)) + \int_0^\zeta p(\zeta - \lambda) f(\lambda, r(\lambda))d\lambda]d\zeta.$$

Then

$$\begin{aligned}\|(\Lambda_1 r)(\xi)\| &\leq \widetilde{W}\|r_0 - n(r)\| + \widetilde{W}\|r_1 - m(r)\| \\ &\leq \widetilde{W}(\|r_0\| + \|r_1\| + l_1 z + \|n(0)\| + l_2 z + \|m(0)\|)\end{aligned}$$

and

$$\begin{aligned}\|(\Lambda_2 r)(\xi)\| \leq & \widetilde{W}\int_0^{\xi}\|g(\zeta,r(\zeta))+\int_0^{\zeta}p(\zeta-\lambda)f(\lambda,r(\lambda))d\lambda]d\zeta\|+\widetilde{W}\int_0^{\xi}\|\mathcal{B}r(\zeta)\|d\zeta \\ \leq & \widetilde{W}\int_0^{\xi}\nu_z(\zeta)d\zeta+\widetilde{W}\int_0^{\xi}\|\mathcal{B}r(\zeta)\|d\zeta \\ \leq & \widetilde{W}\int_0^{\xi}\nu_z(\zeta)d\zeta+d\widetilde{W}\mathcal{M}\Big(\|s_0\|+l_1z+\|n(0)\|+\widetilde{W}\|u_0\|+\widetilde{W}l_1z+\widetilde{W}\|n(0)\|+\widetilde{W}\|u_1\| \\ & +\widetilde{W}l_2z+\widetilde{W}\|m(0)\|+\widetilde{W}\int_0^{d}\nu_z(\zeta)d\zeta\Big).\end{aligned} \tag{3.15}$$

Using (3.14), we deduce that $\|(\Lambda_1 r)(\xi)+(\Lambda_2 s)(\xi)\| \leq z$. That is, for any $r, s \in \mathcal{B}_z$, $\Lambda_1 r+\Lambda_2 s \in \mathcal{B}_z$. Next, for any $r, s \in \mathcal{B}_z$, we have

$$\begin{aligned}\|(\Lambda_1 r)(\xi)-(\Lambda_1 s)(\xi)\| \leq & \mathcal{T}_\gamma(\xi)(r_0-n(r))+Q_\gamma(\xi)(r_1-m(r))-\mathcal{T}_\gamma(\xi)(r_0-n(s))-Q_\gamma(\xi)(r_1-m(s)) \\ \leq & \widetilde{W}\|n(s)-n(r)\|+\widetilde{W}\|m(s)-m(r)\| \\ \leq & \widetilde{W}l_1\|s-r\|+\widetilde{W}l_2\|s-r\| \\ = & \widetilde{W}(l_1+l_2)\|s-r\|.\end{aligned}$$

It follow from (3.12) that Λ_1 is a contraction mapping.

Let $\{r_i\}_{i=1}^{\infty}$ be a sequence in $\mathcal{B}_z$, $r \in \mathcal{B}_z$, and $r_i \to r$ $(i \to \infty)$. Noting that g is continuous on $J \times Y$, we get

$$g(\zeta,r_i(\zeta))+\int_0^{\zeta}p(\zeta-\lambda)f(\lambda,r_i(\lambda))d\lambda \to g(\zeta,r(\zeta))+\int_0^{\zeta}p(\zeta-\lambda)f(\lambda,r(\lambda))d\lambda, \quad i \to \infty. \tag{3.16}$$

For all $\xi \in [0,d]$, we obtain

$$\begin{aligned}\|(\Lambda_2 r_i)(\xi)-(\Lambda_2 r)(\xi)\| \leq & \Big\|\int_0^{\xi}\mathcal{S}_\gamma(\xi-\zeta)[g(\zeta,r_i(\zeta))+\int_0^{\zeta}p(\zeta-\lambda)f(\lambda,r_i(\lambda))d\lambda+\mathcal{B}v_i(\zeta)]d\zeta \\ & -\int_0^{\xi}\mathcal{S}_\gamma(\xi-\zeta)[g(\zeta,r(\zeta))+\int_0^{\zeta}p(\zeta-\lambda)f(\lambda,r(\lambda))d\lambda+\mathcal{B}v(\zeta)]d\zeta\Big\|.\end{aligned}$$

Using (H8), for an arbitrary function $r(\cdot)$, we define the control v by

$$\begin{aligned}v(\xi) = & W^{-}\Big[s_0-n(r)-\mathcal{T}_\gamma(d)(r_0-n(r))-Q_\gamma(d)(r_1-m(r)) \\ & -\int_0^{\xi}\mathcal{S}_\gamma(\xi-\zeta)g(\zeta,r(\zeta))+\int_0^{\zeta}p(\zeta-\lambda)f(\lambda,r(\lambda))d\lambda\Big](\xi).\end{aligned}$$

Then

$$\begin{aligned}\|(\mathcal{B}v_i)(\xi)-(\mathcal{B}v)(\xi)\| \leq & \Big\|\mathcal{B}W^{-}\Big[s_0-n(r_i)-\mathcal{T}_\gamma(d)(r_0-n(r_i))-Q_\gamma(d)(r_1-m(r_i)) \\ & -\int_0^{\xi}\mathcal{S}_\gamma(\xi-\zeta)g(\zeta,r_i(\zeta))+\int_0^{\zeta}p(\zeta-\lambda)f(\lambda,r_i(\lambda))d\lambda\Big](\xi)\end{aligned}$$

$$
\begin{aligned}
& -\mathcal{B}W^{-}\Big[s_0 - n(r) - \mathcal{T}_\gamma(d)(r_0 - n(r)) - \mathcal{Q}_\gamma(d)(r_1 - m(r)) \\
& -\int_0^{\xi} \mathcal{S}_\gamma(\xi-\zeta)g(\zeta, r(\zeta)) + \int_0^{\zeta} p(\zeta-\lambda)f(\lambda, r(\lambda))d\lambda\Big](\xi)\Big\| \\
\leq\ & \mathcal{M}\|n(r) - n(r_i)\| + \mathcal{M}\|\mathcal{T}_\gamma(d)n(r) - \mathcal{T}_\gamma(d)n(r_i)\| \\
& +\mathcal{M}\|\mathcal{Q}_\gamma(d)m(r) - \mathcal{Q}_\gamma(d)m(r_i)\| \\
& +\mathcal{M}\int_0^{d} \mathcal{S}_\gamma(d-\zeta)\|g(\zeta, r(\zeta)) + \int_0^{\zeta} p(\zeta-\lambda)f(\lambda, r(\lambda)) \\
& -g(\zeta, r_i(\zeta)) + \int_0^{\zeta} p(\zeta-\lambda)f(\lambda, r_i(\lambda))\|d\lambda \\
\leq\ & \mathcal{M}l_1\|r_i - r\| + \mathcal{M}\widetilde{W}l_1\|r_i - r\| + \mathcal{M}\widetilde{W}l_2\|r_i - r\| \\
& +\mathcal{M}\widetilde{W}\int_0^{d} \|g(\zeta, r(\zeta)) + \int_0^{\zeta} p(\zeta-\lambda)f(\lambda, r(\lambda)) \\
& -g(\zeta, r_i(\zeta)) + \int_0^{\zeta} p(\zeta-\lambda)f(\lambda, r_i(\lambda))\|d\lambda.
\end{aligned}
$$

By Lebesgue dominated convergence theorem, it is easy to see that

$$\|(\Lambda_2 r_i)(\xi) - (\Lambda_2 r)(\xi)\| \to 0, \text{ as } i \to \infty.$$

Thus we obtain that Λ_2 is continuous. Now, in order to show the compactness of Λ_2, we prove that $\{(\Lambda_2 r)(\xi) : r \in \mathcal{B}_z\}$ is uniformly bounded and relatively compact for all $\xi \in J$, respectively. It follows from (3.15) that $\|(\Lambda_2 r)(\xi)\| \leq C$, where C is a positive constant. For $0 < \xi_1 < \xi_2 \leq d$, we obtain

$$
\begin{aligned}
\|(\Lambda_2 r_i)(\xi_1) - (\Lambda_2 r)(\xi_2)\| \leq\ & \Big\| \int_0^{\xi_2} \mathcal{S}_\gamma(\xi_1-\zeta)[g(\zeta, r_i(\zeta)) + \int_0^{\zeta} p(\zeta-\lambda)f(\lambda, r_i(\lambda))d\lambda + \mathcal{B}v(\zeta)]d\zeta \\
& \int_{\xi_2}^{\xi_1} \mathcal{S}_\gamma(\xi_1-\zeta)[g(\zeta, r_i(\zeta)) + \int_0^{\zeta} p(\zeta-\lambda)f(\lambda, r_i(\lambda))d\lambda + \mathcal{B}v(\zeta)]d\zeta \\
& -\int_0^{\xi_2} \mathcal{S}_\gamma(\xi-\zeta)[g(\zeta, r(\zeta)) + \int_0^{\zeta} p(\zeta-\lambda)f(\lambda, r(\lambda))d\lambda + \mathcal{B}v(\zeta)]d\zeta\Big\| \\
\leq\ & \int_0^{\xi_2} \|\mathcal{S}_\gamma(\xi_1-\zeta) - \mathcal{S}_\gamma(\xi_2-\zeta)\|\|g(\zeta, r(\zeta)) \\
& +\int_0^{\zeta} p(\zeta-\lambda)f(\lambda, r(\lambda))d\lambda + \mathcal{B}v(\zeta)\|d\zeta \\
& +\int_{\xi_1}^{\xi_2} \|\mathcal{S}_\gamma(\xi_1-\zeta)\|\|g(\zeta, r(\zeta)) + \int_0^{\zeta} p(\zeta-\lambda)f(\lambda, r(\lambda))d\lambda + \mathcal{B}v(\zeta)\|d\zeta \\
\leq\ & I_1 + I_2,
\end{aligned}
$$

where

$$I_1 = \int_0^{\xi_2} \|\mathcal{S}_\gamma(\xi_1-\zeta) - \mathcal{S}_\gamma(\xi_2-\zeta)\|\|g(\zeta, r(\zeta)) + \int_0^{\zeta} p(\zeta-\lambda)f(\lambda, r(\lambda))d\lambda + \mathcal{B}v(\zeta)\|d\zeta$$

and

$$I_2 = \int_{\xi_1}^{\xi_2} \|\mathcal{S}_\gamma(\xi_1-\zeta)\|\|g(\zeta, r(\zeta)) + \int_0^{\zeta} p(\zeta-\lambda)f(\lambda, r(\lambda))d\lambda + \mathcal{B}v(\zeta)\|d\zeta.$$

Noting that the continuity of the function $\xi \mapsto \|\mathcal{S}_\gamma(\xi)\|$ for $\xi \in (0, d]$, we have $\lim_{\xi_2 \to \xi_1} I_i = 0$. So, by the Arzelà-Ascoli theorem, Λ_2 is compact. Then Theorem 2.5 and Definition 3.3 allow us to conclude that (1.1) is controllable on J. □

4. Example

Example 4.1. *To demonstrate applicability of our main results, we consider the following fractional differential equation of the form:*

$$\begin{aligned} &{}^{c}D^{\gamma} r(\xi, h) = \frac{\partial^2 r(\xi, h)}{\partial h^2} + \frac{e^{\xi |r(\xi,h)|}}{(24 + e^{\xi})(1 + |r(\xi, h)|)} + \int_0^{\xi} e^{\xi-\zeta} \frac{e^{\zeta}}{\sqrt{48} + |r(\xi, h)|} d\zeta + \mathcal{B}v(\xi) \\ &\omega(q, 0) - \frac{|r(\xi, h)|}{8 + \|r(\xi, h)|} = 0, \quad \frac{dr(\xi, h)}{d\xi}|_0 - \frac{|r(\xi, h)|}{8 + \|r(\xi, h)|} = 0, \\ &r(\xi, 0) = r(\xi, \pi) = 0, \quad r'(\xi, 0) = r'(\xi, \pi) = 0, \end{aligned} \tag{4.1}$$

where, $1 < \gamma < 2, \xi \in J = [0, 1], 0 \le h \le \pi$. *Let* $Y = L^2([0, \pi])$, *then the system (4.1) is controllable on J.*

Proof. As $Y = L^2([0, \pi])$, then the positive cone of Y is $P = \{r \in C(J, Y) : r(\xi, h) \ge 0,\ i.e.(\xi, h) \in J \times Y\}$. The operator $\mathcal{A} : D(\mathcal{A}) \subset Y \to Y$ is given by

$$\mathcal{A}h = h \text{ with } D(\mathcal{A}) := \{h \in Y : h', h'' \in Y, h(0) = h(\pi) = 0\}.$$

$\mathcal{B} : D(\mathcal{B}) \subset Y \to Y$ is a bounded linear operator. In [11], obviously it is known that $\mathcal{A}$ is infinitesimal generator of the semigroup $\{\mathcal{T}(\xi)\}_{\xi \ge 0}$ on Y. Also, $\mathcal{A}$ has discrete spectrum with eigenvalues $-m^2, m \in N$, and associate normalized eigenfunctions given by $z_m(h) = (\frac{\pi}{2})^{\frac{1}{2}} \sin(mh)$ and also $\{z_m : m \in N\}$,

$$K(\xi) = \sum_{m=1}^{\infty} e^{-m^2} \langle h, z_m \rangle z_m, \quad for\, h \in Y, \xi \ge 0.$$

From this representation it follows that $K(\xi)$ is compact for every $\xi > 0$ and that $\|K(\xi)\| \le e^{-\xi}$ for all $\xi \ge 0$ [33].

As indicated in [34], the operator $\mathcal{A} = \Delta$ is a sectorial of type $(W, \vartheta, \gamma, \nu)$ and generates compact γ−resolvent families $\{\mathcal{T}_\gamma(\xi)\}_{\xi \ge 0}$, $\{Q_\gamma(\xi)\}_{\xi \ge 0}$, and $\{\mathcal{S}_\gamma(\xi)\}_{\xi \ge 0}$. In view of the fact that it has proved in [35] that the operator $\mathcal{A} = \Delta$ is an m−accretive on Y with dense domain, (H2) is satisfied,

$$g(\xi, r) = \tfrac{e^{\xi}|r|}{(24+e^{\xi})(1+|r|)}, \ f(\xi, r) = \tfrac{e^{\xi}}{\sqrt{48}+|r|)},$$

$$n(r) = -\tfrac{|r|}{8+|r|}, \ m(r) = -\tfrac{|r|}{8+|r|}, \ p(\xi - \zeta) = e^{\xi-\zeta}.$$

Via the estimation on the norms of operators of Theorems 2.19 and 2.20, we can get $\widetilde{W} = 3$ (see [9]). Furthermore, for $\xi \in J, r, u \in R$ we have

$$\|g(\xi, r) - g(\xi, u)\| = \frac{e^{\xi}}{24 + e^{\xi}} \left\| \frac{r}{1 + r} - \frac{u}{1 + u} \right\| \le \frac{e^{\xi}}{24 + e^{\xi}} \|r - u\| \le \frac{1}{8} \|r - v\|,$$

$$\|f(\xi, r) - f(\xi, u)\| = e^{\xi} \left\| \frac{1}{\sqrt{48} + r} - \frac{1}{\sqrt{48} + u} \right\| \le \frac{e^{\xi}}{48} \|r - u\| \le \frac{1}{16} \|r - v\|.$$

There, we have $l_1 = \frac{1}{8}$, $l_2 = \frac{1}{16}$. Meanwhile,

$$\|g(\xi, r)\| \leq \frac{e^{\xi}}{24 + e^{\xi}} < \frac{1}{8}, \quad \|f(\xi, r)\| \leq \frac{e^{\xi}}{\sqrt{48}} < \frac{3}{\sqrt{48}} + \frac{\sqrt{3}}{4}.$$

Hence, $\|\nu_1\| = \frac{1}{8}$, $\|\mu_1\| = \frac{\sqrt{3}}{4}$, which shows that (H3) and (H6) are satisfied. We have

$$\max \int_0^{\xi} |p(\xi - \zeta)| d\zeta = \max_{\xi \in [0,1]} \int_0^{\xi} e^{\xi - \zeta} d\zeta = \max_{\xi \in [0,1]} e^{\xi} - 1 \leq 2.$$

For $r \in R$, we have

$$\|r_0 - n(r)\| \leq \left\| \frac{r}{8+r} \right\| \leq \frac{1}{8}, \quad \|r_1 - m(r)\| \leq \frac{1}{8}.$$

Then assumption (H5) is satisfied.

Consequently,

$$\widetilde{W} K(l_1 + pl_2) = 3\left(\frac{1}{8} + 2 \times \frac{1}{16}\right) = \frac{3}{4} < 1.$$

So, all the conditions of Theorem 3.4 are satisfied. Hence, the fractional differential system (1.1) is controllable on J. □

5. Conclusions

In this paper, we discussed the existence of a positive mild solution and the controllability of the system (1.1). The existence and controllability results are proved by applying the fact related to fractional calculus, fixed point theory, and materials related to existence and controllability under some specific conditions. First, we have proved the existence of positive mild solutions, then continued to prove the controllability of fractional differential evolution equations of order $1 < \gamma < 2$ with nonlocal conditions. In the end, for demonstration of theory, we have presented an example. Based on this research work, in the future, we will discuss the existence and controllability of the system (1.1) of higher orders.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. R. Gul, M. Sarwar, K. Shah, T. Abdeljawad, F. Jarad, Qualitative analysis of implicit Dirichlet boundary value problem for Caputo-Fabrizio fractional differential equations, *J. Funct. Space.*, **2020** (2020), 4714032. https://doi.org/10.1155/2020/4714032
2. M. B.Zada, M.Sarwar, C. Tunc, Fixed point theorems in b-metric spaces and their applications to non-linear fractional differential and integral equations, *J. Fixed Point Theory Appl.*, **20**, (2018), 25. https://doi.org/10.1007/s11784-018-0510-0
3. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier Science, 2006.
4. Y. Zhou, J. R. Wang, L. Zhang, *Basic theory of fractional differential equations*, Singapore: World Scientific, 2016.
5. K. B. Oldham, J. Spanier, *The fractional calculus theory and applications of differentiation and integration to arbitrary order*, New York: Academic Press, 1974.
6. I. Podlubny, *Fractional differential equations, mathematics in science and engineering*, New York: Academic Press, 1999.
7. L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.*, **162** (1991), 494–505. https://doi.org/10.1016/0022-247X(91)90164-U
8. G. M. Mophou, G. M. N'Guerekata, Existence of mild solution for some fractional differential equations with nonlocal conditions, *Semigroup Forum,* **79** (2009), 315–322. https://doi.org/10.1007/s00233-008-9117-x
9. X. Shu, Q. Wang, The existence and uniqueness of mild solutions for fractonal differential equations with nonlocal conditions of order $1 < \alpha < 2$, *Comput. math Appl.*, **64** (2012), 2100–2110. https://doi.org/10.1016/j.camwa.2012.04.006
10. X. Wang, X. Shu, The existence of positive mild solutions for fractional differential evolution equations with nonlocal conditions of order $1 < \alpha < 2$, *Adv. Differ. Equ.*, **2015** (2015), 159. https://doi.org/10.1186/s13662-015-0461-3
11. K. Balachandran, R. Sakthivel, Existence of solutions of neutral functional integrodifferential equation in Banach spaces, *Proc. Indian Acad. Sci. Math. Sci.*, **109** (1999), 325–332. https://doi.org/10.1007/BF02843536
12. G. Arthi, K. Balachandran, Controllabibity of damped second-order neutral integro-differential systems with nonlocal conditions, *J. Control Theorey Appl.*, **11** (2013), 186–192.
13. M. M. Raja, V. Vijayakumar, R. Udhayakumar, Results on the existence and controllability of fractional integro-differential system of order $1 < r < 2$ via measure of noncompactness, *Chaos Soliton. Fract.*, **139** (2020), 110299. https://doi.org/10.1016/j.chaos.2020.110299
14. J. R. Wang, Y. Zhou, Existence and controllability results for fractional semilinear differential inclusions, *Nonlinear Anal. Real World Appl.*, **12** (2011), 3642–3653. https://doi.org/10.1016/j.nonrwa.2011.06.021

15. Z. Liu, B. Zeng, Existence and controllability for fractional evolution inclusions of Clarke's subdifferential type, *Appl. Math. Comput.*, **257** (2015), 178–189. https://doi.org/10.1016/j.amc.2014.12.057

16. S. Ji, G. Li, M. Wang, Controllability of impulsive differential systems with nonlocal conditions, *Appl. Math. Comput.*, **217** (2011), 6981–6989. https://doi.org/10.1016/j.amc.2011.01.107

17. N. I. Mahmudov, S. Zorlu, Approximate controllability of fractional integro-differential equations involving nonlocal initial conditions, *Bound. Value Probl.*, **2013** (2013), 118. https://doi.org/10.1186/1687-2770-2013-118

18. L. Gorniewicz, S. K. Ntouyas, D. O'Regan, Existence and controllability results for first and second order functioal semilinear differential inclusions with nonlocal conditions, *Numer. Funct. Anal. Optim.*, **28** (2007), 53–82. https://doi.org/10.1080/01630560600883093

19. Y. Guo, X. B. Shu, F. Xu, C. Yang, HJB equation for optimal control system with random impulses, *Optimization*, **2022** (2022), 1–25. https://doi.org/10.1080/02331934.2022.2154607

20. D. Chendrayan, U. Ramalingam, V. Vijaykumar, A. Shukla, K. S. Nisar, New discussion regarding approximate controllability for Sobolev-type fractional stochastic hemivariational inequalitities of order $r \in (1,2)$, *Commun. Nonlinear Sci. Numer. Simul.*, **116** (2023), 106891. https://doi.org/10.1016/j.cnsns.2022.106891

21. M. M. Raja, A. Shukla, J. J. Nieto, V. Vijaykumar, K. S. Nisar, A note on the existence and controllability results for fractional integhro differential inclusions of order $r \in (1,2]$, *Qual. Theory Dyn. Syst.*, **21** (2022), 150. https://doi.org/10.1007/s12346-022-00681-z

22. D. Chendrayan, V. Vijaykumar, U. Ramalingam, A. Shukla, K. S. Nisar, Controllability discussion for fractional stochastic Volterra-Fredholm integro-differential systems of order $1 < r < 2$, *Int. J. Nonlinear Sci. Numer.*, 2022. https://doi.org/10.1515/ijnsns-2021-0479

23. U. Arora, V. Vijaykumar, A. Shukla, M. Sajid, K. S. Nisar, A discussion on controllability of nonlocal fractional semilinear equations of order $1 < r < 2$ with monotonic nonlinearity, *J. King Saud. Univ. Sci.*, **34** (2022), 102295. https://doi.org/10.1016/j.jksus.2022.102295

24. M. Mohan Raja, V. Vijaykumar, A. Shukla, K. S. Nisar, H. M. Baskonus, On the approximate controllability results for fractional integrodifferential systems of order $1 < r < 2$ with sectorial operators, *J. Comput. Appl. Math.* **415** (2022), 114492. https://doi.org/10.1016/j.cam.2022.114492

25. Y. Ma, M. M. Raja, K. S. Nisar, A. Shukla, V. Vijaykumar, Results on controllability for Sobolev type fractional differential equations of order $1 < r < 2$ with finite delay, *AIMS Math.*, **7** (2022), 10215–10233. https://doi.org/10.3934/math.2022568

26. L. Shu, X. B. Shu, J. Mao, Approximate controllability and existence of mild solutions for Reemann-Liouville fractional stochastic evolution equations with nonlocal conditions of order $1 < \alpha < 2$, *Fract. Calc. Appl. Anal.*, **22** (2019), 1086–1112. https://doi.org/10.1515/fca-2019-0057

27. G. Rajchakit, A. Pratap, R. Raja, J. Cao, J. Alzabut, C. Huang, Hybrid control scheme for projective lag synchronization of riemann-liouville sense fractional order memristive bam neuralnetworks with mixed delays, *Mathematics*, **7** (2019), 759. https://doi.org/10.3390/math7080759

28. T. Abdeljawad, J. Alzabut, On Riemann-Liouville fractional q -difference equations and their application to retarded logistic type model, *Math. Method. Appl. Sci.*, **41** (2018), 8953–8962. https://doi.org/10.1002/mma.4743

29. D. Guo, V. Lakshmikantham, *Nonlinear problems in abstract cones*, Boston: Academic Press, 1988.

30. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, New York: Springer, 2012.

31. E. Bazhlekova, Fractional evolution equations in Banach spaces, Ph.D thesis, *Eindhoven University of Technology*, 2001.

32. J. W. Hanneken, D. M. Vaught, B. N. Narahari Achar, Enumeration of the real zeros of the Mittag-Leffler function Eα(z) $1 < \alpha < 2$, *Adv. Fract. Calc. Theor. Dev. Appl.*, **2007** (2007), 15–26.

33. E. Hernández, R. Sakthivel, S. T. Aki, Existence results for impulsive evolution differential equations with state-dependent delay, *Electron. J. Differ. Equ.*, **2008** (2008), 28.

34. Z. Denton, P. W. Ng, A. S. Vastsala, Quasi linearization method via lower and upper solutions for Riemann-Liouville fractional differential equations, *Nonlinear Dyn. Syst. Theory*, **11** (2011), 239–251.

35. P. Bénilan, Equations d́evolution dans un espace de Banach quelconque et appications, *These, Publications Math. Orsay, Univ. Paris-Sud.*, 1972.

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