Research article

Concentration of solutions for double-phase problems with a general nonlinearity

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Abstract: In this paper, we study the following problems with a general nonlinearity:

\[
\begin{align*}
-\Delta \rho u - \Delta \sigma u + V(\varepsilon x)(|u|^{\rho - 2}u + |u|^{\sigma - 2}u) &= f(u), \quad \text{in } \mathbb{R}^N, \\
 u &\in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \quad \text{in } \mathbb{R}^N,
\end{align*}
\]

where \(\varepsilon > 0\) is a small parameter, \(2 \leq p < q < N\), the potential \(V\) is a positive continuous function having a local minimum. \(f : \mathbb{R} \rightarrow \mathbb{R}\) is a \(C^1\) subcritical nonlinearity. Under some proper assumptions of \(V\) and \(f\), we obtain the concentration of positive solutions with the local minimum of \(V\) by applying the penalization method for above equation. We must note that the monotonicity of \(\frac{f(s)}{s^{p+1}}\) and the so-called Ambrosetti-Rabinowitz condition are not required.

Keywords: double-phase problems; penalization method; variational methods

Mathematics Subject Classification: 35A15, 35B38, 35J60

1. Introduction and main results

In this paper, we investigate the concentration of positive solutions for the following double-phase problems with a general nonlinearity:

\[
\begin{align*}
-\Delta \rho u - \Delta \sigma u + V(\varepsilon x)(|u|^{\rho - 2}u + |u|^{\sigma - 2}u) &= f(u), \quad \text{in } \mathbb{R}^N, \\
 u &\in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \quad \text{in } \mathbb{R}^N,
\end{align*}
\]

where \(\varepsilon > 0\) is a small parameter, \(2 \leq p < q < N\), the potential \(V\) is a positive continuous function having a local minimum. \(f : \mathbb{R} \rightarrow \mathbb{R}\) is a \(C^1\) subcritical nonlinearity.

The content of this paper is closely related to the double phase problems, we briefly introduce the development of this research. It is common knowledge that the first contributions to this field
are due to Ball [9], in relationship with problems in nonlinear elasticity and composite materials. The double-phase problem (1.1) is motivated by numerous local and nonlocal models arising in mathematical physics. For example, we can refer to Born-Infeld equation [11, 12] which appears in electromagnetism, electrostatics and electrodynamics as a model based on a modification of Maxwell’s Lagrangian density:

\[-\text{div} \left( \frac{\nabla u}{(1 - 2|\nabla u|^2)^{1/2}} \right) = h(u) \text{ in } \Omega.\]

In fact, according to the Taylor formula, we obtain that

\[(1 - x)^{-1/2} = 1 + \frac{x}{2} + \frac{3}{2 \cdot 2^2} x^2 + \frac{5!!}{3! \cdot 2^3} x^3 + \cdots + \frac{(2n - 3)!!}{(n - 1)!2^{n-1}} x^{n-1} + \cdots \text{ for } |x| < 1.\]

Taking \( x = 2|\nabla u|^2 \) and adopting the first order approximation, we get a particular case of the problem (1.1) for \( p = 2 \) and \( q = 4 \).

When \( p = q \), the problem (1.1) becomes \( p \)-Laplace equation:

\[-\varepsilon^p \Delta_p u + V(x)|u|^{p-2} u = f(u) \text{ in } \mathbb{R}^N.\] (1.2)

Elliptic problems like (1.2), in the semilinear case which corresponds to \( p = 2 \), arise from the problem of obtaining standing waves for the nonlinear Schrödinger equations given by

\[i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2} \Delta \psi - V(x)\psi + f(\psi) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,\]

where \( i \) is the imaginary unit and \( \hbar \) is the Planck constant. Further backgrounds for these equations can be found in [13, 41] and references therein. Gloss in [23] studied existence and asymptotic behavior of positive solutions by using penalization for quasilinear elliptic equations of (1.2). Note that, \( f \) is a subcritical nonlinearity without Ambrosetti-Rabinowitz condition (AR condition in short):

\[0 < \theta \int_0^u f(s) ds \leq f(u)u \text{ for } \theta \in (p, p^*).\]

In [21], by using a variational approach based on the local mountain-pass theorem, the author proved the existence and concentration of positive bound states of the equation involving critical growth:

\[f(u) = g(u) + u^{p-1} \text{ in (1.2). However this } g \text{ requires the (AR) condition and the monotonicity of } f(s).\]

In [24], He and Li studied the following elliptic problem:

\[\left\{ \begin{array}{l}
-\varepsilon^p \Delta_p u + V(z)|u|^{p-2} u - f(u) = 0 \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega, \quad u > 0 \text{ in } \Omega, \quad N > p > 2,
\end{array} \right.\]

where \( \Omega \) is a possibly unbounded domain in \( \mathbb{R}^N \) with empty or smooth boundary, \( \varepsilon \) is a small parameter. \( f \in C^1(\mathbb{R}^+, \mathbb{R}) \) is of subcritical and \( V : \mathbb{R}^N \rightarrow \mathbb{R} \) is a locally Hölder continuous function. As a result, they obtained the existence and concentration of weak solutions by penalization method.

When \( \varepsilon = 1 \) in problem (1.1), the main interest in this general class of problems has been due to the fact that they arise from applications in physics and related sciences, such as biophysics, plasma physics and chemical reaction, see for example [16]. In last decade, many authors paid their attention to seek positive solutions, bounded states solutions, multiple solutions, see for instance [15, 33, 36, 43] and the
which was proved under different conditions by using Morse theory in terms of critical groups. The corresponding eigenvalue problem of the double phase operator with Dirichlet boundary condition was analyzed by Colasuonno-Squassina [17] who proved the existence and properties of related variational eigenvalues. According to variational methods, Liu-Dai [34] treated double phase problems and proved existence and multiplicity results.

Ambrosio in [4] dealt with the following problem

\[ (-\Delta)_p^s u + (-\Delta)^q u + |u|^{p-2}u + |u|^{q-2}u = \lambda h(x)f(u) + |u|^{q'-2}u \quad \text{in } \mathbb{R}^N. \]

Using suitable variational arguments and concentration-compactness lemma, the authors proved the existence of a nontrivial non-negative solution for \( \lambda \) sufficiently large. Note that [4] dealt with the constant potential, and then in [27], under proper assumptions, Isernia proved the existence of a positive solution and a negative ground state solution for the following class of fractional \( p&\ell \)-Laplacian problems with potentials vanishing at infinity:

\[ (-\Delta)_p^s u + (-\Delta)^q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = K(x)f(u) \quad \text{in } \mathbb{R}^N. \]

Recently, Alves, Ambrosio and Isernia [1] by applying minimax theorems and the Ljusternik-Schnirelmann theory, they investigated the existence, multiplicity and concentration of nontrivial solutions for (1.1) provided that \( \varepsilon \) is sufficiently small. Costa and Figueiredo [19] used Mountain Pass Theorem and the penalization arguments introduced by Del Pino and Felmer’s associated to Lions’ Concentration and Compactness Principle to overcome the lack of compactness, and then showed existence and concentration results for (1.1). Ambrosio and Rădulescu in [7] considered the following class of fractional problems with unbalanced growth:

\[
\begin{aligned}
(-\Delta)_p^s u + (-\Delta)^q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) &= \lambda f(u), \\
\quad u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N), u > 0,
\end{aligned}
\]

(1.3)

Applying suitable variational and topological arguments, they obtained multiple positive solutions for \( \varepsilon > 0 \) that were sufficiently small as well as related concentration properties, in relationship with the set where the potential \( V \) attains its minimum. In [45], Zhang et al. investigated the following perturbed double phase problem with competing potentials:

\[
\begin{aligned}
-\varepsilon^p \Delta_p u - \varepsilon^q \Delta_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) &= K(x)f(u), \\
\quad u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), u > 0,
\end{aligned}
\]

where \( (1 < p < q < N) \). The authors assumed that the potentials \( V, K \) and the nonlinearity \( f \) are continuous but are not necessarily of class \( C^1 \). Under some natural hypotheses, using topological and variational tools from Nehari manifold analysis and Ljusternik-Schnirelmann category theory, they studied the existence of positive ground state solutions. Moreover, they determined two concrete sets related to the potentials \( V \) and \( K \) as the concentration positions and described the concentration of ground state solutions as \( \varepsilon \to 0 \). Zhang et al. in [49] studied the following double phase problem

\[
\begin{aligned}
(-\Delta)_p^s u + (-\Delta)^q u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) &= \lambda f(u) + |u|^{q'-2}u, \\
\quad u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N), u > 0,
\end{aligned}
\]

in \( \mathbb{R}^N \).
They established the existence of multiple positive solutions as well as related concentration properties. In [48], Zhang et al. considered the singularly perturbed double phase problems with nonlocal reaction, they got the concentration result. For more results of the existence and concentration of solutions, we refer to [8, 30, 46, 47] and the references therein.

It is worth mentioning that all the works above assumed that the nonlinearity satisfied Ambrosetti-Rabinowitz condition, so the authors can use Nehari manifold to obtain the concentration and multiplicity properties of solutions. [7, 49] are associated with fractional, but this it also true for $s = 1$.

On the other hand, the nonlinearities in these equations are not general, so by being motivated by the above works, it is quite natural to ask if $f(u)$ is a general nonlinearity which satisfies Berestycki-Lions type assumptions, does the same results established for double phase problem? In the present paper, we give an affirmative answer to this question.

Before stating our main result, we shall introduce the main hypotheses. Assume that the potential $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function fulfilling the following conditions which are always called del Pino-Felmer type [20] conditions.

(V1) $V \in C(\mathbb{R}^N, \mathbb{R})$ such that $V_1 := \inf_{x \in \mathbb{R}^N} V(x) > 0$.

(V2) There exists a bounded open set $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 := \inf_{x \in \Lambda} V(x) < \min_{x \notin \Lambda} V(x)$$

with $V_0 > 0$, and $0 \in M := \{x \in \Lambda : V(x) = V_0\}$.

Moreover, the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(t) = 0$ for $t \leq 0$, and satisfies the following hypotheses:

(f1) $\lim_{t \to 0} \frac{f(t)}{t^{p-1}} = 0$;

(f2) There exists $\nu \in (q, q^*)$ such that $\lim_{|t| \to +\infty} \frac{f(t)}{t^{p-1}} < \infty$;

(f3) There exists $T > 0$ such that $F(T) > \frac{V_0}{p} T^p + \frac{V_0}{q} T^q$.

Theorem 1.1. Assume that (V1) – (V2) and (f1) – (f3) are satisfied. Then, for small $\varepsilon > 0$, there exists a positive solution $u_\varepsilon$ to (1.1) such that $u_\varepsilon$ has a maximum point $x_\varepsilon$ satisfying

$$\lim_{\varepsilon \to 0} \text{dist} (x_\varepsilon, M) = 0$$

and for any such $x_\varepsilon$, the function $v_\varepsilon(x) = u_\varepsilon(\varepsilon x + x_\varepsilon)$ converges uniformly as $\varepsilon \to 0$ (up to a subsequence) to a least energy solution of

$$\begin{cases}
-\Delta_p u - \Delta_q u + V_0(|u|^{p-2} u + |u|^{q-2} u) = f(u), & \text{in } \mathbb{R}^N, \\
u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), & \text{in } \mathbb{R}^N.
\end{cases}$$

Moreover, we have

$$u_\varepsilon(x) \leq C_1 e^{-C_2 |x - x_\varepsilon|} \text{ for all } x \in \mathbb{R}^N, \ C_1, \ C_2 > 0.$$
We use a truncation approach to prove our result. The main difficulties in the proof of Theorem 1.1 lie in two aspects:

(1) The nonlinearity \( f(u) \) does not satisfy (AR) condition and the fact that the function \( \frac{f(u)}{u^q} \) is not increasing for \( u > 0 \) prevent us from obtaining a bounded Palais-Smale sequence and using the Nehari manifold, respectively. Moreover, the arguments in [20] can not be applied in this paper;

(2) The unboundedness of the domain \( \mathbb{R}^N \) leads to the lack of compactness.

As we will see later, the above two aspects prevent us from using the variational method in a standard way. In order to get over the above two difficulties, inspired by [13, 25], we recover the compactness by penalization method which was first introduced in [14].

The plan of this paper is the following. In Section 2, we define some function spaces. Section 3 is devoted to study ground state solution for the limit problem of (1.1), and we give the proof of Theorem 1.1 in the last section.

2. Variational setting

In this section, we fix the notations and recall some results for the uses later.

Let \( u : \mathbb{R}^N \rightarrow \mathbb{R} \). For \( 1 < p < q \), let us define \( D^{1,p}(\mathbb{R}^N) = C^\infty(\mathbb{R}^N)^{\nabla^p} \). We denote the following fractional Sobolev space

\[
W^{1,p}(\mathbb{R}^N) := \{ u : |u|_p^p < +\infty, |\nabla u|_p^p < +\infty \}
\]
equipped with the natural norm

\[
||u||_{W^{1,p}(\mathbb{R}^N)} := \left( |\nabla u|_p^p + |u|_p^p \right)^{1/p},
\]
where \( | \cdot |_p^p := \int_{\mathbb{R}^N} | \cdot |^p \, dx \).

For all \( u, v \in W^{1,p}(\mathbb{R}^N) \),

\[
\langle u, v \rangle_{W^{1,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv) \, dx.
\]

In this work we need to introduce the following Banach space

\[
X = W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)
\]
equipped with the norm

\[
||u||_X := ||u||_{W^{1,p}(\mathbb{R}^N)} + ||u||_{W^{1,q}(\mathbb{R}^N)}.
\]

Note that \( W^{1,r}(\mathbb{R}^N) \) is a separable reflexive Banach space for all \( r \in (1, +\infty) \), and so \( X \) is a separable reflexive Banach space.

For any fixed \( \varepsilon \geq 0 \), we also introduce the following Banach space

\[
X_\varepsilon := \left\{ u \in X : \int_{\mathbb{R}^N} V(\varepsilon x)(|u|^p + |u|^q) \, dx < +\infty \right\}
\]
equipped with the norm

\[
||u||_{X_\varepsilon} := ||u||_{V,p} + ||u||_{V,q},
\]
where $\|u\|_{V_0,t} = \int_{\mathbb{R}^N} (|\nabla u'| + V(\varepsilon x)|u'|) \, dx$ for all $t \in \{p, q\}$. When $V(x) = V_0$, we denote the following Banach space

$$X_0 := \left\{ u \in X : \int_{\mathbb{R}^N} V_0(|u|^p + |u|^q) \, dx < +\infty \right\}$$

equipped with the norm

$$\|u\|_{X_0} := \|u\|_{V_0,p} + \|u\|_{V_0,q},$$

where $\|u\|_{V_0,t} = \int_{\mathbb{R}^N} (|\nabla u'| + V_0|u'|) \, dx$ for all $t \in \{p, q\}$. Finally, we consider

$$X_{\text{rad}}(\mathbb{R}^N) := \{ u \in X_0 : u(x) = u(|x|) \}.$$

**Lemma 2.1.** (see [42, Theorem 2.8]) (General Minimax Principle) Let $X$ be a Banach space. Let $M_0$ be a closed subspace of the metric space $M$ and $\Gamma_0 \subset C(M_0, X)$. Define

$$\Gamma := \{ \gamma \in C(M, X) : \gamma|_{M_0} \in \Gamma_0 \}.$$

If $\varphi \in C^1(X, \mathbb{R})$ satisfies

$$a := \sup_{\gamma \in \Gamma_0} \sup_{u \in M_0} \varphi(\gamma(u)) < c := \inf_{\gamma \in \Gamma} \sup_{u \in M} \varphi(\gamma(u)) < \infty,$$

then, for every $\varepsilon \in (0, (c - a)/2)$, $\delta > 0$ and $\gamma \in \Gamma$ such that $\sup_{M} \varphi \circ \gamma \leq c + \varepsilon$, there exists $u \in X$ such that

(a) $c - 2\varepsilon \leq \varphi(u) \leq c + 2\varepsilon,$

(b) $\text{dist}(u, \gamma(M)) \leq 2\delta,$

(c) $\|\varphi'(u)\| \leq \frac{8\varepsilon}{\delta}.$

### 3. The limiting problem

First of all, in order to make functional of the limiting problem equation to be $C^1$ and let it is a meaningful functional on $X_0$, we modify $f$ as in [10]. Let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be define as follows:

(i) if $f(t) > 0$ for all $t \geq \overline{T}$, put $\tilde{f}(t) := f(t),$

(ii) if there exists $\tau_0 \geq \overline{T}$ such that $f(\tau_0) = 0$, we put

$$\tilde{f}(t) := \begin{cases} f(t), & \text{for } t < \tau_0, \\ 0, & \text{for } t \geq \tau_0, \end{cases}$$

where $\overline{T} := \sup\{t \in [0, T] : f(t) > V_0 t^{p-1} + V_0 t^{q-1}\}$.

It is clear that $\tilde{f}$ satisfies the same assumptions as $f$ and

$$0 \leq \lim_{t \to \infty} \frac{\tilde{f}(t)}{t^p} \leq \lim_{t \to \infty} \frac{\tilde{f}(t)}{t^p} < \infty.$$
At the same time, note that, if (ii) occurs and \( u \) is a solution to (1.1) with \( \tilde{f}(t) \), we can use \( (u - \tau_0) \), as test function to obtain that \( u \leq \tau_0 \) in \( \mathbb{R}^N \), then \( u \) is solution to (1.1) with \( f(t) \). From now on, we replace \( f \) by \( \tilde{f} \) and keep the same notation \( f(t) \).

In this section we focus on the following limiting problem associated with (1.1):

\[
\begin{cases}
-\Delta_p u - \Delta_q u + V_0(|u|^{p-2}u + |u|^{q-2}u) = f(u), &\text{in } \mathbb{R}^N, \\
u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), &\text{in } \mathbb{R}^N.
\end{cases}
\]  

We define the energy functional for the limiting problem (3.1) by

\[
I_{V_0}(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla u|^q \, dx + \frac{1}{p} \int_{\mathbb{R}^N} V_0 |u|^p \, dx + \frac{1}{q} \int_{\mathbb{R}^N} V_0 |u|^q \, dx - \int_{\mathbb{R}^N} F(u) \, dx.
\]

In view of [38], if \( u \in X_0 \) is a weak solution to problem (3.1), then we have the following Pohožáev identity:

\[
P_{V_0}(u) = \frac{N - p}{p} \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \frac{N - q}{q} \int_{\mathbb{R}^N} |\nabla u|^q \, dx + \frac{N}{p} \int_{\mathbb{R}^N} V_0 |u|^p \, dx + \frac{N}{q} \int_{\mathbb{R}^N} V_0 |u|^q \, dx - N \int_{\mathbb{R}^N} F(u) \, dx.
\]

**Lemma 3.1.** \( I_{V_0} \) possesses the Mountain-Pass geometry.

**Proof.** By \((f_1)\) and \((f_2)\), for all \( t \in \mathbb{R} \) we get

\[
|f(t)| \leq \varepsilon |t|^{p-1} + C_\varepsilon |t|^{q-1}
\]

and

\[
|F(t)| \leq \frac{\varepsilon}{p} |t|^{p-1} + \frac{C_\varepsilon}{q} |t|^{q}.
\]

So, for \( 2 \leq p < q < N \), we have

\[
I_{V_0}(u) \geq \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + V_0 |u|^p) \, dx + \frac{1}{q} \int_{\mathbb{R}^N} (|\nabla u|^q + V_0 |u|^q) \, dx - \frac{\varepsilon}{p} |u|^p_{p} - \frac{C_\varepsilon}{q} |u|^q_{q}
\]

\[
= \frac{1}{q} \int_{\mathbb{R}^N} (|\nabla u|^q + V_0 |u|^q) \, dx + \frac{V_0 - \varepsilon}{p V_0} \int_{\mathbb{R}^N} (|\nabla u|^p + V_0 |u|^p) \, dx - \frac{C_\varepsilon}{q} |u|^q_{q}
\]

\[
= \frac{1}{Q} \int_{\mathbb{R}^N} (|\nabla u|^q + V_0 |u|^q) \, dx + \int_{\mathbb{R}^N} (|\nabla u|^q + V_0 |u|^q) \, dx - \frac{C_\varepsilon}{q} |u|^q_{q}
\]

\[
= \frac{1}{Q} (||u||_{V_0,p}^{q} + ||u||_{V_0,q}^{q}) - C ||u||_{X_0}^{q},
\]

where we used \( \varepsilon = (\frac{1}{p} - \frac{1}{q}) p V_0 \). Hence, there exist \( \rho, \delta > 0 \) such that

\[
I_{V_0}(u) \geq \frac{1}{Q} (||u||_{V_0,p}^{q} + ||u||_{V_0,q}^{q}) - C ||u||_{X_0}^{q}
\]

\[
\geq \frac{1}{2q-1} ||u||_{X_0}^{q} - C ||u||_{X_0}^{q}
\]
for \( \|u\|_{X_0} = \rho \).

Now, for all \( R > 0 \), we define

\[
w_R(x, y) := \begin{cases} 
T & \text{if } (x) \in B_R^+(0), \\
T \left( R + 1 - \sqrt{|x|} \right) & \text{if } (x) \in B_{R+1}^+(0) \setminus B_R^+(0), \\
0 & \text{if } (x) \in \mathbb{R}^N \setminus B_{R+1}^+(0).
\end{cases}
\]

It is easy to see that \( w_R \in X_{\text{rad}}(\mathbb{R}^N) \). We note that, according to \((f_3)\), for \( R > 0 \) large enough it holds

\[
\int_{\mathbb{R}^N} \left[ F(w_R(x)) - \frac{V_0}{p} w_R^p(x) - \frac{V_0}{q} w_R^q(x) \right] dx \geq 0.
\]

Now, fix such an \( R > 0 \) and consider \( w_{R, \theta}(x) := w_R \left( \frac{x}{\rho} \right) \). Then,

\[
I_{V_0}(w_{R, \theta}) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla u|^q dx - \frac{V_0}{p} w_R^p(x) - \frac{V_0}{q} w_R^q(x)
\]

\[
\to -\infty \text{ as } \theta \to \infty.
\]

This ends the proof. \( \square \)

Hence, according to Lemma 3.1, we can define the Mountain-Pass level of \( I_{V_0} \) by

\[
c_{V_0} := \inf_{\gamma \in \Gamma_{V_0}} \sup_{t \in [0, 1]} I_{V_0}(\gamma(t)),
\]

where the set of paths is defined as

\[
\Gamma_{V_0} := \{ \gamma \in C([0, 1], X_0) : \gamma(0) = 0 \text{ and } I_{V_0}(\gamma(1)) < 0 \}.
\]

Obviously, \( c_{V_0} > 0 \). Moreover, similar to \([3]\), we note that

\[
c_{V_0} = c_{V_0, \text{rad}},
\]

where

\[
c_{V_0, \text{rad}} := \inf_{\gamma \in \Gamma_{V_0, \text{rad}}} \max_{t \in [0, 1]} I_{V_0}(\gamma(t))
\]

and

\[
\Gamma_{V_0, \text{rad}} := \{ \gamma \in C([0, 1], X_{\text{rad}}(\mathbb{R}^N)) : \gamma(0) = 0, I_{V_0}(\gamma(1)) < 0 \}.
\]

Next, we will construct a (PS) sequence \( \{w_n\}_{n=1}^\infty \) for \( I_{V_0} \) at the level \( c_{V_0} \) that satisfies \( I'_{V_0}(w_n) \to 0 \) as \( n \to \infty \), i.e.,

**Proposition 3.1.** There exists a sequence \( \{w_n\}_{n=1}^\infty \) in \( X_0 \) such that, as \( n \to \infty \),

\[
I_{V_0}(w_n) \to c_{V_0}, \quad I'_{V_0}(w_n) \to 0, \quad P_{V_0}(w_n) \to 0.
\]
Proof. Let we define $\tilde{I}_V(\theta, u) := (I_V \circ \Phi)(\theta, u)$ for $(\theta, u) \in \mathbb{R} \times X_{\text{rad}} \left( \mathbb{R}_+^N \right)$, where $\Phi(\theta, u) := u \left( \frac{\theta}{\varepsilon} \right)$. Here $\mathbb{R} \times X_{\text{rad}} \left( \mathbb{R}_+^N \right)$ is equipped with the standard norm

$$\|(\theta, u)\|_{\mathbb{R} \times X_0} := \left( |\theta|^2 + \|u\|^2_{X_0} \right)^{\frac{1}{2}}.$$  

According to Lemma 3.1 that $\tilde{I}_V$ has a mountain pass geometry, so we can define the mountain pass level of $\tilde{I}_V$

$$\tilde{c}_V := \inf_{\gamma \in \tilde{I}_V} \max_{t \in [0, 1]} \tilde{I}_V(\gamma(t)),$$

where

$$\tilde{I}_V := \left\{ \gamma \in C \left( [0, 1], \mathbb{R} \times X_{\text{rad}} \left( \mathbb{R}_+^N \right) \right) : \gamma(0) = (0), \tilde{I}_V(\gamma(1)) < 0 \right\}.$$

It is easy to prove that $\tilde{c}_V = c_V$ (see [6, 26]). Then according to Lemma 2.1, we obtain that there exists a sequence $(\theta_n, u_n) \subset \mathbb{R} \times X_{\text{rad}} \left( \mathbb{R}_+^N \right)$ such that, as $n \to \infty$,

(i) $(I_V \circ \Phi)(\theta_n, u_n) \to c_V$,

(ii) $(I_V \circ \Phi)'(\theta_n, u_n) \to 0$ in $(\mathbb{R} \times X_{\text{rad}} \left( \mathbb{R}_+^N \right))'$,

(iii) $\theta_n \to 0$.

In fact, we only take $\varepsilon = \varepsilon_n = \frac{1}{n^2}, \delta = \delta_n = \frac{1}{n}$ in Lemma 2.1, (i) and (ii) follow by (a) and (c) in Lemma 2.1. In view of (3.2) and (3.3), for $\varepsilon = \varepsilon_n := \frac{1}{n^2}$, it is easy to find that $\gamma_n \in \Gamma_{V_0}$ such that

$$\sup_{t \in [0, 1]} I_{V_0}(\gamma_n(t)) \leq c_{V_0} + \frac{1}{n^2}.$$  

We define $\gamma_n(t) := (0, \gamma_n(t))$, then we have

$$\sup_{t \in [0, 1]} (I_V \circ \Phi)(\gamma_n(t)) = \sup_{t \in [0, 1]} I_{V_0}(\gamma_n(t)) \leq c_{V_0} + \frac{1}{n^2}.$$  

From (b) of Lemma 2.1, there exists $(\theta_n, u_n) \in \mathbb{R} \times X_0$ such that

$$\text{dist}((\theta_n, u_n), (0, \gamma_n(t))) \leq \frac{2}{n},$$

which implies that (iii) holds true. Here, we used the notation

$$\text{dist}((\theta, u), A) := \inf_{(\tau, v) \in \mathbb{R} \times X_0} \left( |\theta - \tau|^2 + \|u - v\|_{X_0}^2 \right)^{\frac{1}{2}}$$

for $A \subset \mathbb{R} \times X_0$. Now, for $(h, w) \in \mathbb{R} \times X_0$, it holds

$$\left( (I_V \circ \Phi)'(\theta_n, u_n), (h, w) \right) = \left( I_{V_0}'(\Phi(\theta_n, u_n)), \Phi'(\theta_n, w) \right) + P_{V_0}(\Phi(\theta_n, u_n)) h.$$  

Then, taking $h = 1$ and $w = 0$ in (3.5), we obtain that

$$P_{V_0}(\Phi(\theta_n, u_n)) \to 0.$$  

Moreover, for all $v \in X_0$, we only take $w(x, y) = v \left( e^{\theta_n x}, e^{\theta_n y} \right)$ and $h = 0$ in (3.5), it follows from (ii) and (iii) that

$$\left( I_{V_0}'(\Phi(\theta_n, u_n)), v \right) = o(1) \left\| v \left( e^{\theta_n x}, e^{\theta_n y} \right) \right\|_{X_0} = o(1) \|v\|_{X_0}.$$  

Therefore, $w_n := \Phi(\theta_n, u_n)$ is the sequence that fulfills the desired properties.  

\hfill $\Box$
Lemma 3.2. Every sequence \((w_n)\) satisfying (3.4) is bounded in \(X_0\).

Proof. According to (3.4), it is easy to see that
\[
c_{V_0} + o_n(1) = I_{V_0}(w_n) - \frac{1}{N} P_{V_0}(w_n)
= \frac{1}{p} \int_{\mathbb{R}^N} |\nabla w_n|^p dx + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla w_n|^q dx + \frac{1}{p} \int_{\mathbb{R}^N} V_0|w_n|^p dx + \frac{1}{q} \int_{\mathbb{R}^N} V_0|w_n|^q dx
- \int_{\mathbb{R}^N} F(w_n) dx - \frac{1}{N} \left( \frac{N-p}{p} \int_{\mathbb{R}^N} |\nabla w_n|^p dx + \frac{N-q}{q} \int_{\mathbb{R}^N} |\nabla w_n|^q dx \right)
+ \frac{N}{p} \int_{\mathbb{R}^N} V_0|w_n|^p dx + \frac{N}{q} \int_{\mathbb{R}^N} V_0|w_n|^q dx
= \frac{1}{N} \left( \int_{\mathbb{R}^N} |\nabla w_n|^p dx + \int_{\mathbb{R}^N} |\nabla w_n|^q dx \right).
\]

So we get that \(\int_{\mathbb{R}^N} |\nabla w_n|^p dx\) and \(\int_{\mathbb{R}^N} |\nabla w_n|^q dx\) are bounded in \(\mathbb{R}\). On the other hand, \(P(w_n) = o_n(1)\) and \((f_1) - (f_2)\) yield
\[
\frac{N-p}{p} \int_{\mathbb{R}^N} |\nabla w_n|^p dx + \frac{N-q}{q} \int_{\mathbb{R}^N} |\nabla w_n|^q dx + \frac{N}{p} \int_{\mathbb{R}^N} V_0|w_n|^p dx + \frac{N}{q} \int_{\mathbb{R}^N} V_0|w_n|^q dx
= N \int_{\mathbb{R}^N} F(w_n) dx + o_n(1)
\leq N \delta |w_n(\cdot, 0)|_p^p + NC_0 |w_n(\cdot, 0)|_q^q + o_n(1).
\]

Choosing \(\delta > 0\) sufficiently small and using the boundedness of \(\left(\left|w_n(\cdot, 0)\right|_p\right)\), we can deduce that \(\left(\left|w_n\right|_p\right)\) and \(\left(\left|w_n\right|_q\right)\) are bounded in \(\mathbb{R}\). In conclusion, \((w_n)\) is bounded in \(X_0\). \(\Box\)

The following lemma is a version of Lions’ concentration-compactness lemma.

Lemma 3.3. (see [42]) Let \(2 \leq p < \xi < q^*\). Assume \(\{u_n\}\) is a bounded sequence in \(X_0\) which satisfies
\[
\lim_{N \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n(x)|^\xi dx = 0
\]
for some \(R > 0\). Then \(u_n \to 0\) in \(L^\xi(\mathbb{R}^N)\) for \(\xi \in (p, q^*)\).

Lemma 3.4. There exist a sequence \((x_n) \subset \mathbb{R}^N\) and constants \(R > 0, \beta > 0\) such that
\[
\int_{\Gamma^{2}_n(x_n)} w_n^2(x) dx \geq \beta,
\]
where \((w_n)\) is the sequence given in Proposition 3.1.

Proof. By contradiction, we assume that the thesis is not true. Then, according to Lemma 3.3, we deduce that
\[
w_n(\cdot) \to 0 \text{ in } L^\xi(\mathbb{R}^N), \quad \forall \xi \in (p, q^*). \tag{3.6}
\]
Consequently, by using $(f_1)-(f_2)$, we have that
\[ \int_{\mathbb{R}^N} f(w_n(x))w_n(x)dx = o_n(1). \]

According to $\left\langle I'_{V_0}(w_n), w_n \right\rangle = o_n(1)$, we can obtain that
\[ \int_{\mathbb{R}^N} |\nabla w_n|^pdx + \int_{\mathbb{R}^N} |\nabla w_n|^qdx + \int_{\mathbb{R}^N} V_0|w_n|^pdx + \int_{\mathbb{R}^N} V_0|w_n|^qdx - \int_{\mathbb{R}^N} f(w_n)w_ndx = o_n(1), \]
and so we deduce that $\|w_n\|_{X_0} \to 0$. Therefore, $I_{V_0}(w_n) \to 0$ and this leads to a contradiction because $c_{V_0} > 0$. □

Now we define
\[ T_{V_0} := \left\{ u \in X(\mathbb{R}^N) \setminus \{0\} : I'_{V_0}(u) = 0, \max_{x \in \mathbb{R}^N} u(x) = u(0) \right\}, \]
\[ b_{V_0} := \inf_{u \in T_{V_0}} I_{V_0}(u), \]
and
\[ S_{V_0} := \{ u \in T_{V_0} : I_{V_0}(u) = b_{V_0} \}. \]

**Lemma 3.5.** There exists $u \in S_{V_0}$.

**Proof.** Assume that $(w_n)$ is the sequence given by Proposition 3.1. Let $\widetilde{w}_n(x) := w_n(x + x_n)$, where $x_n$ is given by Lemma 3.4. Due to Lemma 3.3, $(w_n)$ is bounded in $X_{rad}(\mathbb{R}^N)$, that is $\|w_n\|_{X_{rad}(\mathbb{R}^N)} \leq C$ for all $n \in \mathbb{N}$. Going if necessary to a subsequence, we can assume that $\widetilde{w}_n \rightharpoonup \widetilde{w}$ in $X_{rad}(\mathbb{R}^N)$ for some $\widetilde{w} \in X_{rad}(\mathbb{R}^N) \setminus \{0\}$ and we obtain that
\[ \widetilde{w}_n(x) \to \widetilde{w}(x) \text{ in } L^\xi(\mathbb{R}^N) \text{ for any } \xi \in (p, q^*). \]

So
\[ \int_{\mathbb{R}^N} f(\widetilde{w}_n)\widetilde{w}_n \to \int_{\mathbb{R}^N} f(\widetilde{w})\widetilde{w}. \tag{3.7} \]

Moreover, $\widetilde{w}$ satisfies
\[ (-\Delta)_p \widetilde{w} + (-\Delta)_q \widetilde{w} + V_0(|\widetilde{w}|^{p-2}\widetilde{w} + |\widetilde{w}|^{q-2}\widetilde{w}) = f(\widetilde{w}) \text{ in } \mathbb{R}^N. \tag{3.8} \]

Therefore, we have
\[ \frac{1}{p} \int_{\mathbb{R}^N} |\nabla \widetilde{w}|^pdx + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla \widetilde{w}|^qdx + \frac{1}{p} \int_{\mathbb{R}^N} V_0|\widetilde{w}|^pdx + \frac{1}{q} \int_{\mathbb{R}^N} V_0|\widetilde{w}|^qdx = \int_{\mathbb{R}^N} F(\widetilde{w})dx, \]
and
\begin{align*}
\frac{N-p}{p} \int_{\mathbb{R}^N} |\nabla \widetilde{w}|^pdx + \frac{N-q}{q} \int_{\mathbb{R}^N} |\nabla \widetilde{w}|^qdx + \frac{N}{p} \int_{\mathbb{R}^N} V_0|\widetilde{w}|^pdx + \frac{N}{q} \int_{\mathbb{R}^N} V_0|\widetilde{w}|^qdx \\
= N \int_{\mathbb{R}^N} F(\widetilde{w})dx.
\end{align*}

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From (3.7) we can see that

\[
\int_{\mathbb{R}^N} |\nabla \tilde{w}|^p \, dx + \int_{\mathbb{R}^N} |\nabla \tilde{w}|^q \, dx + \int_{\mathbb{R}^N} V_0 |\tilde{w}|^p \, dx + \int_{\mathbb{R}^N} V_0 |\tilde{w}|^q \, dx \\
\leq \liminf_{n \to \infty} \left[ \int_{\mathbb{R}^N} |\nabla \tilde{w}_n|^p \, dx + \int_{\mathbb{R}^N} |\nabla \tilde{w}_n|^q \, dx + \int_{\mathbb{R}^N} V_0 |\tilde{w}_n|^p \, dx + \int_{\mathbb{R}^N} V_0 |\tilde{w}_n|^q \, dx \right]
\]

\[
\leq \limsup_{n \to \infty} \left[ \int_{\mathbb{R}^N} |\nabla \tilde{w}_n|^p \, dx + \int_{\mathbb{R}^N} |\nabla \tilde{w}_n|^q \, dx + \int_{\mathbb{R}^N} V_0 |\tilde{w}_n|^p \, dx + \int_{\mathbb{R}^N} V_0 |\tilde{w}_n|^q \, dx \right]
\]

\[
= \limsup_{n \to \infty} \int_{\mathbb{R}^N} f(w_n) \tilde{w}_n \, dx
\]

\[
= \limsup_{n \to \infty} \int_{\mathbb{R}^N} f(\tilde{w}_n) \tilde{w}_n \, dx
\]

\[
= \int_{\mathbb{R}^N} \nabla \tilde{w} \, dx
\]

which implies that \( |\tilde{w}_n|_{X_0} \to |\tilde{w}|_{X_0} \) and thus \( \tilde{w}_n \to \tilde{w} \) in \( X_0 \). Therefore, by \( I_{V_0}(w_n) = I_{V_0}(\tilde{w}_n) \to c_{V_0} \) and \( I'_{V_0}(w_n) = I'_{V_0}(\tilde{w}_n) \to 0 \), we have that \( I_{V_0}(\tilde{w}) = c_{V_0} \) and \( I'_{V_0}(\tilde{w}) = 0 \). Since \( \tilde{w} \neq 0 \), we deduce that \( c_{V_0} \geq b_{V_0} \).

Now, let \( w \in X_0 \setminus \{0\} \) be any solution to (3.1). Define

\[
w_t(x) := \begin{cases} w(\frac{x}{t}) & \text{for } t > 0, \\ 0 & \text{for } t = 0. \end{cases}
\]

Using the fact that \( w \) satisfies the Pohozaev identity, we get

\[
I_{V_0}(w_t(x)) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla w_t(x)|^p \, dx + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla w_t(x)|^q \, dx + \frac{1}{p} \int_{\mathbb{R}^N} V_0 |w_t(x)|^p \, dx
\]

\[
+ \frac{1}{q} \int_{\mathbb{R}^N} V_0 |w_t(x)|^q \, dx - \int_{\mathbb{R}^N} F(w_t(x)) \, dx
\]

\[
= \frac{t^{N-p}}{p} \int_{\mathbb{R}^N} |\nabla w|^p \, dx + \frac{t^{N-q}}{q} \int_{\mathbb{R}^N} |\nabla w|^q \, dx + \frac{1}{p} \int_{\mathbb{R}^N} V_0 |w|^p \, dx
\]

\[
+ \frac{1}{q} \int_{\mathbb{R}^N} V_0 |w|^q \, dx - t^{N} \int_{\mathbb{R}^N} F(w) \, dx
\]

\[
= \frac{t^{N-p}}{p} \int_{\mathbb{R}^N} |\nabla w|^p \, dx + \frac{t^{N-q}}{q} \int_{\mathbb{R}^N} |\nabla w|^q \, dx - \frac{N-p}{N} t^{N} \int_{\mathbb{R}^N} |\nabla w|^p \, dx
\]

\[
- \frac{N-q}{N} t^{N} \int_{\mathbb{R}^N} |\nabla w|^q \, dx,
\]

and differentiating with respect to \( t \) we obtain

\[
\frac{d}{dt} I_{V_0}(w_t(x)) = \frac{N-p}{p} t^{N-p-1} \int_{\mathbb{R}^N} |\nabla w|^p \, dx + \frac{N-q}{q} t^{N-q-1} \int_{\mathbb{R}^N} |\nabla w|^q \, dx
\]
\[
-\frac{N-p}{p} r^{N-1} \int_{\mathbb{R}^N} |\nabla w|^p \, dx - \frac{N-q}{q} r^{N-1} \int_{\mathbb{R}^N} |\nabla w|^q \, dx \\
= -\frac{N-p}{p} r^{N-p-1} (1 - t^p) \int_{\mathbb{R}^N} |\nabla w|^p \, dx + \frac{N-q}{q} r^{N-q-1} (1 - t^q) \int_{\mathbb{R}^N} |\nabla w|^q \, dx,
\]
so we obtain that

\[
\frac{d}{dr} I_{V_0}(w_t(x)) > 0 \quad \forall t \in (0, 1), \quad \frac{d}{dr} I_{V_0}(w_t(x)) < 0 \quad \forall t \in (1, \infty),
\]
which implies that

\[
\max_{t \geq 0} I_{V_0}(w_t(x)) = I_{V_0}(w_1(x)) = I_{V_0}(w).
\]

Therefore, we have that

\[
I_{V_0}(w_t(x)) = -\frac{t^{N-p}}{p} \int_{\mathbb{R}^N} |\nabla w|^p \, dx + \frac{t^{N-q}}{q} \int_{\mathbb{R}^N} |\nabla w|^q \, dx - \frac{N-p}{Np} t^N \int_{\mathbb{R}^N} |\nabla w|^p \, dx \\
-\frac{N-q}{Nq} t^N \int_{\mathbb{R}^N} |\nabla w|^q \, dx \rightarrow -\infty,
\]
as \(t \rightarrow \infty\). After a suitable scale change in \(t\), we obtain that \(w_t(x) \in \Gamma_{V_0}\). By the definition of \(c_{V_0}\), we see that \(I_{V_0}(w_t(x)) \geq c_{V_0}\). Since \(w\) is arbitrary, we have that \(b_{V_0} \geq c_{V_0}\) and this implies that \(b_{V_0} = c_{V_0}\).

Choosing \(w^- = \min\{w, 0\}\) as test function of (3.1) we can deduce that \(w \geq 0\) in \(\mathbb{R}^N\). By \((f_1)-(f_2)\) and using a Moser iteration argument (see [6]), we obtain that \(w \in L^{\infty}(\mathbb{R}^N)\). According to Corollary 2.1 in Ambrosio and Rădulescu [7], we can see that \(w \in C^\sigma(\mathbb{R}^N)\) for some \(\sigma \in (0, 1)\). Similar to the proof of Theorem 1.1-(ii) in Jarohs [28], we obtain that \(w > 0\) in \(\mathbb{R}^N\). Note that, the methods of [6] and [7] are still applicable to this article, so they are directly quoted here.

\[\square\]

**Remark 3.1.** For \(m > 0\), we use the notation

\[
I_m(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla u|^q \, dx + \frac{m}{p} \int_{\mathbb{R}^N} |u|^p - \frac{m}{q} \int_{\mathbb{R}^N} |u|^q \, dx - \int_{\mathbb{R}^N} F(u) \, dx,
\]
and denote by \(c_m\) the corresponding mountain pass level. It is standard to verify that if \(m_1 > m_2\) then \(c_{m_1} > c_{m_2}\).

In what follows, we aim to prove that \(S_{V_0}\) is compact in \(X_0\).

**Lemma 3.6.** \(S_{V_0}\) is compact in \(X_0\).

**Proof.** For any \(U \in S_{V_0}\), we have that

\[
c_{V_0} + c_0(1) = I_{V_0}(U) - \frac{1}{N} P_{V_0}(U) \\
= \frac{1}{p} \int_{\mathbb{R}^N} |\nabla U|^p \, dx + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla U|^q \, dx + \frac{1}{p} \int_{\mathbb{R}^N} V_0 |U|^p \, dx + \frac{1}{q} \int_{\mathbb{R}^N} V_0 |U|^q \, dx \\
- \int_{\mathbb{R}^N} F(U) \, dx - \frac{1}{N} \left( \frac{N-p}{p} \int_{\mathbb{R}^N} |\nabla U|^p \, dx + \frac{N-q}{q} \int_{\mathbb{R}^N} |\nabla U|^q \, dx \right) \\
+ \frac{N}{p} \int_{\mathbb{R}^N} V_0 |U|^p \, dx + \frac{N}{q} \int_{\mathbb{R}^N} V_0 |U|^q \, dx - N \int_{\mathbb{R}^N} F(U) \, dx
\]
completes the proof that Similar to the proof of Lemma 3.5, we can check that

\[ J \in \text{AIMS Mathematics} \]

find that

\[ \exists U \]

Lemma 3.5, we have that

\[ \text{From } (3.10), \]

\[ U \]

\[ S \]

\[ \text{have } \]

\[ \text{fact, by using } \]

\[ \text{According to } \lim \]

\[ \text{For any sequence } \{U_k\} \subset S_{V_0}, \text{ up to a subsequence, we can assume that there is a } U_0 \in X_0 \text{ such that } \]

\[ U_k \to U_0 \text{ in } X_0 \]

and \( U_0 \) satisfies

\[ -\Delta_p U_0 - \Delta_q U_0 + V_0(|U_0|^{p-2}U_0 + |U_0|^{q-2}U_0) = f(U_0), \text{ in } \mathbb{R}^N, \quad U_0 \geq 0. \]

Next, we will prove that \( U_0 \) is nontrivial. Note that, up to a subsequence, we have

\[ U_k \to U_0 \text{ in } L_{\text{loc}}^q(\mathbb{R}^N), \quad t \in (p, q'). \]

From (3.10), \( \{U_k\} \) is uniformly integrable in any bounded domain in \( \mathbb{R}^N \). By Lemma 2.2 (i) in [25], \( \|U_k\|_{L_{\text{loc}}^q(\mathbb{R}^N)} \leq C \). In view of [31], \( 3\alpha \in (0, 1) \) such that \( \|U_k\|_{C^{1,\alpha}(\mathbb{R}^N)} \leq C \). Due to \( \{U_k\} \subset S_{V_0}, \text{ by Lemma } 3.5, \) we have that \( U_k > 0 \). It is easy to prove that \( \lim \inf_{k \to \infty} \|U_k\|_{\infty} > 0 \) because of \( \lim_{t \to 0} \frac{f(t)}{t^{p-1}} = 0 \). In fact, by using \( U_k \) satisfies (3.1), we have that

\[ -\Delta_p U_k - \Delta_q U_k + V_0(|U_k|^{p-2}U_k + |U_k|^{q-2}U_k) = f(U_k), \]

that is

\[ \frac{N - p}{p} \int_{\mathbb{R}^N} |\nabla U_k|^p dx + \frac{N - q}{q} \int_{\mathbb{R}^N} |\nabla U_k|^q dx + \frac{V_0N}{p} \int_{\mathbb{R}^N} |U_k|^p dx + \frac{V_0N}{q} \int_{\mathbb{R}^N} |U_k|^q dx = \]

\[ = N \int_{\mathbb{R}^N} F(U_k) dx. \]

According to \( \lim_{t \to 0} \frac{f(t)}{t^{p-1}} = 0 \), we obtain that, \( \forall \epsilon > 0, \exists \delta > 0 \) such that

\[ f(t) < \epsilon t^{p-1} \text{ for } |t| < \delta, \]

then \( F(U_k) < \frac{\epsilon}{p} |U_k|^p \). Assume by contradiction, we have \( \liminf_{k \to \infty} \|U_k\|_{\infty} = 0 \), then for \( \delta \) given above, we have \( |U_k| < \delta \). Therefore,

\[ \frac{N - p}{p} \int_{\mathbb{R}^N} |\nabla U_k|^p dx + \frac{N - q}{q} \int_{\mathbb{R}^N} |\nabla U_k|^q dx \]

\[ = N \int_{\mathbb{R}^N} F(U_k) dx - \frac{V_0N}{p} \int_{\mathbb{R}^N} |U_k|^p dx - \frac{V_0N}{q} \int_{\mathbb{R}^N} |U_k|^q dx < 0, \]

which leads to a contradiction. Noting that \( U_0(0) = \|U_k\|_{\infty}, \) we know that \( U_0 \neq 0 \). Therefore, we can find that \( \exists C_0 > 0 \) such that \( U_k(0) \geq C_0 > 0 \), then \( U_0(0) \geq C_0 > 0 \), this means that \( U_0 \) is nontrivial. Similar to the proof of Lemma 3.5, we can check that \( J_V(0) = c_V \) and \( U_k \to U_0 \text{ in } X_0 \). This completes the proof that \( S_{V_0} \) is compact in \( X_0 \).\[ \square \]
4. Proof of Theorem 1.1

The energy functional corresponding to (1.1) is

$$I_\varepsilon(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + V(\varepsilon x)|u|^p)dx + \frac{1}{q} \int_{\mathbb{R}^N} (|\nabla u|^q + V(\varepsilon x)|u|^q)dx - \int_{\mathbb{R}^N} F(u)dx.$$  

We define

$$\chi_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in \Lambda / \varepsilon \\ \varepsilon^{-1} & \text{if } x \notin \Lambda / \varepsilon \end{cases}$$

and

$$Q_\varepsilon(v) = \left( \int_{\mathbb{R}^N} \chi_\varepsilon v^p - 1 \right)_+.$$

Finally, set $J_\varepsilon : X_\varepsilon \to \mathbb{R}$ be given by

$$J_\varepsilon(v) = I_\varepsilon(v) + Q_\varepsilon(v).$$

Note that this type of penalization was first introduced in [14]. It is standard to prove that $J_\varepsilon \in C^1(X_\varepsilon, \mathbb{R})$. In order to find solutions of (1.1) which concentrate around the local minimum of $V$ in $\Lambda$ as $\varepsilon \to 0$, we only look for the critical points of $J_\varepsilon$ for which $Q_\varepsilon$ is zero.

Let $c_{\varepsilon_0} = I_{\varepsilon_0}(U)$ for $U \in S_{\varepsilon_0}$ and $10\delta = \text{dist} \left[ M, \mathbb{R}^N \setminus \Lambda \right]$, we fix a $\beta \in (0, \delta)$ and a cut-off function $\varphi \in C_c^\infty \left( \mathbb{R}^N \right)$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ for $|x| \leq \beta$, $\varphi(x) = 0$ for $|x| \geq 2\beta$ and $|\nabla \varphi| \leq \frac{C}{\beta}$. Also, setting $\varphi_\varepsilon(y) = \varphi(\varepsilon y)$. We will look for a solution of (1.1) near the set

$$Y_\varepsilon := \left\{ \varphi(\varepsilon x - x')U(x - (x'/\varepsilon)) : x' \in M^0, U \in S_{\varepsilon_0} \right\}$$

for sufficiently small $\varepsilon > 0$, where $M^0 := \left\{ y \in \mathbb{R}^N : \inf_{z \in M} |y - z| \leq \beta \right\}$. Moreover, for $A \subset X_\varepsilon$, we use the notation

$$A^\varepsilon := \left\{ u \in X_\varepsilon : \inf_{v \in A} ||u - v||_{X_\varepsilon} \leq \varepsilon \right\}.$$

For $U \in S_{\varepsilon_0}$ arbitrary but fixed, we define $W_{\varepsilon,t}(x) := \varphi(\varepsilon x)U \left( \frac{x - t}{\varepsilon_0} \right)$, we will show that $J_\varepsilon$ possesses the Mountain-Pass geometry.

Let $U_t(x) := U(\frac{x}{t})$, similar to the proof of Lemma 3.1, we obtain that

$$I_{\varepsilon_0}(U_t) = \frac{t^{N-p}}{p} \int_{\mathbb{R}^N} |\nabla U|^p dx + \frac{t^{N-q}}{q} \int_{\mathbb{R}^N} |\nabla U|^q dx + \frac{t^N}{p} \int_{\mathbb{R}^N} V_0|U|^p dx + \frac{t^N}{q} \int_{\mathbb{R}^N} V_0|U|^q dx - t^N \int_{\mathbb{R}^N} F(U)dx \rightarrow -\infty \text{ as } t \rightarrow \infty,$$

so there exists $t_0 > 0$ such that $I_{\varepsilon_0}(U_{t_0}) < -3$.

It is easy to check that $Q_\varepsilon(W_{\varepsilon,t_0}) = 0$, then from the Dominated Convergence Theorem we have, for $\varepsilon > 0$ small,

$$J_\varepsilon(W_{\varepsilon,t_0}) = I_\varepsilon(W_{\varepsilon,t_0})$$
where we used $\epsilon$.

Hence, we can define the Mountain-Pass value of $J$.

Lemma 4.1. There holds

$$\lim_{\epsilon \to 0} c_\epsilon \leq c_{V_0}.$$

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Proof. Denote $W_{e,0} = \lim_{t \to 0} W_{e,t}$ in $X_e$ sense, then it is easy to see that $W_{e,0} = 0$. Therefore, let $\gamma(s) := W_{e,s_0} (0 \leq s \leq 1)$, we obtain that $\gamma(s) \in \Gamma_e$, then

$$c_e \leq \max_{\varepsilon \in [0,1]} J_e(\gamma(s)) = \max_{\varepsilon \in [0,1]} J_e(W_{e,\varepsilon}) ,$$

and we only need to prove that

$$\lim_{\varepsilon \to 0} \max_{\varepsilon \in [0,1]} J_e(W_{e,\varepsilon}) \leq c_{V_0}.$$

In fact, similar to (4.1), we obtain that

$$\max_{\varepsilon \in [0,1]} J_e(W_{e,\varepsilon}) = \max_{\varepsilon \in [0,1]} I_{V_0}(U_1^*) + o(1)
\leq \max_{\varepsilon \in [0,1]} I_{V_0}(U_1^*) + o(1) = I_{V_0}(U_1^*) + o(1) = c_{V_0} + o(1).$$

This finishes the proof. \(\square\)

Lemma 4.2. There holds

$$\lim_{\varepsilon \to 0} c_e \geq c_{V_0}.$$

Proof. Assuming by contradiction that $\lim_{\varepsilon \to 0} c_e < c_{V_0}$, then there exist $\delta_0 > 0, \epsilon_n \to 0$ and $\gamma_n \in \Gamma_e$ such that $J_{e_n}(\gamma_n(s)) < c_{V_0} - \delta_0$ for $s \in [0, 1]$. We could fix an $\epsilon_n$ such that

$$\frac{1}{p} V_{0\epsilon_n} \left( 1 + (1 + c_{V_0})^{1/2} \right) < \min \{ \delta_0, 1 \}. \quad (4.2)$$

Due to $J_{e_n}(\gamma_n(0)) = 0$ and $I_{e_n}(\gamma_n(1)) \leq J_{e_n}(\gamma_n(1)) = J_{e_n}(W_{e_n,0}) < -2$, we can look for an $s_n \in (0, 1)$ such that $I_{e_n}(\gamma_n(s)) \geq -1$ for $s \in [0, s_n]$ and $I_{e_n}(\gamma_n(s_n)) = -1$. Moreover, for any $s \in [0, s_n]$, we have that

$$Q_{e_n}(\gamma_n(s)) = J_{e_n}(\gamma_n(s)) - I_{e_n}(\gamma_n(s)) \leq 1 + c_{V_0} - \delta_0,$$

this implies that

$$\int_{\mathbb{R}^N \setminus \Lambda \setminus A_{\varepsilon_n}} \gamma_n(s) dx \leq \varepsilon_n \left( 1 + (1 + c_{V_0})^{1/2} \right) \text{ for } s \in [0, s_n].$$

So for $s \in [0, s_n]$, we have

$$J_{e_n}(\gamma_n(s)) = I_{V_0}(\gamma_n(s)) + \frac{1}{p} \int_{\mathbb{R}^N} (V(\varepsilon_n x) - V_0) \gamma_n^p(s) dx + \frac{1}{q} \int_{\mathbb{R}^N} (V(\varepsilon_n x) - V_0) \gamma_n^q(s) dx$$

$$\geq I_{V_0}(\gamma_n(s)) + \frac{1}{p} \int_{\mathbb{R}^N \setminus \Lambda \setminus A_{\varepsilon_n}} (V(\varepsilon_n x) - V_0) \gamma_n^p(s) dx + \frac{1}{q} \int_{\mathbb{R}^N \setminus \Lambda \setminus A_{\varepsilon_n}} (V(\varepsilon_n x) - V_0) \gamma_n^q(s) dx$$

$$\geq I_{V_0}(\gamma_n(s)) + \frac{1}{p} \int_{\mathbb{R}^N \setminus \Lambda \setminus A_{\varepsilon_n}} (V(\varepsilon_n x) - V_0) \gamma_n^p(s) dx$$

$$\geq I_{V_0}(\gamma_n(s)) - \frac{1}{p} V_{0\epsilon_n} \left( 1 + (1 + c_{V_0})^{1/2} \right).$$
Then
\[ I_{V_0} (\gamma_n (s_n)) \leq I_{e_n} (\gamma_n (s_n)) + \frac{1}{p} V_0 e_n \left( 1 + (1 + c_{V_0})^{1/2} \right) \]
\[ = -1 + \frac{1}{p} V_0 e_n \left( 1 + (1 + c_{V_0})^{1/2} \right) < 0, \]
and recalling (3.2), we obtain that
\[ \max_{s \in [0, s_n]} I_{V_0} (\gamma_n (s)) \geq c_{V_0}. \]
Therefore, we get that
\[ c_{V_0} - \delta_0 \geq \max_{s \in [0, 1]} J_{e_n} (\gamma_n (s)) \geq \max_{s \in [0, 1]} I_{e_n} (\gamma_n (s)) \geq \max_{s \in [0, s_n]} I_{V_0} (\gamma_n (s)) \]
\[ \geq \max_{s \in [0, s_n]} I_{V_0} (\gamma_n (s)) - \frac{1}{p} V_0 e_n \left( 1 + (1 + c_{V_0})^{1/2} \right), \]
that is \( 0 < \delta_0 \leq \frac{1}{p} V_0 e_n \left( 1 + (1 + c_{V_0})^{1/2} \right), \) which contradicts (4.2). As desired. \( \square \)

By using Lemma 4.1 and Lemma 4.2, we have
\[ \lim_{\varepsilon \to 0} \left( \max_{s \in [0, 1]} J_{\varepsilon} (\gamma_{\varepsilon} (s)) - c_{\varepsilon} \right) = 0, \]
where \( \gamma_{\varepsilon} (s) = W_{\varepsilon, s_0} \) for \( s \in [0, 1] \). Denote
\[ \tilde{c}_{\varepsilon} := \max_{s \in [0, 1]} J_{\varepsilon} (\gamma_{\varepsilon} (s)), \]
it is easy to see that \( c_{\varepsilon} \leq \tilde{c}_{\varepsilon} \) and
\[ \lim_{\varepsilon \to 0} c_{\varepsilon} = \lim_{\varepsilon \to 0} \tilde{c}_{\varepsilon} = c_{V_0}. \]
Now define
\[ J_{\alpha}^e = \{ u \in X_e \mid J_e (u) \leq \alpha \}, \]
and for a set \( A \subset X_e \) and \( \alpha > 0 \), let \( \alpha^A \equiv \{ u \in X_e \mid \inf_{v \in A} \| u - v \|_e \leq \alpha \} \).

**Lemma 4.3.** Let \( \{ \varepsilon_i \}_{i=1}^\infty \) be such that \( \lim_{i \to \infty} \varepsilon_i = 0 \) and \( \{ u_{\varepsilon_i} (\cdot) \} \subset Y_{\varepsilon_i}^i \) such that
\[ \lim_{i \to \infty} J_{\varepsilon_i} (u_{\varepsilon_i} (\cdot)) \leq c_{V_0} \text{ and } \lim_{i \to \infty} J_{\varepsilon_i}' (u_{\varepsilon_i} (\cdot)) = 0. \]

Then, for sufficiently small \( d > 0 \), there exists, up to a subsequence, \( \{ y_i \}_{i=1}^\infty \subset \mathbb{R}^N, x \in M, U \in S_{V_0} \) such that
\[ \lim_{i \to \infty} |\varepsilon_i y_i - x| = 0 \text{ and } \lim_{i \to \infty} \| u_{\varepsilon_i} (\cdot) - \varphi_{\varepsilon_i} (\cdot - y_i) U (\cdot - y_i) \|_{X_{\varepsilon_i}} = 0. \]

**Proof.** For convenience’ sake, we write \( \varepsilon \) for \( \varepsilon_i \). According to the compactness of \( S_{V_0} \) and \( M^\theta \), there exist \( Z \in S_{V_0} \) and \( x \in M^\theta \) such that
\[ \| u_{\varepsilon} (\cdot) - \varphi_{\varepsilon} (\cdot - \frac{x}{\varepsilon}) Z (\cdot - \frac{x}{\varepsilon}) \|_{X_{\varepsilon}} \leq 2d \quad (4.3) \]
for small $\varepsilon > 0$. Note that, we denote $u^1_\varepsilon (\cdot) = \varphi_\varepsilon (\cdot - \frac{x}{\varepsilon})u_\varepsilon (\cdot)$ and $u^2_\varepsilon = u_\varepsilon - u^1_\varepsilon$.

As a first step in the proof of this lemma we will check that

$$J_\varepsilon (u_\varepsilon) \geq J_\varepsilon (u^1_\varepsilon) + J_\varepsilon (u^2_\varepsilon) + O(\varepsilon). \quad (4.4)$$

Suppose there exist $x_\varepsilon \in B(\frac{x}{\varepsilon}, \frac{2\beta}{\varepsilon}) \setminus B(\frac{x}{\varepsilon}, \frac{\beta}{\varepsilon})$ and $R > 0$ satisfying $\liminf_{\varepsilon \to 0} \int_{B(x_\varepsilon, R)} (u_\varepsilon)^2 dy > 0$. Going if necessary to a subsequence, we can assume that $\varepsilon x_\varepsilon \to x_0$ with $x_0$ in the closure of $B(x, 2\beta) \setminus B(x, \beta)$ and that $u_\varepsilon (\cdot + x_\varepsilon) \rightharpoonup \tilde{W}$ in $X_\varepsilon$ for some $\tilde{W} \in X_\varepsilon$. Moreover, note that $\tilde{W}$ satisfies

$$(-\Delta)_p \tilde{W} + (-\Delta)_q \tilde{W} + V(x_0)(|\tilde{W}|^{p-2}\tilde{W} + |\tilde{W}|^{q-2}\tilde{W}) = f(\tilde{W}) \in X_\varepsilon.$$

According to definition, $I_{V(x_0)}(\tilde{W}) \geq c_{V(x_0)}$. For large $R > 0$, by using Fatou’s lemma, we also have that

$$\liminf_{\varepsilon \to 0} \int_{B(x_\varepsilon, R)} |\nabla u_\varepsilon|^p dy \geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \tilde{W}|^p dy \quad (4.5)$$

and

$$\liminf_{\varepsilon \to 0} \int_{B(x_\varepsilon, R)} |\nabla u_\varepsilon|^q dy \geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \tilde{W}|^q dy. \quad (4.6)$$

Now, recalling from Remark 3.1 that $c_a > c_b$ if $a > b$, we see that $c_{V(x_0)} \geq c_{V_0}$ because of $V(x_0) \geq V_0$. According to Pohozaev identity, for any $U \in S_{V_0}$,

$$\frac{1}{N} \left( \int_{\mathbb{R}^N} |\nabla U|^p dx + \int_{\mathbb{R}^N} |\nabla U|^q dx \right) = I_{V_0}(U). \quad (4.7)$$

Thus, from (4.5), (4.6) and (4.7) we get that

$$\liminf_{\varepsilon \to 0} \int_{B(x_\varepsilon, R)} |\nabla u_\varepsilon|^p dy + \liminf_{\varepsilon \to 0} \int_{B(x_\varepsilon, R)} |\nabla u_\varepsilon|^q dy \geq \frac{N}{2} I_{V(x_0)}(\tilde{W}) \geq \frac{N}{2} c_{V_0} > 0.$$

Then, taking $d > 0$ sufficiently small, we get a contradiction with (4.3), so there does not exist such a sequence $\{x_\varepsilon\}_\varepsilon$ and we deduce from a result of Lemma 3.3 that

$$\lim_{\varepsilon \to 0} \int_{B(x/\varepsilon, 2\beta/\varepsilon) \setminus B(x/\varepsilon, \beta/\varepsilon)} |u^t_\varepsilon|^p dy = 0,$$

where $t \in (p, q^*)$. As a consequence, we can deduce using $(f_1)$, $(f_2)$ that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \left( F(u_\varepsilon) - F(u^1_\varepsilon) - F(u^2_\varepsilon) \right) dy = 0.$$

At this point, we write

$$J_\varepsilon (u_\varepsilon) \geq J_\varepsilon (u^1_\varepsilon) + J_\varepsilon (u^2_\varepsilon) - \int_{\mathbb{R}^N} \left( F(u_\varepsilon) - F(u^1_\varepsilon) - F(u^2_\varepsilon) \right) dy + O(\varepsilon).$$

Hence, the inequality (4.4) holds.
Next, we estimate $J_\epsilon(u_\epsilon^2)$. Due to $\{u_{\epsilon,k}\}$ is bounded, it is easy to see from (4.3) that $\|u_\epsilon^2\|_\epsilon \leq 4d$ for small $\epsilon > 0$. By using Sobolev’s inequality, for some $C > 0$, we have that

$$J_\epsilon(u_\epsilon^2) \geq I_\epsilon(u_\epsilon^2)$$
$$\geq \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u_\epsilon^2|^p + V_\epsilon |u_\epsilon^2|^p)dx + \frac{1}{q} \int_{\mathbb{R}^N} (|\nabla u_\epsilon^2|^q + V_\epsilon |u_\epsilon^2|^q)dx - \frac{\epsilon}{p} |u_\epsilon^2|_p^p - C_\epsilon |u_\epsilon^2|_q^q$$
$$\geq \frac{1}{2p} \int_{\mathbb{R}^N} (|\nabla u_\epsilon^2|^p + V_\epsilon |u_\epsilon^2|^p)dx + \frac{1}{q} \int_{\mathbb{R}^N} (|\nabla u_\epsilon^2|^q + V_\epsilon |u_\epsilon^2|^q)dx - C_\epsilon |u_\epsilon^2|_q^q$$
$$\geq \frac{1}{2q} \int_{\mathbb{R}^N} (|\nabla u_\epsilon^2|^p + V_\epsilon |u_\epsilon^2|^p)dx + \frac{1}{2q} \int_{\mathbb{R}^N} (|\nabla u_\epsilon^2|^q + V_\epsilon |u_\epsilon^2|^q)dx - C_\epsilon |u_\epsilon^2|_q^q$$
$$\geq \frac{1}{2q} \|u_\epsilon^2\|_\epsilon^p - C \|u_\epsilon^2\|_\epsilon^q$$
$$\geq \|u_\epsilon^2\|_\epsilon^p \left( \frac{1}{2q} - C(4d)^{q-q} \right).$$

In particular, taking $d > 0$ small enough, we can assume that $J_\epsilon(u_\epsilon^2) \geq 0$.

Now let $W_\epsilon(y) = u_\epsilon^2(y + \frac{x}{\epsilon})$. Going if necessary to a subsequence, we can assume that, $W_\epsilon \rightharpoonup W$ in $X_\epsilon$ for some $W$. Moreover $W$ satisfies

$$(-\Delta)_p W(y) + (-\Delta)_q W(y) + V(x)(|W(y)|^{p-2}W(y) + |W(y)|^{q-2}W(y)) = f(W(y)), \ y \in \mathbb{R}^N.$$ 

According to the maximum principle, we obtain that $W$ is positive. Let us prove that $W_\epsilon \rightharpoonup W$ in $X_\epsilon$. Suppose there exist $R > 0$ and a sequence $\{z_\epsilon\}_\epsilon$ with $z_\epsilon \in B(\frac{x}{\epsilon}, \frac{2R}{\epsilon})$ satisfying

$$\liminf_{\epsilon \to 0} |z_\epsilon - \frac{x}{\epsilon}| = \infty \quad \text{and} \quad \liminf_{\epsilon \to 0} \int_{B(z_\epsilon,R)} (u_\epsilon^2)^2 dy > 0.$$ 

We can assume that $\epsilon z_\epsilon \to z_0 \in \Lambda$ as $\epsilon \to 0$. Then we have $\tilde{W}_\epsilon(y) = u_\epsilon^2(y + z_\epsilon)$ converges weakly to $\tilde{W}$ in $X_\epsilon$ satisfying

$$(-\Delta)_p \tilde{W} + (-\Delta)_q \tilde{W} + V(z_0)(|\tilde{W}|^{p-2}\tilde{W} + |\tilde{W}|^{q-2}\tilde{W}) = f(\tilde{W}), \ y \in \mathbb{R}^N.$$ 

At this point as before we get a contradiction, then by using $(f_1), (f_2)$ and Lemma 3.3 we obtain that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^N} F(W_\epsilon) dx \to \int_{\mathbb{R}^N} F(W) dx. \quad (4.8)$$

It follows from the weak convergence of $W_\epsilon$ to $W$ in $X_\epsilon$ that

$$\lim_{\epsilon \to 0} J_\epsilon(u_\epsilon^2) \geq \liminf_{\epsilon \to 0} I_\epsilon(u_\epsilon^2)$$
$$= \liminf_{\epsilon \to 0} \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla W_\epsilon(y)|^p + V_\epsilon |W_\epsilon(y)|^p)dx + \frac{1}{q} \int_{\mathbb{R}^N} (|\nabla W_\epsilon(y)|^q + V_\epsilon |W_\epsilon(y)|^q)dx$$

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Moreover, by using weak lower semi-continuity, we prove the strong convergence of $u_\varepsilon$ and $u_\varepsilon^2$. Proof.\[\lim_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon) \leq c_{V_0}, \ J_\varepsilon(u_\varepsilon^2) \geq 0\] and because of (4.4), we have
\[\limsup_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon) \leq c_{V_0}.\] (4.10)

Combining (4.9) and (4.10), we obtain that $J_\varepsilon(W) = c_{V_0}$. Similar to [29], we can obtain that $x \in M$. At this point it is clear that $W(y) = U(y - z)$ with $U \in S_{V_0}$ and $z \in \mathbb{R}^N$.

Finally, by using (4.8) and (4.10) and the fact that $V(y) \geq V_0$ on $\Lambda$, it follows from (4.9) that
\[\int_{\mathbb{R}^N} (|\nabla W|^p + V_0|W|^p) \, dy \geq \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} (|\nabla u_\varepsilon^1(y)|^p + V(\varepsilon y)|u_\varepsilon^1(y)|^p) \, dy\]
\[\geq \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} (|\nabla u_\varepsilon^1(y)|^p + V_0|u_\varepsilon^1(y)|^p) \, dy\]
\[\geq \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} (|\nabla W_\varepsilon(y)|^p + V_0|W_\varepsilon(y)|^p) \, dy\]

and
\[\int_{\mathbb{R}^N} (|\nabla W|^q + V_0|W|^q) \, dy \geq \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} (|\nabla u_\varepsilon^2(y)|^q + V(\varepsilon y)|u_\varepsilon^2(y)|^q) \, dy\]
\[\geq \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} (|\nabla u_\varepsilon^2(y)|^q + V_0|u_\varepsilon^2(y)|^q) \, dy\]
\[\geq \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} (|\nabla W_\varepsilon(y)|^q + V_0|W_\varepsilon(y)|^q) \, dy.\]

Moreover, by using weak lower semi-continuity, we prove the strong convergence of $u_\varepsilon^1$ to $W$ in $X_\varepsilon$. In particular, setting $y_\varepsilon = x/\varepsilon + z$ we obtain $u_\varepsilon^1 \to \varphi_\varepsilon (\cdot - y_\varepsilon) U (\cdot - y_\varepsilon)$ strongly in $X_\varepsilon$. This means that $u_\varepsilon^1 \to \varphi_\varepsilon (\cdot - y_\varepsilon) U (\cdot - y_\varepsilon)$ strongly in $X_\varepsilon$.

In order to conclude the proof of the Lemma, it suffices to show that $u_\varepsilon^2 \to 0$ in $X_\varepsilon$. Now, using (4.4), $\lim_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon^1) = c_{V_0}$ and the estimation of $J_\varepsilon(u_\varepsilon^2)$, we have that for some $C > 0$
\[c_{V_0} \geq \lim_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon) \geq c_{V_0} + \|u_\varepsilon^2\|_{X_\varepsilon}^q \left(\frac{1}{2q} - C(4d)^{q/2 - q}\right)\]

This proves that $u_\varepsilon^2 \to 0$ in $X_\varepsilon$, which completes the proof. \[\square\]

**Lemma 4.4.** For sufficiently small $d_1 > d_2 > 0$, there exist constants $\omega > 0$ and $\varepsilon_0 > 0$ such that $|J_\varepsilon'(u)| \geq \omega$ for $u \in J_{F_\varepsilon} \cap (Y_{F_\varepsilon} \setminus Y_{F_\varepsilon}^{d_1})$ and $\varepsilon \in (0, \varepsilon_0)$.

**Proof.** To the contrary, we can suppose that for small $d_1 > d_2 > 0$, there exist $\{\varepsilon_i\}_{i=1}^{\infty}$ with $\lim_{i \to \infty} \varepsilon_i = 0$ and $u_{\varepsilon_i} \in Y_{F_\varepsilon} \setminus Y_{F_\varepsilon}^d$ satisfying $\lim_{i \to \infty} J_\varepsilon(u_{\varepsilon_i}) \leq c_{V_0}$ and $\lim_{i \to \infty} |J_\varepsilon'(u_{\varepsilon_i})| = 0$. Note that, for convenience’ sake, we write $\varepsilon$ for $\varepsilon_i$. By using Lemma 4.3, there exists $\{y_\varepsilon\}_{\varepsilon} \subset \mathbb{R}^N$ such that for some $U \in S_{V_0}$ and $x \in M$, \[\lim_{\varepsilon \to 0} |y_\varepsilon - x| = 0\] and \[\lim_{\varepsilon \to 0} \|u_\varepsilon - \varphi_\varepsilon (\cdot - y_\varepsilon) U (\cdot - y_\varepsilon)\| = 0.\]
According to the definition of $Y_\varepsilon$, we obtain that $\lim_{\varepsilon \to 0} \text{dist}(u_\varepsilon, Y_\varepsilon) = 0$. This contradicts that $u_\varepsilon \notin Y^{d_2}_\varepsilon$, and completes the proof.

According to Lemma 4.4, we fix a $d > 0$ and corresponding $\omega > 0$ and $\varepsilon_0 > 0$ such that $|J'_\varepsilon(u)| \geq \omega$ for $u \in J^\varepsilon \cap (Y^{d_2}_\varepsilon \setminus Y^{d_2}_0)$ and $\varepsilon \in (0, \varepsilon_0)$. Then, we obtain the following Lemma.

**Lemma 4.5.** There exists $\alpha > 0$ such that for sufficiently small $\varepsilon > 0$, $J_\varepsilon(\gamma_\varepsilon(s)) \geq c_\varepsilon - \alpha$ implies that $\gamma_\varepsilon(s) \in Y^{d_2}_\varepsilon$ where $\gamma_\varepsilon(s) = W_{c_\varepsilon}(s)$.

**Proof.** Due to $\text{supp}(\gamma_\varepsilon(s)) \subset M^{2p}_s$ for each $s \in [0, 1]$, it follows that $J_\varepsilon(\gamma_\varepsilon(s)) = I_\varepsilon(\gamma_\varepsilon(s))$. Moreover, we see from a change of variables that

$$I_\varepsilon(\gamma_\varepsilon(s)) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla \gamma_\varepsilon(s)|^p + V_\varepsilon |\gamma_\varepsilon(s)|^p)dx + \frac{1}{q} \int_{\mathbb{R}^N} (|\nabla \gamma_\varepsilon(s)|^q + V_\varepsilon |\gamma_\varepsilon(s)|^q)dx$$

$$= \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla \gamma_\varepsilon(s)|^p + V_0 |\gamma_\varepsilon(s)|^p)dx + \frac{1}{q} \int_{\mathbb{R}^N} (|\nabla \gamma_\varepsilon(s)|^q + V_0 |\gamma_\varepsilon(s)|^q)dx$$

$$+ \frac{1}{p} \int_{\mathbb{R}^N} (V_\varepsilon(x) - V_0) |\gamma_\varepsilon(s)|^pdx + \frac{1}{q} \int_{\mathbb{R}^N} (V_\varepsilon(x) - V_0) |\gamma_\varepsilon(s)|^qdx$$

$$- \int_{\mathbb{R}^N} F(\gamma_\varepsilon(s))dx$$

$$= \frac{(s_{0})^{N-p}}{p} \int_{\mathbb{R}^N} |\nabla U|^pdx + \frac{(s_{0})^{N-q}}{q} \int_{\mathbb{R}^N} |\nabla U|^qdx + \frac{(s_{0})^{N}}{p} \int_{\mathbb{R}^N} V_0|U|^pdx$$

$$+ \frac{(s_{0})^{N}}{q} \int_{\mathbb{R}^N} V_0|U|^qdx - (s_{0})^{N} \int_{\mathbb{R}^N} F(U)dx + O(\varepsilon).$$

Then by using the Pohozaev identity, we have that

$$J_\varepsilon(\gamma_\varepsilon(s)) = I_\varepsilon(\gamma_\varepsilon(s))$$

$$= \frac{(s_{0})^{N-p}}{p} \int_{\mathbb{R}^N} |\nabla U|^pdx + \frac{(s_{0})^{N-q}}{q} \int_{\mathbb{R}^N} |\nabla U|^qdx - \frac{N-p}{Np} (s_{0})^{N} \int_{\mathbb{R}^N} |\nabla U|^pdx$$

$$- \frac{N-q}{Nq} (s_{0})^{N} \int_{\mathbb{R}^N} |\nabla U|^qdx + O(\varepsilon)$$

$$= \left(\frac{(s_{0})^{N-p}}{p} - \frac{N-2}{Np} (s_{0})^{N}\right) \int_{\mathbb{R}^N} |\nabla U|^pdx$$

$$+ \left(\frac{(s_{0})^{N-q}}{q} - \frac{N-q}{Nq} (s_{0})^{N}\right) \int_{\mathbb{R}^N} |\nabla U|^qdx + O(\varepsilon).$$

Note that

$$c_{V_0} = \max_{t \in (0, \infty)} \left(\frac{t^{N-p}}{p} - \frac{N-2}{Np} t^{N}\right) \int_{\mathbb{R}^N} |\nabla U|^pdx + \left(\frac{t^{N-q}}{q} - \frac{N-q}{Nq} t^{N}\right) \int_{\mathbb{R}^N} |\nabla U|^qdx$$
and \( \lim_{\varepsilon \to 0} c_{\varepsilon} = c_{V_0} \). Then, since, denoting \( g_1(t) = \frac{\varepsilon^{p-q}}{p} - \frac{N-p}{Np} t^{N} \), \( g_2(t) = \frac{\varepsilon^{q}}{q} - \frac{N-q}{Nq} t^{N} \),

\[
g'_1(t) = \begin{cases} > 0 & \text{for } t \in (0, 1), \\ 0 & \text{for } t = 1, \\ < 0 & \text{for } t > 1, \end{cases} \quad \text{and} \quad g'_2(t) = \begin{cases} > 0 & \text{for } t \in (0, 1), \\ 0 & \text{for } t = 1, \\ < 0 & \text{for } t > 1. \end{cases}
\]

Then we have \( g''_1(1) = p - N < 0 \) and \( g''_2(1) = q - N < 0 \), the conclusion follows. \( \square \)

**Lemma 4.6.** For sufficiently fixed small \( \varepsilon > 0 \), there exists a sequence \( \{u_n\}_{n=1}^\infty \subset Y^d_\varepsilon \cap J^\varepsilon_\gamma \) such that \( J'_\varepsilon (u_n) \to 0 \) as \( n \to \infty \).

**Proof.** According to Lemma 4.5, there exists \( \alpha > 0 \) such that for sufficiently small \( \varepsilon > 0 \), \( J_\varepsilon (\gamma_\varepsilon(s)) \geq c_{\varepsilon} - \alpha \) implies that \( \gamma_\varepsilon(s) \in Y^d_\varepsilon \). If Lemma 4.6 does not hold for sufficiently small \( \varepsilon > 0 \), there exists \( \alpha > 0 \) such that \( J'_\varepsilon(u) \geq \alpha \varepsilon \) on \( Y^d_\varepsilon \cap J^\varepsilon_\gamma \). Also we know from Lemma 4.4 that there exists \( \omega > 0 \), independent of \( \varepsilon > 0 \), such that \( |J'_\varepsilon(u)| \geq \omega \) for \( u \in J^\varepsilon_\gamma \cap (Y^d_\varepsilon \setminus Y^d_\varepsilon) \). Thus, recalling that \( \lim_{\varepsilon \to 0} (c_\varepsilon - c_{\varepsilon}) = 0 \), by a deformation argument, for sufficiently small \( \varepsilon > 0 \), it is possible to construct a path \( \gamma \in \Gamma_\varepsilon \) satisfying \( J_\varepsilon(\gamma(s)) < c_{\varepsilon}, s \in [0, 1] \). This contradiction proves the Lemma. \( \square \)

**Lemma 4.7.** For sufficiently small fixed \( \varepsilon > 0 \), \( J_\varepsilon \) has a critical point \( u_{\varepsilon} \in Y^d_\varepsilon \cap J^\varepsilon_\gamma \).

**Proof.** Let \( \varepsilon > 0 \) be fixed, small enough. According to Lemma 4.6, there exists a sequence \( \{u_{n,\varepsilon}\}_{n=1}^\infty \subset Y^d_\varepsilon \cap J^\varepsilon_\gamma \) such that \( J'_\varepsilon(u_{n,\varepsilon}) \to 0 \) as \( n \to \infty \). Since \( Y^d_\varepsilon \) is bounded, we can assume that \( u_{n,\varepsilon} \rightharpoonup u_{\varepsilon} \) in \( X_\varepsilon \) as \( n \to \infty \). Similar to [14, Proposition 3], we obtain that

\[
\lim_{R \to \infty} \sup_{n \geq 1} \int_{|x| \geq R} \left( |\nabla u_{n,\varepsilon}|^p + V_\varepsilon |u_{n,\varepsilon}|^p \right) dx = 0 \quad (4.11)
\]

and

\[
\lim_{R \to \infty} \sup_{n \geq 1} \int_{|x| \geq R} \left( |\nabla u_{n,\varepsilon}|^q + V_\varepsilon |u_{n,\varepsilon}|^q \right) dx = 0, \quad (4.12)
\]

which immediately implies that \( u_{n,\varepsilon} \to u_\varepsilon \) in \( L^r(\mathbb{R}^N) (p \leq r < q^*) \) as \( n \to \infty \). Moreover, by using \((f_1) - (f_2)\), we have sup \( \|f(u_{n,\varepsilon})\| < \infty \). Then, for any \( \varphi \in C^0_0(\mathbb{R}^N) \),

\[
\int_{\mathbb{R}^N} f(u_{n,\varepsilon})(u_{n,\varepsilon} - u_\varepsilon) \varphi dx \to 0 \quad \text{as} \quad n \to \infty.
\]

Then, similar to [23, Proposition 5.3], \( u_{n,\varepsilon} \to u_\varepsilon \) strongly in \( X_\varepsilon \) as \( n \to \infty \). Thus, \( J'_\varepsilon(u_\varepsilon) = 0 \) in \( X_\varepsilon \) and \( u_\varepsilon \in Y^d_\varepsilon \cap J^\varepsilon_\gamma \). This completes the proof. \( \square \)

Next, we use a Moser iteration argument [35] to obtain a fundamental \( L^\infty \)-estimate.

**Lemma 4.8.** Let \( \{u_n\} \) be the sequence defined as in Lemma 4.3. Then, \( J_{\varepsilon_n}(u_n) \to c_{V_0} \) in \( \mathbb{R} \) as \( n \to \infty \), and there is some sequence \( \{\hat{y}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N \) such that \( \hat{u}_n(\cdot) := u_n(\cdot + \hat{y}_n) \in L^\infty(\mathbb{R}^N) \) and \( |\hat{u}_n|_{L^\infty(\mathbb{R}^N)} \leq C \) for all \( n \in \mathbb{N} \).
Proof. Proceeding as in the proof of Lemma 4.1 and Lemma 4.2, we know that $J_{ε_n}(u_n) → c_{V_0}$ in $ℝ$ as $n → ∞$. Then, we can use Lemma 4.3 to deduce that there is a sequence $\{\hat{y}_n\}_{n∈ℕ} \subset ℝ^N$ such that

$$\hat{u}_n(·) := u_n(· + \hat{y}_n) → \hat{u}(·) \in X_0 \text{ and } y_n := ε_n\hat{y}_n → y_0 \in M \text{ as } n → ∞.$$

For any $L > 0$ and $β > 1$ we introduce the function

$$ψ(\hat{u}_n) := \hat{u}_n\hat{u}^{α(β-1)} \in X_{ε_n}, \text{ where } \hat{u}_{n,L} := \min \{\hat{u}_n, L\}.$$

Choosing $ψ(\hat{u}_n)$ as test function, we have

$$\int _{ℝ^N}|∇\hat{u}|^{p-1}\hat{u}_n(x)dx + \int _{ℝ^N}|∇\hat{u}|^{q-1}\hat{u}_n(x)dx$$

$$+ \int _{ℝ^N}V(ε_nx + y_n)\hat{u}_n^{\beta-2}\hat{u}_nψ(\hat{u}_n)dx + \int _{ℝ^N}V(ε_nx + y_n)|\hat{u}_n|^{q-2}\hat{u}_nψ(\hat{u}_n)dx$$

$$= \int _{ℝ^N}f(ε_nx + y_n, \hat{u}_n)ψ(\hat{u}_n)dx.$$

According to the growth of $f$, we see that for any $σ > 0$ there exists $C_σ > 0$ such that

$$|f(t)| ≤ σ|t|^p + C_σ|t|^q$$

for all $t ∈ ℝ$.

Using $(V_1)$ and taking $σ ∈ (0, V_0)$, together with the above relations, we can conclude that

$$\int _{ℝ^N}|∇\hat{u}_n|^{p-1}\hat{u}_n(x)dx + \int _{ℝ^N}|∇\hat{u}_n|^{q-1}\hat{u}_n(x)dx ≤ C \int _{ℝ^N}|\hat{u}_n|^{q}\hat{u}_n^{α(β-1)}dx$$

(4.13)

for some constant $C > 0$.

Now, let us introduce the following functions

$$φ(t) := \frac{|t|^q}{q} \quad \text{and} \quad Υ(t) := \int _0^t(ψ'(τ))^\frac{1}{q}dτ.$$

We first observe that $ψ$ is an increasing function, so we have that

$$(a - b)(ψ(a) - ψ(b)) ≥ 0 \quad \text{for all } a, b ∈ ℝ.$$

(4.14)

Then by using (4.14) and the Jensen inequality, we can obtain that

$$φ'(a - b)(ψ(a) - ψ(b)) ≥ |Υ(a) - Υ(b)|^q \quad \text{for all } a, b ∈ ℝ.$$

(4.15)

Obviously, we have

$$Υ(\hat{u}_n) ≥ \frac{1}{β}\hat{u}_n\hat{u}^{α(β-1)}.$$

(4.16)

Therefore, by using (4.13), (4.14), (4.15) and (4.16), we can look for some constant $C > 0$ such that

$$|\hat{u}_n\hat{u}_n^{α(β-1)}|^q ≤ Cβ^q \int _{ℝ^N}\hat{u}_n^q\hat{u}_n^{α(β-1)}dx.$$

(4.17)
Choose $\beta = \frac{q^*}{q}$ and let $R > 0$ large enough. According to $\hat{u}_n \to \hat{u}$ in $X_0$ as $n \to \infty$ with the Hölder inequality, we can obtain that there exists some constant $C > 0$ such that

$$\left[ \int_{\mathbb{R}^N} \left( \hat{u}_n \hat{u}_{n,L}^{q^*-q} \right)^{q^*} \right]^\frac{q}{q^*} \leq C \beta^q \int_{\mathbb{R}^N} R^{q^*-q} \hat{u}_n^q \, dx + C \epsilon \left[ \int_{\mathbb{R}^N} \left( \hat{u}_n \hat{u}_{n,L}^{q^*-q} \right)^{q^*} \right]^\frac{q}{q^*} .$$

We choose a fixed $\epsilon \in (0, 1/C)$ and deduce that

$$\left[ \int_{\mathbb{R}^N} \left( \hat{u}_n \hat{u}_{n,L}^{q^*-q} \right)^{q^*} \right]^\frac{q}{q^*} \leq C \beta^q \int_{\mathbb{R}^N} R^{q^*-q} \hat{u}_n^q \, dx + C \epsilon \left[ \int_{\mathbb{R}^N} \left( \hat{u}_n \hat{u}_{n,L}^{q^*-q} \right)^{q^*} \right]^\frac{q}{q^*} .$$

In the above inequality, we pass to the limit as $L \to +\infty$ and we can obtain $\hat{u}_n \in L^{\frac{q^*}{q}}(\mathbb{R}^N)$.

Due to $0 \leq \hat{u}_{n,L} \leq \hat{u}_n$, then in (4.17) we pass to the limit as $L \to +\infty$ and we obtain that

$$|\hat{u}_n|_{L^{\frac{q^*}{q}}} \leq C \beta^q \int_{\mathbb{R}^N} R^{q^*-q} \hat{u}_n^q \, dx .$$

The fact means that

$$\left( \int_{\mathbb{R}^N} \hat{u}_n^{\beta_m q^*} \, dx \right)^\frac{1}{\beta_m q^*} \leq \left( C^{1/q} \beta \right)^{\frac{1}{\beta_m q^*}} \left[ \int_{\mathbb{R}^N} \hat{u}_n^{q^* + (q(\beta_m - 1))} \, dx \right]^{\frac{1}{q(\beta_m - 1)}} .$$

Next, we consider the sequence $\{\beta_m\}_{m \geq 1} \subset \mathbb{R}(m \in \mathbb{N})$ which satisfies the following relation:

$$q^* + q(\beta_{m+1} - 1) = \beta_m q^* \quad \text{and} \quad \beta_1 = \frac{q^*}{q} .$$

It follows that

$$\beta_{m+1} = \beta_m \beta_1 = \beta_1 - 1 + 1 ,$$

and so we have that

$$\lim_{m \to \infty} \beta_m = +\infty .$$

Define

$$T_m := \left( \int_{\mathbb{R}^N} \hat{u}_n^{\beta_m q^*} \, dx \right)^\frac{1}{\beta_m q^*} ,$$

then we have

$$T_{m+1} \leq \left( C^{1/q} \beta_{m+1} \right)^{\frac{1}{\beta_{m+1} q^*}} T_m .$$

Obviously, by using a standard iteration argument we obtain that

$$T_{m+1} \leq \prod_{k=1}^{m} \left( C^{1/q} \beta_{k+1} \right)^{\frac{1}{\beta_{k+1} q^*}} T_1 \leq \tilde{C} T_1 , \text{ where } \tilde{C} \text{ is independent of } m .$$

According to the above inequality we pass to the limit as $m \to \infty$ and then we deduce that $|\hat{u}_n|_{L^{\infty}(\mathbb{R}^N)} \leq C$ uniformly in $n \in \mathbb{N}$.

\[ \square \]
**Proof of Theorem 1.1.** According to Lemma 4.7, there exist \( d > 0 \) and \( \varepsilon_0 > 0 \) such that, for \( \varepsilon \in (0, \varepsilon_0) \), \( J_\varepsilon \) has a critical point \( u_\varepsilon \in Y^d_\varepsilon \cap \Gamma^d_\varepsilon \). Since \( u_\varepsilon \) satisfies

\[
(-\Delta)_p u_\varepsilon + (-\Delta)_q u_\varepsilon + V(\varepsilon x)(|u_\varepsilon|^{p-2} u_\varepsilon + |u_\varepsilon|^{q-2} u_\varepsilon) = f(u_\varepsilon) + 4\left(\int_{\mathbb{R}^N} \chi_\varepsilon u_\varepsilon^p dx - 1\right)\chi_\varepsilon u_\varepsilon \text{ in } \mathbb{R}^N
\]

and \( f(t) = 0 \) for \( t \leq 0 \), we have that \( u_\varepsilon > 0 \) in \( \mathbb{R}^N \). Moreover, by elliptic estimates through Moser iteration scheme, that is Lemma 4.8, we obtain that \( \{\|u_\varepsilon\|_{L^\infty}\}_\varepsilon \) is bounded. Now by using Lemma 4.3, we have

\[
\lim_{\varepsilon \to 0} \frac{1}{p} \left( \int_{\mathbb{R}^N \setminus M^{\varepsilon}_0} |\nabla u_\varepsilon|^p + V_\varepsilon (u_\varepsilon) p \right) + \frac{1}{q} \left( \int_{\mathbb{R}^N \setminus M^{\varepsilon}_0} |\nabla u_\varepsilon|^q + V_\varepsilon (u_\varepsilon) q \right) = 0,
\]

and thus, by elliptic estimates (see [22]), we have that

\[
\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^\infty(\mathbb{R}^N \setminus M^{\varepsilon}_0)} = 0.
\]

Similar to [44], it is easy to check that there exist \( C, c > 0 \), independent of \( u \in S_{V_0} \) such that

\[
u(x) \leq C \exp(-c|x|).
\]

In fact, by using the Radial Lemma ([10], Radial Lemma A.IV) we obtain

\[
u(x) \leq C \frac{\|u\|_{L^p}}{|x|^{N/p}} \quad \text{for all } x \neq 0,
\]

where \( C = C(N, p) \). Thus \( \lim_{|x| \to \infty} \nu(x) = 0 \) uniformly for \( u \in S_{V_0} \). By the comparison principle there exist \( C, c > 0 \), independent of \( u \in S_{V_0} \) such that

\[
u(x) \leq C \exp(-c|x|) \quad \text{for all } x \in \mathbb{R}^N.
\]

According to a comparison principle, for some \( C, c > 0 \), we obtain that

\[
u_\varepsilon(x) \leq C \exp\left(-c \text{ dist} \left( x, M^{\varepsilon}_0 \right) \right).
\]

This implies that \( Q_\varepsilon(u_\varepsilon) = 0 \) and thus \( u_\varepsilon \) satisfies (1.1). Finally let \( x_\varepsilon \) be a maximum point of \( u_\varepsilon \). By Lemma 3.6 and Lemma 4.3, we readily deduce that \( \varepsilon x_\varepsilon \to x \) for some \( x \in M \) as \( \varepsilon \to 0 \), and that for some \( C, c > 0 \)

\[
u_\varepsilon(x) \leq C \exp(-c|x - x_\varepsilon|).
\]

This completes the proof.

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Conflict of interest

The authors declare that they have no competing interests concerning the publication of this article.

References


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