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Research article

Adaptive inertial Yosida approximation iterative algorithms for split variational inclusion and fixed point problems

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Abstract: In this paper, we present self-adaptive inertial iterative algorithms involving Yosida approximation to investigate a split variational inclusion problem (SVIP) and common solutions of a fixed point problem (FPP) and SVIP in Hilbert spaces. We analyze the weak convergence of the proposed iterative algorithm to explore the approximate solution of the SVIP and strong convergence to estimate the common solution of the SVIP and FPP under some mild suppositions. A numerical example is demonstrated to validate the theoretical findings, and comparison of our iterative methods with some known schemes is outlined.

Keywords: split variational inclusion; fixed point problem; Yosida approximation; algorithms; weak convergence; strong convergence
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1. Introduction

Among the most significant generalizations of convex feasibility problems, the split feasibility problem (in short, SFP) was developed by Censor and Elfving [10]. Inverse problems related to phase retrievals and medical image reconstruction, signal recovery, computer tomography and radiation therapy treatment planning can be modeled as SFP. For more details, see [9, 11, 12, 17] and references therein.

Byrne [9] studied the CQ method along with many iterative algorithms and their convergence to approximate the solution of the SFP. Further, various feasible sets have been considered in the study of the SFP. Consequently, Moudafi [24] brought into existence the concept of the split monotone variational inclusion problem (SMVIP), stated below:

$$\text{Find } x^* \in H_1 \text{ such that } 0 \in F_1(x^*) + M_1(x^*) \text{ and } 0 \in F_2(Bx^*) + M_2(Bx^*), \tag{1.1}$$

where H_1, H_2 are Hilbert spaces; B^* is the adjoint operator of $B : H_1 \to H_2$; $M_1 : H_1 \to 2^{H_1}$, $M_2 : H_2 \to 2^{H_2}$ are set-valued maximal monotone mappings; $F_1 : H_1 \to H_1$ and $F_2 : H_2 \to H_2$ are two single-valued mappings. Following the CQ method due to Byrne, Moudafi proposed the following scheme: For arbitrary $z_0 \in H_1$ and $\mu > 0$, compute

$$z_{n+1} = U[z_n + \gamma B^*(V - I)Bz_n], \tag{1.2}$$

where $\gamma \in (0, 1/R)$, R is the spectral radius of B^*B, $U = R_\mu^{M_1}(I - \mu F_1)$, and $V = R_\mu^{M_2}(I - \mu F_2)$; $R_\mu^{M_1}, R_\mu^{M_2}$ are resolvents of M_1 and M_2, respectively.

Currently, we intend to inspect the split variational inclusion problems (SVIP), which can be obtained by putting $F_1 = F_2 = 0$ in the SMVIP:

$$\text{Find } x^* \in H_1 \text{ such that } 0 \in M_1(x^*) \text{ and } 0 \in M_2(Bx^*). \tag{1.3}$$

Byrne et al. [8], investigated the SVIP by employing the scheme:

$$z_{n+1} = R_\mu^{M_1}[z_n + \gamma B^*(R_\mu^{M_2} - I)Bz_n], \ \forall n \geq 1 \ \mu > 0, \tag{1.4}$$

and showed that the weak limit leads to the solution of the SVIP. Later, Kazmi and Rizvi [21] looked into the common solution of the SVIP and FPP of a non-expansive mapping using the following method:

$$\begin{cases} u_n = R_\mu^{M_1}[z_n + \gamma B^*(R_\mu^{M_2} - I)Bz_n], \\ z_{n+1} = \beta_n f(u_n) + (1 - \beta_n)Tu_n, \end{cases} \tag{1.5}$$

where $T : H_1 \to H_1$ is a nonexpansive mapping, f is a contraction mapping with constant $\alpha \in (0, 1)$, $\mu > 0$, $\gamma \in (0, \frac{1}{\|B\|^2})$, $\beta_n \in (0, 1)$ is a real sequence satisfying $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, and $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$. Sitthithakerngkiet et al. [32], proposed a hybrid viscosity algorithm to estimate the common solution of an SVIP and a countable family of non-expansive mappings:

$$\begin{cases} u_n = R_\mu^{M_1}[z_n + \gamma B^*(R_\mu^{M_2} - I)Bz_n], \\ z_{n+1} = \beta_n \xi f(u_n) + (1 - \beta_n D)T_n u_n, \end{cases} \tag{1.6}$$

where $T_n : H_1 \to H_1$ is a sequence nonexpansive mapping, f is a contraction mapping with constant $\alpha \in (0, 1)$, D is a strongly bounded linear operator with constant $\bar{\gamma}$, such that $0 < \xi < \frac{\bar{\gamma}}{\alpha}$, $\mu > 0$, $\gamma \in (0, \frac{1}{\|B\|^2})$, and $\beta_n \in (0, 1)$ is a real sequence satisfying $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Very recently, Akram et al. [2] modified the Algorithm 1.5 and investigated the common solution of the SVIP and FPP:

$$\begin{cases} u_n = z_n - \gamma[(I - R^{M_1}_{\mu_1})z_n + B^*(I - R^{B_2}_{\mu_2})Bz_n], \\ x_{n+1} = \beta_n f(z_n) + (1 - \beta_n)T(u_n), \end{cases} \tag{1.7}$$

where f is an α-contraction mapping, $\gamma = \frac{1}{1+\|B\|^2}$, $\beta_n \in (0, 1)$ satisfying $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Investigating solutions of split problems and FPP and their applications in Banach and Hilbert spaces is interesting, and have been studied by many authors [6, 13, 16, 19, 20, 27, 30, 34] and references therein.

In all the abovementioned methods, step size depends on the operator norm $\|B\|$, which is computationally expensive. This drawback was resolved by employing a new iterative scheme involving self-adaptive step size. López et al. [22] proposed the iterative method to explore the SFP such that step size is not determined by the matrix norm as follows:

$$z_{n+1} = P_C[I - \gamma_n B^*(I - P_Q)Bz_n], \ \forall n \geq 1, \tag{1.8}$$

where $\gamma_n = \frac{\sigma_n f(z_n)}{\|\nabla f(z_n)\|^2}$ with $f(x) = \frac{1}{2}\|(I - P_Q Bx\|^2, \nabla f(x) = B^*(I - P_Q)Bx$, $n \geq 0$ and $0 < \sigma_n < 4, \inf \sigma_n(4 - \sigma_n) > 0$, and P_C and P_Q are the orthogonal projections on the closed convex sets C and Q, respectively. Moudafi [26] solved the SFP without prior calculation of operator norm. Dilshad et al. studied the SVIP [15] and SMVIP [14] without prior estimation of the norm of bounded linear operator.

It is notable that set-valued monotone operators can be regularized into single-valued monotone operators by the Yosida approximation, which is a useful tool for investigating variational inclusions and their systems in linear as well as nonlinear spaces. For a given monotone mapping M with parameter $\mu > 0$, the Yosida approximation operator is defined as $J^M_\mu = \frac{1}{\mu}(I - R^M_\mu)$, where R^M_μ is the resolvent of M. Several authors have utilized Yosida approximation of monotone mappings to approximate the solution of variational inclusions, systems of variational inclusions, and split variational inclusions. For more details, see [1, 3, 4, 14].

To accelerate the convergence of iterative methods, Polyak [29] introduced an inertial iterative scheme known as the heavy ball method and applied it to investigate smooth convex optimization problems. Due to its convergence properties in smooth optimization, many scholars have been used this method widely by adding an inertial term to their algorithms to accelerate the convergence rate. Alvarez and Attouch [5] composed an inertial algorithm to solve the null point problem of monotone operator M and obtained the weak convergence. They combined the inertial term with their algorithm for arbitrary z_0, z_1 and $\theta_n \in [0, 1)$ defined as follows:

$$z_{n+1} = J^M_{\mu_n}[z_n + \theta_n(z_n - z_{n-1})], \ n \geq 1, \tag{1.9}$$

where $J^M_{\mu_n}$ is the resolvent of monotone operator M, and $\mu_n > 0$. More related work can seen in [25, 31, 33] and references therein.

Motivated by the abovementioned discussion and following the work reported in [2], we propose a new iterative algorithm for SVIP by adding an inertial term to accelerate the convergence, using Yosida approximation of M_1 and M_2 instead of their resolvents and a new stepsize η_n (defined in Section 3) in place of γ so that the implementation of the algorithm does not require the pre-calculated norm of bounded linear operator $\|B\|$. Further, we extend the proposed algorithm for solving SVIP and FPP of a nonexpansive mapping. We analyze the weak and strong convergences of the proposed inertial methods in Hilbert spaces. Finally, an illustrative example is constructed to show the convergence of the considered iterative procedures and a comparison with other well known results.

2. Preliminaries

From now onward, H refers to a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Strong and weak convergences will be denoted by $\rightarrow$ and $\rightharpoonup$, respectively. For all $u, v \in H$, an operator $T : H \rightarrow H$ is called a contraction if $\|Tu - Tv\| \leq \kappa\|u - v\|, \kappa \in [0, 1)$; firmly nonexpansive if $\|Tu - Tv\|^2 \leq \langle u - v, Tu - Tv\rangle$; and τ-inverse strongly monotone if there exists $\tau > 0$ such that $\langle Tu - Tv, u - v\rangle \geq \tau\|Tu - Tv\|^2$. If $\kappa = 1$, then T is nonexpansive. For all $u, v, w \in H, \xi, \zeta, \varsigma \in [0, 1]$ with $\xi + \zeta + \varsigma = 1$, the following characteristic inequality and equality hold:

$$\|\zeta u + \xi v + \varsigma w\|^2 = \zeta\|u\|^2 + \xi\|v\|^2 + \varsigma\|w\|^2 - \zeta\xi\|u - v\|^2 - \xi\varsigma\|v - w\|^2 - \varsigma\zeta\|u - w\|^2, \tag{2.1}$$

and

$$\|v + w\|^2 \leq \|v\|^2 + 2\langle w, v + w\rangle. \tag{2.2}$$

Definition 2.1. [21] Let $u \in H$, the projection of u onto $K \subset H$, be defined by

$$\|u - P_K u\| \leq \|u - v\|, \ \forall v \in K.$$

$P_K u$ also satisfies the following inequality:

$$\|P_K u - P_K v\|^2 \leq \langle u - v, P_K u - P_K v\rangle, \forall u, v \in H,$$

$$\text{and} \quad P_K u = w \Leftrightarrow \langle u - w, v - w\rangle \geq 0, \ v \in K. \tag{2.3}$$

Definition 2.2. [7] A mapping $M : H \rightarrow 2^H$ is monotone if $\langle u - v, x - y\rangle \geq 0, \forall u \in M(x)$ and $v \in M(y)$. The resolvent associated with M is defined by $R^M_\mu = [I + \mu M]^{-1}$, which is single-valued as well as firmly nonexpansive, and the Yosida approximation of M is defined by $J^M_\mu = \frac{1}{\mu}[I - R^M_\mu]$.

Lemma 2.1. *[35] If $\{w_n\}$ is a nonnegative real sequence satisfying*

$$w_{n+1} \leq (1 - \psi_n)w_n + \varphi_n \geq 0,$$

where $\{\psi_n\}$ is a sequence in $(0, 1)$, and $\{\varphi_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \psi_n = \infty$,
(ii) $\limsup_{n\to\infty} \frac{\varphi_n}{\psi_n} \leq 0$ *or* $\limsup_{n\to\infty} |\varphi_n| < \infty$,

then $\lim_{n\to\infty} w_n = 0$.

Lemma 2.2. *[18] If I is an identity mapping, and $\tau > 0$, then $T : H \to H$ is τ-inverse strongly monotone if and only if $I - \tau T$ is firmly nonexpansive.*

Lemma 2.3. *[28] Let $K(\neq \emptyset) \subset H$ and $\{u_n\}$ be a bounded sequence in H such that*

(i) $\lim_{n\to\infty} \|u_n - p\|$ *exists for every* $p \in K$,
(ii) $\omega_w(u_n) \subset K$.

Then, there exists $s^* \in K$ such that $u_n \rightharpoonup s^*$ as $n \to \infty$.

Lemma 2.4. *[23] Let $\{\Upsilon_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Upsilon_{n_k}\}$ of $\{\Upsilon_n\}$ such that $\Upsilon_{n_k} < \Upsilon_{n_k+1}$ for all $k \geq 0$. Also, consider the sequence of integers $\{\gamma(n)\}_{n\geq n_0}$ defined by*

$$\gamma(n) = \max\{k \leq n : \Upsilon_k \leq \Upsilon_{k+1}\}.$$

Then, $\{\gamma(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty} \gamma(n) = \infty$, and for all $n \geq n_0$,

$$\max\{\Upsilon_{\gamma(n)}, \Upsilon_n\} \leq \Upsilon_{\gamma(n)+1}.$$

Lemma 2.5. *[23] Let $\{\psi_n\}$ be a nonnegative real sequence such that*

(i) $\psi_{n+1} - \psi_n \leq \phi_n(\psi_n - \psi_{n-1}) + \varepsilon_n$;
(ii) $\sum_{n=1}^{\infty} \varepsilon_n < \infty$;
(iii) $\phi_n \in [0, \kappa]$, *where* $\kappa \in [0, 1)$.

Then, $\{\psi_n\}$ is convergent, and $\sum_{n=1}^{\infty} (\psi_{n+1} - \psi_n) < \infty$, where $[t]_+ = \max\{t, 0\}$ for any $t \in \mathbb{R}$.

3. Main results

Next, we propose two inertial self-adaptive iterative methods based on the Yosida approximation operators.

Assumption 3.1. ($\mathbf{A_1}$) Let Θ denotes the solution set of Problem (1.3) such that $\Theta \neq \emptyset$ and $J_{\mu_1}^{M_1}$ and $J_{\mu_2}^{M_2}$ be Yosida approximation operators associated with set-valued maximal monotone mappings $M_1 : H_1 \to 2^{H_1}$ and $M_2 : H_2 \to 2^{H_2}$, respectively.

($\mathbf{A_2}$) Let $T : H_1 \to H_1$ be a nonexpansive mapping such that $\text{Fix(T)}\cap\Theta \neq \emptyset$.

Algorithm 3.1. Step 0: Choose $\phi \in [0, 1)$, $\mu > \frac{1}{2}$, $\mu = \min\{\mu_1, \mu_2\}$, and $\{\delta_n\}$ is a positive sequence such that $\sum_{n=1}^{\infty} \delta_n < \infty$.

Step 1: Given arbitrary z_0 and z_1, for $n \geq 1$, choose $0 < \phi_n < \bar{\phi}_n$, where

$$\bar{\phi}_n = \begin{cases} \min\{\frac{\delta_n}{\|z_n - z_{n-1}\|}, \quad \phi\}, & \text{if } z_n \neq z_{n-1}, \\ \phi, & \text{otherwise.} \end{cases} \tag{3.1}$$

Compute

$$u_n = z_n + \phi_n(z_n - z_{n-1}), \tag{3.2}$$

$$z_{n+1} = u_n - \eta_n[J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n)], \tag{3.3}$$

where

$$\eta_n = \begin{cases} \frac{\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2}{\|J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n)\|^2}, & \text{if } \|J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n)\| \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Stopping Criteria: Stop if $z_{n+1} = z_n = u_n$; otherwise, go to step 1.

Algorithm 3.2. Step 0: Choose $\phi \in [0, 1)$, $\mu > \frac{1}{2}$, $\mu = \min\{\mu_1, \mu_2\}$, a positive sequence $\{\delta_n\}$ such that $\sum_{n=1}^{\infty} \delta_n < \infty$. Let $\{\varphi_n\}, \{\psi_n\}$ are real sequences in $(0, 1)$ satisfying

$$\lim_{n\to\infty} \psi_n = 0, \ \sum_{n=0}^{\infty} \psi_n = \infty, \ \lim_{n\to\infty} (1 - \varphi_n - \psi_n)\varphi_n > 0. \tag{3.5}$$

Step 1: Given arbitrary z_0 and z_1, for $n \geq 1$, choose $0 < \phi_n < \bar{\phi}_n$ where

$$\bar{\phi}_n = \begin{cases} \min\{\frac{\delta_n}{\|z_n - z_{n-1}\|}, \quad \phi\}, & \text{if } z_n \neq z_{n-1}, \\ \phi, & \text{otherwise.} \end{cases} \tag{3.6}$$

Compute

$$u_n = z_n + \phi_n(z_n - z_{n-1}), \tag{3.7}$$

$$v_n = u_n - \eta_n[J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n)], \tag{3.8}$$

$$z_{n+1} = (1 - \varphi_n - \psi_n)v_n + \varphi_n T(v_n), \tag{3.9}$$

where

$$\eta_n = \begin{cases} \frac{\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2}{\|J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n)\|^2}, & \text{if } \|J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n)\| \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.10}$$

Stopping Criteria: Stop if $z_{n+1} = z_n = v_n = u_n$; otherwise, go to step 1.

Remark 3.1. From the selection of $\phi_n \in [0, 1)$ in Algorithm 3.2, it can be easily observed that

$$\lim_{n\to\infty} \phi_n \|z_n - z_{n-1}\| = 0.$$

Lemma 3.1. *If* $\lim_{n\to\infty} \frac{(\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2)^2}{\|J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n)\|^2} = 0$, *then* $\lim_{n\to\infty} \|J_{\mu_1}^{M_1}(u_n)\| = \lim_{n\to\infty} \|J_{\mu_2}^{M_2}(Bu_n)\| = 0$.

Proof. We have

$$\begin{aligned} 0 = \frac{(\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2)^2}{\|J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n)\|^2} &\geq \frac{(\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2)^2}{2[\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|B^* J_{\mu_2}^{M_2}(Bu_n)\|^2]} \\ &\geq \frac{(\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2)^2}{2[\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|B^*\|^2 \|J_{\mu_2}^{M_2}(Bu_n)\|^2]} \\ &\geq \frac{(\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2)^2}{2\max\{1, \|B^*\|^2\}[\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2]} \end{aligned}$$

$$\geq \frac{\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2}{2\max\{1, \|B^*\|^2\}}$$
$$\geq 0.$$

Taking the limit $n \to \infty$ on both sides, we get the desired result. □

Remark 3.2. It is known to us that R_μ^M and $[I - R_\mu^M]$ are firmly nonexpansive (1-inverse strongly monotone) if M is maximal monotone (Corollary 23.10, [7]). Therefore, by (Lemma 1(v), [4]), the Yosida approximation operator $J_\mu^M = \frac{1}{\mu}[I - R_\mu^M]$ is μ-inverse strongly monotone.

The following essential lemma can be proved by employing the definitions of resolvents and Yosida approximation operators of monotone mappings.

Lemma 3.2. *If $R_{\mu_1}^{M_1}$ and $J_{\mu_1}^{M_1}$ are the resolvent and Yosida approximation operator of monotone mapping M_1, the following assertions are equivalent.*

(i) $s^* \in H_1$ *is the solution of* $(M_1)^{-1}(0)$,
(ii) $R_{\mu_1}^{M_1}(s^*) = s^*$,
(iii) $J_{\mu_1}^{M_1}(s^*) = 0$.

Proposition 3.1. *Suppose that Assumptions 3.1(A_1) holds. If $z_{n+1} = z_n = u_n$ in Algorithm 3.1, then the sequence $z_n \in \Theta$.*

Proof. Let $z_{n+1} = z_n = u_n$. If $\|J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n)\| = 0$, then

$$0 = \|J_{\mu_1}^{M_1}(z_n) + B^* J_{\mu_2}^{M_2}(Bz_n)\|^2 \geq 2\|J_{\mu_1}^{M_1}(z_n)\|^2 + 2\|B^* J_{\mu_2}^{M_2}(Bz_n)\|^2 \geq 0,$$

which yield $\|J_{\mu_1}^{M_1}(z_n)\| = 0$ and $\|B^* J_{\mu_2}^{M_2}(Bz_n)\| = 0$. Boundedness of the operator B^*, implies $\|J_{\mu_2}^{M_2}(Bz_n)\| = 0$; consequently, $z_n \in \Theta$. If $\|J_{\mu_1}^{M_1}(z_n) + B^* J_{\mu_2}^{M_2}(Bz_n)\| \neq 0$, then using (3.3), we have

$$\frac{\|J_{\mu_1}^{M_1}(z_n)\|^2 + \|J_{\mu_2}^{M_2}(Bz_n)\|^2}{\|J_{\mu_1}^{M_1}(z_n) + B^* J_{\mu_2}^{M_2}(Bz_n)\|^2}[J_{\mu_1}^{M_1}(z_n) + B^* J_{\mu_2}^{M_2}(Bz_n)] = 0.$$

Taking the norm on both sides, we obtain

$$\frac{\|J_{\mu_1}^{M_1}(z_n)\|^2 + \|J_{\mu_2}^{M_2}(Bz_n)\|^2}{\|J_{\mu_1}^{M_1}(z_n) + B^* J_{\mu_2}^{M_2}(Bz_n)\|} = 0.$$

From Lemma 3.1 , we deduce $\|J_{\mu_1}^{M_1}(z_n)\| = \|J_{\mu_2}^{M_2}(Bz_n)\| = 0$, and Lemma 2.3 implies that $z_n \in \Theta$. □

Theorem 3.1. *Suppose the (A_1) holds of Assumptions 3.1. Then, the sequence $\{z_n\}$ obtained from Algorithm 3.1 converges weakly to $s^* \in \Theta$.*

Proof. Let $s^* \in \Theta$, and using (3.3) and (2.2), we get

$$\begin{aligned}\|z_{n+1} - s^*\|^2 &= \|u_n - \eta_n[J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n)] - s^*\|^2 \\ &\leq \|u_n - s^*\|^2 + \eta_n^2\|J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n)\|^2 \\ &\quad - 2\eta_n\langle J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n), \quad u_n - s^*\rangle. \end{aligned} \tag{3.11}$$

For $s^* \in \Theta$, by Lemma 3.2, we have $J_{\mu_1}^{M_1}(s^*) = 0$ and $J_{\mu_2}^{M_2}(Bs^*) = 0$. Since $J_{\mu_1}^{M_1}$ is μ_1-inverse strongly monotone (Remark 3.2), we have

$$\begin{aligned}
&\langle J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n), \quad u_n - s^*\rangle \\
&= \langle J_{\mu_1}^{M_1}(u_n), \quad u_n - s^*\rangle + \langle B^* J_{\mu_2}^{M_2}(Bu_n), u_n - s^*\rangle \\
&= \langle J_{\mu_1}^{M_1}(u_n) - J_{\mu_1}^{M_1}(s^*), \quad u_n - s^*\rangle + \langle B^* J_{\mu_2}^{M_2}(Bu_n) - J_{\mu_2}^{M_2}(Bs^*), u_n - s^*\rangle \\
&= \langle J_{\mu_1}^{M_1}(u_n) - J_{\mu_1}^{M_1}(s^*), \quad u_n - s^*\rangle + \langle J_{\mu_2}^{M_2}(Bu_n) - J_{\mu_2}^{M_2}(Bs^*), B(u_n) - B(s^*)\rangle \\
&\geq \mu_1 \|J_{\mu_1}^{M_1}(u_n)\|^2 + \mu_2 \|J_{\mu_2}^{M_2}(Bu_n)\|^2 \\
&\geq \min\{\mu_1, \mu_2\}\{\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2\} \\
&\geq \mu\{\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2\}, \qquad (3.12)
\end{aligned}$$

and

$$\begin{aligned}
\eta_n^2 \|J_{\mu_1}^{M_1}(u_n) \quad &+ \quad B^* J_{\mu_2}^{M_2}(Bu_n)\|^2 - 2\eta_n \langle J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n), \quad u_n - s^*\rangle \\
&\leq \quad \frac{(\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2)^2}{\|J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n)\|^2} - 2\mu \frac{(\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2)^2}{\|J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n)\|^2} \\
&= \quad (1 - 2\mu) \frac{(\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2)^2}{\|J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n)\|^2}. \qquad (3.13)
\end{aligned}$$

From (3.11)–(3.13), we achieve

$$\|z_{n+1} - s^*\|^2 \quad \leq \quad \|u_n - s^*\|^2 + (1 - 2\mu) \frac{(\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2)^2}{\|J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n)\|^2}. \qquad (3.14)$$

From (2.2), (3.2) and using the Cauchy-Schwarz inequality, we observe that

$$\begin{aligned}
\|u_n - s^*\|^2 \quad &= \quad \|z_n - \phi_n(z_n - z_{n-1}) - s^*\|^2 \\
&= \quad \|z_n - s^* + \phi_n^2(z_n - z_{n-1})\|^2 \\
&= \quad \|z_n - s^*\|^2 + \phi_n^2 \|z_n - z_{n-1}\|^2 + 2\phi_n^2 \langle z_n - z_{n-1}, z_n - s^*\rangle \\
&\leq \quad \|z_n - s^*\|^2 + \phi_n^2 \|z_n - z_{n-1}\|^2 + 2\phi_n^2 \|z_n - z_{n-1}\| \|z_n - s^*\|.
\end{aligned}$$

Since

$$2\|z_n - z_{n-1}\| \|z_n - s^*\| = \|z_n - z_{n-1}\|^2 + \|z_n - s^*\|^2 - \|(z_n - z_{n-1}) - (z_n - s^*)\|^2$$

and $\phi_n^2 \leq \phi_n$, therefore

$$\begin{aligned}
\|u_n - s^*\|^2 \quad &\leq \quad \|z_n - s^*\|^2 + \phi_n \|z_n - z_{n-1}\|^2 + \phi_n\{\|z_n - z_{n-1}\|^2 + \|z_n - s^*\|^2 \\
&\qquad - \|(z_n - z_{n-1}) - (z_n - s^*)\|^2\} \\
&= \quad \|z_n - s^*\|^2 + 2\phi_n \|z_n - z_{n-1}\|^2 + \phi_n\{\|z_n - s^*\|^2 - \|z_{n-1} - s^*\|^2\}. \qquad (3.15)
\end{aligned}$$

Thus, taking (3.14) and (3.15) into account, we acquire

$$\|z_{n+1} - s^*\| \quad \leq \quad \|z_n - s^*\|^2 + 2\phi_n \|z_n - z_{n-1}\|^2 + \phi_n\{\|z_n - s^*\|^2 - \|z_{n-1} - s^*\|^2\}$$

$$+(1-2\mu)\frac{(\|J_{\mu_1}^{M_1}(u_n)\|^2+\|J_{\mu_2}^{M_2}(Bu_n)\|^2)^2}{\|J_{\mu_1}^{M_1}(u_n)+B^*J_{\mu_2}^{M_2}(Bu_n)\|^2}. \tag{3.16}$$

Since $\mu > \frac{1}{2}$, we get

$$\|z_{n+1}-s^*\| \leq \|z_n-s^*\|^2+2\phi_n\|z_n-z_{n-1}\|^2+\phi_n\{\|z_n-s^*\|^2-\|z_{n-1}-s^*\|^2\}. \tag{3.17}$$

Consequently, by Lemma 2.5 $\{\|z_n-s^*\|\}$ is convergent, and $\sum_{n=1}^{\infty}(\|z_{n+1}-s^*\|-\|z_n-s^*\|)<\infty$. From (3.16), we infer

$$\lim_{n\to\infty}\frac{(\|J_{\mu_1}^{M_1}(u_n)\|^2+\|J_{\mu_2}^{M_2}(Bu_n)\|^2)^2}{\|J_{\mu_1}^{M_1}(u_n)+B^*J_{\mu_2}^{M_2}(Bu_n)\|^2}=0.$$

From Lemma 3.1, we obtain

$$\lim_{n\to\infty}\|J_{\mu_1}^{M_1}(u_n)\|=\lim_{n\to\infty}\|J_{\mu_2}^{M_2}(Bu_n)\|=0. \tag{3.18}$$

Convergence of $\{\|z_n-s^*\|\}$ implies that $\{z_n\}$ is bounded. Let $x^*\in\omega_w(z_n)$ and $\{x_{n_k}\}$ be a subsequence of $\{z_n\}$ such that $x_{n_k}\to x^*$. From (3.2), and Remark 3.1, we have

$$\|u_n-z_n\|=\phi_n\|z_n-z_{n-1}\|=0 \quad \text{as} \quad n\to\infty.$$

Therefore, there will exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k}\to x^*$. Hence, from (3.18), we get

$$\|J_{\mu_1}^{M_1}(x^*)\|=\lim_{k\to\infty}\|J_{\mu_1}^{M_1}(u_{n_k})\|=0 \quad \text{and} \quad \|J_{\mu_2}^{M_2}(Bx^*)\|=\lim_{k\to\infty}\|J_{\mu_2}^{M_2}(Bu_{u_{n_k}})\|=0.$$

This implies that $x^*\in M_1^{-1}(0)$ and $Bx^*\in M_2^{-1}(0)$. □

Theorem 3.2. *Suppose that Assumption 3.1 holds. Then, the sequence* $\{z_n\}$ *produced by Algorithm 3.2 converges strongly to* $s^*=P_{\text{Fix}(T)\cap\Theta}(0)$.

Proof. Let $s^*\in\text{Fix}(T)\cap\Theta$. From (3.7) and (3.8), following the steps of Theorem 3.1, we achieve

$$\|v_n-s^*\|^2 = \|u_n-s^*\|^2+(1-2\mu)\frac{(\|J_{\mu_1}^{M_1}(u_n)\|^2+\|J_{\mu_2}^{M_2}(Bu_n)\|^2)^2}{\|J_{\mu_1}^{M_1}(u_n)+B^*J_{\mu_2}^{M_2}(Bu_n)\|^2}. \tag{3.19}$$

Since $\mu > \frac{1}{2}$, we get

$$\|v_n-s^*\|\leq\|u_n-s^*\|. \tag{3.20}$$

Assumption $\sum_{n=1}^{\infty}\phi_n\|z_n-z_{n-1}\|<\infty$ implies that there exists a number K_1 such that $\phi_n\|z_n-z_{n-1}\|\leq K_1$. By combining (3.7), (3.8) and using (3.20), we achieve

$$\begin{aligned}\|z_{n+1}-s^*\| &= \|(1-\varphi_n-\psi_n)v_n+\varphi_nT(v_n)-s^*\|\\ &\leq (1-\varphi_n-\psi_n)\|v_n-s^*\|+\varphi_n\|T(v_n)-s^*\|+\psi_n\|s^*\|\\ &\leq (1-\varphi_n-\psi_n)\|u_n-s^*\|+\varphi_n\|v_n-s^*\|+\psi_n\|s^*\|\end{aligned}$$

$$\begin{aligned}
&\leq (1-\psi_n)\|u_n - s^*\| + \psi_n\|s^*\| \\
&\leq (1-\psi_n)[\|z_n - s^*\| + \phi_n\|z_n - z_{n-1}\|] + \psi_n\|s^*\| \\
&\leq (1-\psi_n)\|z_n - s^*\| + \psi_n K_1 + \psi_n\|s^*\| \\
&\leq \max\{\|z_n - s^*\|, \|\|s^*\| + K_1\},
\end{aligned}$$

implying that $\{\|z_n - s^*\|\}$ is bounded, and hence $\{z_n\}$, $\{u_n\}$, $\{v_n\}$ are also bounded. Furthermore, using (3.9), (3.15), (2.1) and nonexpansive property of T, we have

$$\begin{aligned}
\|z_{n+1} - s^*\|^2 &= \|(1-\varphi_n-\psi_n)v_n + \varphi_n T(v_n) - s^*\|^2 \\
&= \|(1-\varphi_n-\psi_n)v_n + \varphi_n(T(v_n) - s^*) + \psi_n(-s^*)\|^2 \\
&\leq (1-\varphi_n-\psi_n)\|v_n - s^*\|^2 + \varphi_n\|T(v_n) - s^*\|^2 + \psi_n\| - s^*)\|^2 \\
&\quad -\varphi_n(1-\varphi_n-\psi_n)\|T(v_n) - v_n\|^2 \\
&\leq (1-\varphi_n-\psi_n)\|v_n - s^*\|^2 + \varphi_n\|v_n - s^*\|^2 + \psi_n\| - s^*)\|^2 \\
&\quad -\varphi_n(1-\varphi_n-\psi_n)\|T(v_n) - v_n\|^2 \\
&\leq (1-\psi_n)\|v_n - s^*\|^2 + \psi_n\| - s^*\|^2 \\
&\quad -\varphi_n(1-\varphi_n-\psi_n)\|T(v_n) - v_n\|^2 \\
&\leq (1-\psi_n)\Big[\|u_n - s^*\|^2 + (1-2\mu)\frac{(\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2)^2}{\|J_{\mu_1}^{M_1}(u_n) + B^*J_{\mu_2}^{M_2}(Bu_n)\|^2}\Big] \\
&\quad +\psi_n\| - s^*\|^2 - \varphi_n(1-\varphi_n-\psi_n)\|T(v_n) - v_n\|^2 \\
&\leq \|z_n - s^*\|^2 + 2\phi\|z_n - z_{n-1}\|^2 + \phi_n\{\|z_n - s^*\|^2 - \|z_{n-1} - s^*\|^2\} \\
&\quad +\psi_n\| - s^*\|^2 - \varphi_n(1-\varphi_n-\psi_n)\|T(v_n) - v_n\|^2 \\
&\quad +(1-2\mu)\frac{(\|J_{\mu_1}^{M_1}u_n\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2)^2}{\|J_{\mu_1}^{M_1}u_n + B^*J_{\mu_2}^{M_2}(Bu_n)\|^2}. \qquad (3.21)
\end{aligned}$$

Since $\mu > \frac{1}{2}$, then (3.21) can be written as

$$\begin{aligned}
&\varphi_n(1-\varphi_n-\psi_n)\|T(v_n) - v_n\|^2 + (2\mu-1)\frac{(\|J_{\mu_1}^{M_1}u_n\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2)^2}{\|J_{\mu_1}^{M_1}u_n + B^*J_{\mu_2}^{M_2}(Bu_n)\|^2} \leq \phi_n\{\|z_n - s^*\|^2 \\
&-\|z_{n-1} - s^*\|^2\} + \|z_{n+1} - s^*\|^2 - \|z_n - s^*\|^2 + 2\phi\|z_n - z_{n-1}\|^2 + \psi_n\|s^*\|^2. \qquad (3.22)
\end{aligned}$$

Now, we discuss two possibilities:

Case I. If the sequence $\|z_n - s^*\|$ is non-increasing, then there exists $m \geq 0$ so that $\|z_{n+1} - s^*\| \leq \|z_n - s^*\|$ for each $n \geq m$. Hence, $\lim_{n\to\infty} \|z_{n+1} - s^*\|$ exists, and

$$\lim_{n\to\infty}\{\|z_{n+1} - s^*\| - \|z_n - s^*\|\} = 0. \qquad (3.23)$$

Since $\mu > \frac{1}{2}$, $\inf \varphi_n(1-\varphi_n-\psi_n) > 0$ and $\psi_n \to 0$. It follows from (3.22) that

$$\lim_{n\to\infty} \|T(v_n) - v_n\| = 0, \quad \lim_{n\to\infty} \frac{(\|J_{\mu_1}^{M_1}(u_n)\|^2 + \|J_{\mu_2}^{M_2}(Bu_n)\|^2)^2}{\|J_{\mu_1}^{M_1}(u_n) + B^*J_{\mu_2}^{M_2}(Bu_n)\|^2} = 0. \qquad (3.24)$$

We deduce from Lemma 3.1 that

$$\lim_{n\to\infty} \|J_{\mu_1}^{M_1}(u_n)\| = \|J_{\mu_2}^{M_2}(Bu_n)\| = 0. \tag{3.25}$$

Thus, from (3.8), we obtain

$$\lim_{n\to\infty} \|v_n - u_n\| = 0. \tag{3.26}$$

Since $\|v_n - u_n\| \to 0, \|T(v_n) - v_n\| \to 0, \|u_n - z_n\| \to 0$ and $\psi_n \to 0$ as $n \to \infty$, we have

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \|(1-\varphi_n-\psi_n)v_n + \varphi_n T(v_n) - z_n\| \\
&= (1-\psi_n)(v_n - u_n) + \varphi_n(T(v_n) - v_n) + (1-\psi_n)(u_n - z_n) + \psi_n(z_n) \\
&\le (1-\psi_n)\|v_n - u_n\| + \varphi_n\|T(v_n) - v_n\| + (1-\psi_n)\|u_n - z_n\| + \psi_n\|z_n\| \\
&\le (1-\psi_n)\|v_n - u_n\| + \varphi_n\|T(v_n) - v_n\| \\
&\quad +(1-\psi_n)\phi_n\|z_n - z_{n-1}\| + \psi_n\|z_n\| \to 0.
\end{aligned} \tag{3.27}$$

By setting $x_n = (1-\varphi_n)v_n + \varphi_n T(v_n)$, estimate

$$\begin{aligned}
\|x_n - s^*\| &\le \|(1-\varphi_n)v_n + \varphi_n T(v_n) - s^*\| \\
&= (1-\varphi_n)\|v_n - s^*\| + \varphi_n\|T(v_n) - s^*\| \\
&\le \|v_n - s^*\|.
\end{aligned} \tag{3.28}$$

Let $K_3 = \sup_{n\ge1}\{2\|z_n - z_{n-1}\| + \|z_n - s^*\| + \|z_{n-1} - s^*\|\}$, and using (3.9), (3.18) and (3.20), we get

$$\begin{aligned}
\|z_{n+1} - s^*\|^2 &= \|(1-\psi_n)x_n + \varphi_n\psi_n(Tv_n - v_n) - s^*\|^2 \\
&\le \|(1-\psi_n)(x_n - s^*) + \varphi_n\psi_n(Tv_n - v_n) - \psi_n(s^*)\|^2 \\
&\le (1-\psi_n)\|x_n - s^*\|^2 + 2\langle\varphi_n\psi_n(Tv_n) - v_n - \psi_n(s^*), \quad z_{n+1} - s^*\rangle \\
&\le (1-\psi_n)\|v_n - s^*\|^2 + 2\psi_n\langle\varphi_n((Tv_n) - v_n) - s^*, \quad z_{n+1} - s^*\rangle \\
&\le (1-\psi_n)\|u_n - s^*\|^2 + 2\psi_n\langle\varphi_n(Tv_n - v_n) - s^*, \quad z_{n+1} - s^*\rangle \\
&\le (1-\psi_n)\Big[\|z_n - s^*\|^2 + 2\phi_n\|z_n - z_{n-1}\|^2 + \phi_n\{\|z_n - s^*\|^2 - \|z_{n-1} - s^*\|^2\}\Big] \\
&\quad +2\psi_n\langle\varphi_n((Tv_n) - v_n), \quad z_{n+1} - s^*\rangle + 2\psi_n\langle -s^*, \quad z_{n+1} - s^*\rangle \\
&\le (1-\psi_n)\|z_n - s^*\|^2 + \phi_n\|z_n - z_{n-1}\|\{2\|z_n - z_{n-1}\| + \|z_n - s^*\| + \|z_{n-1} - s^*\|\} \\
&\quad +2\psi_n\langle\varphi_n((Tv_n) - v_n), \quad z_{n+1} - s^*\rangle + 2\psi_n\langle -s^*, \quad z_{n+1} - s^*\rangle \\
&\le (1-\psi_n)\|z_n - s^*\|^2 + K_3\phi_n\|z_n - z_{n-1}\| + 2\psi_n\langle\varphi_n((Tv_n) - v_n), \quad z_{n+1} - s^*\rangle \\
&\quad +2\psi_n\langle -s^*, \quad z_{n+1} - s^*\rangle.
\end{aligned} \tag{3.29}$$

Since

$$\lim_{n\to\infty} \phi_n\|z_n - z_{n-1}\| = 0, \quad \lim_{n\to\infty}(Tv_n - v_n) = 0, \quad \lim_{n\to\infty} \psi_n = 0,$$

using property (2.3), we have

$$\limsup_{n\to\infty}\langle -s^*, \quad z_{n+1} - s^*\rangle = \max_{\bar{p}\in \mathrm{Fix(T)}\cap\Theta}\langle -s^*, \quad \bar{p} - s^*\rangle \le 0.$$

Hence, by applying Lemma 2.1, $\|z_n - s^*\|$ converges to 0, that is, $\{z_n\}$ converges strongly to $s^* = P_{\mathrm{Fix(T)}\cap\Theta}(0)$. Further, using the property of metric projection, we have

$$\langle s^*, p - s^*\rangle \geq 0,\ \ \forall p \in \mathrm{Fix}(T) \cap \Theta,$$

which implies that $\langle s^*, p\rangle \geq \|z^*\|^2$, that is, $\|z^*\| \leq \|p\|$, which means that z^* is the minimum norm element of $\mathrm{Fix}(T) \cap \Theta$.

Case II. If the sequence $\|z_n - s^*\|$ is not nonincreasing, then there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\|z_{n_k} - s^*\|^2 \leq \|z_{n_k+1} - s^*\|^2$. Without loss of generality, we can define a subsequence $\gamma(n) = \max\{m \leq n : \|z_m - p\| \leq \|z_{m+1} - p\|\}$, and $\gamma(n) \to \infty$ as $n \to \infty$. It follows from (3.21), that

$$\begin{aligned}
\|z_{\gamma(n)} - s^*\|^2 &\leq \|z_{\gamma(n+1)} - s^*\|^2 \\
&\leq \|z_{\gamma(n)} - s^*\|^2 + 2\phi_{\gamma(n)}\|z_{\gamma(n)} - z_{\gamma(n-1)}\|^2 + \phi_{\gamma(n)}\{\|z_{\gamma(n)} - s^*\|^2 - \|z_{\gamma(n)-1} - s^*\|^2\} \\
&\quad +\psi_{\gamma(n)}\|s^*\|^2 - \varphi_{\gamma(n)}(1 - \varphi_{\gamma(n)} - \psi_{\gamma(n)})\|Tv_{\gamma(n)} - u_{\gamma(n)}\|^2 \\
&\quad -(2\mu - 1)(1 - \varphi_{\gamma(n)} - \psi_{\gamma(n)})\frac{(\|J_{\mu_1}^{M_1}(u_{\gamma(n)})\|^2 + \|J_{\mu_2}^{M_2}(Bu_{\gamma(n)})\|^2)^2}{\|J_{\mu_1}^{M_1}(u_{\gamma(n)}) + B^* J_{\mu_2}^{M_2}(Bu_{\gamma(n)})\|^2}.
\end{aligned} \tag{3.30}$$

That is,

$$\begin{aligned}
\psi_{\gamma(n)}\|s^*\|^2 &+ 2\phi_{\gamma(n)}\|z_{\gamma(n)} - z_{\gamma(n-1)}\|^2 + \phi_{\gamma(n)}\{\|z_{\gamma(n)} - s^*\|^2 - \|z_{\gamma(n)-1} - s^*\|^2\} \\
&\geq \varphi_{\gamma(n)}(1 - \varphi_{\gamma(n)} - \psi_{\gamma(n)})\|Tv_{\gamma(n)} - v_{\gamma(n)}\|^2 \\
&\quad +(2\mu - 1)(1 - \varphi_{\gamma(n)} - \psi_{\gamma(n)})\frac{(\|J_{\mu_1}^{M_1}(u_{\gamma(n)})\|^2 + \|J_{\mu_2}^{M_2}(Bu_{\gamma(n)})\|^2)^2}{\|J_{\mu_1}^{M_1}(u_{\gamma(n)}) + B^* J_{\mu_2}^{M_2}(Bu_{\gamma(n)})\|^2},
\end{aligned} \tag{3.31}$$

or

$$\begin{aligned}
\psi_{\gamma(n)}\|s^*\|^2 &+ \phi_{\gamma(n)}\|z_{\gamma(n)} - z_{\gamma(n-1)}\|\{2\|z_{\gamma(n)} - z_{\gamma(n)-1}\| + \|z_{\gamma(n)} - s^*\| + \|z_{\gamma(n)-1} - s^*\|\} \\
&\geq \varphi_{\gamma(n)}(1 - \varphi_{\gamma(n)} - \psi_{\gamma(n)})\|Tv_{\gamma(n)} - v_{\gamma(n)}\|^2 \\
&\quad +(2\mu - 1)(1 - \varphi_{\gamma(n)} - \psi_{\gamma(n)})\frac{(\|J_{\mu_1}^{M_1}(u_{\gamma(n)})\|^2 + \|J_{\mu_2}^{M_2}(Bu_{\gamma(n)})\|^2)^2}{\|J_{\mu_1}^{M_1}(u_{\gamma(n)}) + B^* J_{\mu_2}^{M_2}(Bu_{\gamma(n)})\|^2}.
\end{aligned} \tag{3.32}$$

$\psi_{\gamma(n)} \to 0$ and $\phi_{\gamma(n)}\|z_{\gamma(n)} - z_{\gamma(n)-1}\|^2 \to 0$ as $\gamma_{(n)} \to \infty$. Therefore,

$$\lim_{n\to\infty} \|Tv_{\gamma(n)} - v_{\gamma(n)}\|^2 = 0, \qquad \lim_{n\to\infty} \frac{(\|J_{\mu_1}^{M_1}(u_{\gamma(n)})\|^2 + \|J_{\mu_2}^{M_2}(Bu_{\gamma(n)})\|^2)^2}{\|J_{\mu_1}^{M_1}(u_{\gamma(n)}) + B^* J_{\mu_2}^{M_2}(Bu_{\gamma(n)})\|^2} = 0, \tag{3.33}$$

and using the proof in Case I, we also have

$$\lim_{n\to\infty} \sup\langle -s^*, \quad z_{\gamma(n)+1} - s^*\rangle = \max_{\bar{p}\in\omega_W(z_{\gamma(n)})} \langle -s^*, \quad \bar{p} - s^*\rangle \leq 0. \tag{3.34}$$

Also, using (3.29), we have

$$\begin{aligned}
\|z_{\gamma(n)+1} - s^*\|^2 &\leq (1 - \psi_{\gamma(n)})\{\|z_{\gamma(n)} - s^*\|^2 + 2\phi_{\gamma(n)}\|z_{\gamma(n)} - z_{\gamma(n)-1}\|^2 + \phi_{\gamma(n)}\{\|z_{\gamma(n)} - s^*\|^2 \\
&\quad -\|z_{\gamma(n)-1} - s^*\|^2\} - 2\psi_{\gamma(n)}\{\langle\varphi_{\gamma(n)}(Tv_{\gamma(n)} - v_{\gamma(n)}) - s^*, \quad z_{\gamma(n)+1} - s^*\rangle
\end{aligned}$$

$$+\langle s^*, \quad z_{\gamma(n)+1} - s^*\rangle\}. \tag{3.35}$$

Hence,

$$\begin{aligned} \|z_{\gamma(n)} - s^*\|^2 \leq & \frac{K_4(1-\psi_{\gamma(n)})}{\psi_{\gamma(n)}}\phi_{\gamma(n)}\|z_{\gamma(n)} - z_{\gamma(n)-1}\| \\ & -2\psi_{\gamma(n)}\{\langle\varphi_{\gamma(n)}(Tv_{\gamma(n)} - v_{\gamma(n)}) - s^*, \quad z_{\gamma(n)+1} - s^*\rangle \\ & +\langle s^*, \quad z_{\gamma(n)+1} - s^*\rangle\}, \end{aligned} \tag{3.36}$$

where $K_4 = \sup_{\gamma(n)\geq 1}\{2\|z_{\gamma(n)} - z_{\gamma(n)-1}\| + \|z_{\gamma(n)} - s^*\| + \|z_{\gamma(n)-1} - s^*\|\}$. Combining (3.34)–(3.36), we obtain

$$\limsup_{n\to\infty}\|z_{\gamma(n)} - s^*\|^2 = 0, \quad \text{and} \quad \text{hence} \quad \lim_{n\to\infty}\|z_{\gamma(n)} - s^*\|^2 = 0. \tag{3.37}$$

Making use of (3.35), we obtain

$$\limsup_{n\to\infty}\|z_{\gamma(n)+1} - s^*\|^2 = 0, \limsup_{n\to\infty}\|z_{\gamma(n)} - s^*\|^2 = 0. \tag{3.38}$$

Thus, $\lim_{n\to\infty}\|z_{\gamma(n)+1} - s^*\|^2 = 0$. Applying Lemma 2.4, we have

$$0 \leq \|z_n - s^*\|^2 \leq \max\{\|z_{\gamma(n)} - s^*\|^2, \|z_n - s^*\|^2\} \leq \|z_{\gamma(n)+1} - s^*\|^2 \to 0.$$

Consequently, $z_n \to s^* = P_{\mathrm{Fix}(T)\cap\Theta}(0)$, which is the minimum norm element of $\mathrm{Fix}(T) \cap \Theta$. □

Corollary 3.1. *Let H_1, H_2, M_1, M_2, T, B, B^*, μ_1, μ_2, ϕ_n and η_n be the same as considered in Theorem 3.2. If $\{\varphi_n\}$ is a sequence in $(0,1)$ such that*

$$\lim_{n\to\infty}(1-\varphi_n)\varphi_n > 0, \tag{3.39}$$

then, the sequence $\{z_n\}$ generated by

$$\begin{aligned} u_n &= z_n + \phi_n(z_n - z_{n-1}), \\ v_n &= u_n - \eta_n[J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n)], \\ z_{n+1} &= (1-\varphi_n)v_n + \varphi_n T(v_n), \end{aligned}$$

converges strongly to $z \in \mathrm{Fix}(T) \cap \Theta$.

If $T = I$, the identity mapping, and $\psi = 0$, then we acquire the following corollary for SVIP.

Corollary 3.2. *Let H_1, H_2, M_1, M_2, B, B^*, μ_1, μ_2, ϕ_n and η_n be the same as considered in Theorem 3.2. If $\{\varphi_n\}$ is a sequence in $(0,1)$ such that*

$$\lim_{n\to\infty}(1-\varphi_n)\varphi_n > 0, \tag{3.40}$$

then, the sequence $\{z_n\}$ generated by

$$\begin{aligned} u_n &= z_n + \phi_n(z_n - z_{n-1}), \\ v_n &= u_n - \eta_n[J_{\mu_1}^{M_1}(u_n) + B^* J_{\mu_2}^{M_2}(Bu_n)], \\ z_{n+1} &= (1-\varphi_n)v_n + \varphi_n(v_n), \end{aligned} \tag{3.41}$$

converges strongly to $z \in \Theta$.

4. Numerical example

Let $H_1 = H_2 = \mathbb{R}$. Define the monotone operators M_1 and M_2 as $M_1(x) = \frac{x}{2} + 3$ and $M_2(x) = x + 1$ and a nonexpansive mapping $T : H_1 \to H_1$ by $T(x) = \frac{x-6}{2}$. The bounded linear operator $B : H_1 \to H_2$ is defined as $B(x) = \frac{x}{6}$. It can be easily seen that $\mathrm{Fix}T \cap \Theta = \{-6\}$.

Since

$$\langle M_1(x) - M_1(y),\ x - y\rangle = \langle \frac{x}{2} + 3 - \frac{y}{2} - 3,\ x - y\rangle = \frac{1}{2}\|x - y\|^2 = 2\|M_1(x) - M_1(y)\|^2,$$

and

$$\langle M_2(x) - M_2(y),\ x - y\rangle = \langle x + 1 - y - 1,\ x - y\rangle = \|x - y\|^2 = \|M_2(x) - M_2(y)\|^2,$$

imply that M_1 is 2-inverse strongly monotone and M_2 is 1-inverse strongly monotone. The resolvents of M_1 and M_2 for $\mu_1 > 0, \mu_2 > 0$ are calculated as

$$R^{M_1}_{\mu_1}(x) = [I + \mu_1 M_1]^{-1}(x) = \frac{2x - 6\mu_1}{2 + \mu_1},$$

and

$$R^{M_2}_{\mu_2}(x) = [I + \mu_2 M_2]^{-1}(x) = \frac{x - \mu_2}{1 + \mu_2}.$$

Hence, the Yosida approximations of M_1 and M_2 are

$$J^{M_1}_{\mu_1}(x) = \frac{1}{\mu_1}[I - R^{M_1}_{\mu_1}](x) = \frac{x + 6}{2 + \mu_1},$$

and

$$J^{M_2}_{\mu_2}(x) = \frac{1}{\mu_2}[I - R^{M_2}_{\mu_2}](x) = \frac{x + 1}{1 + \mu_2}.$$

For Algorithm 3.2, we choose $\varphi_n = \frac{3n}{5n+1}$ and $\psi = \frac{1}{n+1}$ satisfying the condition (3.5). We use the maximum number of iterations 50 as stopping criterion. Parameter ϕ_n is generated randomly in $(0, \bar{\phi}_n)$, where $\bar{\phi}_n$ is calculated by (3.6). The behaviours of the sequences $\{z_n\}, \{v_n\}$ and $\{u_n\}$ are recorded in Figures 1–4, using three different cases of parameters, as listed below:

Case (I): $z_0 = 0, z_1 = 0; \mu_1 = 1, \mu_2 = 2; \phi = 0.1; \delta_n = \frac{1}{(1+n)^{1.2}}$.

Case (II): $z_0 = 5, z_1 = -5; \mu_1 = 5, \mu_2 = 10; \phi = 0.2; \delta_n = \frac{1}{(1+n)^{1.5}}$.

Case (III): $z_0 = 4, z_1 = 8; \mu_1 = 10, \mu_2 = 20; \phi = 0.9; \delta_n = \frac{1}{(1+n)^{2}}$.

Observations:

- In Figures 1–3, it can be observed that the behaviours of $\{z_n\}, \{v_n\}$ and $\{u_n\}$ is consistent irrespective of the choice of parameters using all three cases.
- From Figure 4, we can see that the Algorithm 3.2 converges to the same solution with appropriate choice of parameters.

Furthermore, we compare our Algorithm 3.2, with some known iterative schemes, which are (1.5) introduced by Kazmi and Rizvi [21], (1.6) by Sitthithakerngkiet et al. [32] (in short, Sitthi) and (1.7) by Akram et al. [2]. The parameters are selected as follows:

We choose $f(x) = \frac{x}{2}$, $T(x) = \frac{x-6}{2}$, $\gamma = 0.5$, $\beta_n = \frac{1}{(n+1)^{0.6}}$ for (1.5)–(1.7); $\xi = \frac{1}{2}$, $D = 1$, $T_n = T$, for all $n \in N$ for (1.6); $\varphi_n = \frac{3n}{5n+2}$, $\psi = \frac{1}{n+2}$, $\delta_n = \frac{1}{(1+n)^2}$, $\gamma = 0.9$ and $\phi = 0.8$ for Algorithm 3.2. We define $D_n = \|z_n - s^*\|$ to measure the error of n^{th} iteration step for all algorithms.

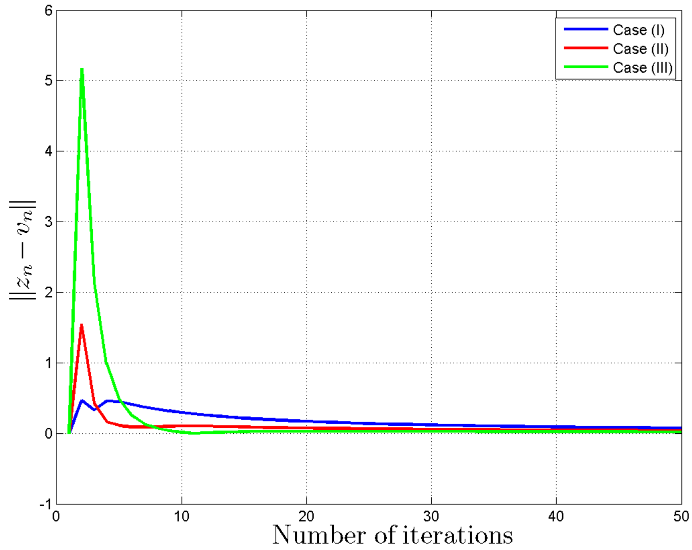

Figure 1. Numerical behavior of $\|z_n - v_n\|$ with different parameters.

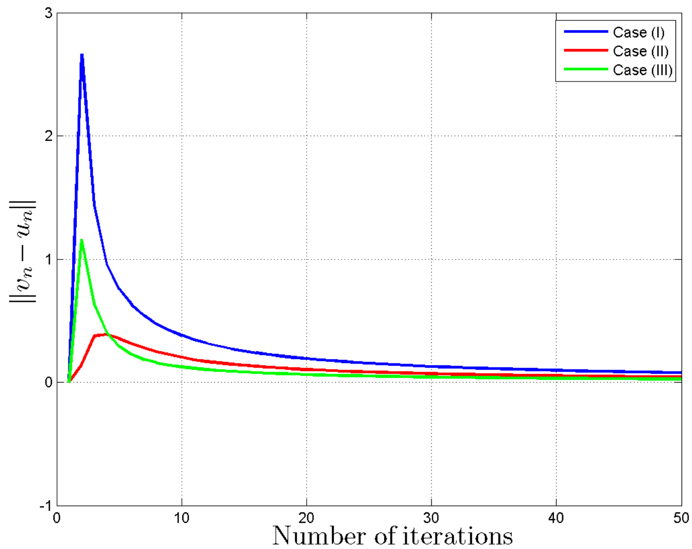

Figure 2. Numerical behavior of $\|v_n - u_n\|$ with different parameters.

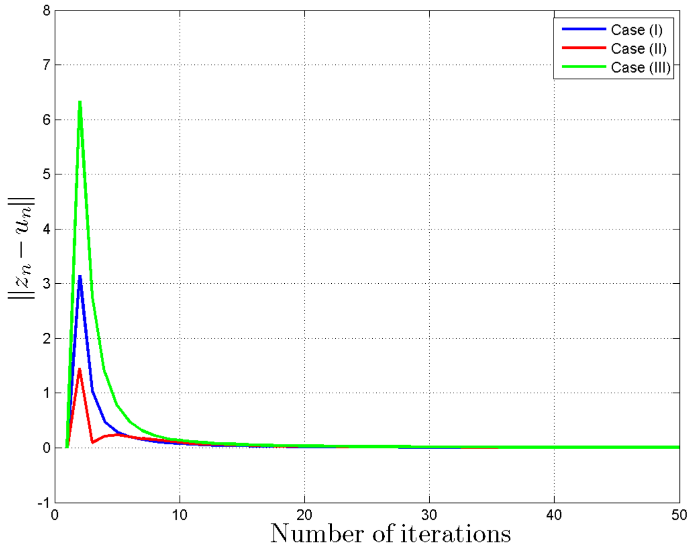

Figure 3. Numerical behavior of $||z_n - u_n||$ with different parameters.

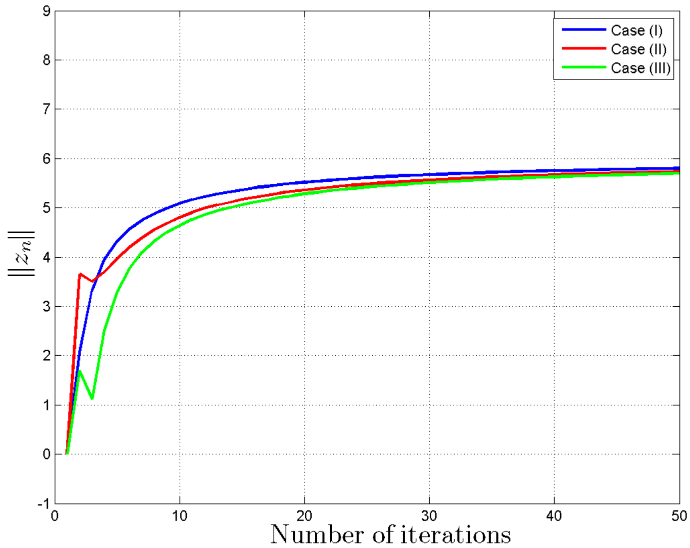

Figure 4. Numerical behavior of $||z_n||$ with different parameters.

Observations:

- The importance of our Algorithm 3.2 is that its implementation does not require the calculation of the norm of $||B||$ or the spectral radius of B^*B. In other schemes (1.5)–(1.7), it is mandatory to estimate $||B||$ to know the stepsize γ, which is expensive to calculate in general. In Algorithm 3.2, γ is chosen by itself without knowing the value of $||B||$.
- From Tables 1 and 2, and Figures 5 and 6, we observed that the value of the error D_n is less than

other algorithms with some fixed parameters or fixed initial values.

- In Table 3, we observed that by fixing D_n, the sequence obtained in Algorithm 3.2 converges to the solution in fewer steps in comparison to other algorithms. These results are independent of the size of initial values and other parameters.

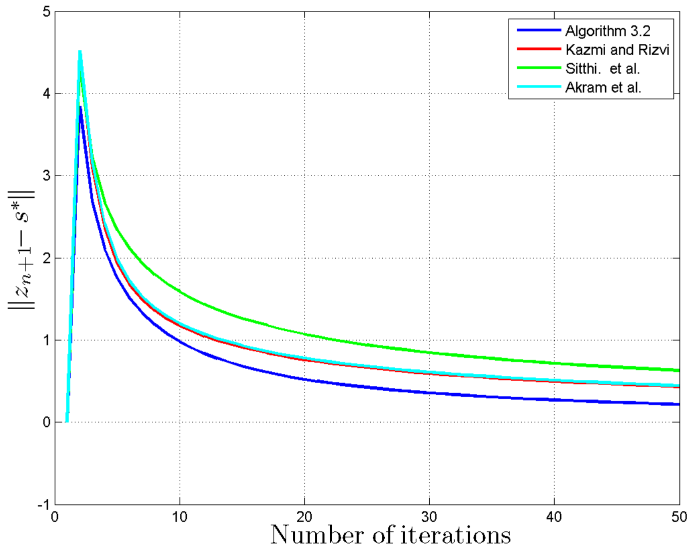

Figure 5. Numerical behavior of all algorithms with fixed $z_1 = -10, \mu_1 = \mu_2 = 1 = \mu = 1, \gamma = 0.5, \phi = 0.9$ and $m = 100$.

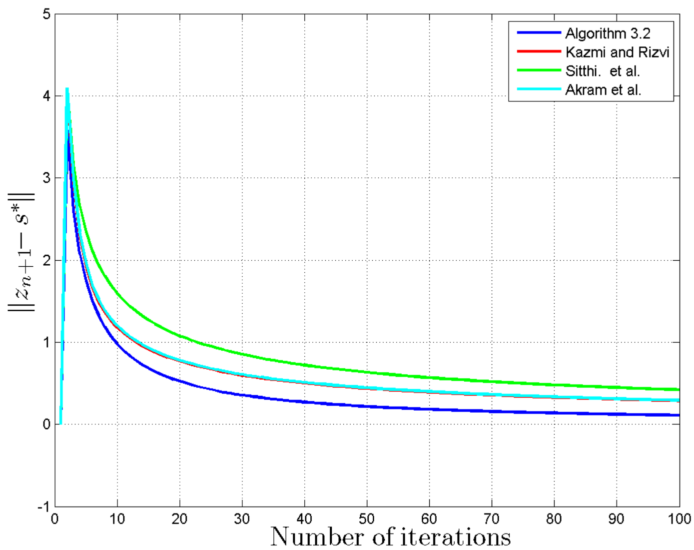

Figure 6. Numerical behavior of all algorithms for fixed $z_1 = -10, \mu_1 = \mu_2 = \mu = 1, \gamma = 0.5$, $\phi = 0.9$ and stopping criteria $m = 100$.

Table 1. Numerical results of algorithms with different parameters and fixed $z_1 = 1$ and $\phi = 0.9$.

Iter.	*Algorithms*	Algorithm 3.2		Kazmi and Rizvi [21]		Sitthi. et al. [32]		Akram et al. [2]	
m	*Parameters*	D_n	CPU(s)	D_n	CPU(s)	D_n	CPU(s)	D_n	CPU(s)
10	$\mu_1 = \mu_2 = \mu = 0.7, \gamma = 0.75$	8.70E-1	3.90E-6	1.24E+0	1.50E-6	1.67E+0	1.40E-6	1.26E+0	1.20E-6
20	$\mu_1 = \mu_2 = \mu = 1, \gamma = 1.5$	5.04E-1	3.90E-6	7.61E-1	1.10E-6	1.07E+0	1.10E-6	7.80E-1	1.10E-6
30	$\mu_1 = 3, \mu_2 = \mu = 3, \gamma = 4$	4.17E-1	8.80E-6	4.95E-1	2.50E-6	7.13e-1	2.10E-6	5.18E-1	2.50E-6
40	$\mu_1 = 1, \mu_2 = \mu = 4, \gamma = 5$	3.36E-1	4.10E-6	3.98E-1	1.20E-6	5.78E-1	1.20E-6	4.18E-1	1.10E-6
100	$\mu_1 = 1, \mu_2 =, \mu = 8, \gamma = 10$	1.57E-1	4.90E-6	2.10E-1	1.20E-6	3.09E-1	1.20E-6	2.22E-1	1.10E-6
500	$\mu_1 = 1, \mu_2 = \mu = 20, \gamma = 30$	3.59E-2	5.00E-6	7.35E-2	1.80E-6	1.10E-1	1.70E-6	7.90E-2	1.30E-6

Table 2. Numerical results of algorithms with different initial values by fixing $\mu_1 = \mu_2 = \mu = 1, \gamma = 0.5$ and $\phi = 0.9$.

Iter.	*Algorithms*	Algorithm 3.2		Kazmi and Rizvi [21]		Sitthi. et al. [32]		Akram et al. [2]	
m	*InitialValues*	D_n	CPU(s)	D_n	CPU(s)	D_n	CPU(s)	D_n	CPU(s)
$m = 25$	$z_0 = 0.0$	4.11E-01	4.10E-06	6.68E-01	1.10E-06	9.47E-01	1.10E-06	6.80E-01	1.10E-06
$m = 50$	$z_0 = 2.5$	2.14E-01	4.30E-06	4.37E-01	1.20E-06	6.32E-01	1.10E-06	4.45E-01	1.10E-06
$m = 75$	$z_0 = 5.0$	1.45E-01	1.08E-05	3.41E-01	1.20E-06	4.98E-01	1.10E-06	3.48E-01	1.10E-06
$m = 100$	$z_0 = -10$	1.09E-01	4.40E-06	2.87E-01	1.10E-06	4.20E-01	1.10E-06	2.92E-01	1.10E-06
$m = 200$	$z_0 = 15.0$	5.52E-02	4.90E-06	1.88E-01	1.30E-06	2.78E-01	1.30E-06	1.92E-01	1.30E-06

Table 3. Comparison table of all algorithms by fixing D_n and $z_1 = 1, \mu_1 = \mu_2 = \mu = 3, \phi = 0.9$.

D_n		Algorithm 3.2	Kazmi and Rizvi [21]	Sitthi. et al. [32]	Akram et al. [2]
10^{-3}	Iteration	105	175	179	143
	Time/Sec	1.18E-005	8.60E-006	2.80E-006	1.87e-004
10^{-4}	Iteration	335	795	755	645
	Time/Sec	5.90E-006	3.20E-006	1.60E-006	1.12e-003
10^{-5}	Iteration	1061	3658	3186	2956
	Time/Sec	8.10E-006	7.80E-006	2.60E-006	8.02E-003

5. Conclusions

We have presented inertial self-adaptive iterative techniques involving Yosida approximation operators. Weak and strong convergences of the proposed schemes are analyzed to investigate the solution of SVIP and common solution of SVIP and FPP, respectively, with some appropriate assumptions in which calculation of step size does not require any pre-calculation of the norm of bounded linear operator B. Our results refine and enhance many well-known results studied in the field. We have given a numerical example showing the usefulness of the proposed methods and comparison with some known results.

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Conflict of interest

The authors declare no conflicts of interest.

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