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Research article

Boundedness of some operators on grand Herz spaces with variable exponent

Mehvish Sultan[1], **Babar Sultan**[2], **Ahmad Aloqaily**[3,4,*] **and Nabil Mlaiki**[3]

[1] Department of Mathematics , Capital University Of Science and Technology, Islamabad, Pakistan
[2] Department of Mathematics, Quaid-I-Azam University, Islamabad 45320, Pakistan
[3] Department of Mathematics and Sciences, Prince Sultan University. Riyadh, 11586 Saudi Arabia
[4] School of computer, Data and Mathematical Sciences, Western Sydney University, Australia-Sydney-2150

* **Correspondence:** Email: maloqaily@psu.edu.sa.

Abstract: Our aim in this paper is to prove boundedness of an intrinsic square function and higher order commutators of fractional integrals on grand Herz spaces with variable exponent $\dot{K}^{a(\cdot),u),\theta}_{s(\cdot)}(\mathbb{R}^n)$ by applying some properties of variable exponent.

Keywords: Intrinsic square function; fractional integrals; BMO spaces; grand Lebesgue spaces; grand Herz spaces
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1. Introduction

Function spaces and operator theory are most important tools in harmonic analysis. There is a vast literature dealing with variable exponent spaces, some instances of these works are in [1–7]. In recent times, variable exponent function spaces has witnessed tremendous progress. In fact, it is widely recognized that variable exponent function spaces play an important role in partial differential equations and applied mathematics.

The problem on the boundedness of an intrinsic square function on Lebesgue spaces is considered by [8]. The first generalization of Herz spaces with variable exponent is given by Izuki [9]. He proved boundedness of sublinear operators in these spaces. Herz-Morrey spaces is the generalization of Herz spaces with variable exponent. This class of function spaces is initially defined by the author [10]. In [11], variable parameters were used to define continual Herz spaces, and demonstrated the boundedness of sublinear operators in these spaces.

The idea of grand Morrey spaces introduced in [12] and took considerable amount of attention of researchers, author also proved boundedness of class of integral operators in newly defined grand Morrey spaces. Grand Herz spaces with variable exponent was introduced in [13]. Inspired by the concept, in this article we demonstrated the boundedness of an intrinsic square function and higher order commutators of fractional integral operator in grand Herz spaces with variable exponent.

We divided this article into different sections. Apart from introduction, a section is dedicated to basic lemmas and definitions. One section is for boundedness of intrinsic square function on grand Herz spaces with variable exponent. Last section contains the boundedness of higher order commutators of fractional integral operator in grand Herz spaces with variable exponent.

2. Preliminaries

For this section we refer to [14–18].

Definition 2.1. If H is a measurable set in $\mathbb{R}^n$ and $p(\cdot) : H \to [1, \infty)$ is a measurable function.

(a) Lebesgue space with variable exponent $L^{p(\cdot)}(H)$ can be defined as

$$L^{p(\cdot)}(H) = \left\{ f \textit{ measurable} : \int_H \left(\frac{|f(y)|}{\gamma} \right)^{p(y)} dy < \infty \textit{ where } \gamma \textit{ is a constant} \right\}.$$

Norm in $L^{p(\cdot)}(H)$ can be defined as,

$$\|f\|_{L^{p(\cdot)}(H)} = \inf \left\{ \gamma > 0 : \int_H \left(\frac{|f(y)|}{\gamma} \right)^{p(y)} dy \leq 1 \right\}.$$

(b) The space $L^{p(\cdot)}_{\text{loc}}(H)$ can be defined as

$$L^{p(\cdot)}_{\text{loc}}(H) := \left\{ f : f \in L^{p(\cdot)}(G) \textit{ for all compact subsets } G \subset H \right\}.$$

We use these notations in this paper:

(i) The Hardy-Littlewood maximal operator $\mathcal{M}$ for $f \in L^1_{\text{loc}}(H)$ is defined as

$$\mathcal{M}f(y) := \sup_{s>0} s^{-n} \int\limits_{B(y,s)} |f(y)| dy \quad (y \in H),$$

where $B(y, s) := \{x \in H : |y - x| < s\}$.

(ii) The set $\mathfrak{P}(H)$ is consists of all measuable functions $p(\cdot)$ satisfying

$$p_- := \operatorname*{ess\,inf}_{h \in H} p(h) > 1, \quad p_+ := \operatorname*{ess\,sup}_{h \in H} p(h) < \infty. \tag{2.1}$$

(iii) $\mathfrak{P}^{\log} = \mathfrak{P}^{\log}(H)$ is the class of functions $p \in \mathfrak{P}(H)$ satisfying (2.1) and log-condition defined as,

$$|\Omega(z_1) - \Omega(z_2)| \leq \frac{C(\Omega)}{-\ln|z_1 - z_2|}, \quad |z_1 - z_2| \leq \frac{1}{2}, \quad z_1, z_2 \in H. \tag{2.2}$$

(iv) Let H is unbounded, $\mathfrak{P}_\infty(H)$ and $\mathfrak{P}_{0,\infty}(H)$ are the subsets of $\mathfrak{P}(H)$ and values are in $[1,\infty)$ satisfying following conditions respectively

$$|\Omega(z_1) - \Omega_\infty| \leq \frac{C}{\ln(e + |z_1|)}, \tag{2.3}$$

where $\Omega_\infty \in (1,\infty)$.

$$|\Omega(z_1) - \Omega_0| \leq \frac{C}{\ln|z_1|}, |z_1| \leq \frac{1}{2}, \tag{2.4}$$

in the case of homogenous Herz spaces.

(v) Let H is bounded, then $\mathfrak{P}_\infty(H)$ and $\mathfrak{P}_{0,\infty}(H)$ are the subsets of $\mathfrak{P}(H)$.

(vi) Let H is unbounded, then $\mathfrak{P}_\infty(H)$ are the subsets of exponents in $L^\infty(H)$ and its values are in $[1,\infty]$ satisfying both conditions (2.2) and (2.3), respectively and $\mathfrak{P}_\infty^{\log}(H)$ is the set of exponent $p \in \mathfrak{P}_\infty(H)$ satisfying condition (2.1).

(vii) $\mathcal{B}(H)$ is the collection of $p(\cdot) \in H$ satisfying the condition that M is bounded on $L^{p(\cdot)}(H)$.

(viii) $\chi_l = \chi_{R_l}$, $R_l = D_l \setminus D_{l-1}$, $D_l = D(0, 2^l) = \{z_1 \in \mathbb{R}^n : |z_1| < 2^l\}$ for all $l \in \mathbb{Z}$.

C is a constant, its value varies from line to line and independent of main parameters involved.

Lemma 2.1. *[11] Let $D > 1$ and $\omega \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$. Then*

$$\frac{1}{t_0} s^{\frac{n}{\omega(0)}} \leq \|\chi_{R_{s,D_s}}\|_{\omega(\cdot)} \leq t_0 s^{\frac{n}{\omega(0)}}, \text{ for } 0 < s \leq 1, \tag{2.5}$$

and

$$\frac{1}{t_\infty} s^{\frac{n}{\omega_\infty}} \leq \|\chi_{R_{s,D_s}}\|_{\omega(\cdot)} \leq t_\infty s^{\frac{n}{\omega_\infty}}, \text{ for } s \geq 1, \tag{2.6}$$

respectively, where $t_0 \geq 1$ and $t_\infty \geq 1$ is depending on D but not depending on s.

Lemma 2.2. *[15] [Generalised Hölder's inequality] Assume that H is a measurable subset of $\mathbb{R}^n$, and $1 \leq p_-(H) \leq p_+(H) \leq \infty$. Then*

$$\|\boldsymbol{f}\boldsymbol{g}\|_{L^{r(\cdot)}(H)} \leq \|\boldsymbol{f}\|_{L^{p(\cdot)}(H)} \|\boldsymbol{g}\|_{L^{q(\cdot)}(H)}$$

holds, where $\boldsymbol{f} \in L^{p(\cdot)}(H)$, $\boldsymbol{g} \in L^{q(\cdot)}(H)$ and $\frac{1}{r(z)} = \frac{1}{p(z)} + \frac{1}{q(z)}$ for every $z \in H$.

Definition 2.2. [*BMO space*] A BMO function is a locally integrable function u whose mean oscillation given by $\frac{1}{|Q|}\int_Q |u(y) - u_Q| dy$ is bounded. Mathematically,

$$\|u\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |u(y) - u_Q| dy < \infty.$$

Lemma 2.3. *[19] Let k is a positive integers. Then for all $b \in BMO(\mathbb{R}^n)$ and all $j, i \in \mathbb{Z}$ for $j > i$,*

$$C^{-1}\|b\|^k_{BMO(\mathbb{R}^n)} \leq \sup_{D:ball} \frac{1}{\|\chi_D\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b - b_D)^k \chi_D\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{2.7}$$

$$\leq C\|b\|^k_{BMO(\mathbb{R}^n)}, \tag{2.8}$$

$$\|(b - b_{D_i})^k \chi_{D_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C(j - i)^k \|b\|^k_{BMO(\mathbb{R}^n)} \|\chi_{D_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{2.9}$$

Lemma 2.4. *[19] Let $r(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$; then for all balls D in $\mathbb{R}^n$,*

$$\frac{1}{|D|} \|\chi_D\|_{L^{r(\cdot)}(\mathbb{R}^n)} \|\chi_D\|_{L^{r'(\cdot)}(\mathbb{R}^n)} \leq C. \tag{2.10}$$

2.1. Variable exponent Herz spaces

In this section we will define variable exponent Herz spaces.

Definition 2.3. Let $u, v \in [1,\infty)$, $\zeta \in \mathbb{R}$, the classical versions of homogenous and non-homogenous Herz spaces, can be defined by the norms,

$$\|g\|_{K^{\zeta}_{u,v}(\mathbb{R}^n)} := \|g\|_{L^u(D(0,1))} + \left\{ \sum_{l\in\mathbb{N}} 2^{l\zeta v} \left(\int\limits_{F_{2^{l-1},2^l}} |g(y)|^u dy \right)^{\frac{v}{u}} \right\}^{\frac{1}{v}}, \tag{2.11}$$

$$\|g\|_{\dot{K}^{\zeta}_{u,v}(\mathbb{R}^n)} := \left\{ \sum_{l\in\mathbb{Z}} 2^{l\zeta v} \left(\int\limits_{F_{2^{l-1},2^l}} |g(y)|^u dy \right)^{\frac{v}{u}} \right\}^{\frac{1}{v}}, \tag{2.12}$$

respectively, where $F_{t,\tau}$ stands for the annulus $F_{t,\tau} := D(0,\tau)\backslash D(0,t)$.

Definition 2.4. Let $u \in [1,\infty)$, $\zeta \in \mathbb{R}$ and $v(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$. The homogenous Herz space $\dot{K}^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)$ can be defined as

$$\dot{K}^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{v(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{2.13}$$

where

$$\|g\|_{\dot{K}^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)} = \left(\sum_{l=-\infty}^{l=\infty} \|2^{l\zeta} g\chi_l\|^u_{L^{v(\cdot)}} \right)^{\frac{1}{u}}.$$

Definition 2.5. Let $u \in [1,\infty)$, $\zeta \in \mathbb{R}$ and $v(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$. The non-homogenous Herz space $K^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)$ can be defined as

$$K^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{v(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{K^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{2.14}$$

where

$$\|g\|_{K^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{l=\infty} \|2^{l\zeta} g\chi_l\|^u_{L^{v(\cdot)}} \right)^{\frac{1}{u}} + \|g\|_{L^{v(\cdot)}(D(0,1))}.$$

3. Grand Herz spaces with variable exponent

Next we define Grand Herz spaces with variable exponent.

Definition 3.1. *[20]* Let $a(\cdot) \in L^\infty(\mathbb{R}^n)$, $u \in [1,\infty)$, $v : \mathbb{R}^n \to [1,\infty)$, $\theta > 0$. A grand Herz spaces with variable exponent $\dot{K}^{a(\cdot),u),\theta}_{v(\cdot)}$ is defined by

$$\dot{K}^{a(\cdot),u),\theta}_{v(\cdot)} = \left\{ g \in L^{v(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}^{a(\cdot),u),\theta}_{v(\cdot)}} < \infty \right\},$$

where

$$\begin{aligned} \|g\|_{\dot{K}^{a(\cdot),u),\theta}_{v(\cdot)}} &= \sup_{\psi>0} \left(\psi^{\theta} \sum_{k\in\mathbb{Z}} 2^{ka(\cdot)u(1+\psi)} \|g\chi_k\|^{u(1+\psi)}_{L^{v(\cdot)}} \right)^{\frac{1}{u(1+\psi)}} \\ &= \sup_{\psi>0} \psi^{\frac{\theta}{u(1+\psi)}} \|g\|_{\dot{K}^{a(\cdot),u(1+\psi)}_{v(\cdot)}}. \end{aligned}$$

4. Boundedness of an intrinsic square function

In this section, we show that an intrinsic square function is bounded on $\dot{K}^{a(\cdot),u),\theta}_{s(\cdot)}(\mathbb{R}^n)$. First we will define intrinsic square function $S_\zeta f(z_1)$.

Definition 4.1. Let $z_1 \in \mathbb{R}^n$, the set $\Gamma(z_1)$ is defined as,

$$\Gamma(z_1) :\equiv \{(z_2, t) \in \mathbb{R}^{n+1}_+ : |z_1 - z_2| < t\},$$

where $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$. Let $0 < \zeta \leq 1$ is a constant. The set C_ζ consists of all functions ϕ defined on $\mathbb{R}^n$ such that

(i) $\operatorname{supp} \phi \subset \{|z_1| \leq 1\}$,
(ii) $\int_{\mathbb{R}^n} \phi(z_1)dz_1 = 0$,
(iii) $|\phi(z_1) - \phi(z_1')| \leq |z_1 - z_1'|^\zeta$ for $z_1, z_1' \in \mathbb{R}^n$.

For every $(z_2, t) \in \mathbb{R}^{n+1}_+$ we write $\phi_t(z_2) = t^{-n}\phi(z_2/t)$. Then we define a maximal function for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

$A_\zeta f(z_2, t) := \sup_{\phi \in C_\zeta} |f * \phi_t(z_2)|$, where $(z_2, t) \in \mathbb{R}^{n+1}_+$. Using above, we define the intrinsic square function with order ζ by

$$S_\zeta f(z_1) := \left(\int \int_{\Gamma(z_1)} A_\zeta f(z_2, t)^2 \frac{dz_2 dt}{t^{n+1}} \right)^{1/2}.$$

An intrinsic square function S_ζ is bounded in variable Lebesgue spaces $L^{p(\cdot)}$, for more detail see [8].

Theorem 4.1. *Let $1 < u < \infty$, $a(\cdot), s(\cdot) \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$, and ζ be such that*

(i) $\frac{-n}{s(0)} < a(0) < \frac{n}{s'(0)}$,
(ii) $\frac{-n}{s_\infty} < a_\infty < \frac{n}{s'_\infty}$.

Suppose that an intrinsic square function S_ζ bounded on Lebesgue spaces will be bounded on $\dot{K}^{a(\cdot),u),\theta}_{s(\cdot)}(\mathbb{R}^n)$.

Proof. Let $f \in \dot{K}^{a(\cdot),u),\theta}_{s(\cdot)}(\mathbb{R}^n)$, we have

$$\begin{aligned}
\|S_\zeta f\|_{\dot{K}^{a(\cdot),u),\theta}_{s(\cdot)}(\mathbb{R}^n)} &= \sup_{\psi>0} \left(\psi^\theta \sum_{k\in\mathbb{Z}} 2^{ka(\cdot)u(1+\psi)} \|\chi_k S_\zeta f\|^{u(1+\psi)}_{L^{s(\cdot)}} \right)^{\frac{1}{u(1+\psi)}} \\
&\leq \sup_{\psi>0} \left(\psi^\theta \sum_{k\in\mathbb{Z}} 2^{ka(\cdot)u(1+\psi)} \left(\sum_{l=-\infty}^{\infty} \|\chi_k S_\zeta f(\chi_l)\|^{u(1+\psi)}_{L^{s(\cdot)}} \right) \right)^{\frac{1}{u(1+\psi)}} \\
&\leq \sup_{\psi>0} \left(\psi^\theta \sum_{k\in\mathbb{Z}} 2^{ka(\cdot)u(1+\psi)} \left(\sum_{l=-\infty}^{k-2} \|\chi_k S_\zeta (f\chi_l)\|_{L^{s(\cdot)}} \right)^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}} \\
&+ \sup_{\psi>0} \left(\psi^\theta \sum_{k\in\mathbb{Z}} 2^{ka(\cdot)u(1+\psi)} \left(\sum_{l=k-1}^{k+1} \|\chi_k S_\zeta (f\chi_l)\|_{L^{s(\cdot)}} \right)^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}}
\end{aligned}$$

$$+\sup_{\psi>0}\left(\psi^{\theta}\sum_{k\in\mathbb{Z}}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=k+2}^{\infty}\|\chi_k S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$
$$=: E_1+E_2+E_3.$$

As operator S_{ζ} is bounded on Lebesgue space $L^{s(\cdot)}(\mathbb{R}^n)$ so for E_2,

$$E_2\leq\sup_{\psi>0}\left(\psi^{\theta}\sum_{k\in\mathbb{Z}}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=k-1}^{k+1}\|S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$
$$\leq\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=k-1}^{k+1}\|S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$
$$+\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=k-1}^{k+1}\|S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$
$$=: E_{21}+E_{22}.$$

By using the fact $2^{ka(z_1)}=2^{ka(0)}$, $k<0$, $z_1\in R_k$ implies that

$$\|2^{ka(\cdot)}f\chi_k\|_{L^{s(\cdot)}}=2^{ka(0)}\|f\chi_k\|_{L^{s(\cdot)}},$$

$$E_{21}\leq\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=k-1}^{k+1}\|S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$
$$\leq C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\psi)}\left(\sum_{l=k-1}^{k+1}\|f\chi_l\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$
$$\leq C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k\in\mathbb{Z}}2^{ka(\cdot)u(1+\psi)}\|f\chi_k\|_{L^{s(\cdot)}}^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}=C\|f\|_{\dot{K}^{a(\cdot),u),\theta}_{s(\cdot)}(\mathbb{R}^n)}.$$

For E_{22}, we use the fact $2^{ka(z_1)}=2^{ka_{\infty}}$, $k\geq 0$, $z_1\in R_k$, we get,

$$E_{22}\leq\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=k-1}^{k+1}\|S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$
$$\leq C\|\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{ka_{\infty}u(1+\psi)}\left(\sum_{l=k-1}^{k+1}\|f\chi_l\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$
$$\leq C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{ka_{\infty}u(1+\psi)}\|f\chi_k\|_{L^{s(\cdot)}}^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$

$$
\begin{aligned}
&\leq C \sup_{\psi>0}\left(\psi^{\theta}\sum_{k\in\mathbb{Z}} 2^{ka(\cdot)u(1+\psi)}\|f\chi_k\|_{L^{s(\cdot)}}^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&= C\|f\|_{\dot{K}_{s(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)}.
\end{aligned}
$$

For E_1, for $k \in \mathbb{Z}$, $l \leq k-2$, $z_1 \in R_k$ and $(z_2, t) \in \Gamma(z_1)$. For $\phi \in C_\zeta$ we have

$$
\begin{aligned}
|f(\chi_l) * \phi_t(z_2)| &= \left|\int_{R_l} \phi_t(z_2) f(x)dx\right|\\
&\leq Ct^{-n}\int_{\{x\in R_l : |z_2-x|<t\}} |f(x)|dx.
\end{aligned}
$$

$x \in R_l$ with $|z_2 - x| < t$ we obtain

$$
\begin{aligned}
t =& \frac{1}{2}(t+t) > \frac{1}{2}(|z_1 - z_2| + |z_2 - x|) \geq \frac{1}{2}|z_1 - x| \geq \frac{1}{2}(|z_1| - |x|)\\
\geq& \frac{1}{2}(|z_1| - 2^{t}) \geq \frac{1}{2}(|z_1| - 2^{k-2}) \geq \frac{1}{2}(|z_1| - 2^{-1}|z_1|) = \frac{|z_1|}{4}.
\end{aligned}
$$

As a result we get

$$
\begin{aligned}
&|S_\zeta(f\chi_l)(z_1)|\\
&= \left(\int\int_{\Gamma(z_1)}\left(\sup_{\phi\in C_\zeta}|f\chi_l * \phi_t(z_2)|^2\frac{dz_2dt}{t^{n+1}}\right)^2\right)^{1/2}\\
&\leq C\left(\int_{\frac{|z_1|}{4}}^{\infty}\int_{\{z_2:|z_1-z_2|<t\}}\left(\frac{1}{t^n}\int_{\{x\in R_l:|z_-x|<t\}}|f(x)|dx\right)^2\frac{dz_2dt}{t^{n+1}}\right)^{1/2}\\
&\leq C\left(\int_{R_l}|f(x)|dx\right)\left(\int_{\frac{|z_1|}{4}}^{\infty}\left(\int_{\{z_2:|z_1-z_2|<t\}}dz_2\right)\frac{dt}{t^{3n+1}}\right)^{1/2}\\
&= C\left(\int_{R_l}|f(x)|dx\right)\left(\int_{\frac{|z_1|}{4}}^{\infty}\frac{dt}{t^{2n+1}}\right)^{1/2}\\
&= C\left(\int_{R_l}|f(x)|dx\right)|z_1|^{-n}.
\end{aligned}
$$

Note that $z_1 \in R_k$ and $|z_1| > 2^{k-1}$, implies that $|z_1|^{-n} \leq 2^{-kn}$, by using Hölder's inequality,

$$
S_\zeta(f\chi_l)(z_1)| \leq C2^{-kn}\left(\int_{R_l}|f(x)|dx\right)
$$

$$\leq C2^{-kn}\|f\chi_k\|_{L^{s(\cdot)}}\|\chi_l\|_{L^{s'(\cdot)}}.$$

By using Lemma (2.1) we have

$$2^{-kn}\|\chi_k\|_{L^{s(\cdot)}}\|\chi_l\|_{L^{s'(\cdot)}} \leq C2^{-kn}2^{\frac{kn}{s(0)}}2^{\frac{ln}{s'(0)}} \leq C2^{\frac{(l-k)n}{s'(0)}}. \tag{4.1}$$

Splitting E_1 by means of Minkowski's inequality we have

$$\begin{aligned} E_1 \leq & \sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=-\infty}^{k-2}\|\chi_k S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}} \\ & + \sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=-\infty}^{k-2}\|\chi_k S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}} \\ & =: E_{11} + E_{12}. \end{aligned}$$

Applying above results to E_{11} we can get

$$\begin{aligned} E_{11} \leq & \sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=-\infty}^{k-2}\|\chi_k S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}} \\ \leq & C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=-\infty}^{k-2}\|\chi_k\|_{L^{s(\cdot)}}2^{-kn}\|f\chi_l\|_{L^{s(\cdot)}}\|\chi_l\|_{L^{s'(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}} \\ \leq & C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=-\infty}^{k-2}\|\chi_k\|_{L^{s(\cdot)}}2^{-kn}\|f\chi_l\|_{L^{s(\cdot)}}\|\chi_l\|_{L^{s'(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}. \end{aligned}$$

Let $b = \frac{n}{s'(0)} - a(0)$,

$$E_{11} \leq C\sup_{\psi>0}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2}2^{a(0)l}\|f\chi_l\|_{L^{s(\cdot)}}2^{b(l-k)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}, \tag{4.2}$$

by using Hölder's inequality, Fubini's theorem for series and the inequality $2^{-u(1+\psi)} < 2^{-u}$ we get,

$$\begin{aligned} E_{11} \leq & C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2}2^{a(0)u(1+\psi)l}\|f\chi_l\|_{L^{s(\cdot)}}^{u(1+\psi)}2^{bu(1+\psi)(l-k)/2}\sum_{l=-\infty}^{k-2}2^{b(u(1+\psi))'(l-k)/2}\right)^{\frac{u(1+\psi)}{(u(1+\psi))'}}\right)^{\frac{1}{u(1+\psi)}} \\ = & C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}\sum_{l=-\infty}^{k-2}2^{a(0)u(1+\psi)l}\|f\chi_l\|_{L^{s(\cdot)}}^{u(1+\psi)}2^{bu(1+\psi)(l-k)/2}\right)^{\frac{1}{u(1+\psi)}} \\ = & C\sup_{\psi>0}\left(\psi^{\theta}\sum_{l=-\infty}^{-1}2^{a(0)u(1+\psi)l}\|f\chi_l\|_{L^{s(\cdot)}}^{u(1+\psi)}\sum_{k=l+2}^{-1}2^{bu(1+\psi)(l-k)/2}\right)^{\frac{1}{u(1+\psi)}} \end{aligned}$$

$$
\begin{aligned}
&< C\sup_{\psi>0}\left(\psi^{\theta}\sum_{l=-\infty}^{-1}2^{a(0)u(1+\psi)l}\|f\chi_l\|_{L^{s(\cdot)}}^{u(1+\psi)}\sum_{k=l+2}^{-1}2^{bp(l-k)/2}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\sup_{\psi>0}\left(\psi^{\theta}\sum_{l=-\infty}^{-1}2^{a(0)u(1+\psi)l}\|f\chi_l\|_{L^{s(\cdot)}}^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&= C\sup_{\psi>0}\left(\psi^{\theta}\sum_{l\in\mathbb{Z}}2^{a(\cdot)u(1+\psi)l}\|f\chi_l\|_{L^{s(\cdot)}}^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\|f\|_{\dot{K}_{s(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)}.
\end{aligned}
$$

Now for E_{12} using Minkowski's inequality we have

$$
\begin{aligned}
E_{12} \le &\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=-\infty}^{-1}\|\chi_k S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&+\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=0}^{k-2}\|\chi_k S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&=: A_1 + A_2.
\end{aligned}
$$

The estimate for A_2 can be obtained by similar way to E_{11} by replacing $s'(0)$ with s'_∞ and using the fact $\frac{n}{s'_\infty} - a_\infty > 0$. For A_1 using Lemma (2.1) we have

$$
2^{-kn}\|\chi_k\|_{L^{s(\cdot)}}\|\chi_l\|_{L^{s'(\cdot)}} \le C2^{-kn}2^{\frac{kn}{s_\infty}}2^{\frac{ln}{s'(0)}} \le C2^{\frac{-kn}{s'_\infty}}2^{\frac{ln}{s'(0)}}. \tag{4.3}
$$

As $a_\infty - \frac{n}{s'_\infty} < 0$ we have

$$
\begin{aligned}
A_1 \le &\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{ka_\infty u(1+\psi)}\left(\sum_{l=-\infty}^{-1}\|\chi_k S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\sup_{\epsilon>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{ka_\infty u(1+\psi)}\times\left(\sum_{l=-\infty}^{-1}2^{\frac{-kn}{s_\infty}}2^{\frac{ln}{s'(0)}}\|f\chi_l\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\sup_{\psi>0}\left(\epsilon^{\theta}\sum_{k=0}^{\infty}2^{\frac{k\alpha-kn}{s'_\infty}u(1+\psi)}\times\left(\sum_{l=-\infty}^{-1}2^{\frac{ln}{s'(0)}}\|f\chi_l\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\sup_{\psi>0}\left(\psi^{\theta}\left(\sum_{l=-\infty}^{-1}2^{\frac{ln}{s'(0)}}\|f\chi_l\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\sup_{\psi>0}\left(\psi^{\theta}\left(\sum_{l=-\infty}^{-1}2^{\frac{ln}{s'(0)}-a(0)l}\|f\chi_l\|_{L^{s(\cdot)}}2^{a(0)l}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}
\end{aligned}
$$

Now by applying Hölder's inequality and using the fact that $\frac{n}{s'(0)} - a(0) > 0$ we have

$$\begin{aligned}
A_1 \leq & C \sup_{\psi>0} \left(\psi^{\theta} \left(\sum_{l=-\infty}^{-1} 2^{\frac{ln}{s'(0)} - a(0)l} \|f\chi_l\|_{L^{s(\cdot)}} 2^{a(0)l} \right)^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}} \\
\leq & C \sup_{\psi>0} \left(\psi^{\theta} \sum_{l=-\infty}^{-1} 2^{a(0)lu(1+\psi)} \|f\chi_l\|^{u(1+\psi)}_{L^{s(\cdot)}} \left(\sum_{l=-\infty}^{-1} 2^{(\frac{ln}{s'(0)} - a(0)l)(u(1+\psi))'} \right)^{\frac{u(1+\psi)}{(u(1+\psi))'}} \right)^{\frac{1}{u(1+\psi)}} \\
\leq & C \sup_{\psi>0} \left(\psi^{\theta} \sum_{l\in\mathbb{Z}} 2^{a(\cdot)lu(1+\psi)} \|f\chi_l\|^{u(1+\psi)}_{L^{s(\cdot)}} \right)^{\frac{1}{u(1+\psi)}} \\
\leq & C\|f\|_{\dot{K}^{a(\cdot),u),\theta}_{s(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Now we estimate E_3, we take $k \in \mathbb{Z}$, $l \geq k-2$, $z_1 \in R_k$ and $(z_2, t) \in \Gamma(z_1)$. For $x \in R_l$, $\phi \in C_\zeta$ with $|z_2 - x| < t$ we obtain

$$\begin{aligned}
t = \frac{1}{2}(t+t) > \frac{1}{2}(|z_1 - z_2| + |z_2 - x|) \geq & \frac{1}{2}|z_1 - x| \geq \frac{1}{2}(|x| - |z_1|) \\
\geq & \frac{1}{2}(2^{l-1} - 2^k) \geq 2^{l-3}.
\end{aligned}$$

As a result we get

$$\begin{aligned}
& |S_\zeta(f\chi_l)(z_1)| \\
& = \left(\int\int_{\Gamma(z_1)} \left(\sup_{\phi\in C_\zeta} |f\chi_l * \phi_t(z_2)|^2 \frac{dz_2 dt}{t^{n+1}} \right)^2 \right)^{1/2} \\
& \leq C \left(\int\int_{\Gamma(z_1)} \left(\frac{1}{t^n} \int_{\{x\in R_l : |z_2 - x| < t\}} |f(x)| dx \right)^2 \frac{dz_2 dt}{t^{n+1}} \right)^{1/2} \\
& \leq C \left(\int_{R_l} |f(x)| dx \right) \left(\int_{2^{l-3}}^{\infty} \left(\int_{\{z_2 : |z_1 - z_2| < t\}} dz_2 \right) \frac{dt}{t^{3n+1}} \right)^{1/2} \\
& = C \left(\int_{R_l} |f(x_1)| dx_1 \right) \left(\int_{2^{l-3}}^{\infty} \frac{dt}{t^{2n+1}} \right)^{1/2} \\
& = C \left(\int_{R_l} |f(x)| dx \right) |B_l|^{-1} \\
& \leq C 2^{-ln} \|f\chi_l\|_{L^{s(\cdot)}} \|\chi_l\|_{L^{s'(\cdot)}}.
\end{aligned}$$

Splitting E_3 by using Minkowski's inequality we have

$$E_3 \leq \sup_{\psi>0} \left(\psi^{\theta} \sum_{k\in\mathbb{Z}} 2^{ka(\cdot)u(1+\psi)} \left(\sum_{l=k+2}^{\infty} \|\chi_k S_\zeta(f\chi_l)\|_{L^{s(\cdot)}} \right)^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}}$$

$$
\begin{aligned}
&\leq \sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=k+2}^{\infty}\|\chi_k S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&+\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=k+2}^{\infty}\|\chi_k S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&=:E_{31}+E_{32}.
\end{aligned}
$$

For E_{32} Lemma (2.1) yields

$$
2^{-ln}\|\chi_k\|_{L^{s(\cdot)}}\|\chi_l\|_{L^{s'(\cdot)}} \leq C2^{-ln}2^{\frac{kn}{s_\infty}}2^{\frac{ln}{s'_\infty}} \leq C2^{\frac{(k-l)n}{s_\infty}}, \tag{4.4}
$$

we get

$$
\begin{aligned}
E_{32} &\leq \sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=k+2}^{\infty}\|\chi_k S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\leq C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=k+2}^{\infty}\|\chi_k\|_{L^{s(\cdot)}}2^{-ln}\cdot\|f\chi_l\|_{L^{s(\cdot)}}\|\chi_l\|_{L^{s'(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\leq C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}\left(\sum_{l=k+2}^{\infty}2^{a_\infty l}\|f\chi_l\|_{L^{s(\cdot)}}2^{d(k-l)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}},
\end{aligned}
$$

where $d = \frac{n}{s_\infty} + \alpha_\infty > 0$. Then we use Hölder's theorem for series and $2^{-u(1+\psi)} < 2^{-u}$ to obtain

$$
\begin{aligned}
E_{32} \leq & C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}\left(\sum_{l=k+2}^{\infty}2^{\alpha_\infty u(1+\psi)l}\|f\chi_l\|_{L^{s(\cdot)}}^{u(1+\psi)}2^{du(1+\psi)(k-l)/2}\right)\right.\\
&\left.\times\left(\sum_{l=k+2}^{\infty}2^{d(u(1+\psi))'(k-l)/2}\right)^{\frac{u(1+\psi)}{(u(1+\psi))'}}\right)^{\frac{1}{u(1+\psi)}}\\
\leq & C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}\sum_{l=k+2}^{\infty}2^{a_\infty u(1+\psi)l}\|f\chi_l\|_{L^{s(\cdot)}}^{u(1+\psi)}2^{du(1+\psi)(k-l)/2}\right)^{\frac{1}{u(1+\psi)}}\\
\leq & C\sup_{\psi>0}\left(\psi^{\theta}\sum_{l=0}^{\infty}2^{a_\infty u(1+\psi)l}\|f\chi_l\|_{L^{s(\cdot)}}^{u(1+\psi)}\sum_{k=0}^{l-2}2^{du(1+\psi)(k-l)/2}\right)^{\frac{1}{u(1+\psi)}}\\
\leq & C\sup_{\psi>0}\left(\psi^{\theta}\sum_{l\in\mathbb{Z}}2^{a_\infty u(1+\psi)l}\|f\chi_l\|_{L^{s(\cdot)}}^{u(1+\psi)}\sum_{k=-\infty}^{l-2}2^{dp(k-l)/2}\right)^{\frac{1}{u(1+\psi)}}\\
\leq & C\sup_{\psi>0}\left(\psi^{\theta}\sum_{l\in\mathbb{Z}}2^{a(\cdot)u(1+\psi)l}\|f\chi_l\|_{L^{s(\cdot)}}^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
\leq & C\|f\|_{\dot{K}_{s(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)}.
\end{aligned}
$$

Now for E_{31} using Monkowski's inequality we have

$$
\begin{aligned}
E_{31} \le& \sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=k+2}^{-1}\|\chi_k S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&+\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{ka(\cdot)u(1+\psi)}\left(\sum_{l=0}^{\infty}\|\chi_k S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
=:& B_1+B_2.
\end{aligned}
$$

The estimate for B_1 can be obtained by similar way to E_{32} by replacing s_∞ with $s(0)$ and using the fact that $\frac{n}{s(0)}+a(0)>0$. For B_2 using Lemma (2.1) we have

$$
2^{-ln}\|\chi_k\|_{L^{s(\cdot)}}\|\chi_l\|_{L^{s'(\cdot)}} \le C2^{-ln}2^{\frac{kn}{s(0)}}2^{\frac{ln}{s'_\infty}} \le C2^{\frac{kn}{s(0)}}2^{\frac{-ln}{s_\infty}} \tag{4.5}
$$

$$
\begin{aligned}
B_2 \le& \sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\psi)}\left(\sum_{l=0}^{\infty}\|\chi_k S_{\zeta}(f\chi_l)\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
\le& C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\psi)}\times\left(\sum_{l=0}^{\infty}2^{-ln}2^{\frac{kn}{s(0)}}2^{\frac{ln}{s'_\infty}}\|f\chi_l\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
\le& C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{ka(0)u(1+\psi)}\times\left(\sum_{l=0}^{\infty}2^{\frac{kn}{s(0)}}2^{\frac{-ln}{s_\infty}}\|f\chi_l\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
\le& C\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=-\infty}^{-1}2^{k(a(0)+n)/s(0)u(1+\psi)}\times\left(\sum_{l=0}^{\infty}2^{\frac{-ln}{s_\infty}}\|f\chi_l\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
\le& C\sup_{\psi>0}\left(\psi^{\theta}\left(\sum_{l=0}^{\infty}2^{\frac{-ln}{s_\infty}}\|f\chi_l\|_{L^{s(\cdot)}}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
\le& C\sup_{\psi>0}\left(\psi^{\theta}\left(\sum_{l=0}^{\infty}2^{la_\infty}\|f\chi_l\|_{L^{s(\cdot)}}2^{l(ns_\infty+a_\infty)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}.
\end{aligned}
$$

Now by applying Hölder's inequality and using the fact that $\frac{n}{s_\infty}+a_\infty>0$ we have

$$
\begin{aligned}
B_2 \le& C\sup_{\psi>0}\left(\psi^{\theta}\left(\sum_{l=0}^{\infty}2^{la_\infty u(1+\psi)}\|f\chi_l\|_{L^{s(\cdot)}}^{u(1+\psi)}\right)^{u(1+\psi)}\right.\\
&\left.\times\left(\sum_{l=0}^{\infty}2^{l(ns_\infty+a_\infty)u(1+\psi)}\right)^{\frac{u(1+\psi)}{u(1+\psi)'}}\right)^{\frac{1}{u(1+\psi)}}\\
\le& C\sup_{\psi>0}\left(\psi^{\theta}\left(\sum_{l\in\mathbb{Z}}2^{a_\infty lu(1+\psi)}\|f\chi_l\|_{L^{s(\cdot)}}^{u(1+\psi)}\right)\right)^{\frac{1}{u(1+\psi)}}
\end{aligned}
$$

$$\leq C\|f\|_{\dot{K}^{a(\cdot),u),\theta}_{s(\cdot)}(\mathbb{R}^n)}.$$

Combining the estimates for E_1, E_2 and E_3 yields

$$\|S_\zeta f\|_{\dot{K}^{a(\cdot),u),\theta}_{s(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{\dot{K}^{a(\cdot),u),\theta}_{s(\cdot)}(\mathbb{R}^n)}$$

which ends the proof.

5. Boundedness of higher order commutators of fractional integral operator

5.1. Higher order commutators of fractional integral operator

Let b be a locally integrable function, $0 < \zeta < n$, and $m \in \mathbb{N}$; the higher order commutators of fractional integrable operator $I^m_{\zeta,b}$ are defined by

$$I^m_{\zeta,b}f(x) = \int_{\mathbb{R}^n} \frac{[b(x) - b(y)]^m}{|x - y|^{n-\zeta}} f(y)dy. \tag{5.1}$$

When $m = 0$ then $I^0_{\zeta,b} = I_\zeta$ and for, $m = 1$ $I^1_{\zeta,b} = [b, I_\zeta]$. According to Hardy-Littlewood-Sobolev theorem the fractional integral operator I_ζ is bounded operator from Lebesgue spaces $L^{p_1}(\mathbb{R}^n)$ to $L^{p_2}(\mathbb{R}^n)$ when $0 < p_1 < p_2 < \infty$ and $1/p_1 - 1/p_2 = \zeta/n$.

Lemma 5.1. *[19] Suppose that $q_1(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$ satisfies (2.2), (2.3), $0 < \zeta < n/(q_1)_+$ and $1/q_1(x) - 1/q_2(x) = \zeta/n$, $b \in BMO(\mathbb{R}^n)$ then*

$$\|I^m_{\zeta,b}(f)\|_{q_2(\cdot)} \leq C\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{q_1(\cdot)}. \tag{5.2}$$

Theorem 5.1. *Let $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$, $\alpha, q_2 \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$, $1/q_1(x) - 1/q_2(x) = \zeta/n$. If $0 < r < \min\{1/(q_1)_+, 1(q_2')_+\}$, $0 < \zeta < nr$, and*

$$\frac{-n}{q_{1\infty}} < a_\infty < \frac{n}{q'_{1\infty}}, \quad \frac{-n}{q_1(0)} < a(0) < \frac{n}{q'_1(0)}.$$

Suppose that $I^m_{\zeta,b}$ is higher order commutators of fractional integral operator bounded on Lebesgue spaces will be bounded from $\dot{K}^{a(\cdot),p),\theta}_{q_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{a(\cdot),p),\theta}_{q_2(\cdot)}(\mathbb{R}^n)$.

Proof. Let $f \in \dot{K}^{a(\cdot),p),\theta}_{q_2(\cdot)}(\mathbb{R}^n)$, and $f(x) = \sum_{l=-\infty}^{\infty} f(x)\chi_l(x) = \sum_{l=-\infty}^{\infty} f_l(x)$, we have

$$\|I^m_{\zeta,b}(f)\|_{\dot{K}^{a(\cdot),p),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} = \sup_{\epsilon>0}\left(\epsilon^\theta \sum_{k\in\mathbb{Z}} 2^{ka(\cdot)p(1+\epsilon)}\|\chi_k I^m_{\zeta,b}f\|^{p(1+\epsilon)}_{q_2(\cdot)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$\leq \sup_{\epsilon>0}\left(\epsilon^\theta \sum_{k\in\mathbb{Z}} 2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=-\infty}^{\infty} \|\chi_k I^m_{\zeta,b}f(\chi_l)\|^{p(1+\epsilon)}_{q_2(\cdot)}\right)\right)^{\frac{1}{p(1+\epsilon)}}$$

$$\leq \sup_{\epsilon>0}\left(\epsilon^\theta \sum_{k\in\mathbb{Z}} 2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=-\infty}^{k-2} \|\chi_k I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$
\begin{aligned}
&+\sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k\in\mathbb{Z}}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=k-1}^{k+1}\|\chi_k I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&+\sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k\in\mathbb{Z}}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=k+2}^{\infty}\|\chi_k I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&=:E_1+E_2+E_3.
\end{aligned}
$$

As operator $I^m_{\zeta,b}$ is bounded on Lebesgue space $q_2(\cdot)$ so for E_2,

$$
\begin{aligned}
E_2 &\le\sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k\in\mathbb{Z}}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=k-1}^{k+1}\|I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&\le\sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=k-1}^{k+1}\|I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&+\sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k=0}^{\infty}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=k-1}^{k+1}\|I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&:=E_{21}+E_{22}.
\end{aligned}
$$

Here we use the fact $2^{k\alpha(x)}=2^{k\alpha(0)}$, $k<0$, $x\in R_k$ equivalent to say that

$$
\|2^{ka(\cdot)}f\chi_k\|_{q_1(\cdot)}=2^{k\alpha(0)}\|f\chi_k\|_{q_1(\cdot)},
$$

$$
\begin{aligned}
E_{21} &\le\sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=k-1}^{k+1}\|I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&\le C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(0)p(1+\epsilon)}\left(\sum_{l=k-1}^{k+1}\|f\chi_l\|_{q_1(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&\le C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(0)p(1+\epsilon)}\|f\chi_k\|^{p(1+\epsilon)}_{q_1(\cdot)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&\le C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k\in\mathbb{Z}}2^{ka(\cdot)p(1+\epsilon)}\|f\chi_k\|^{p(1+\epsilon)}_{q_1(\cdot)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&=C\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{\dot{K}^{a(\cdot),p),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

For E_{22}, we use the fact $2^{k\alpha(x)}=2^{k\alpha_\infty}$, $k<0$, $x\in R_k$, we have,

$$
E_{22}\le C\sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k=0}^{\infty}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=k-1}^{k+1}\|I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}
$$

$$
\begin{aligned}
&\le C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0}\left(\epsilon^\theta \sum_{k=0}^{\infty} 2^{ka_\infty p(1+\epsilon)}\left(\sum_{l=k-1}^{k+1}\|f\chi_l\|_{q_1(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&\le C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0}\left(\epsilon^\theta \sum_{k=0}^{\infty} 2^{ka_\infty p(1+\epsilon)}\|f\chi_k\|^{p(1+\epsilon)}_{q_1(\cdot)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&\le C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0}\left(\epsilon^\theta \sum_{k\in\mathbb{Z}} 2^{ka(\cdot)p(1+\epsilon)}\|f\chi_k\|^{p(1+\epsilon)}_{q_1(\cdot)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&= C\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{\dot{K}^{a(\cdot),p),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

For E_1, we use the facts that, if $x \in R_k$, $y \in R_l$ and $l \le k-2$, then $|x-y| \sim |x| \sim 2^k$, we get

$$
E_1 \le \sup_{\epsilon>0}\left(\epsilon^\theta \sum_{k\in\mathbb{Z}} 2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=-\infty}^{k-2}\|\chi_k I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}},
$$

by using size condition and Hölder's inequality, we get

$$
\begin{aligned}
&|I^m_{\zeta,b}(f\chi_l)(x).\chi_k(x)| \le C\int_{R_l}\frac{|f_l(y)|[b(x)-b(y)]^m}{|x-y|^{n-\zeta}}dy.\chi_k(x)\\
&\le C2^{k(\zeta-n)}\int_{R_l}|f_l(y)|b(x)-b(y)|^m dy.\chi_k(x)\\
&\le C2^{k(\zeta-n)}\left(|b(x)-b_{B_l}|^m\int_{R_l}|f_l(y)|dy+\int_{R_l}|f_l(y)||b(y)-b_{B_l}|^m dy\right).\chi_k(x)\\
&\le C2^{k(\zeta-n)}\|f_l\|_{q_1(\cdot)}\left(|b(x)-b_{B_l}|^m\|\chi_l\|_{q_1'(\cdot)}+\|((b-b_{B_l})^m\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\right).\chi_k(x).
\end{aligned}
$$

Thus, from Lemma (2.3), and the fact that $\|\chi_l\|_{L^{s(\cdot)}} \le \|\chi_{B_l}\|_{L^{s(\cdot)}}$, we get

$$
\begin{aligned}
&\||I^m_{\zeta,b}(f\chi_l)(x).\chi_k(x)\|_{q_2(\cdot)}\\
&\quad\le C2^{k(\zeta-n)}\|f_l\|_{q_1(\cdot)}(\|(b-b_{B_l})^m\chi_k\|_{q_2(\cdot)}\|\chi_l\|_{q_1'(\cdot)}\\
&\quad+\|((b-b_{B_l})^m\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{q_2(\cdot)})\\
&\quad\le C2^{k(\zeta-n)}\|f_l\|_{q_1(\cdot)}((k-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_{B_k}\|_{q_2(\cdot)}\|\chi_l\|_{q_1'(\cdot)}\\
&\quad+\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_{B_l}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{q_2(\cdot)})\\
&\quad\le C2^{k(\zeta-n)}(k-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|f_l\|_{q_1(\cdot)}\|\chi_{B_l}\|_{q_1'(\cdot)}\|\chi_{B_k}\|_{q_2(\cdot)}.
\end{aligned}
$$

It is known, see e.g., [19] that $\chi_{B_k}(x) \le C2^{-k\zeta}I_\zeta(\chi_{B_k})(x)$, which yields

$$
\begin{aligned}
\|\chi_{B_k}\|_{q_2(\cdot)} &\le C2^{-k\zeta}\|I_\zeta(\chi_{B_k})\|_{q_2(\cdot)}\\
&\le C2^{-k\zeta}\|\chi_{B_k}\|_{q_1(\cdot)}.
\end{aligned}
$$

As a result we get,

$$\begin{aligned}&2^{k(\zeta-n)}\|\chi_{B_l}\|_{q_1'(\cdot)}\|\chi_{B_k}\|_{q_2(\cdot)}\\&\quad\le 2^{k(\zeta-n)}\|\chi_{B_l}\|_{q_1'(\cdot)}.2^{-k\zeta}\|\chi_{B_k}\|_{q_1(\cdot)}\\&\quad\le 2^{-kn}\|\chi_{B_l}\|_{q_1'(\cdot)}\|\chi_{B_k}\|_{q_1(\cdot)}.\end{aligned}$$

Consequently, we have

$$2^{-kn}\|\chi_{B_l}\|_{q_1'(\cdot)}\|\chi_{B_k}\|_{q_1(\cdot)}\le C2^{-kn}2^{\frac{kn}{q_1(0)}}2^{\frac{ln}{q_1'(0)}}\le C2^{\frac{(l-k)n}{q_1'(0)}}, \tag{5.3}$$

splitting E_1 by means of Minkowski's inequality we have

$$\begin{aligned}E_1&\le\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=-\infty}^{k-2}\|\chi_k I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\&+\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=0}^{\infty}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=-\infty}^{k-2}\|\chi_k I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\&:=E_{11}+E_{12},\end{aligned}$$

applying above results to E_{11},

$$\begin{aligned}E_{11}&\le\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=-\infty}^{k-2}\|\chi_k I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\&\le C\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=-\infty}^{k-2}2^{\frac{(l-k)n}{q_1'(0)}}(k-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|f\chi_l\|_{q_1(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\&\le C\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=-\infty}^{k-2}2^{\frac{(l-k)n}{q_1'(0)}}(k-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|f\chi_l\|_{q_1(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}},\end{aligned}$$

let $b:=\frac{n}{q_1'(0)}-\alpha(0)$, applying Hölder's inequality, Fubini's theorem for series and $2^{-p(1+\epsilon)}<2^{-p}$ we get,

$$\begin{aligned}&\le C\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2}2^{\alpha(0)l}(k-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|f\chi_l\|_{q_1(\cdot)}2^{b(l-k)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\&\le C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2}2^{\alpha(0)l}\|f\chi_l\|_{q_1(\cdot)}(k-l)^m2^{b(l-k)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\&\le C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2}2^{\alpha(0)p(1+\epsilon)l}\|f\chi_l\|^{p(1+\epsilon)}_{q_1(\cdot)}(k-l)^{p(1+\epsilon)m/2}\right)\right.\end{aligned}$$

$$\times 2^{bp(1+\epsilon)(l-k)/2}\left(\sum_{l=-\infty}^{k-2}(k-l)^{p(1+\epsilon)'m/2}2^{bp(1+\epsilon)'(l-k)/2}\right)^{\frac{p(1+\epsilon)}{p(1+\epsilon)'}}\Bigg)^{\frac{1}{p(1+\epsilon)}}$$

$$=C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\sum_{l=-\infty}^{k-2}2^{\alpha(0)p(1+\epsilon)l}\|f\chi_l\|^{p(1+\epsilon)}_{q_1(\cdot)}(k-l)^{p(1+\epsilon)m/2}2^{bp(1+\epsilon)(l-k)/2}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$=C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{l=-\infty}^{-1}2^{a(\cdot)p(1+\epsilon)l}\|f\chi_l\|^{p(1+\epsilon)}_{q_1(\cdot)}\sum_{k=l+2}^{-1}(k-l)^{p(1+\epsilon)m/2}2^{bp(1+\epsilon)(l-k)/2}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$<C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{l=-\infty}^{-1}2^{\alpha(0)p(1+\epsilon)l}\|f\chi_l\|^{p(1+\epsilon)}_{q_1(\cdot)}\sum_{k=l+2}^{-1}(k-l)^{pm/2}2^{bp(l-k)/2}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{l=-\infty}^{-1}2^{\alpha(0)p(1+\epsilon)l}\|f\chi_l\|^{p(1+\epsilon)}_{q_1(\cdot)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$=C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{l\in\mathbb{Z}}2^{a(\cdot)p(1+\epsilon)l}\|f\chi_l\|^{p(1+\epsilon)}_{q_1(\cdot)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{\dot{K}^{a(\cdot),p),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.$$

Now for E_{12} using Minkowski's inequality we have

$$E_{12}\leq\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=0}^{\infty}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=-\infty}^{-1}\|\chi_kI^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$+\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=0}^{\infty}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=0}^{k-2}\|\chi_kI^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$:=A_1+A_2.$$

The estimate for A_2 follows in a similar manner to E_{11} with $q_1'(0)$ replaced by q'_{1_∞} and using the fact $b:=\frac{n}{q_{1'_\infty}}-a_\infty>0$. For A_1 using Lemma (2.1) we have

$$2^{-kn}\|\chi_{B_k}\|_{q_1(\cdot)}\|\chi_{B_l}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\leq C2^{-kn}2^{\frac{kn}{q_{1\infty}}}2^{\frac{ln}{q_1'(0)}}\leq C2^{\frac{-kn}{q'_{1\infty}}}2^{\frac{ln}{q_1'(0)}},\tag{5.4}$$

as $a_\infty-\frac{n}{q'_{1_\infty}}<0$ we have

$$A_1\leq C\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=0}^{\infty}2^{ka_\infty p(1+\epsilon)}\left(\sum_{l=-\infty}^{-1}\|\chi_kI^m_{\zeta,b}f\chi_l\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=0}^{\infty}2^{ka_\infty p(1+\epsilon)}\left(\sum_{l=-\infty}^{-1}(k-l)^m2^{(\frac{ln}{q_1'(0)})}2^{(\frac{-kn}{q'_{1\infty}})}\|f\chi_l\|_{q_1(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$
\begin{aligned}
&\le C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k=0}^{\infty}\left(\sum_{l=-\infty}^{-1}(k-l)^m 2^{l(\frac{n}{q'_1(0)}-a_\infty)}2^{k(-\frac{n}{q'_{1\infty}}+a_\infty)}2^{la_\infty}\|f\chi_l\|_{q_1(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&\le C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k=0}^{\infty}\left(\sum_{l=-\infty}^{-1}2^{a_\infty lp(1+\epsilon)}((k-l)^m 2^{l(\frac{n}{q'_1(0)}-a_\infty)}2^{k(-\frac{n}{q'_{1\infty}}+a_\infty)})^{p(1+\epsilon)/2}\right)\|f\chi_l\|^{p(1+\epsilon)}_{q_1(\cdot)}\right.\\
&\qquad\qquad \left.\times\left(\sum_{l=-\infty}^{-1}((k-l)^m 2^{l(\frac{n}{q'_1(0)}-a_\infty)}2^{k(-\frac{n}{q'_{1\infty}}+a_\infty)})^{p(1+\epsilon)'/2}\right)^{\frac{p(1+\epsilon)}{p(1+\epsilon)}}\right)^{\frac{1}{p(1+\epsilon)}}\\
&\le C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k=0}^{\infty}\sum_{l=-\infty}^{-1}2^{a_\infty lp(1+\epsilon)}\|f\chi_l\|^{p(1+\epsilon)}_{q_1(\cdot)}\left((k-l)^m 2^{l(\frac{n}{q'_1(0)}-a_\infty)}2^{k(-\frac{n}{q'_{1\infty}}+a_\infty)}\right)^{p(1+\epsilon)/2}\right)^{\frac{1}{p(1+\epsilon)}}\\
&\le C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{l=-\infty}^{-1}2^{a_\infty lp(1+\epsilon)}\|f\chi_l\|^{p(1+\epsilon)}_{q_1(\cdot)}\times\sum_{k=l}^{\infty}((k-l)^m 2^{l(\frac{n}{q'_1(0)}-a_\infty)}2^{k(-\frac{n}{q'_{1\infty}}+a_\infty)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&\le C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{l=-\infty}^{-1}2^{a_\infty lp(1+\epsilon)}\|f\chi_l\|^{p(1+\epsilon)}_{q_1(\cdot)}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{\dot{K}^{a(\cdot),p),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Now we estimate E_3, for each $k\in\mathbb{Z}$ and $l\ge k+2$ and a.e. $x\in R_k$; splitting E_3 by using Minkowski's inequality and using similar method of E_1, E_2 we have

$$
\begin{aligned}
E_3 &\le \sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k\in\mathbb{Z}}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=k+2}^{\infty}\|\chi_k I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&\le \sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=k+2}^{\infty}\|\chi_k I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&+ \sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k=0}^{\infty}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=k+2}^{\infty}\|\chi_k I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&:= E_{31}+E_{32}.
\end{aligned}
$$

For E_{31} by using Monkowski's inequality

$$
\begin{aligned}
E_{31} &\le \sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=k+2}^{\infty}\|\chi_k I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
E_{31} &\le \sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=k+2}^{-1}\|\chi_k I^m_{\zeta,b}(f\chi_l)\|_{q_1(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&+ \sup_{\epsilon>0}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=0}^{\infty}\|\chi_k I^m_{\zeta,b}(f\chi_l)\|_{q_1(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}
\end{aligned}
$$

$$:=B_1 + B_2.$$

The estimate for B_1 follows in a similar manner to E_{32} with q_{1_∞} replaced by $q_1(0)$ and using the facts that $\frac{n}{q_1(0)} + \alpha(0) > 0$, $\frac{n}{q_{1_\infty}} + a_\infty > 0$, For B_2 using Lemma (2.1) and Hölder's inequality we have,

$$2^{-ln}\|\chi_k\|_{q_1(\cdot)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \leq C2^{-ln}2^{\frac{kn}{q_1(0)}}2^{\frac{ln}{q'1_\infty}} \leq C2^{\frac{kn}{q_1(0)}}2^{\frac{-ln}{q_{1\infty}}}, \tag{5.5}$$

$$\begin{aligned}
B_2 \leq & C\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}2^{ka(0)p(1+\epsilon)}\left(\sum_{l=0}^{\infty}\|\chi_k I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
\leq & C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}2^{ka(0)p(1+\epsilon)}\left(\sum_{l=0}^{\infty}(k-l)^m2^{-l(\frac{n}{q_{1\infty}})}2^{k(\frac{n}{q_1(0)})}\|f\chi_l\|_{q_1(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
\leq & C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l=0}^{\infty}2^{a(0)}(k-l)^m2^{-l(\frac{n}{q_{1\infty}}-a(0))}2^{k(\frac{n}{q_1(0)}+a(0))}\|f\chi_l\|_{q_1(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
\leq & C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l=0}^{\infty}2^{a(0)(p(1+\epsilon))}((k-l)^m2^{-l(\frac{n}{q_{1\infty}}-a(0))}2^{k(\frac{n}{q_1(0)}+a(0)})^{p(1+\epsilon)/2}\right)\right.\\
& \left.\times\|f\chi_l\|^{p(1+\epsilon)}_{q_1(\cdot)}\left(\sum_{l=0}^{\infty}((k-l)^m2^{-l(\frac{n}{q_{1\infty}}-a(0))}2^{k(\frac{n}{q_1(0)}+a(0)})^{p(1+\epsilon)'/2}\right)^{\frac{p(1+\epsilon)}{p(1+\epsilon)'}}\right)^{\frac{1}{u(1+\epsilon)}}\\
\leq & C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}\sum_{l=0}^{\infty}2^{a(0)(p(1+\epsilon))}\|f\chi_l\|^{p(1+\epsilon)/2}_{q_1(\cdot)}\left((k-l)^m2^{-l(\frac{n}{q_{1\infty}}-a(0))}2^{k(\frac{n}{q_1(0)}+a(0))}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
\leq & C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{l=0}^{\infty}2^{a(0)(p(1+\epsilon))}\|f\chi_l\|^{p(1+\epsilon)}_{q_1(\cdot)}\sum_{k=-\infty}^{l}\left((k-l)^m2^{-l(\frac{n}{q_{1\infty}}-a(0))}2^{k(\frac{n}{q_1(0)}+a(0))}\right)^{p(1+\epsilon)/2}\right)^{\frac{1}{p(1+\epsilon)}}\\
\leq & C\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{\dot{K}^{a(\cdot),p),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Finally we will estimate E_{32},

$$\begin{aligned}
|I^m_{\zeta,b}(f\chi_l)(x).\chi_k(x)| \leq & C\int_{R_l}\frac{|f_l(y)|[b(x)-b(y)]^m}{|x-y|^{n-\zeta}}dy.\chi_k(x)\\
\leq & C2^{l(\zeta-n)}\int_{R_l}|f_l(y)|b(x)-b(y)|^m dy.\chi_k(x)\\
\leq & C2^{l(\zeta-n)}\left(|b(x)-b_{B_k}|^m\int_{R_l}|f_l(y)|dy+\int_{R_l}|f_l(y)||b(y)-b_{B_k}|^m dy\right).\chi_k(x)\\
\leq & C2^{l(\zeta-n)}\|f_l\|_{q_1(\cdot)}\left(|b(x)-b_{B_l}|^m\|\chi_l\|_{q_1'(\cdot)}+\|((b-b_{B_l})^m\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\right).\chi_k(x).
\end{aligned}$$

Thus, from Lemma (2.3), and the fact that $\|\chi_l\|_{L^{s(\cdot)}} \leq \|\chi_{B_l}\|_{L^{s(\cdot)}}$, we get

$$\||I^m_{\zeta,b}(f\chi_l)(x).\chi_k(x)\|_{q_2(\cdot)}$$

$$
\begin{aligned}
&\leq C2^{l(\zeta-n)}\|f_l\|_{q_1(\cdot)}(\|(b-b_{B_k})^m\chi_k\|_{q_2(\cdot)}\|\chi_l\|_{q_1'(\cdot)}\\
&+\|((b-b_{B_k})^m\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{q_2(\cdot)})\\
&\leq C2^{l(\zeta-n)}\|f_l\|_{q_1(\cdot)}(\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_{B_k}\|_{q_2(\cdot)}\|\chi_l\|_{q_1'(\cdot)}\\
&+(l-k)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_{B_l}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{q_2(\cdot)})\\
&\leq C2^{l(\zeta-n)}(l-k)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|f_l\|_{q_1(\cdot)}\|\chi_{B_l}\|_{q_1'(\cdot)}\|\chi_{B_k}\|_{q_2(\cdot)}.
\end{aligned}
$$

It is known, see e.g., [19], that $\chi_{B_l}(x) \leq C2^{-l\zeta}I_\zeta(\chi_{B_l})(x)$, which yields

$$
\begin{aligned}
\|\chi_{B_l}\|_{q_2(\cdot)} &\leq C2^{-l\zeta}\|I_\zeta(\chi_{B_l})\|_{q_2(\cdot)}\\
&\leq C2^{-l\zeta}\|\chi_{B_l}\|_{q_1(\cdot)}.
\end{aligned}
$$

As a result we get,

$$
\begin{aligned}
&2^{l(\zeta-n)}\|\chi_{B_l}\|_{q_1'(\cdot)}\|\chi_{B_k}\|_{q_2(\cdot)}\\
&\quad\leq 2^{l(\zeta-n)}\|\chi_{B_l}\|_{q_1'(\cdot)}.2^{-k\zeta}\|\chi_{B_k}\|_{q_1(\cdot)}\\
&\quad\leq 2^{-ln}\|\chi_{B_l}\|_{q_1'(\cdot)}\|\chi_{B_k}\|_{q_1(\cdot)}.
\end{aligned}
$$

For E_{32} Lemma (2.2) yields

$$
2^{-ln}\|\chi_{B_k}\|_{q_1(\cdot)}\|\chi_{B_l}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \leq C2^{-ln}2^{\frac{kn}{q_{1\infty}}}2^{\frac{ln}{q_{1\infty}'}} \leq C2^{\frac{(k-l)n}{q_{1\infty}}}, \tag{5.6}
$$

we get

$$
\begin{aligned}
E_{32} &\leq C\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=0}^{\infty}2^{ka(\cdot)p(1+\epsilon)}\left(\sum_{l=k+2}^{\infty}\|\chi_k I^m_{\zeta,b}(f\chi_l)\|_{q_2(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=0}^{\infty}\left(\sum_{l=k+2}^{\infty}(l-k)^m2^{a_\infty l}\|f\chi_l\|_{q_1(\cdot)}2^{d(k-l)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}},
\end{aligned}
$$

where $d = \frac{n}{q_{1\infty}} + a_\infty > 0$. Then we use Hölder's theorem for series and $2^{-p(1+\epsilon)} < 2^{-p}$ to obtain

$$
\begin{aligned}
E_{32} &\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=0}^{\infty}\left(\sum_{l=k+2}^{\infty}2^{a_\infty p(1+\epsilon)l}\|f\chi_l\|^{p(1+\epsilon)}_{q_1(\cdot)}(l-k)^{mp(1+\epsilon)/2}\right)\right.\\
&\qquad\left.\times 2^{dp(1+\epsilon)(k-l)/2}\left(\sum_{l=k+2}^{\infty}(l-k)^{mp(1+\epsilon)/2}2^{dp(1+\epsilon)'(k-l)/2}\right)^{\frac{p(1+\epsilon)}{p(1+\epsilon)'}}\right)^{\frac{1}{p(1+\epsilon)}}\\
&\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{k=0}^{\infty}\sum_{l=k+2}^{\infty}2^{a_\infty p(1+\epsilon)l}\|f\chi_l\|^{p(1+\epsilon)}_{q_1(\cdot)}(l-k)^{mp(1+\epsilon)/2}2^{dp(1+\epsilon)(k-l)/2}\right)^{\frac{1}{p(1+\epsilon)}}\\
&\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\left(\epsilon^\theta\sum_{l=0}^{\infty}2^{a_\infty p(1+\epsilon)l}\|f\chi_l\|^{p(1+\epsilon)}_{q_1(\cdot)}\sum_{k=0}^{l-2}(l-k)^{mp(1+\epsilon)/2}2^{dp(1+\epsilon)(k-l)/2}\right)^{\frac{1}{p(1+\epsilon)}}
\end{aligned}
$$

$$
\begin{aligned}
&<C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0}\left(\epsilon^\theta \sum_{l\in\mathbb{Z}} 2^{a_\infty p(1+\epsilon)l}\|f\chi_l\|^{p(1+\epsilon)}_{q_1(\cdot)} \sum_{k=-\infty}^{l-2}(l-k)^{mp(1+\epsilon)/2}2^{dp(k-l)/2}\right)^{\frac{1}{p(1+\epsilon)}}\\
&=C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0}\left(\epsilon^\theta \sum_{l\in\mathbb{Z}} 2^{a(\cdot)p(1+\epsilon)l}\|f\chi_l\|^{p(1+\epsilon)}_{q_1(\cdot)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&\le C\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{\dot{K}^{a(\cdot),p),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Combining the estimates for E_1, E_2 and E_3 yields

$$\|I^m_{\zeta,b}f\|_{\dot{K}^{a(\cdot),p),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} \le C\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{\dot{K}^{a(\cdot),p),\theta}_{q_1(\cdot)}(\mathbb{R}^n)},$$

which ends the proof.

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Conflicts of interest

The authors declare no conflict of interest.

References

1. B. Sultan, F. M. Azmi, M. Sultan, T. Mahmood, N. Mlaiki, N. Souayah, Boundedness of fractional integrals on grand weighted Herz-Morrey spaces with variable exponent, *Fractal Fract.*, **6** (2022), 660. https://doi.org/10.3390/fractalfract6110660
2. B. Sultan, M. Sultan, M. Mehmood, F. Azmi, M. A. Alghafli, N. Mlaiki, Boundedness of fractional integrals on grand weighted Herz spaces with variable exponent, *AIMS Math.*, **8** (2023), 752–764. https://doi.org/10.3934/math.2023036
3. V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral operators in non-standard function spaces*, Birkhuser, 2016. https://doi.org/10.1007/978-3-319-21018-6
4. B. Sultan, F. Azmi, M. Sultan, M. Mehmood, N. Mlaiki, Boundedness of Riesz potential operator on grand Herz-Morrey spaces, *Axioms*, **11** (2022), 583. https://doi.org/10.3390/axioms11110583
5. A. Hussain, M. Asim, F. Jarad, Variable λ-central Morrey space estimates for the fractional Hardy operators and commutators, *J. Math.*, **2022** (2022), 5855068. https://doi.org/10.1155/2022/5855068
6. M. Asim, A. Hussain, Weighted variable Morrey-Herz estimates for fractional Hardy operators, *J. Inequal. Appl.*, **2022** (2022), 2. https://doi.org/10.1186/s13660-021-02739-z
7. A. Hussain, M. Asim, M. Aslam, F. Jarad, Commutators of the fractional Hardy operator on weighted variable Herz-Morrey spaces, *J. Funct. Space.*, **2021** (2021), 9705250. https://doi.org/10.1155/2021/9705250

8. K. P. Ho, Intrinsic square functions on Morrey and block spaces with variable exponents, *Bull. Malays. Math. Sci. Soc.*, **40** (2017), 995–1010. https://doi.org/10.1007/s40840-016-0330-6
9. M. Izuki, Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization, *Anal. Math.*, **36** (2010), 33–50. https://doi.org/10.1007/s10476-010-0102-8
10. M. Izuki, Boundedness of vector-valued sublinear operators on Herz-Morrey spaces with variable exponents, *Math. Sci. Res. J.*, **13** (2009), 243–253.
11. S. Samko, Variable exponent Herz spaces, *Mediterr. J. Math.*, **10** (2013), 2007–2025. https://doi.org/10.1007/s00009-013-0285-x
12. A Meskhi, Integral operators in grand Morrey spaces, 2010, arXiv: 1007.1186.
13. H. Nafis, H. Rafeiro, M. Zaighum, A note on the boundedness of sublinear operators on grand variable Herz spaces, *J. Inequal. Appl.*, **2020** (2020), 1. https://doi.org/10.1186/s13660-019-2265-6
14. O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czech. Math. J.*, **41** (1991), 592–618.
15. D. Cruz-Uribe, A. Fiorenza, J. M. Martell, C. Perez, The boundedness of classical operators on variable L^p spaces, *Ann. Acad. Sci. Fenn. M.*, **31** (2006), 239–264.
16. L. Diening, P. Harjulehto, P. Hst, M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, Berlin: Springer, 2011. https://doi.org/10.1007/978-3-642-18363-8
17. A. Ajaib, A. Hussain, Weighted CBMO estimates for commutators of matrix Hausdorff operator on the Heisenberg group, *Open Math.*, **18** (2020), 496–511. https://doi.org/10.1515/math-2020-0175
18. A. Hussain, A. Ajaib, Some results for the commutators of generalized Hausdorff operator, *J. Math. Inequal.*, **13** (2019), 1129–1146. https://doi.org/10.7153/jmi-2019-13-80
19. W. L. Jiang, W. J. Zhao, Boundedness for higher order commutators of fractional integrals on variable exponent Herz-Morrey spaces, *Mediterr. J. Math.*, **14** (2017), 198. https://doi.org/10.1007/s00009-017-1002-y
20. H. Nafis, H. Rafeiro, M. A. Zaighum, Boundedness of the Marcinkiewicz integral on grand Herz spaces, *J. Math. Inequal.*, **15** (2021), 739–753. https://doi.org/10.7153/jmi-2021-15-52

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