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Research article

Different types of multifractal measures in separable metric spaces and their applications

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Abstract: The properties of various fractal and multifractal measures and dimensions have been under extensive study in the real-line and higher-dimensional Euclidean spaces. In non-Euclidean spaces, it is often impossible to construct non-trivial self-similar or self-conformal sets, etc. We consider in the present paper the proper way to phrase the definitions for use in general metric spaces. We investigate the relative Hausdorff measures $\mathscr{H}_\mu^{q,t}$ and the relative packing measures $\mathscr{P}_\mu^{q,t}$ defined in a separable metric space. We give some product inequalities which are a consequence of a new version of density theorems for these measures. Moreover, we prove that $\mathscr{H}_\mu^{q,t}$ and $\mathscr{P}_\mu^{q,t}$ can be expressed as Henstock-Thomson variation measures. The question of the weak-Vitali property arises in this context.

Keywords: generalized Hausdorff measure; generalized packing measure; Henstock-Thomson variation; regularities; densities; doubling measures
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1. Introduction and preliminaries

Multifractal theory was first introduced by Mandelbrot in [43, 44]. In the multifractal analysis of measures, the study of the behavior of the measure is usually transformed into a study of sets related to the local behavior of such measures called level sets and defined according to the so-called Holder regularity of the measure. The focus thus may somehow forget about the measure and its point-wise character and falls in set theory and the suitable coverings that permit the computation of the Hausdorff dimension. However, some geometric sets are essentially known by means of measures that are supported by them, i.e., given a set E and a measure μ, the quantity $\mu(E)$ may be computed as the maximum value $\mu(F)$ for all subsets $F \subset E$. So, contrarily to the previous idea, we mathematically forget the geometric structure of E and focus instead on the properties of the measure μ. The set E is thus partitioned into α-level sets $X_\mu(\alpha)$ relatively to the regularity exponent of μ. This makes the inclusion of the measure μ into the computation of the Hausdorff (or fractal) dimension and thus into the definition of the Hausdorff measure a necessity to understand more the geometry of the set simultaneously with the behavior of the measure that is supported on. One step ahead in this direction has been conducted by Olsen in [50] where the author introduced multifractal generalizations of the fractal dimensions such as Hausdorff, packing and Bouligand ones by considering general variants of measures.

Then Olsen established the multifractal formalism in [50] and proved some density theorems for the multifractal Hausdorff measure $\mathscr{H}_\mu^{q,t}$ and the multifractal packing measure $\mathscr{P}_\mu^{q,t}$ in $\mathbb{R}^d$, $d \geq 1$. These measures have been investigated by a large number of authors [6, 19, 26, 28, 32, 34, 51, 52, 54, 57]. The measure $\mathscr{H}_\mu^{q,t}$ is of course a multifractal generalization of the centered Hausdorff measure, whereas $\mathscr{P}_\mu^{q,t}$ is a multifractal generalization of the packing measure. Moreover, the developments showed that to get a valid variant of the multifractal formalism does not require the application of radius power-laws equivalent measures. This leads one to think about a general framework where the restriction of the function on balls may be any measure which is not equivalent to power-laws r^α and develop a new multifractal analysis (see also [49, 66]). In particular, J. Cole introduced in [16] a general formalism for the multifractal analysis of one probability measure μ with respect to another measure ν. More specifically, he calculated, for $\alpha \geq 0$, the size of the set

$$E(\alpha) = \left\{ x \in supp\,\mu \cap supp\,\nu;\ \ \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log \nu(B(x,r))} = \alpha \right\},$$

where $supp\,\mu$ is the topologic support of μ. For this, Cole introduced a generalized Hausdorff and packing measures $\mathscr{H}_{\boldsymbol{\mu}}^{q,t}$ and $\mathscr{P}_{\boldsymbol{\mu}}^{q,t}$ respectively, where $\boldsymbol{\mu} = (\mu, \nu)$. The relative multifractal dimensions b and B defined by these measures were used to give estimates for the multifractal spectrum of a measure. In several recent papers on multifractal analysis, this type of multifractal analysis has re-emerged as mathematicians and physicists have begun to discuss the idea of performing multifractal analysis with respect to an arbitrary reference measure, see for example [2, 7, 19, 20, 42, 49, 59–61] see also [8–10]. In [22, 37], the authors prove a modification of this type of analysis. Instead of studying sets of points with a local dimension which is given with respect to the Lebesgue measure, they studied sets of points with a local dimension given with respect to a non-atomic probability measure ν and checks an auxiliary condition. Actually, it is very natural to study this formalism of multifractal analysis for what differs slightly from what was introduced in [16]. The difference between the two types is that we used centered ν-δ-coverings and centered ν-δ-packings rather than centered δ-coverings and δ-packings. These relative multifractal measures and dimensions have been used for other purposes as well, for example, [19, 59] and have recently become an object of study themselves, see [22, 42, 60, 61]. Its intuitive connection to statistical mechanics has been a major theme in the development of multifractal analysis of one measure with respect to another. The use of thermodynamic formalism in the context of the code space is the focus of this analysis. It introduces topological pressure, Gibbs states, and entropy in particular, and it derives the variational principle, which connects pressure and entropy. We have focused our attention on what, in our opinion, are the key historical advances in the field because there is a wealth of material on the subject and it is surely conceivable to produce a book on it in many volumes.

Balls in the space $\mathbb{R}^d$ obtained from the usual Euclidean norm possess certain nice regularity properties: the diameter of a ball is twice its radius, and open and closed balls of the same radius have the same diameter. In arbitrary metric spaces, the possible absence of such regularity properties means that the usual measure construction based on diameters can lead to packing measures with undesirable features. We will show that, under some new definitions, the fundamental properties of Euclidean measures carry over to general metric spaces. In this paper, we will investigate the measures $\mathscr{H}_\mu^{q,t}$ and $\mathscr{P}_\mu^{q,t}$ in a general metric space. In particular, we prove that they are regular in section 2. In section 3.1, we will prove that these measures can be expressed as Henstock-Thomas "variation" measures. As an applicaton, we prove that $\mathscr{H}_\mu^{q,t} \leq \mathscr{P}_\mu^{q,t}$ provided that μ and ν satisfying the doubling condition in a general metric space (see definition in section 2).

Regular sets are defined by density with respect to the Hausdorff measure [17, 23, 27, 46–48], to packing measure [56, 63, 64] or to Hewitt-Stromberg measure [3, 4, 11, 39–41]. Tricot et al. [56, 63] managed to show that a subset of $\mathbb{R}^d$ has an integer Hausdorff and packing dimension if it is strongly regular. Then, the results of [56] were improved to a generalized Hausdorff measure in a Polish space by Mattila and Mauldin in [47]. Later, Baek [12] used the multifractal density theorems [50, 53] to prove the decomposition theorem for the regularities of a generalized centered Hausdorff measure and a generalized packing measure in a Euclidean space which enables him to split a set into regular and irregular parts. In addition, he extended the Olsen's density theorem to any measurable set. Later, these results have been improved in some different contexts in [21, 22, 58, 59]. In the present paper, we will formulate a new version of the density theorem given in [23, 50, 56] in section 5.1. As an application, we will study the generalized Hausdorff and packing measures of cartesian product sets by means of the measure of their components. Furthermore, we will set up in section 5.2 a necessary and sufficient condition for which we have $\mathscr{H}_\mu^{q,t}(E) = \mathscr{P}_\mu^{q,t}(E)$: such set E is said to be strong regular.

We end this section with some useful definitions. Let $(\mathscr{X}, \rho)$ be a separable metric space and consider nonempty subset E of $\mathscr{X}$. The diameter of E is defined by

$$\operatorname{diam}(E) = \sup\Big\{\rho(x,y);\ x, y \in E\Big\}.$$

We define the closed ball with center x and radius $r > 0$ by

$$B(x,r) = \Big\{y \in \mathscr{X};\ \rho(x,y) \leq r\Big\}.$$

In most "regular" spaces, such as Euclidean space $\mathbb{R}^d$, an open or closed ball has one center and one radius, in particular, $r = \operatorname{diam}(B)/2$; however, in general, neither the radius nor center of a ball need be unique. For convenience, take $\mathscr{X} = \Big\{(x,y) \in \mathbb{R}^2;\ x \leq 0\Big\} \bigcup B$, where $B = \{(2,0),(3,0)\}$, with the subspace topology inherited from $\mathbb{R}^2$. Let $a = (2,0)$ and $b = (3,0)$, then,

$$B(a, r_1) = B(b, r_2) = \{a, b\}$$

for any $r_1 \in (1,2)$ and $r_2 \in [1,3)$. In particular $\operatorname{diam}(B) = 1$. Therefore, in general metric space, the center x or radius r of a ball are not uniquely determined by the sets $B(x,r)$ so we emphasize a center and radius are given as the constituent.

Definition 1. A constituent π is a collection of ordered pairs (x,r), where $x \in \mathscr{X}$ and $r > 0$. It represents the closed ball centered at x with radius r. Let $\varepsilon > 0$, π is said to be ε-fine if $r \leq \varepsilon$ for all $(x,r) \in \pi$. Moreover, π is said to be fine cover of $E \subseteq \mathscr{X}$ if, for every $x \in E$ and every $\delta > 0$, there exists $r > 0$ such that $r < \delta$ and $(x,r) \in \pi$.

We consider a collection of constituents $\pi = \{(x_i, r_i)\}$ with $x_i \in E$ and $r_i > 0$, then in several spaces (such Euclidean space $\mathbb{R}^d$) we have π is a packing of E if, and only if, π is a relative-packing of E, i.e., for all $(x, r) \neq (x', r') \in \pi$ we have

$$\rho(x, x') > r + r' \iff B(x, r) \cap B(x', r') = \emptyset.$$

Clearly this is not the case in general metric space and so we may consider a variant definition of packing measure. In addition, we can also relax the condition on ball relative-packings, and consider families of balls $\{(x_i, r_i)\}$ centered in E such that the intersection of any two of them contains no point x_i, which we will called weak-packing of E. This gives a three different generalized packing measures : $\mathscr{P}_\mu^{q,t}$, $\mathscr{R}_\mu^{q,t}$ and $\mathscr{W}_\mu^{q,t}$ respectively.

Let $\Theta \in \mathscr{P}(\mathscr{X})$, we say that Θ has the weak-Vitali property (respectively, relative-Vitali, strong-Vitali) if, for any Borel set $E \subseteq \mathscr{X}$ with $\Theta(E) < \infty$ and any fine cover β of E, there exists a countable weak-packing $\pi \subset \beta$ of E (respectively, relative-packing, packing) such that

$$\Theta\left(E \backslash \bigcup_{(x,r)\in\pi} B(x, r)\right) = 0.$$

It's clear that if a measure Θ has the strong-Vitali property then Θ has the relative-Vitali property and if Θ has the relative-Vitali property then Θ has the weak-Vitali property. Moreover, if $\mathscr{X}$ is the Euclidean space $\mathbb{R}^d$ then every finite Borel measure has the strong-Vitali property [13, 24]. Unfortunately, the strong Vitali property (and even the weak-Vitali property) fails for some measures in some metric spaces. For this, we will assume this property when required which is not a restrictive assumption in this paper. The interested reader is referred to [30, 31, 38] for more discussion.

2. General relative multifractal measures

Let $(\mathscr{X},\rho)$ be a separable metric space and denote by $\mathscr{P}(\mathscr{X})$ the set of finite positive Borel measures on $\mathscr{X}$. For $\mu \in \mathscr{P}(\mathscr{X})$ and $a > 1$, we write

$$P_a(\mu) = \limsup_{r\searrow 0}\left(\sup_{x\in supp\,\mu} \frac{\mu(B(x, ar))}{\mu(B(x, r))}\right).$$

We say that the measure μ satisfies the doubling condition if there exists $a > 1$ such that $P_a(\mu) < \infty$. It is easily seen that the exact value of the parameter a is unimportant:

$$P_a(\mu) < \infty, \quad \text{for some} \;\; a > 1 \quad \text{if and only if} \;\; P_a(\mu) < \infty, \quad \text{for all} \;\; a > 1.$$

Also, we denote by $\mathscr{P}_0(\mathscr{X})$ the family of finite positive Borel measures on $\mathscr{X}$ which satisfy the doubling condition. We can cite classical examples of doubling measures, self-similar measures, and self-conformal ones [50].

While the definitions of the generalized packing measure and generalized Hausdorff measure are well-known, we have, nevertheless, decided to briefly recall the definitions below. Since we are working in separable metric spaces, the different definitions that appear in the literature may not all agree and for this reason it is useful to state precisely the definition that we are using. In this paper we denote $\boldsymbol{\mu} = (\mu, \nu)$ where $\mu, \nu \in \mathscr{P}(\mathscr{X})$.

2.1. The construction of the general multifractal measures

Now we will consider possible generalizations of the definition. Let $E \subseteq \mathscr{X}$ and $\delta > 0$, a collection β of constituents is a (centered) δ-cover of E if $x \in E$, $r < \delta$ for all $(x, r) \in \beta$ and $E \subseteq \bigcup_{(x,r)\in\beta} B(x, r)$. We write

$$\mathscr{H}^{q,t}_{\mu,\delta}(E) = \inf\left\{ \sum_{(x,r)\in\beta} \mu(B(x, r))^q \nu(B(x, r))^t \;\middle|\; \beta \text{ is a } \delta\text{-cover of } E \right\},$$

$$\mathscr{H}^{q,t}_{\mu,0}(E) = \sup_{\delta>0} \mathscr{H}^{q,t}_{\mu,\delta}(E) = \lim_{\delta\to 0} \mathscr{H}^{q,t}_{\mu,\delta}(E),$$

with the conventions $0^q = \infty$ for $q \leq 0$ and $0^q = 0$ for $q > 0$. The function $\mathscr{H}^{q,t}_{\mu,0}$ is sub-additive but not increasing. For this, we will use the following modification:

$$\mathscr{H}^{q,t}_{\mu}(E) = \sup_{F\subseteq E} \mathscr{H}^{q,t}_{\mu,0}(F).$$

The function $\mathscr{H}^{q,t}_{\mu}$ is a metric outer measure. In addition [16], there exists a unique number $\dim^q_\mu(E) \in [-\infty, +\infty]$, such that

$$\mathscr{H}^{q,t}_{\mu}(E) = \begin{cases} \infty & \text{if} \quad t < \dim^q_\mu(E), \\ 0 & \text{if} \quad \dim^q_\mu(E) < t. \end{cases}$$

We give here a multifractal extension of dimension of measure: We define for $\Theta \in \mathscr{P}(\mathscr{X})$,

$$\dim^q_\mu(\Theta) = \inf_E \left\{ \dim^q_\mu(E);\;\; \Theta(\mathscr{X} \setminus E) = 0 \right\}.$$

Remark 2.1. For any sets $E, F \subseteq \mathscr{X}$, we have

$$\mathscr{H}^{q,t}_{\mu,0}(E \cup F) \leq \mathscr{H}^{q,t}_{\mu,0}(E) + \mathscr{H}^{q,t}_{\mu,0}(F)$$

and we have the equality if $\rho(E, F) > 0$.

Remark 2.2. If $(\mathscr{X}, \rho)$ is not separable, for small enough $\delta > 0$ there is no countable cover by sets of diameter less than δ. So the infinimum in the definition of Hausdorff's outer measure is over the empty set and then it is $+\infty$. So the limit for δ going to zero is also $+\infty$. So that for a non-separable set $\mathscr{X}$ for any $q, t \in \mathbb{R}$ the Hausdorff measure is $\mathscr{H}^{q,t}_{\mu}(\mathscr{X}) = +\infty$ and the Hausdorff dimension of $\mathscr{X}$ is $+\infty$.

Let $E \subseteq \mathscr{X}$ and $\delta > 0$, a collection of constituents π is a δ-packing of E if, and only if, for all $(x, r) \neq (x', r') \in \pi$ we have

$$\rho(x, x') > r + r' \tag{2.1}$$

and $r < \delta$, for all $(x, r) \in \pi$. We denote by $\Upsilon_\delta(E)$ the set of all δ-packing of E. Let $q, t \in \mathbb{R}$ and $\mu \in \mathscr{P}(\mathscr{X})$. We write for $E \neq \emptyset$,

$$\mathscr{P}^{q,t}_{\mu,\delta}(E) \;\; = \;\; \sup\left\{ \sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^t;\; (x_i, r_i)_i \in \Upsilon_\delta(E) \right\},$$

$$\mathscr{P}^{q,t}_{\mu,0}(E) \;=\; \inf_{\delta>0} \mathscr{P}^{q,t}_{\mu,\delta}(E) = \lim_{\delta\to 0} \mathscr{P}^{q,t}_{\mu,\delta}(E).$$

The function $\mathscr{P}^{q,t}_{\mu,0}$ is increasing but not sub-additive. By applying now the standard construction [55, 65, 67], we obtain the generalized packing measure defined as follows

$$\mathscr{P}^{q,t}_{\mu}(E) = \inf\left\{\sum_{i=1}^{\infty} \mathscr{P}^{q,t}_{\mu,0}(E_i);\quad E \subseteq \bigcup_{i=1}^{\infty} E_i\right\},$$

if $E = \emptyset$ then $\mathscr{P}^{q,t}_{\mu}(\emptyset) = 0$. The function $\mathscr{P}^{q,t}_{\mu}$ is of course a multifractal generalization of the packing measure $\mathscr{P}^t$ [36, 56]. In addition [16], there exists a unique number $\mathrm{Dim}^q_\mu(E) \in [-\infty, +\infty]$, such that

$$\mathscr{P}^{q,t}_{\mu}(E) = \begin{cases} \infty & \text{if} \quad t < \mathrm{Dim}^q_\mu(E), \\ \\ 0 & \text{if} \quad \mathrm{Dim}^q_\mu(E) < t. \end{cases}$$

We give the multifractal extension of dimension of measure: For $\Theta \in \mathscr{P}(\mathscr{X})$, we define

$$\mathrm{Dim}^q_\mu(\Theta) = \inf_{E} \left\{\mathrm{Dim}^q_\mu(E);\;\; \Theta(\mathscr{X} \setminus E) = 0\right\}.$$

Note that a δ-packing π of a set E may be interpreted in Euclidean space as: $B(x, r) \bigcap B(x', r') = \emptyset$ for all $(x, r) \neq (x', r') \in \pi$. Since this is not the case in general metric space, we may consider a new generalized measure. A collection of constituents π is a δ-relative-packing of E if, and only if, for all $(x, r) \neq (x', r') \in \pi$ we have

$$B(x, r) \bigcap B(x', r') = \emptyset \tag{2.2}$$

and $r < \delta$, for all $(x, r) \in \pi$. We denote by $\widetilde{\Upsilon}_\delta(E)$ the set of all δ-relative-packing of E. Let $q, t \in \mathbb{R}$, and $\mu, \nu \in \mathscr{P}(\mathscr{X})$. We write for $E \neq \emptyset$,

$$\begin{aligned} \mathscr{R}^{q,t}_{\mu,\delta}(E) &= \sup\left\{\sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^t;\; (x_i, r_i)_i \in \widetilde{\Upsilon}_\delta(E)\right\}, \\ \mathscr{R}^{q,t}_{\mu,0}(E) &= \inf_{\delta>0} \mathscr{R}^{q,t}_{\mu,\delta}(E) = \lim_{\delta\to 0} \mathscr{R}^{q,t}_{\mu,\delta}(E). \end{aligned}$$

The function $\mathscr{R}^{q,t}_{\mu,0}$ is increasing but not sub-additive. Similarly, by applying a standard construction, we obtain the generalized relative-packing measure defined by

$$\mathscr{R}^{q,t}_{\mu}(E) = \inf\left\{\sum_{i=1}^{\infty} \mathscr{R}^{q,t}_{\mu,0}(E_i);\quad E \subseteq \bigcup_{i=1}^{\infty} E_i\right\},$$

if $E = \emptyset$ then $\mathscr{R}^{q,t}_{\mu}(\emptyset) = 0$. The function $\mathscr{R}^{q,t}_{\mu}$ is a generalization of the (b)-packing measure introduced in [25]. We can also relax the condition on ball relative-packings, and consider families of balls centered in E such that the intersection of any two of them contains no point x_i. More precisely, $(x_i, r_i)_i$, $x_i \in E$ and $r_i > 0$, is a δ-weak-packing of E if and only if, for all $i, j = 1, 2, \ldots$, we have

$$i \neq j \implies \rho(x_i, x_j) > \max(r_i, r_j)$$

and $r_i < \delta$. We denote by $\widetilde{\widetilde{\Upsilon}}_\delta(E)$ the set of all δ-weak-packing of E. Similarly, the weak-packing h-measure $\mathscr{W}_\mu^{q,t}$ is defined by

$$\begin{aligned}
\mathscr{W}_{\mu,\delta}^{q,t}(E) &= \sup\left\{\sum_i \mu(B(x_i,r_i))^q \nu(B(x_i,r_i))^t;\ (x_i,r_i)_i \in \widetilde{\widetilde{\Upsilon}}_\delta(E)\right\},\\
\mathscr{W}_{\mu,0}^{q,t}(E) &= \inf_{\delta>0} \mathscr{W}_{\mu,\delta}^{q,t}(E) = \lim_{\delta\to 0} \mathscr{W}_{\mu,\delta}^{q,t}(E),\\
\mathscr{W}_{\mu}^{q,t}(E) &= \inf\left\{\sum_{i=1}^{\infty} \mathscr{W}_{\mu,0}^{q,t}(E_i);\quad E \subseteq \bigcup_{i=1}^{\infty} E_i\right\}
\end{aligned}$$

if $E \neq \emptyset$ and $\mathscr{W}_\mu^{q,t}(\emptyset) = 0$. The function $\mathscr{W}_\mu^{q,t}$ is a generalization of the weak-packing measure $\mathscr{W}^t$ [33].

Remark 2.3. It is clear that in Euclidean space we have $\mathscr{R}_\mu^{q,t} = \mathscr{P}_\mu^{q,t} \le \mathscr{W}_\mu^{q,t}$, but in a general metric space, we only have: every packing is a relative-packing, and every relative-packing is a weak-packing which implies that

$$\mathscr{P}_\mu^{q,t} \le \mathscr{R}_\mu^{q,t} \le \mathscr{W}_\mu^{q,t}.$$

In the next proposition, we will prove that the three definitions agree within a constant γ provided that $\mu, \nu \in \mathscr{P}_0(\mathscr{X})$. Nevertheless, this doubling assumption on μ and ν does not matter under a suitable condition on $\mathscr{X}$ (see Section 5.3).

Proposition 2.4. *Let $\mu, \nu \in \mathscr{P}_0(\mathscr{X})$ and $q, t \in \mathbb{R}$. Then, there exists a constant γ such that*

$$\mathscr{W}_\mu^{q,t} \le \gamma \mathscr{P}_\mu^{q,t}. \tag{2.3}$$

Proof. If (x_i, r_i) is a δ-weak-packing of E then $(x_i, r_i/2)$ is a δ-packing of E and we get the right side of the inequality (2.3) since $\mu, \nu \in \mathscr{P}_0(\mathscr{X})$. □

Remark 2.5. If μ coincides with ν and is equal to the Lebesgue measure then the multifractal measures reduce to the classical measures introduced in [23].

2.2. Regularities of the general multifractal measures

We will prove, in this short section, that the generalized fractal measures are regular, that is, for any subset $E \subseteq \mathscr{X}$, there exists a Borel subset B such that

$$E \subseteq B \quad \text{and} \quad \Gamma(E) = \Gamma(B),$$

where $\Gamma \in \left\{\mathscr{W}_\mu^{q,t}, \mathscr{P}_\mu^{q,t}, \mathscr{H}_\mu^{q,t}\right\}$. In the following proposition, we give the result for $\mathscr{P}_\mu^{q,t}$ and $\mathscr{W}_\mu^{q,t}$. This is done by proving, for all $E \subseteq \mathscr{X}$, the closure theorem, that is, $\mathscr{P}_{\mu,0}^{q,t}(E) = \mathscr{P}_{\mu,0}^{q,t}(\overline{E})$ and $\mathscr{W}_{\mu,0}^{q,t}(E) = \mathscr{W}_{\mu,0}^{q,t}(\overline{E})$, where $\overline{E}$ is the closure of E. The closure theorem may fail when we consider the relative packing measure [23, Example 5.18].

Proposition 2.6. *Let $\mu, \nu \in \mathscr{P}(\mathscr{X})$ and $q, t \in \mathbb{R}$. Then $\mathscr{W}_\mu^{q,t}$ and $\mathscr{P}_\mu^{q,t}$ are regular measures on $\mathscr{X}$.*

Proof. First we claim that for any set $E \subseteq \mathscr{X}$ we have

$$\mathscr{W}_{\mu}^{q,t}(E) = \inf\left\{\sum_{i=1}^{\infty} \mathscr{W}_{\mu,0}^{q,t}(\overline{E}_i); \quad E \subseteq \bigcup_{i=1}^{\infty} E_i\right\}. \tag{2.4}$$

Therefore, for any positive integer n, we may choose sets $\{E_{ni}\}_i$ such that $E \subseteq \bigcup_{i=1}^{\infty} E_{ni}$ and

$$\sum_i \mathscr{W}_{\mu,0}^{q,t}(\overline{E}_{ni}) \le \mathscr{W}_{\mu}^{q,t}(E) + \frac{1}{n}.$$

Put $B = \cap_n \cup_i \overline{E}_{ni}$, then the set B is Borel with $E \subseteq B$. In addition, for any integer n we have

$$\mathscr{W}_{\mu}^{q,t}(E) \le \mathscr{W}_{\mu}^{q,t}(B) \le \mathscr{W}_{\mu}^{q,t}(\cup_i \overline{E}_{ni}) \le \sum_i \mathscr{W}_{\mu}^{q,t}(\overline{E}_{ni}) \le \mathscr{W}_{\mu}^{q,t}(E) + \frac{1}{n}.$$

Now, we will prove (2.4), for this, we only have to prove that $\mathscr{W}_{\mu,0}^{q,t}(E) = \mathscr{W}_{\mu,0}^{q,t}(\overline{E})$. Since the function $\mathscr{W}_{\mu,0}^{q,t}$ is monotonic, we need only prove that

$$\mathscr{W}_{\mu,0}^{q,t}(E) \ge \mathscr{W}_{\mu,0}^{q,t}(\overline{E}).$$

Let $\epsilon > 0$, $\delta > 0$ and consider $\pi = \{(x_i, r_i)\}$ to be a δ-weak-packing of $\overline{E}$. For each i, by continuity we can choose $\eta_i > 0$ and $y_i \in E$ such that $\rho(y_i, x_i) < \eta_i$. It follows that $\{(y_i, r_i - \frac{1}{2}\eta_i)\}$ is a δ-weak-packing of E. We want $\eta_i < r_i$ as well. Then

$$\mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^t - \frac{\epsilon}{2^i} \le \mu\left(B(y_i, r_i - \frac{1}{2}\eta_i)\right)^q \nu\left(B(y_i, r_i - \frac{1}{2}\eta_i)\right)^t$$

and so

$$\begin{aligned}\sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^t &\le \sum_i \mu\left(B(y_i, r_i - \frac{1}{2}\eta_i)\right)^q \nu\left(B(y_i, r_i - \frac{1}{2}\eta_i)\right)^t + \epsilon \\ &\le \mathscr{W}_{\mu,\delta}^{q,t} + \epsilon.\end{aligned}$$

Hence $\mathscr{W}_{\mu,\delta}^{q,t}(\overline{E}) \le \mathscr{W}_{\mu,\delta}^{q,t}(E) + \epsilon$. Letting ϵ and δ to 0 to get the desire result. □

Remark 2.7. As a standard consequence of the regularity, we have

$$\text{if } E_n \nearrow E \text{ then } \mathscr{W}_{\mu}^{q,t}(E_n) \to \mathscr{W}_{\mu}^{q,t}(E).$$

The following result proves that $\mathscr{H}_{\mu}^{q,t}$ is Borel regular measure. This is done by the construction of new multifractal fractal measure $\widetilde{\mathscr{H}}_{\mu}^{q,t}$, in a similar manner to $\mathscr{H}_{\mu}^{q,t}$ but using the class of all covering balls in the definition rather than the class of all centered balls. The idea is to prove that $\widetilde{\mathscr{H}}_{\mu}^{q,t}$ is regular and $\widetilde{\mathscr{H}}_{\mu}^{q,t}$ is comparable to $\mathscr{H}_{\mu}^{q,t}$. This result has been studied in [20].

Theorem 2.8. *[20] Let $\mu, \nu \in \mathscr{P}_0(\mathscr{X})$ and $q, t \in \mathbb{R}$. Then $\mathscr{H}_{\mu}^{q,t}$ is regular. Moreover, if $q, t \le 0$, then this measure is regular even without doubling condition on μ and ν.*

3. Variation multifractal measures and centered densities theorems

3.1. Fine and full variations

We will prove that the generalized Hausdorff, packing and weak-packing measures can be expressed as Henstock-Thomas "variation" measures. Note that these measures have been introduced in [61] in Euclidean space, but here we will define the "variation" measures in general metric space.

Let $E \subseteq \mathscr{X}$ and β is a collection of constituents such that $x \in E$ for each $(x,r) \in \beta$. Recall that β is said to be fine cover of E if, for every $x \in E$ and every $\delta > 0$, there exists $r > 0$ such that $r < \delta$ and $(x,r) \in \beta$.

Lemma 3.1. *[23, Theorem 3.1] Let $\mathscr{X}$ be a metric space, $E \subseteq \mathscr{X}$ and β be a fine cover of E. Then there exists either*

1. *an infinite packing $\{(x_i, r_i)\} \subseteq \beta$ of E such that* $\inf r_i > 0$,
   or
2. *a countable closed ball packing $\{(x_i, r_i)\} \subseteq \beta$ such that for all $n \in \mathbb{N}$,*

$$E \subseteq \bigcup_{i=1}^{n} \overline{B}(x_i, r_i) \cup \bigcup_{i=n+1}^{\infty} \overline{B}(x_i, 3r_i).$$

Definition 2. Let $\mu, \nu \in \mathscr{P}(\mathscr{X})$ and $q, t \in \mathbb{R}$. If β is a fine cover of $E \neq \emptyset$, we define

$$\mathsf{H}^{q,t}_{\beta,\mu}(E) = \sup\left\{ \sum_{(x,r)\in\pi} \mu(B(x,r))^q \nu(B(x,r))^t \right\},$$

where the supremum is over all (closed balls) packing with $\pi \subseteq \beta$, that is $\rho(x,x') > r + r'$ for all $(x,r), (x',r') \in \pi$ with $(x,r) \neq (x',r')$. The fine variation on E is defined by

$$\mathsf{H}^{q,t}_{\mu}(E) = \inf\left\{ \mathsf{H}^{q,t}_{\beta,\mu}(E) : \beta \quad \text{is a fine cover of} \quad E \right\}$$

and $\mathsf{H}^{q,t}_{\mu}(\emptyset) = 0$.

Definition 3. Let $E \subseteq \mathscr{X}$, π be a collection of constituents and Δ be a gauge function for E, that is a function $\Delta : E \to (0,\infty)$. π is said to be Δ-fine if $r < \Delta(x)$ for all $(x,r) \in \pi$.

Let Δ be a gauge function for a set $E \subseteq \mathscr{X}$. We write,

$$\mathscr{W}^{q,t}_{\Delta,\mu}(E) = \sup\left\{ \sum_{(x,r)\in\pi} \mu(B(x,r))^q \ \nu(B(x,r))^t \right\},$$

where the supremum is over all Δ-fine weak-packings π of E. As Δ decreases pointwise, the value $\mathscr{W}^{q,h}_{\Delta,\mu}(E)$ decreases. For the limit, we write

$$\mathscr{W}^{q,t}_{*,\mu}(E) = \inf_{\Delta} \mathscr{W}^{q,t}_{\Delta,\mu}(E),$$

where the infimum is over all gauges Δ for E. Similarly, we define

$$\mathscr{P}^{q,t}_{*,\mu}(E) = \inf_{\Delta} \mathscr{P}^{q,t}_{\Delta,\mu}(E),$$

where we use in the definition of $\mathscr{P}^{q,t}_{\Delta,\mu}$ the Δ-fine packings.

Proposition 3.2. *Let $\mu, \nu \in \mathscr{P}(\mathscr{X})$ and $q, t \in \mathbb{R}$. Then $\mathsf{H}_{\mu}^{q,t}$, $\mathscr{W}_{*,\mu}^{q,t}$ and $\mathscr{P}_{*,\mu}^{q,t}$ are metric outer measures on $\mathscr{X}$ and then they are measures on the Borel algebra.*

Proof. See Propositions 3.11 and 3.15 in [23]. □

The measure $\mathsf{H}_{\mu}^{q,t}$ is absolutely continuous with respect to $\mathscr{H}_{\mu}^{q,t}$ and we write $\mathsf{H}_{\mu}^{q,t} \ll \mathscr{H}_{\mu}^{q,t}$, that is, $\mathsf{H}_{\mu}^{q,t} = 0$ for every Borel set with $\mathscr{H}_{\mu}^{q,t}(E) = 0$. More precisely, we have the following lemma which generalize Lemma 2.3 in [15] in Euclidean space.

Lemma 3.3. *Let $\mu, \nu \in \mathscr{P}(\mathscr{X})$, $q, t \in \mathbb{R}$ and E a Borel subset of $\mathscr{X}$. Assume that $\mathscr{H}_{\mu}^{q,t}(E) = 0$ then $\mathsf{H}_{\mu}^{q,t}(E) = 0$.*

Proof. Let $\epsilon > 0$, since for each positive integer n we have $\mathscr{H}_{\mu,1/n}^{q,t} = 0$, then we can find a centered cover $(x_{in}, r_{in})_i$ of E such that $r_{in} \le 1/n$ and

$$\sum_i \mu(B(x_{in}, r_{in}))^q \nu(B(x_{in}, r_{in}))^t \le \frac{\epsilon}{2^n}.$$

Now, for each n and i, we consider

$$\beta_{in} := \{(y, r_{in}) : \ \rho(y, x_{in}) \le r_{in}\}$$

and put $\beta = \bigcup_{i,n} \beta_{in}$. Then β is a fine cover of E. Let $\pi \subseteq \beta$ be a packing. Since all elements of β_{in} contain x_{in}, there is at most one element of β_{in} in π. Hence,

$$\begin{aligned}\sum_{(x,r)\in\pi} \mu(B(x,r))^q \nu(B(x,r))^t &\le \sum_n \sum_i \mu(B(x_{in}, r_{in}))^q \nu(B(x_{in}, r_{in}))^t \\ &\le \sum_n \frac{\epsilon}{2^n} = \epsilon.\end{aligned}$$

Taking the supremum over all packings $\pi \subseteq \beta$ gives $\mathsf{H}_{\mu}^{q,t}(E) \le \mathsf{H}_{\beta,\mu}^{q,t}(E) \le \epsilon$, and so $\mathsf{H}_{\mu}^{q,t}(E) = 0$. □

In the next, we will prove that the fine variation $\mathsf{H}_{\mu}^{q,t}$ can be compared to the multifractal Hausdorff measure measure $\mathscr{H}_{\mu}^{q,t}$. Note that, we do not make any assumption on μ or ν. First, we give the following definition.

Definition 4. For $x \in \mathscr{X}$, $q, t \in \mathbb{R}$, and $\mu, \nu, \Theta \in \mathscr{P}(\mathscr{X})$. The lower and upper (q, t)-density of Θ with respect to μ and ν at $x \in supp\,\mu \cap supp\,\nu$, are defined respectively as follows

$$\underline{D}_{\mu}^{q,t}(x, \Theta) = \liminf_{r\to 0} \frac{\Theta(B(x,r))}{\mu(B(x,r))^q \nu(B(x,r))^t} \tag{3.1}$$

and

$$\overline{D}_{\mu}^{q,t}(x, \Theta) = \limsup_{r\to 0} \frac{\Theta(B(x,r))}{\mu(B(x,r))^q \nu(B(x,r)^t}. \tag{3.2}$$

If $\underline{D}_{\mu}^{q,t}(x, \Theta) = \overline{D}_{\mu}^{q,t}(x, \Theta)$ we denote $D_{\mu}^{q,t}(x, \Theta)$ the commune value. The main densities result in this section (Theorem 3.7 and 3.9) links the quantities $\Theta(E)$ and the generalized fractal measures via the lower or upper q-density of μ and ν. This connection, is made through the use of certain vitali property of Θ.

Proposition 3.4. *For all Borel sets $E \subseteq \mathscr{X}$, we have $\mathsf{H}_{\mu}^{q,t}(E) \le \mathscr{H}_{\mu}^{q,t}(E)$.*

Proof. We may clearly assume that $\mathscr{H}_{\mu}^{q,t}(E) < \infty$. Fix $a > 1$ and let Θ denote the restriction of $\mathscr{H}_{\mu}^{q,t}$ to E, i.e., $\Theta(A) = \mathscr{H}_{\mu}^{q,t}(A \cap E)$, for all $A \subseteq \mathscr{X}$. Write

$$F = \left\{x \in E,\ \overline{D}_{\mu}^{q,t}(x,\Theta) \le a^{-3}\right\} \quad \text{and} \quad G = \left\{x \in E,\ \overline{D}_{\mu}^{q,t}(x,\Theta) > a^{-3}\right\},$$

where $\overline{D}_{\mu}^{q,t}(x,\Theta)$ is defined in (3.2). First consider the set F, we will prove that $\mathsf{H}_{\mu}^{q,t}(F) = 0$. For $n \in \mathbb{N}$, we set

$$F_n = \left\{x \in F,\ \frac{\Theta(B(x,r))}{\mu(B(x,r))^q \nu(B(x,r))^t} < a^{-2},\ \text{for all}\ \ r < 1/n\right\}.$$

In the next, we will prove that $\mathscr{H}_{\mu}^{q,t}(F_n) = 0$. Let $\delta < 1/n$ and β be a δ-cover of F_n, then

$$\begin{aligned}\sum_{(x,r)\in\beta} \mu(B(x,r))^q \nu(B(x,r))^t &\ge a^2 \sum_{(x,r)\in\beta} \Theta(B(x,r)) \ge a^2 \Theta\left(\bigcup_{(x,r)\in\beta} B(x,r)\right)\\ &\ge a^2 \nu(F_n) = a^2 \mathscr{H}_{\mu}^{q,t}(F_n).\end{aligned}$$

Hence $\mathscr{H}_{\mu,\delta}^{q,t}(F_n) \ge a^2 \mathscr{H}_{\mu}^{q,t}(F_n)$, which implies that

$$\mathscr{H}_{\mu}^{q,t}(F_n) \ge \mathscr{H}_{\mu,0}^{q,t}(F_n) \ge a^2 \mathscr{H}_{\mu}^{q,t}(F_n).$$

Now, since $a > 1$ and $\mathscr{H}_{\mu}^{q,t}(F_n) \le \mathscr{H}_{\mu}^{q,t}(E) < \infty$, we have $\mathscr{H}_{\mu}^{q,t}(F_n) = 0$. Finally, since $F_n \nearrow F$, this implies that $\mathscr{H}_{\mu}^{q,t}(F) = 0$ and therefore, by Lemma 3.3 we have $\mathsf{H}_{\mu}^{q,t}(F) = 0$.

Next, we consider the set G, we will prove that

$$\mathsf{H}_{\mu}^{q,t}(G) \le a^4 \mathscr{H}_{\mu}^{q,t}(E). \tag{3.3}$$

Since $a^{-4} < a^{-3}$, the set

$$\beta = \left\{(x,r):\ \ x \in G,\ \ \frac{\Theta(B(x,r))}{\mu(B(x,r))^q \nu(B(x,r))^t} > a^{-4}\right\}$$

is a fine cover of G. Let $\pi \subset \beta$ be a packing, then

$$\begin{aligned}\sum_{(x,r)\in\pi} \mu(B(x,r))^q \nu(B(x,r))^t &\le a^4 \sum_{(x,r)\in\pi} \Theta(B(x,r)) = a^4 \Theta\left(\bigcup_{(x,r)\in\pi} B(x,r)\right)\\ &= a^4 \mathscr{H}_{\mu}^{q,t}\left(\bigcup_{(x,r)\in\pi} B(x,r) \cap E\right) \le a^4 \mathscr{H}_{\mu}^{q,t}(E).\end{aligned}$$

Since this is true for all packing π, we conclude that $\mathsf{H}_{\beta,\mu}^{q,t}(G) \le a^4 \mathscr{H}_{\mu}^{q,t}(E)$, which implies (3.3). Finally, we have

$$\mathsf{H}_{\mu}^{q,t}(E) \le \mathsf{H}_{\mu}^{q,t}(F) + \mathsf{H}_{\mu}^{q,t}(G) \le 0 + a^4 \mathscr{H}_{\mu}^{q,t}(E).$$

Taking the infimum over all countable $a > 1$ to obtain $\mathsf{H}_{\mu}^{q,t}(E) \le \mathscr{H}_{\mu}^{q,t}(E)$.

∎

Identifying the generalized packing (or weak-packing) measure with the full variation does not require any assumptions (such as doubling condition or Vitali property) but to get the equality $\mathsf{H}_{\mu}^{q,t} = \mathscr{H}_{\mu}^{q,t}$, extra assumption is needed.

Theorem 3.5. *Let $q, t \in \mathbb{R}$ and $\mu, \nu \in \mathscr{P}(\mathscr{X})$. Then for all Borel sets $E \subseteq \mathscr{X}$, we have*

1. $\mathscr{W}_{*,\mu}^{q,t}(E) = \mathscr{W}_{\mu}^{q,t}(E)$ *and* $\mathscr{P}_{*,\mu}^{q,t}(E) = \mathscr{P}_{\mu}^{q,t}(E)$.
2. *If $\mu, \nu \in \mathscr{P}_0(\mathscr{X})$ then* $\mathsf{H}_{\mu}^{q,t}(E) = \mathscr{H}_{\mu}^{q,t}(E)$.

Proof.

1. We will only prove the first equality and the other is similar. Let $E \subseteq \mathscr{X}$ and $\delta > 0$. Then, the constant function $\Delta(x) = \delta$ is a gauge for E. Therefore,

$$\mathscr{W}_{\mu,0}^{q,t}(E) = \inf_{\delta>0} \mathscr{W}_{\mu,\delta}^{q,t}(E) \geq \mathscr{W}_{*,\mu}^{q,t}(E).$$

If $E = \bigcup_n E_n$ then, since $\mathscr{W}_{*,\mu}^{q,t}$ is an outer measure, we have

$$\mathscr{W}_{*,\mu}^{q,t}(E) \leq \sum_{n=1}^{\infty} \mathscr{W}_{*,\mu}^{q,t}(E_n) \leq \sum_{n=1}^{\infty} \mathscr{W}_{\mu,0}^{q,t}(E_n).$$

Since, this is true for all countable covers of E, we get

$$\mathscr{W}_{\mu}^{q,t}(E) \geq \mathscr{W}_{*,\mu}^{q,t}(E).$$

Now we will prove $\mathscr{W}_{*,\mu}^{q,t}(E) \geq \mathscr{W}_{\mu}^{q,t}(E)$. Let Δ be a gauge on a set E and consider, for each positive integer n, the set

$$E_n = \left\{ x \in E;\ \Delta(x) \geq \frac{1}{n} \right\}.$$

For each n,

$$\mathscr{W}_{\Delta,\mu}^{q,t}(E) \geq \mathscr{W}_{\Delta,\mu}^{q,t}(E_n) \geq \mathscr{W}_{\mu,1/n}^{q,t}(E_n) \geq \mathscr{W}_{\mu,0}^{q,t}(E_n) \geq \mathscr{W}_{\mu}^{q,t}(E_n).$$

Since $E_n \nearrow E$ and $\mathscr{W}_{\mu}^{q,t}$ is regular, then, by taking the limit as $n \to \infty$, we get $\mathscr{W}_{\Delta,\mu}^{q,t}(E) \geq \mathscr{W}_{\mu}^{q,t}(E)$. This is true for all gauges Δ, so $\mathscr{W}_{*,\mu}^{q,t}(E) \geq \mathscr{W}_{\mu}^{q,t}(E)$.

2. By using Proposition 3.4, it suffices to prove $\mathsf{H}_{\mu}^{q,t}(E) \geq \mathscr{H}_{\mu}^{q,t}(E)$. We may clearly assume that $\mathsf{H}_{\mu}^{q,t}(E) < \infty$. Let β be a fine cover of E such that $\mathsf{H}_{\beta,\mu}^{q,t}(E) < \infty$. Let $\delta > 0$, then

$$\beta_1 = \Big\{(x, r) \in \beta : r < \delta/3\Big\}$$

is a fine cover of E. Therefore, using Lemma 3.1, we can find a packing $\{(x_n, r_n)\} \subseteq \beta$ such that

$$E \subseteq \bigcup_{i=1}^{n} \overline{B}(x_i, r_i) \cup \bigcup_{i=n+1}^{\infty} \overline{B}(x_i, 3r_i).$$

Note that $\limsup_n r_n > 0$ is impossible, since

$$\sum_i \mu(B(x_i, r_i))^q\ \nu(B(x_i, r_i))^t \leq \mathsf{H}_{\beta,\mu}^{q,t}(E) < \infty.$$

Now, since the measures μ and ν are right-continuous at each r_i, we let $\xi > 1$, and choose $r_i^* > r_i$ so that $r_i^* < \delta/3$ and

$$\sum_i \mu(B(x_i, r_i^*))^q \ \nu(B(x_i, r_i^*))^t < \xi \sum_i \mu(B(x_i, r_i))^q \ \nu(B(x_i, r_i))^t.$$

Thus we get open covers

$$E \subseteq \bigcup_{i=1}^{n} B(x_i, r_i^*) \cup \bigcup_{i=n+1}^{\infty} B(x_i, 3r_i^*). \tag{3.4}$$

Then there exists a constants C_1 and C_2 such that

$$\sum_i \mu(B(x_i, 3r_i^*))^q \ \nu(B(x_i, 3r_i^*))^t$$
$$\leq \begin{cases} C_1C_2 \sum_i \mu(B(x_i, r_i^*))^q \ \nu(B(x_i, r_i^*))^t & ; \ q, t > 0 \text{ and } \mu, \nu \in \mathscr{P}_0(\mathscr{X}) \\ \sum_i \mu(B(x_i, r_i^*))^q \ \nu(B(x_i, r_i^*))^t & ; \ q, t \leq 0. \\ C_2 \sum_i \mu(B(x, r_i^*))^q \ \nu(B(x_i, r_i^*))^t & ; \ q \leq 0, t > 0 \text{ and } \nu \in \mathscr{P}_0(\mathscr{X}) \\ C_1 \sum_i \mu(B(x, r_i^*))^q \ \nu(B(x_i, r_i^*))^t & ; \ q > 0, t \leq 0 \text{ and } \mu \in \mathscr{P}_0(\mathscr{X}). \end{cases}$$

Thus, we have $\sum_i \mu(B(x_i, 3r_i^*))^q \nu(B(x_i, 3r_i^*))^t < \infty$ and, by using (3.4), we get

$$\mathscr{H}_{\mu,\delta}^{q,t}(E) \ \leq \ \sum_{i=1}^{n} \mu(B(x_i, r_i^*))^q \nu(B(x_i, r_i^*))^t + \sum_{i=n+1}^{\infty} \mu(B(x_i, 3r_i^*)^q \nu(B(x, 3r_i^*))^t.$$

Then, for $\epsilon > 0$, we can choose n big enough so that we have

$$\mathscr{H}_{\mu,\delta}^{q,t}(E) \leq \sum_{i=1}^{n} \mu(B(x_i, r_i^*))^q \nu(B(x_i, r_i^*))^t + \epsilon$$

and then

$$\mathscr{H}_{\mu,\delta}^{q,t}(E) \leq \epsilon + \sum_{i=1}^{\infty} \mu(B(x_i, r_i^*))^q \nu(B(x_i, r_i^*))^t \leq \epsilon + \xi \mathsf{H}_{\beta,\mu}^{q,t}(E).$$

Let $\xi \downarrow 1$, $\delta \downarrow 0$ and $\epsilon \to 0$ to get $\mathscr{H}_{\mu,0}^{q,t}(E) \leq \mathsf{H}_{\beta,\mu}^{q,t}(E)$. Now, by take the infimum over all fine cover β we get $\mathscr{H}_{\mu,0}^{q,t}(E) \leq \mathsf{H}_{\mu}^{q,t}(E)$. Take the supremum of this over all subsets to obtain the desire result.

□

In Euclidean space $\mathbb{R}^d$, using the definition, there exists a constant ξ such that $\mathscr{H}_{\mu}^{q,t} \leq \xi \mathscr{P}_{\mu}^{q,t}$. Moreover, we have $\mathscr{H}_{\mu}^{q,t} \leq \mathscr{P}_{\mu}^{q,t}$ provide that $\boldsymbol{\mu} \in \mathscr{P}_0(\mathbb{R}^d)$ [7, 16]. See also for $q = 0$, in Euclidean space [56, Lemma 3.3] or in general metric space [17, Theorem 3.11]. As an applications of Theorem 3.5, we will establish the following results.

Theorem 3.6. *Let $\mu, \nu \in \mathscr{P}(\mathscr{X})$ and $q, t \in \mathbb{R}$ then $\mathsf{H}_{\mu}^{q,t} \leq \mathscr{P}_{\mu}^{q,t}$. In particular if $\mu, \nu \in \mathscr{P}_0(\mathscr{X})$ then, for all Borel sets $E \subseteq \mathscr{X}$, we have*

$$\mathscr{H}_{\mu}^{q,t}(E) \leq \mathscr{P}_{\mu}^{q,t}(E).$$

Proof. According to Theorem 3.5 we will prove $\mathsf{H}_{\mu}^{q,t}(E) \leq \mathscr{P}_{*,\mu}^{q,t}(E)$ for all set $E \subseteq \mathscr{X}$. We consider a gauge function Δ on E and $\beta = \{(x, r);\ r < \Delta(x)\}$. β is a fine cover of E then, for any packing $\pi \subseteq \beta$, we have

$$\sum_{(x,r)\in\pi} \mu(B(x, r))^q \nu(B(x, r))^t \leq \mathscr{P}_{\Delta,\mu}^{q,t}(E).$$

Therefore, by taking the supremum on π, we get $\mathsf{H}_{\mu}^{q,t}(E) \leq \mathsf{H}_{\beta,\mu}^{q,t}(E) \leq \mathscr{P}_{\Delta,\mu}^{q,t}(E)$. Take the infinmum on Δ to get the desire result. □

3.2. Densities theorems

In the following, we establish a new version of the density theorem with respect to the generalized packing and weak-packing measures.

Theorem 3.7. *Let $(\mathscr{X}, \rho)$ be a metric space, $q, t \in \mathbb{R}$, $\mu, \nu, \Theta \in \mathscr{P}(\mathscr{X})$, and E be a Borel subset of $supp\,\mu \cap supp\,\nu$.*

1. *We have*
$$\mathscr{P}_{\mu}^{q,t}(E) \inf_{x\in E} \underline{D}_{\mu}^{q}(x, \Theta) \leq \Theta(E), \tag{3.5}$$
where we take the lefthand side to be 0 if one of the factors is zero.
2. *If Θ has the weak-Vitali property, then*
$$\Theta(E) \leq \mathscr{W}_{\mu}^{q,t}(E)\ \sup_{x\in E} \underline{D}_{\mu}^{q,t}(x, \Theta), \tag{3.6}$$
where we take the righthand side to be ∞ if one of the factors is ∞.
3. *Assume that μ and $\nu \in \mathscr{P}_0(\mathscr{X})$, then there exists a constant $C > 0$ such that*
$$\Theta(E) \leq C\mathscr{P}_{\mu}^{q,t}(E)\ \sup_{x\in E} \underline{D}_{\mu}^{q,t}(x, \Theta), \tag{3.7}$$
where we take the righthand side to be ∞ if one of the factors is ∞.

Proof.

1. We begin with the proof of (3.5). Assume that $\inf_{x\in E} \underline{D}_{\mu}^{q,t}(x, \Theta) > 0$. Choose γ such that $0 < \gamma < \underline{D}_{\mu}^{q,t}(x, \Theta)$ for all $x \in E$. Let $\varepsilon > 0$ be given. Then there is an open set V such that $E \subseteq V$ and $\Theta(V) < \Theta(E) + \varepsilon$. For $x \in E$, let $\Delta(x) > 0$ be so small such that
$$\frac{\Theta(B(x, r))}{\mu(B(x, r))^q\ \nu(B(x, r))^t} > \gamma$$
for all $r < \Delta(x)$ and $\Delta(x) < dis(x, \mathscr{X} \backslash V)$. Then Δ is a gauge for E. Now, consider π to be a Δ-fine packing of E. Then $\bigcup_{(x,r)\in\pi} B(x, r)$ is contained in V and
$$\sum_{(x,r)\in\pi} \mu(B(x, r))^q\ \nu(B(x, r))^t < \frac{1}{\gamma} \sum_{\pi} \Theta(B(x, r)) \leq \frac{1}{\gamma}\Theta(V).$$

This shows that

$$\mathscr{P}_{\mu}^{q,t}(E) \leq \mathscr{P}_{\Delta,\mu}^{q,t}(E) \leq \frac{1}{\gamma}\Theta(V) \leq \frac{1}{\gamma}(\Theta(E)+\varepsilon).$$

Let $\varepsilon \to 0$ to obtain $\gamma\, \mathscr{P}_{\mu}^{q,t}(E) \leq \Theta(E)$. Since γ is arbitrarily close to $\underline{D}_{\mu}^{q,t}(x,\Theta)$ we get the desired result.

2. Suppose that ν has the weak-Vitali property and we will prove (3.6). For this, we may assume that $\sup_{x\in E} \underline{D}_{\mu}^{q,t}(x,\Theta) < \infty$. Let Δ be a gauge on E and $\gamma < \infty$ such that $\underline{D}_{\mu}^{q,t}(x,\Theta) < \gamma$ for all $x \in E$. Then

$$\beta = \left\{ (x,r);\, x \in E,\ r < \Delta(x) \ \text{ and } \ \frac{\Theta(B(x,r))}{\mu(B(x,r))^q\ \nu(B(x,r))^t} \leq \gamma \right\}$$

is a fine cover of E. By the weak-Vitali property, there is a weak-packing $\pi \subseteq \beta$ of E such that

$$\Theta\left(E \setminus \bigcup_{(x,r)\in\pi} B(x,r)\right) = 0.$$

Therefore,

$$\begin{aligned}\Theta(E) &= \Theta\left(E \bigcap \bigcup_{(x,r)\in\pi} B(x,r)\right) \leq \sum_{(x,r)\in\pi} \Theta(B(x,r)) \\ &\leq \gamma \sum_{(x,r)\in\pi} \mu(B(x,r))^q\ \nu(B(x,r))^t.\end{aligned}$$

Thus $\Theta(E) \leq \gamma \mathscr{W}_{\Delta,\mu}^{q,t}(E)$ and, by arbitrariness of Δ, we obtain $\Theta(E) \leq \gamma \mathscr{W}_{\mu}^{q,t}(E)$. Since γ is arbitrarily close to $\underline{D}_{\mu}^{q,t}(x,\Theta)$ we get the desired result.

3. Since μ and $\nu \in \mathscr{P}_0(\mathscr{X})$, then, for small r, there exists two positive constants C_1 and C_2 such that

$$\mu(B(x,3r)) \leq C_1 \mu(B(x,r)) \quad \text{ and } \quad \nu(B(x,3r)) \leq C_2 \nu(B(x,r)).$$

Assume that $\sup_{x\in E} \underline{D}_{\mu}^{q,t}(x,\Theta) < \infty$. Let Δ be a gauge on E and $\gamma < \infty$ such that $\underline{D}_{\mu}^{q,t}(x,\Theta) < \gamma$ for all $x \in E$. We must show that, there exists a constant C such that $\Theta(E) \leq \gamma\, C\, \mathscr{P}_{\mu}^{q,t}(E)$, for this, we must show that $\Theta(E) \leq \gamma C\, \mathscr{P}_{\Delta,\mu}^{q,t}(E)$. We assume that $\mathscr{P}_{\Delta,\mu}^{q,t}(E) < \infty$ and we consider the set

$$\beta = \left\{ (x,r);\, x \in E, r < \Delta(x) \ \text{ and } \ \frac{\Theta(B(x,3r))}{\mu(B(x,3r))^q\ \nu(B(x,3r))^t} \leq \gamma \right\}.$$

Since β is a fine cover of E and $\mathscr{P}_{\Delta,\mu}^{q,t}(E) < \infty$, it follows from Lemma 3.1 that there exists a packing $\{\ (x_i,r_i)\}_i \subseteq \beta$ such that

$$E \subseteq \bigcup_{i=1}^{\infty} B(x_i, 3r_i).$$

We remark that $\limsup_n r_n > 0$ is impossible, since $\sum_i \mu(B(x_i,r_i))^q\ \nu(B(x_i,r_i))^t < \infty$. Hence, if $\mu, \nu \in \mathscr{P}_0(\mathscr{X})$ then

$$\Theta(E) \ \leq\ \sum_i \Theta(B(x_i,3r_i)) \leq \gamma \sum_i \mu(B(x_i,3r_i))^q\ \nu(B(x_i,3r_i))^t$$

$$\leq \gamma \begin{cases} C_1C_2 \sum_i \mu(B(x_i,r_i))^q \ \nu(B(x_i,r_i))^t & ; \ q,t>0 \text{ and } \mu,\nu\in\mathscr{P}_0(\mathscr{X}) \\ \sum_i \mu(B(x_i,r_i))^q \ \nu(B(x_i,r_i))^t & ; \ q,t\leq 0. \\ C_2 \sum_i \mu(B(x_i,r_i))^q \ \nu(B(x_i,r_i))^t & ; \ q\leq 0, t>0 \text{ and } \nu\in\mathscr{P}_0(\mathscr{X}) \\ C_1 \sum_i \mu(B(x_i,r_i))^q \ \nu(B(x_i,r_i))^t & ; \ q>0, t\leq 0 \text{ and } \mu\in\mathscr{P}_0(\mathscr{X}). \end{cases}$$

Take $C = \max(1, C_1, C_2, C_1C_2)$ to get

$$\Theta(E) \leq \gamma \ C \sum_i \mu(B(x_i,r_i))^q \ \mu(B(x_i,r_i))^t.$$

Thus $\Theta(E) \leq \gamma \ C\mathscr{P}^{q,t}_{\Delta,\mu}(E)$. Since γ is arbitrarily close to $\underline{D}^{q,t}_{\mu}(x,\Theta)$ we get the desired result.

□

Remark 3.8.

1. If $\mu,\nu\in\mathscr{P}_0(\mathscr{X})$ then there exists a constant $\gamma>0$ such that

$$\mathscr{W}^{q,t}_{\mu}(E) \inf_{x\in E} \underline{D}^{q,t}_{\mu}(x,\Theta) \leq \gamma\Theta(E), \tag{3.8}$$

where we take the left hand side in (3.8) to be 0 if one of the factors is zero.

2. Similarly, if Θ has the strong-Vitali property, then

$$\Theta(E) \leq \mathscr{P}^{q,t}_{\mu}(E) \ \sup_{x\in E} \underline{D}^{q,t}_{\mu}(x,\Theta), \tag{3.9}$$

where we take the righthand side in (3.9) to be ∞ if one of the factors is ∞.

Now, using the generalized Hausdorff measure in terms of variation measure, we give a new version of the density theorem.

Theorem 3.9. *Let $q,t\in\mathbb{R}$, $\mu,\nu\in\mathscr{P}(\mathscr{X})$ and E be a Borel subset of* $\operatorname{supp}\mu\cap\operatorname{supp}\nu$.

1. *Then*

$$\mathsf{H}^{q,t}_{\mu}(E) \inf_{x\in E} \overline{D}^{q,t}_{\mu}(x,\Theta) \leq \Theta(E). \tag{3.10}$$

2. *Assume that Θ has the strong Vitali property. Then*

$$\Theta(E) \leq \mathsf{H}^{q,t}_{\mu}(E) \sup_{x\in E} \overline{D}^{q,t}_{\mu}(x,\Theta). \tag{3.11}$$

except when the product is 0 *times* ∞.

3. *Assume that $\mu,\nu\in\mathscr{P}_0(\mathscr{X})$. Then*

$$\Theta(E) \leq \mathsf{H}^{q,t}_{\mu}(E) \sup_{x\in E} \overline{D}^{q,t}_{\mu}(x,\Theta).$$

except when the product is 0 *times* ∞.

Proof.

1. Let $a := \inf_{x\in E} \overline{D}_{\mu}^{q,t}(x,\Theta)$. If $a = 0$ there is nothing to prove, so we may assume that $a > 0$. Let γ be a constant and V be an open set such that $\inf_{x\in E} \overline{D}_{\mu}^{q,t}(x,\Theta) > \gamma > 0$ and $E \subseteq V$. It follows that

$$\beta = \left\{(x,r),\ x \in E, ; \frac{\Theta(B(x,r))}{\mu(B(x,r))^q \nu(B(x,r))^t} > \gamma,\ 0 < r < \operatorname{dist}(x, \mathscr{X}\backslash V)\right\}$$

is a fine cover of E. Therefore, for any packing $\pi \subseteq \beta$ we have

$$\begin{aligned}\sum_{(x,r)\in\pi} \mu(B(x,r))^q \nu(B(x,r))^t &< \frac{1}{\gamma}\sum_{(x,r)\in\pi}\Theta(B(x,r)) \\ &= \frac{1}{\gamma}\Theta\left(\bigcup_{(x,r)\in\pi} B(x,r)\right) \leq \frac{1}{\gamma}\Theta(V).\end{aligned}$$

Take the supremum on π to obtain

$$\mathsf{H}_{\mu}^{q,t}(E) \leq \mathsf{H}_{\beta,\mu}^{q,t}(E) \leq \frac{1}{\gamma}\Theta(V).$$

Finally, since γ is arbitrarily less then a, we get the desire result by taking the infimum on V.

2. Clearly we may assume that $\sup_{x\in E} \overline{D}_{\mu}^{q,t}(x,\Theta) < \infty$. Let γ be a constant such that $\sup_{x\in E} \overline{D}_{\mu}^{q,t}(x,\Theta) < \gamma$. For a fine cover β of E we set

$$\beta_1 = \left\{(x,r) \in \beta;\ \frac{\Theta(B(x,r))}{\mu(B(x,r))^q \nu(B(x,r))^t} < \gamma\right\}$$

is also a fine cover of E. Therefore, under our assumption, there exists a packing $\pi \subseteq \beta_1$ such that $\Theta\Big(E\backslash \bigcup_{\pi} B(x,r)\Big) = 0$. Thus,

$$\begin{aligned}\mathsf{H}_{\beta,\mu}^{q,t}(E) &\geq \sum_{(x,r)\in\pi} \mu(B(x,r))^q \nu(B(x,r))^t > \frac{1}{\gamma}\sum_{(x,r)\in\pi}\Theta(B(x,r)) \\ &\geq \frac{1}{\gamma}\Theta\left(\bigcup_{(x,r)\in\pi} B(x,r)\right) \geq \frac{1}{\gamma}\Theta(E).\end{aligned}$$

This holds for all β so $\Theta(E) \leq \gamma \mathsf{H}_{\mu}^{q,t}$. Since, γ is arbitrarily close to $\sup_{x\in E} \overline{D}_{\mu}^{q,t}(x,\Theta)$, we get the desire result.

3. We only have to prove

$$\Theta(E) \leq \mathscr{H}_{\mu}^{q,t}(E) \sup_{x\in E} \overline{D}_{\mu}^{q,t}(x,\Theta).$$

Indeed, by Theorem 3.5, we have in this case $\mathscr{H}_{\mu}^{q,t}(E) = \mathsf{H}_{\mu}^{q,t}(E)$. Clearly we may assume that $\overline{D}_{\mu}^{q,t}(x,\Theta) < \infty$, for all $x \in E$. let γ be a constant such that $\sup_{x\in E} \overline{D}_{\mu}^{q,t}(x,\Theta) < \gamma$. For each integer $n \in \mathbb{N}$, we set

$$E_n = \left\{x \in E;\ \frac{\Theta(B(x,r))}{\mu(B(x,r))^q \nu(B(x,r))^t} < \gamma \ \text{ for all } \ r < \frac{1}{n}\right\}.$$

We consider, for each n, a δ-cover β_n of E_n, where $\delta < \frac{1}{n}$. Therefore,

$$\begin{aligned}\sum_{(x,r)\in\beta_n} \mu(B(x,r))^q \nu(B(x,r))^t &\geq \frac{1}{\gamma}\sum_{(x,r)\in\beta_n} \Theta(B(x,r)) \\ &\geq \frac{1}{\gamma}\Theta\left(\bigcup_{(x,r)\in\beta_n} B(x,r)\right) \geq \frac{1}{\gamma}\Theta(E_n)\end{aligned}$$

and so $\Theta(E_n) \leq \gamma \mathscr{H}^{q,t}_{\mu,\delta}(E_n)$. Therefore

$$\Theta(E_n) \leq \gamma \mathscr{H}^{q,t}_{\mu,0}(E_n) \leq \gamma \mathscr{H}^{q,t}_{\mu}(E).$$

Since $E_n \nearrow E$, then letting $n \to \infty$ we get $\Theta(E) \leq \gamma \mathscr{H}^{q,t}_{\mu}(E)$. It follows from γ is arbitrarily large that

$$\Theta(E) \leq \mathscr{H}^{q,t}_{\mu}(E) \sup_{x\in E} \overline{D}^{q,t}_{\mu}(x,\Theta).$$

and then we get the desire result.

□

4. The equivalence of fractal measures on Moran fractal sets

In this section, we concentrate on the properties of the generalized fractal measures on a class of Moran fractal set. In particular, we give sufficient condition so that these measures are equivalent on these sets satisfying the strong separation condition. We will start by defining the Moran sets. Let $\{n_k\}_k$ and $\{\Phi_k\}_{k\geq 1}$ be respectively two sequences of positive integers and positive vectors such that

$$\Phi_k = (c_{k_1}, c_{k_2}, \ldots, c_{k_{n_k}}), \qquad \sum_{j=1}^{n_k} c_{kj} \leq 1,\ k \in \mathbb{N}. \tag{4.1}$$

For any $m, k \in \mathbb{N}$, such that $m \leq k$, let

$$D_{m,k} = \Big\{(i_m, i_{m+1}, \ldots, i_k) \;\Big|\; 1 \leq i_j \leq n_j, m \leq j \leq k\Big\}$$

and

$$D_k = D_{1,k} = \Big\{(i_1, i_2, \ldots, i_k) \;\Big|\; 1 \leq i_j \leq n_j, 1 \leq j \leq k\Big\}.$$

We also set $D_0 = \emptyset$ and $D = \cup_{k\geq 0} D_k$, Considering $\sigma = (i_1, i_2, \ldots, i_k) \in D_k$, $\tau = (j_{k+1}, j_{k+2}, \ldots, j_m) \in D_{k+1,m}$, we set

$$\sigma * \tau = (i_1, i_2, \ldots, i_k, j_{k+1}, j_{k+2}, \ldots, j_m).$$

Definition 5. [2, 18] Let $\mathscr{X}$ be a complete metric space and $I \subset \mathscr{X}$ a compact set with no empty interior (for convenience, we assume that the diameter of I is 1). The collection $\mathscr{F} = \{I_\sigma \;\big|\; \sigma \in D\}$ of subsets of I is called having Moran structure if

1. for any $(i_1, i_2, \ldots, i_k) \in D_k$, $I_{i_1 i_2 \ldots i_k}$ is similar to I. That is, there exists a similar transformation

$$\begin{aligned} S_{i_1 i_2 \ldots i_k} :\ & \mathscr{X} \to \mathscr{X} \\ & I \mapsto I_{i_1 i_2 \ldots i_k}, \end{aligned}$$

where we assume that $I_\emptyset = I$.

2. For all $k \geq 1$, $(i_1, i_2, \ldots, i_{k-1}) \in D_{k-1}$, $I_{i_1 i_2 \ldots i_k}(i_k \in \{1, 2, \ldots, n_k\})$ are subsets of $I_{i_1 i_2 \ldots i_{k-1}}$ and

$$I^{\circ}_{i_1 i_2 \ldots i_{k-1}, i_k} \cap I^{\circ}_{i_1 i_2 \ldots i_{k-1}, i'_k} = \emptyset, \qquad 1 \leq i_k < i'_k \leq n_k,$$

where I° denotes the interior of I.

3. For all $k \geq 1$ and $1 \leq j \leq n_k$, taking $(i_1, i_2, \ldots, i_{k-1}, j) \in D_k$, we have

$$0 < c_{kj} = c_{i_1 i_2 \ldots i_{k-1} j} = \frac{|I_{i_1 i_2 \ldots i_{k-1} j}|}{|I_{i_1 i_2 \ldots i_{k-1}}|} < 1, \quad k \geq 2,$$

where $|I|$ denotes the diameter of I.

Suppose that $\mathscr{F}$ is a collection of subsets of I having Moran structure. We call $E = \bigcap_{k\geq 1} \bigcup_{\sigma \in D_k} I_{\sigma}$, a Moran set determined by $\mathscr{F}$, and called $\mathscr{F}_k = \{I_{\sigma}, \sigma \in D_k\}$ the k-order fundamental sets of E. I is called the original set of E. We assume $\lim_{k\to+\infty} \max_{\sigma \in D_k} |I_{\sigma}| = 0$. For all $w = (i_1, i_2, \ldots i_k, \ldots) \in D$, we use the abbreviation $w|_k$ for the first k elements of the sequence,

$$I_k(w) = I_{w|_k} = I_{i_1 i_2 \ldots i_k}, \quad \text{and} \quad c_n(w) = c_{i_1 i_2 \ldots i_n}. \tag{4.2}$$

We assume that E witch satisfy the strong separation condition (**SSC**): Let $I_{\sigma*1}, I_{\sigma*2}, \ldots, I_{\sigma*n_{k+1}}$ be the $(k+1)$-order fundamental subsets of $I_{\sigma} \in \mathscr{F}$. We say that I_{σ} satisfies the (**SSC**) if $\operatorname{dist}(I_{\sigma*i}, I_{\sigma*j}) \geq \delta_k |I_{\sigma}|$, for all $i \neq j$, where $(\delta_k)_k$ is a sequence of positive real numbers, such that $0 < \delta = \inf_k \delta_k < 1$.

If $c_{k,1} = c_{k,2} = \ldots, = c_{k,n_k} = c_k$ for all $k \geq 1$ then E is said to be homogeneous Moran set. Let $x \in E$ and $I_{\sigma}(x)$ the unique fundamental subset of level k containing x ($\sigma \in D_k$). It is clear that $|I_{\sigma}(x)| = \prod_{j=1}^{k} c_j$ which implies that $I_{\sigma}(x) \subseteq B(x, r)$, where $\prod_{j=1}^{k} c_j < r \leq \prod_{j=1}^{k-1} c_j$. In the other hand, let

$$N(x, r) = \{\sigma \in D_{k-1}, I_{\sigma} \cap B(x, r) \neq \emptyset\}.$$

Clearly $N(x, r) \leq 2$ and

$$\bigcup_{\sigma \in N(x,r)} I^{\circ}_{\sigma}(x) \subset B(x, r + c_1 \cdots c_{k-1}) \subset B(x, 2c_1 \cdots c_{k-1}). \tag{4.3}$$

Definition 6. We say that two Borel measures μ and ν are equivalent and we write $\mu \sim \nu$ if for any Borel set A, we have $\mu(A) = 0 \Leftrightarrow \nu(A) = 0$.

Through this section, we consider $E \subset I$ to be a Moran set satisfying (SSC), μ and ν be two Borel probability measures on $\mathscr{X}$ and $\Theta \in \mathscr{P}(\mathscr{X})$ such that $supp\, \Theta \subset E$. For $w \in D$, we set

$$\underline{D}_{\mu}^{q,t}(w, \Theta) := \liminf_{n\to+\infty} \frac{\Theta(I_n(w))}{\mu(I_n(w))^q \nu(I_n(w))^t} \quad \text{and} \quad \overline{D}_{\mu}^{q,t}(w, \Theta) := \limsup_{n\to+\infty} \frac{\Theta(I_n(w))}{\mu(I_n(w))^q \nu(I_n(w))^t}.$$

Proposition 4.1. *Assume that $\mu, \nu \in \mathscr{P}_0(\mathscr{X})$ or Θ has the strong-Vitali property.*

1. *Suppose that there exists α, such that*

$$\underline{D}_\mu^{q,t}(w,\Theta) = \begin{cases} 0 & if \quad t < \alpha, \\ \infty & if \quad t > \alpha, \end{cases} \qquad \text{for any } w \in D,$$

then $\mathrm{Dim}_\mu^q(E) = \alpha = \mathrm{Dim}_\mu^q(\Theta)$.
2. *Suppose that for all* $w \in D$ *we have* $0 < \underline{D}_\mu^{q,\alpha}(w,\Theta) < \infty$, *then,*

$$\Theta \llcorner E \sim \mathscr{P}_\mu^{q,\alpha} \llcorner E,$$

where $\Theta_\llcorner E$ *designates the measure* Θ *restricted to* E.

Proof. The proof can be deduced from (3.9), Theorem 3.7 and [2, Theorem 5]. □

Remark 4.2.

1. If $\mathscr{X}$ is the Euclidean space $\mathbb{R}^d$, then every finite Borel measure has the strong-Vitali property and then Proposition 4.3 is Theorem 5 in [2].
2. It follows from (2.3), if $\mu, \nu \in \mathscr{P}_0(\mathscr{X})$, then $0 < \underline{D}_\mu^{q,\alpha}(w,\Theta) < \infty$ which implies that $\theta \llcorner E \sim \mathscr{W}_\mu^{q,\alpha} \llcorner E$.

Proposition 4.3. *Assume that* $\mu, \nu \in \mathscr{P}_0(\mathscr{X})$ *or* Θ *has the strong-Vitali property.*

1. *Suppose that there exists α, such that*

$$\overline{D}_\mu^{q,t}(w,\Theta) = \begin{cases} 0 & if \quad t < \alpha, \\ \infty & if \quad t > \alpha, \end{cases} \qquad \text{for any } w \in D,$$

then $\dim_\mu^q(E) = \alpha = \dim_\mu^q(\Theta)$.
2. *Suppose that for all* $w \in D$ *we have* $0 < \overline{D}_\mu^{q,\alpha}(w,\Theta) < \infty$, *then*

$$\Theta \llcorner E \sim \mathsf{H}_\mu^{q,\alpha} \llcorner E.$$

Proof. The proof can be deduced from Theorem 3.9 and [2, Theorem 6]. □

Remark 4.4. It follows from Theorem (3.5), if $\mu, \nu \in \mathscr{P}_0(\mathscr{X})$, then $0 < \overline{D}_\mu^{q,\alpha}(w,\Theta) < \infty$ which implies that $\Theta \llcorner E \sim \mathscr{H}_\mu^{q,\alpha} \llcorner E$.

Example 4.5. We set, for all $k \geq 1$, the number s_k which satisfies

$$\prod_{i=1}^{k} \sum_{j=1}^{n_i} c_{ij}^{s_k} = 1 \tag{4.4}$$

and write

$$s_* = \liminf_{k\to\infty} s_k \quad \text{and} \quad s^* = \limsup_{k\to\infty} s_k.$$

Assume that $c_* := \inf_{k,j}\{c_{kj}\} > 0$. Now, consider $\mathscr{X} = [0,1]$ and define a measure Θ on $\mathscr{X}$ such that $\Theta(\mathscr{X}) = 1$ and

$$\Theta(I_{\sigma * i}) = \frac{c_{ki}^s}{\sum_{j=1}^{n_k} c_{kj}^s} \Theta(I_\sigma), \quad 1 \leq i \leq n_k \ \text{ and } \ \sigma \in D_k,$$

where $s := \lim_{k\to\infty} s_k \in (0,1)$. It follows that

$$\Theta(I_\sigma) := \frac{c_{1\sigma_1}^s c_{2\sigma_2}^s \cdots c_{k\sigma_k}^s}{\prod_{i=1}^k \sum_{j=1}^{n_i} c_{ij}^s} = \frac{|I_\sigma|^s}{\prod_{i=1}^k \sum_{j=1}^{n_i} c_{ij}^s}.$$

It follows from (4.4) that

$$\begin{aligned}\left|\log \prod_{i=1}^{k}\sum_{j=1}^{n_i} c_{ij}^s\right| &= \left|\log \prod_{i=1}^{k}\sum_{j=1}^{n_i} c_{ij}^s - \log \prod_{i=1}^{k}\sum_{j=1}^{n_i} c_{ij}^{s_k}\right| \le \sum_{i=1}^{k}\left|\log \sum_{j=1}^{n_i} c_{ij}^s - \log \sum_{j=1}^{n_i} c_{ij}^{s_k}\right| \\ &\le \sum_{i=1}^{k} |\log c_*|\,|s_k - s| = k|\log c_*|\,|s_k - s|.\end{aligned}$$

Hence, using the fact that $|I_\sigma| \le k|\log(1-c_*)$ and (4.1), we obtain

$$\frac{\log \prod_{i=1}^k \sum_{j=1}^{n_i} c_{ij}^s}{|\log I_\sigma|^s} \le \frac{k|\log c_*|\,|s_k - s|}{k|\log(1-c_*)|} = \frac{|\log c_*|}{|\log(1-c_*)|}|s_k - s| \longrightarrow 0.$$

Therefore,

$$\lim_{k\to\infty} \frac{\log \Theta(I_\sigma)}{\log |I_\sigma|} = s$$

uniformly on σ. As a consequence there exists a non-increasing function $\xi : \mathbb{N} \longrightarrow \mathbb{R}^*$ such that $\lim_{k\to\infty} \xi(k) = 0$ and for any basis interval I_σ, we have

$$|I_\sigma|^{s+\xi(|\sigma|)} \le \Theta(I_\sigma) \le |I_\sigma|^{s-\xi(|\sigma|)}.$$

Let $\mu = \nu$ be the Lebesgue measure on $[0,1]$ then, for all $w \in D$, we have

$$\lim_{n\to+\infty} \frac{\Theta(I_\sigma)}{\mu(I_\sigma)^q \nu(I_\sigma)^t} = \lim_{n\to+\infty} \frac{\Theta(I_\sigma)}{|I_\sigma|^{q+t}} = \begin{cases} 0 & \text{if } t < s+q, \\ \infty & \text{if } t > s+q. \end{cases}$$

In particular, for $q = 0$, the classical Hausdorff and packing measures $\mathscr{H}^\alpha$ and $\mathscr{P}^\alpha$ satisfy

$$\Theta \llcorner E \sim \mathscr{H}^s \llcorner E \sim \mathscr{P}^s \llcorner E.$$

5. Applications

In this section, we will study the extensions of the following product inequalities for the Hausdorff measure $\mathscr{H}^t$ and the packing measure $\mathscr{P}^t$ in Euclidean space. Fix $s, t \ge 0$ and E, F be two Borel sets in $\mathbb{R}^d$, then there exists a number $c > 0$ such that

$$\mathscr{H}^s(E)\mathscr{H}^t(F) \le c\mathscr{H}^{s+t}(E \times F), \tag{5.1}$$

$$\mathscr{P}^{s+t}(E \times F) \le c\mathscr{P}^s(E)\mathscr{P}^t(F). \tag{5.2}$$

Inequality (5.1) was shown in [14] under certain conditions and later in [45] without any restrictions. Inequality (5.2) is proved in [35] (see also [1, 4, 29, 62] for more investigation of product inequalities for fractal measure). Using the density approach, we will study the generalized Hausdorff and packing measures of Cartesian product sets. The disadvantage of this approach includes the inability to handle sets of measure ∞. Moreover, we will give a necessary and sufficient condition to obtain strong regular and very strong regular sets. Recall that if we let $E \subseteq \mathscr{X}$ be a Borel set, we say that E is strongly regular if $\mathsf{H}_\mu^{q,t}(E) = \mathscr{P}_\mu^{q,t}(E) \in (0,\infty)$ and very strongly regular if $\mathsf{H}_\mu^{q,t}(E) = \mathscr{W}_\mu^{q,t}(E) \in (0,\infty)$. Finally, we give an application of Theorem 3.5.

5.1. Product inequalities

Let $(\mathscr{X}, \rho)$ and $(\mathscr{Y}, \rho')$ be two separable metric spaces. Assume that $\mathscr{X} \times \mathscr{Y}$ is endowed with a metric which is the Cartesian product of the metrics in $\mathscr{X}$ and $\mathscr{Y}$, so that for all $\varepsilon > 0$, $x \in \mathscr{X}$ and $y \in \mathscr{Y}$, we have

$$B((x,y),\varepsilon) = B(x,\varepsilon) \times B(y,\varepsilon).$$

Before giving our first main result in this section, we will start with two useful corollaries of Theorem 3.7.

Corollary 5.1. *Let $(\mathscr{X}, \rho)$ be a metric space, $q, t \in \mathbb{R}$, $\mu, \nu \in \mathscr{P}(\mathscr{X})$ and E be a Borel subset of $supp\,\mu \cap supp\,\nu$.*

1. *If there exists $\Theta \in \mathscr{P}(\mathscr{X})$ such that $\inf_{x\in E} \underline{D}_\mu^{q,t}(x,\Theta) = \gamma > 0$ then*

$$\mathscr{P}_\mu^{q,t}(E) \leq \Theta(E)/\gamma.$$

2. *If there exists $\Theta \in \mathscr{P}(\mathscr{X})$ such that $\sup_{x\in E} \underline{D}_\mu^{q,t}(x,\Theta) = \gamma < +\infty$ and has the weak-Vitali property then*

$$\mathscr{W}_\mu^{q,t}(E) \geq \Theta(E)/\gamma.$$

3. *Assume that $\mu, \nu \in \mathscr{P}_0(\mathscr{X})$. If there exists $\Theta \in \mathscr{P}(\mathscr{X})$ such that $\sup_{x\in E} \underline{D}_\mu^{q,t}(x,\Theta) = \gamma < +\infty$ then*

$$\mathscr{P}_\mu^{q,t}(E) \geq \Theta(E)/\gamma C.$$

For a Borel set $E \subseteq \mathscr{X}$ we denote by $\mathscr{P}_\mu^{q,t}{}_{\llcorner E}$ the measure $\mathscr{P}_\mu^{q,t}$ restricted to E. We can deduce also the following result.

Corollary 5.2. *Let $(\mathscr{X}, \rho)$ be a separable metric space, $q, t \in \mathbb{R}$, $\mu, \nu \in \mathscr{P}(\mathscr{X})$ and E be a Borel subset of $supp\,\mu \cap supp\,\nu$ such that $\mathscr{P}_\mu^{q,t}(E) < \infty$. Let $\Theta = \mathscr{P}_\mu^{q,t}{}_{\llcorner E}$.*

1. *For $\mathscr{P}_\mu^{q,t}$-a.a. $x \in E$, we have $\underline{D}_\mu^{q,t}(x,\Theta) \leq 1$.*
2. *If Θ has the strong-Vitali property, then*

$$\underline{D}_\mu^{q,t}(x,\Theta) = 1, \qquad \mathscr{P}_\mu^{q,t}\text{-a.a. on } E.$$

3. *Assume that $\boldsymbol{\mu} \in \mathscr{P}_0(\mathscr{X})$, then*

$$1/C \leq \underline{D}_\mu^{q,t}(x,\Theta) \leq 1, \quad \mathscr{P}_\mu^{q,t} - a.a. \text{ on } E,$$

where C is the constant defined in (3.7).

Proof. 1. Put the set $F = \left\{x \in E;\ \underline{D}_{\mu}^{q,t}(x,\Theta) > 1\right\}$, and for $m \in \mathbb{N}^*$

$$F_m = \left\{x \in E;\ \underline{D}_{\mu}^{q,t}(x,\Theta) > 1 + \frac{1}{m}\right\}.$$

Therefore $\inf_{x\in F_m} \underline{D}_{\mu}^{q,t}(x,\Theta) \geq 1 + \frac{1}{m}$. We deduce from (3.5) that

$$\left(1 + \frac{1}{m}\right) \mathscr{P}_{\mu}^{q,t}(F_m) \leq \Theta(F_m) = \mathscr{P}_{\mu}^{q,t}(F_m).$$

This implies that $\mathscr{P}_{\mu}^{q,t}(F_m) = 0$. Since $F = \bigcup_m F_m$, we obtain $\mathscr{P}_{\mu}^{q,t}(F) = 0$, i.e.

$$\underline{D}_{\mu}^{q,t}(x,\Theta) \leq 1 \quad \text{for} \quad \mathscr{P}_{\mu}^{q,t}\text{-a.a. } x \in E. \tag{5.3}$$

2. Now consider the set $\widetilde{F} = \left\{x \in E;\ \underline{D}_{\mu}^{q,t}(x,\Theta) < 1\right\}$, and for $m \in \mathbb{N}^*$

$$\widetilde{F}_m = \left\{x \in E;\ \underline{D}_{\mu}^{q,t}(x,\Theta) < 1 - \frac{1}{m}\right\}.$$

Using (3.9), we clearly have

$$\Theta(\widetilde{F}_m) = \mathscr{P}_{\mu}^{q,t}(\widetilde{F}_m) \leq \left(1 - \frac{1}{m}\right) \mathscr{P}_{\mu}^{q,t}(\widetilde{F}_m).$$

This implies that $\mathscr{P}_{\mu}^{q,t}(\widetilde{F}_m) = 0$. Since $F = \bigcup_m \widetilde{F}_m$, we obtain $\mathscr{P}_{\mu}^{q,t}(F) = 0$, i.e.

$$\underline{D}_{\mu}^{q,t}(x,\Theta) \geq 1 \quad \text{for} \quad \mathscr{P}_{\mu}^{q,t}\text{-a.a. } x \in E. \tag{5.4}$$

The statement in (2) now follows from (5.3) and (5.4).

3. The proof of this statement is very similar to the statement (2) when we use the set

$$\widetilde{F} = \left\{x \in E;\ \underline{D}_{\mu}^{q,t}(x,\Theta) < 1/C\right\}$$

and the inequality (3.7) instead of (3.9).

□

For $\boldsymbol{\mu} = (\mu_1,\nu_1) \in \mathscr{P}(\mathscr{X}) \times \mathscr{P}_0(\mathscr{X})$, $\boldsymbol{\nu} = (\mu_2,\nu_2) \in \mathscr{P}_0(\mathscr{Y}) \times \mathscr{P}(\mathscr{Y})$, we define the product of measures $\boldsymbol{\mu} \times \boldsymbol{\nu}$ as follows

$$\boldsymbol{\mu} \times \boldsymbol{\nu} = (\mu_1 \times \mu_2, \nu_1 \times \nu_2).$$

Our main result in this section is the following.

Theorem 5.3. *Let $\boldsymbol{\mu} = (\mu_1,\nu_1) \in \mathscr{P}_0(\mathscr{X}) \times \mathscr{P}_0(\mathscr{X})$, $\boldsymbol{\nu} = (\mu_2,\nu_2) \in \mathscr{P}_0(\mathscr{Y}) \times \mathscr{P}_0(\mathscr{Y})$, $q,t \in \mathbb{R}$. Then, there exists a constant M such that*

$$\mathscr{P}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{q,t}(E \times F) \leq M \mathscr{P}_{\boldsymbol{\mu}}^{q,t}(E)\ \mathscr{P}_{\boldsymbol{\nu}}^{q,t}(F) \tag{5.5}$$

and

$$\mathscr{W}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}^{q,t}(E \times F) \leq M \mathscr{W}_{\boldsymbol{\mu}}^{q,t}(E)\ \mathscr{W}_{\boldsymbol{\nu}}^{q,t}(F) \tag{5.6}$$

for all Borel $E \subseteq \mathscr{X}$ and $F \subseteq \mathscr{Y}$ provided it is true for the "nullset" cases when one of the factors on the right is zero, i.e, $\mathscr{W}_{\boldsymbol{\nu}}^{q,t}(F) = 0$ or $\mathscr{W}_{\boldsymbol{\mu}}^{q,t}(E) = 0$.

Proof. We will only prove the first inequality, the other inequality is similar. If $\mathscr{P}_{\mu}^{q,t}(E) = \infty$ or $\mathscr{P}_{\nu}^{q,t}(F) = \infty$ there is noting to prove, so assume they are both finite. Let Θ_1 be the restriction of $\mathscr{P}_{\mu}^{q,t}$ to E and Θ_2 be the restriction of $\mathscr{P}_{\nu}^{q,t}$ to F. Using Corollary 5.2, there exists $C_1 > 0$ and $C_2 >$ such that $\Theta_1(E) = \Theta_1(\widetilde{E})$ and $\Theta_2(F) = \Theta_2(\widetilde{F})$, where

$$\widetilde{E} = \Big\{x \in E, \quad \underline{D}_{\mu}^{q,t}(x, \Theta_1) \geq 1/C_1\Big\}$$

and

$$\widetilde{F} = \Big\{x \in F, \quad \underline{D}_{\nu}^{q,t}(x, \Theta_2) \geq 1/C_2\Big\}.$$

Now, the product measure $\Theta_1 \times \Theta_2 \in \mathscr{P}_0(\mathscr{X} \times \mathscr{Y})$. For $(x, y) \in \widetilde{E} \times \widetilde{F}$, we have

$$\begin{aligned} & \underline{D}_{\mu\times\nu}^{q,t}\Big((x,y), \Theta_1 \times \Theta_2\Big) \\ = \ & \liminf_{r\to 0} \left[\frac{\Theta_1(B(x,r))}{\mu_1(B(x,r))^q \ \nu_1(B(x,r))^t} \frac{\Theta_2(B(y,r))}{\mu_2(B(y,r))^q \ \nu_2(B(y,r))^t} \right] \\ \geq \ & \underline{D}_{\mu}^{q,t}(x, \Theta_1) \ \underline{D}_{\nu}^{q,t}(y, \Theta_2) \\ \geq \ & 1/(C_1C_2) > 0. \end{aligned}$$

Therefore, setting $M = C_1C_2$ and by Corollary 5.1, we have

$$\begin{aligned} \mathscr{P}_{\mu\times\nu}^{q,t}(\widetilde{E} \times \widetilde{F}) \ & \leq \ M\Theta_1 \times \Theta_2(\widetilde{E} \times \widetilde{F}) = M\Theta_1(\widetilde{E})\Theta_2(\widetilde{F}) \\ & = \ M\Theta_1(E)\Theta_2(F) = M\mathscr{P}_{\mu}^{q,t}(E)\mathscr{P}_{\nu}^{q,t}(F). \end{aligned}$$

By the assumption for the nullset cases, we get the result with $E \times F$. □

Before giving our second main result in this section, we will start with two useful corollaries of Theorem 3.9.

Corollary 5.4. *Let $\mu, \nu, \Theta \in \mathscr{P}(\mathscr{X})$ and $E \subset \mathscr{X}$ be a Borel set.*

1. *Assume that $\mathsf{H}_{\mu}^{q,t}(E) < \infty$ and there exists $\Theta \in \mathscr{P}(\mathscr{X})$ such that $\inf_{x\in E} \overline{D}_{\mu}^{q,t}(x, \Theta) = \gamma > 0$ then*
$$\mathsf{H}_{\mu}^{q,t}(E) \leq \Theta(E)/\gamma.$$
2. *If there exists $\Theta \in \mathscr{P}(\mathscr{X})$ such that $\sup_{x\in E} \overline{D}_{\mu}^{q,t}(x, \Theta) = \gamma < \infty$ and Θ has the strong Vitali property or if μ and $\nu \in \mathscr{P}_0(\mathscr{X})$, then*
$$\mathsf{H}_{\mu}^{q,t}(E) \geq \Theta(E)/\gamma. \tag{5.7}$$

Corollary 5.5. *Let $\mu, \nu \in \mathscr{P}(\mathscr{X})$, $q, t \in \mathbb{R}$ and E be a Borel subset of $\operatorname{supp}\mu \cap \operatorname{supp}\nu$ such that $\mathsf{H}_{\mu}^{q,t}(E) < \infty$. Let $\Theta = \mathsf{H}_{\mu}^{q,t}{}_{\llcorner E}$.*

1. *For $\mathsf{H}_{\mu}^{q,t}$-a.a. $x \in E$, we have $\overline{D}_{\mu}^{q,t}(x, \Theta) \leq 1$.*
2. *If Θ has the strong Vitali property or if μ and $\nu \in \mathscr{P}_0(\mathscr{X})$ then $\overline{D}_{\mu}^{q,t}(x, \Theta) = 1$, $\mathsf{H}_{\mu}^{q,t}$-a.a. on E.*

Our second main result in this section is the following.

Theorem 5.6. *Let $\boldsymbol{\mu} = (\mu_1, \nu_1) \in \mathscr{P}_0(\mathscr{X}) \times \mathscr{P}_0(\mathscr{X})$, $\boldsymbol{\nu} = (\mu_2, \nu_2) \in \mathscr{P}_0(\mathscr{Y}) \times \mathscr{P}_0(\mathscr{Y})$, $q, t \in \mathbb{R}$. For all Borel $E \subseteq \mathscr{X}$ and $F \subseteq \mathscr{Y}$ such that $\mathsf{H}^{q,t}_{\boldsymbol{\mu}}(E) < \infty$ and $\mathsf{H}^{q,t}_{\boldsymbol{\nu}}(F) < \infty$ we have*

$$\mathsf{H}^{q,t}_{\boldsymbol{\mu}}(E)\ \mathsf{H}^{q,t}_{\boldsymbol{\nu}}(F) \le \mathsf{H}^{q,t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(E \times F).$$

Proof. Let Θ_1 be the restriction of $\mathsf{H}^{q,t}_{\boldsymbol{\mu}}$ to E and Θ_2 be the restriction of $\mathsf{H}^{q,t}_{\boldsymbol{\nu}}$ to F. By using Corollary 5.5, we have $\Theta_1(E) = \Theta_1(\widetilde{E})$ and $\Theta_2(F) = \Theta_2(\widetilde{F})$, where

$$\widetilde{E} = \Big\{x \in E, \quad \overline{D}^{q,t}_{\boldsymbol{\mu}}(x, \Theta_1) \le 1\Big\}$$

and

$$\widetilde{F} = \Big\{x \in F, \quad \overline{D}^{q,t}_{\boldsymbol{\nu}}(x, \Theta_2) \le 1\Big\}.$$

Now, the product measure $\Theta_1 \times \Theta_2 \in \mathscr{P}(\mathscr{X} \times \mathscr{Y})$. For $(x, y) \in \widetilde{E} \times \widetilde{F}$, we have

$$\begin{aligned} & \overline{D}^{q,t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}\Big((x, y), \Theta_1 \times \Theta_2\Big) \\ = \ & \limsup_{r\to 0} \left[\frac{\Theta_1(B(x, r))}{\mu_1(B(x, r))^q \nu_1(B(x, r))^t}\ \frac{\Theta_2(B(y, r))}{\mu_2(B(y, r))^q \nu_2(B(y, r))^t}\right] \\ \le \ & \overline{D}^{q,t}_{\boldsymbol{\mu}}(x, \Theta_1)\ \ \overline{D}^{q,t}_{\boldsymbol{\nu}}(y, \Theta_2) \le 1. \end{aligned}$$

Therefore, it follows from (5.7) that

$$\begin{aligned} \mathsf{H}^{q,t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(E \times F) \ &\ge\ \Theta_1 \times \Theta_2(\widetilde{E} \times \widetilde{F}) = \Theta_1(\widetilde{E})\Theta_2(\widetilde{F}) \\ &=\ \Theta_1(E)\Theta_2(F) = \mathsf{H}^{q,t}_{\boldsymbol{\mu}}(E)\mathsf{H}^{q,t}_{\boldsymbol{\nu}}(F). \end{aligned}$$

□

As a direct consequence, we get the following result.

Corollary 5.7. *Let $\boldsymbol{\mu} = (\mu_1, \nu_1) \in \mathscr{P}_0(\mathscr{X}) \times \mathscr{P}_0(\mathscr{X})$, $\boldsymbol{\nu} = (\mu_2, \nu_2) \in \mathscr{P}_0(\mathscr{Y}) \times \mathscr{P}_0(\mathscr{Y})$ and $q, t \in \mathbb{R}$. For $E \subseteq \mathscr{X}$ and $F \subseteq \mathscr{Y}$ such that $\mathscr{H}^{q,t}_{\boldsymbol{\mu}}(E) < \infty$ and $\mathscr{H}^{q,s}_{\boldsymbol{\nu}}(F) < \infty$ we have*

$$\mathscr{H}^{q,t}_{\boldsymbol{\mu}}(E)\ \mathscr{H}^{q,t}_{\boldsymbol{\nu}}(F) \le \mathscr{H}^{q,t}_{\boldsymbol{\mu}\times\boldsymbol{\nu}}(E \times F).$$

5.2. *Strong notion of regularity*

In this section, we formulate a new version of regularity result developed in [5, 15, 17, 21, 22, 50, 56, 58, 59]. More precisely, we give a necessary and sufficient condition to get the equality $\mathsf{H}^{q,t}_{\boldsymbol{\mu}}(E) = \mathscr{P}^{q,t}_{\boldsymbol{\mu}}(E)$. Such a set is called a strong regular. The set E will be called very strong regular if $\mathsf{H}^{q,t}_{\boldsymbol{\mu}}(E) = \mathscr{W}^{q,t}_{\boldsymbol{\mu}}(E)$. In Theorem 5.9 we will characterize these sets.

Theorem 5.8. *Let $\mu, \nu \in \mathscr{P}(\mathscr{X})$ and E be a Borel subset of $\operatorname{supp}\mu \cap \operatorname{supp}\nu$ such that $\mathscr{P}^{q,t}_{\boldsymbol{\mu}}(E) < +\infty$. Let $\Theta_1 = \mathsf{H}^{q,t}_{\boldsymbol{\mu}}{}_{\llcorner E}$ and $\Theta_2 = \mathscr{P}^{q,t}_{\boldsymbol{\mu}}{}_{\llcorner E}$. Assume that Θ_1 has the strong-Vitali property, then the following assertions are equivalent*

1. $\mathsf{H}^{q,t}_{\boldsymbol{\mu}}(E) = \mathscr{P}^{q,t}_{\boldsymbol{\mu}}(E)$.
2. $\underline{D}^{q,t}_{\boldsymbol{\mu}}(x, \Theta_1) = 1 = \overline{D}^{q,t}_{\boldsymbol{\mu}}(x, \Theta_1) \qquad$ *for $\mathscr{P}^{q,t}_{\boldsymbol{\mu}}$-a.a. on E.*

3. $\underline{D}_{\mu}^{q,t}(x,\Theta_2) = 1 = \overline{D}_{\mu}^{q,t}(x,\Theta_2)$ *for* $\mathscr{P}_{\mu}^{q,t}$*-a.a. on* E.

Proof. (1) $\Longrightarrow$ (2) Assume that $\mathscr{P}_{\mu}^{q,t}(E) < \infty$. Notice first that (1) is equivalent to

$$\mathsf{H}_{\mu}^{q,t}(F) = \mathscr{P}_{\mu}^{q,t}(F) \quad \text{for any} \quad F \subseteq E. \tag{5.8}$$

Put the set $F = \left\{x \in E;\ \ \overline{D}_{\mu}^{q,t}(x,\Theta_1) > 1\right\}$. Using Corollary 5.5, we have $\mathsf{H}_{\mu}^{q,t}(F) = 0$ and so, $\mathscr{P}_{\mu}^{q,t}(F) = 0$, i.e.

$$\overline{D}_{\mu}^{q,t}(x,\Theta_1) \leq 1 \quad \text{for} \quad \mathscr{P}_{\mu}^{q,t}\text{-a.a. } x \in E. \tag{5.9}$$

Now consider the set $\widetilde{F} = \left\{x \in E;\ \ \underline{D}_{\mu}^{q,t}(x,\Theta_1) < 1\right\}$, and for $m \in \mathbb{N}^*$

$$\widetilde{F}_m = \left\{x \in E;\ \ \underline{D}_{\mu}^{q,t}(x,\Theta_1) < 1 - \frac{1}{m}\right\}.$$

Using (3.9), we clearly have

$$\mathsf{H}_{\mu}^{q,t}(\widetilde{F}_m) = \mathscr{P}_{\mu}^{q,t}(\widetilde{F}_m) \leq \left(1 - \frac{1}{m}\right)\mathscr{P}_{\mu}^{q,t}(\widetilde{F}_m).$$

This implies that $\mathscr{P}_{\mu}^{q,t}(\widetilde{F}_m) = 0$. As $F = \bigcup_m \widetilde{F}_m$, we obtain $\mathscr{P}_{\mu}^{q,t}(F) = 0$, i.e.

$$\underline{D}_{\mu}^{q,t}(x,E) \geq 1 \quad \text{for} \quad \mathscr{P}_{\mu}^{q,t}\text{-a.a. } x \in E. \tag{5.10}$$

The statement in (2) now follows from (5.9) and (5.10).

(2) $\Longrightarrow$ (1) Consider the set

$$F = \left\{x \in E;\ \ \underline{D}_{\mu}^{q,t}(x,\Theta_1) = 1 = \overline{D}_{\mu}^{q,t}(x,\Theta_1)\right\}.$$

It therefore follows from (3.9), and (3.5) and since, $\underline{D}_{\mu}^{q,t}(x,\Theta_1) = 1 = \overline{D}_{\mu}^{q,t}(x,\Theta_1)$ for $\mathscr{P}_{\mu}^{q,t}$-a.a. $x \in E$ that

$$\mathsf{H}_{\mu}^{q,t}(E) \leq \mathscr{P}_{\mu}^{q,t}(E) = \mathscr{P}_{\mu}^{q,t}(F) \leq \mathsf{H}_{\mu}^{q,t}(F) \leq \mathsf{H}_{\mu}^{q,t}(E).$$

(1) $\Longrightarrow$ (3) From Corollary 5.2 we have $\underline{D}_{\mu}^{q,t}(x,\Theta_2) = 1$ for $\mathscr{P}_{\mu}^{q,t}$-a.a. $x \in E$.
Next, put $\widetilde{F} = \left\{x \in E;\ \ \overline{D}_{\mu}^{q,t}(x,\Theta_2) > 1\right\}$, and for $m \in \mathbb{N}^*$

$$\widetilde{F}_m = \left\{x \in E;\ \ \overline{D}_{\mu}^{q,t}(x,\Theta_2) > 1 + \frac{1}{m}\right\}.$$

We deduce from (3.10) that,

$$\left(1 + \frac{1}{m}\right)\mathsf{H}_{\mu}^{q,t}(\widetilde{F}_m) \leq \mathsf{H}_{\mu}^{q,t}(\widetilde{F}_m) = \mathscr{P}_{\mu}^{q,t}(\widetilde{F}_m).$$

This implies that $\mathsf{H}_{\mu}^{q,t}(\widetilde{F}_m) = 0$. Finally, since $F = \bigcup_m \tilde{F}_m$, we get $\mathsf{H}_{\mu}^{q,t}(F) = \mathscr{P}_{\mu}^{q,t}(F) = 0$, i.e.

$$\overline{D}_{\mu}^{q,t}(x, \Theta_2) \leq 1 \quad \text{for} \quad \mathscr{P}_{\mu}^{q,t}\text{-a.a. } x \in E.$$

(3) $\Longrightarrow$ (1) We consider the set

$$F = \Big\{ x \in E;\ \ \underline{D}_{\mu}^{q,t}(x, \Theta_2) = 1 = \overline{D}\boldsymbol{\mu}^{q,t}(x, \Theta_2) \Big\}.$$

Combining (3.10) and (3.11) shows that

$$\mathsf{H}_{\mu}^{q,t}(E) \leq \mathscr{P}_{\mu}^{q,t}(E) = \mathscr{P}_{\mu}^{q,t}(F) \leq \mathsf{H}_{\mu}^{q,t}(F) \leq \mathsf{H}_{\mu}^{q,t}(E),$$

which proves the desired result. □

Similarly, we obtain the following theorem

Theorem 5.9. *Let $\mu, \nu \in \mathscr{P}(\mathscr{X})$ and E be a Borel subset of $\operatorname{supp}\mu \cap \operatorname{supp}\nu$ such that $\mathscr{W}_{\mu}^{q,t}(E) < +\infty$. Let $\Theta_1 = \mathsf{H}_{\mu}^{q,h}{}_{\llcorner E}$ and $\Theta_2 = \mathscr{W}_{\mu}^{q,h}{}_{\llcorner E}$. Assume that Θ_1 has the weak-Vitali property, then the following assertions are equivalent*

1. $\mathsf{H}_{\mu}^{q,t}(E) = \mathscr{W}_{\mu}^{q,t}(E)$.
2. $\underline{D}_{\mu}^{q,t}(x, \Theta_1) = 1 = \overline{D}_{\mu}^{q,t}(x, \Theta_1)$ *for $\mathscr{W}_{\mu}^{q,t}$-a.a. on E.*
3. $\underline{D}_{\mu}^{q,t}(x, \Theta_2) = 1 = \overline{D}_{\mu}^{q,t}(x, \Theta_2)$ *for $\mathscr{W}_{\mu}^{q,t}$-a.a. on E.*

5.3. The weak-relative packing measure

In the following, we will give an application of Theorem 3.5. First, we will prove the inequality (2.3) without any restriction on μ and ν but we will add a suitable assumption on the metric space $\mathscr{X}$. Then, we will modify slightly the construction of the weak-packing measure $\mathscr{W}_{\mu}^{q,t}$ to obtain a new fractal measure $w_{\mu}^{q,t}$ equal to $\mathscr{P}_{\mu}^{q,t}$. This new measure is obtained by using the class of all weak-packing of a set E such that the intersection of any two balls of them contains no point of E.

Definition 7. A metric space $\mathscr{X}$ is said to be amenable to packing if there exists a constant K such that if $\pi = (x_i, r_i)_i$ is a weak packing of a set E then π can be rearranged such that for any n, there are at most $K - 1$ integers $j \in \{1, \dots, n-1\}$ such that

$$\rho(x_n, x_j) \leq r_n + r_j.$$

Proposition 5.10. *Let $\mu, \nu \in \mathscr{P}(\mathscr{X})$, $q, t \in \mathbb{R}$ and suppose that $\mathscr{X}$ is amenable to packing. Then, there exists a constant K such that*

$$\mathscr{W}_{\mu}^{q,t} \leq K \mathscr{P}_{\mu}^{q,t}. \tag{5.11}$$

Proof. Let π be a Δ-fine weak packing of E. Since $\mathscr{X}$ is amenable to packing, we can distribute the constituents of π into K sequences $\pi_i = \{(x_{ik}, r_{ik})\ k \in \mathbb{N}\} \subseteq \pi$, $1 \leq i \leq K$ such that each i we have π_i is a Δ-fine packing of E and so

$$\sum_{(x,r)\in\pi} \mu(B(x,r))^q \nu(B(x,r))^t \leq \sum_{i=1}^{K} \sum_{(x,r)\in\pi_i} \mu(B(x,r))^q \nu(B(x,r))^t.$$

From which it follows (5.11) by Theorem 3.5. □

Let $E \subseteq \mathscr{X}$. $(x_i, r_i)_i$, $x_i \in E$ and $r_i > 0$, is a centered δ-weak-relative-packing of E if and only if, for all $i, j = 1, 2, \ldots$, we have $r_i \leq \delta$ and for all $i \neq j$,

$$\rho(x_i, x_j) > \max(r_i, r_j) \quad \text{and} \quad B(x_i, r_i) \cap B(x_j, r_j) \cap E = \emptyset.$$

Then, the weak-relative-packing measure $w_\mu^{q,t}$ is defined by

$$w_{\mu,\delta}^{q,t}(E) = \sup\left\{ \sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^t \right\},$$

where the supremum is taken over all δ-weak-relative-packing of E. We write

$$\begin{aligned} w_{\mu,0}^{q,t}(E) &= \inf_{\delta>0} w_{\mu,\delta}^{q,t}(E) = \lim_{\delta\to 0} w_{\mu,\delta}^{q,t}(E), \\ w_\mu^{q,t}(E) &= \inf\left\{ \sum_{i=1}^{\infty} w_{\mu,0}^{q,t}(E_i); \quad E \subseteq \bigcup_{i=1}^{\infty} E_i \right\}. \end{aligned}$$

If $E \neq \emptyset$ and $w_\mu^{q,t}(\emptyset) = 0$. Similarly, we define

$$w_{*,\mu}^{q,t}(E) = \inf_{\Delta} w_{\Delta,\mu}^{q,t}(E),$$

where we use in the definition of $w_{\Delta,\mu}^{q,t}$ the Δ-fine weak-relative-packings. It is clear that $w_{*,\mu}^{q,t}$ is a metric outer regular measure. In addition, we have

$$w_{*,\mu}^{q,t}(E) = w_\mu^{q,t}(E),$$

for all $E \subseteq \mathscr{X}$.

Theorem 5.11. *For any $E \subseteq \mathscr{X}$, $\mu, \nu \in \mathscr{P}(\mathscr{X})$ and $q, t \in \mathbb{R}$. Assume that $\mathscr{X}$ is amenable to packing and every finite Borel measure on $\mathscr{X}$ satisfies the strong Vitali property. Then*

$$\mathscr{P}_\mu^{q,t}(E) = w_\mu^{q,t}(E).$$

Proof. Since any Δ-fine packing π is a Δ-fine weak-relative-packing, we have the first inequality

$$\mathscr{P}_{*,\mu}^{q,t}(E) \leq w_{*,\mu}^{q,t}(E).$$

Now, we will prove the converse inequality. Since, by Proposition 5.10, we have $w_{*,\mu}^{q,t}(E) \leq K \mathscr{P}_{*,\mu}^{q,t}(E)$ and then

$$\mathscr{P}_{*,\mu}^{q,t}(E) = 0 \Longleftrightarrow w_{*,\mu}^{q,t}(E) = 0 \ \text{ and } \ \mathscr{P}_{*,\mu}^{q,t}(E) = \infty \Longleftrightarrow w_{*,\mu}^{q,t}(E) = \infty.$$

Therefore, we may assume that $\mathscr{P}_{*,\mu}^{q,t}(E) < \infty$. Let $\Theta = \mathscr{P}_{*,\mu \llcorner E}^{q,t}$ then, by Corollary 5.2, we have

$$\underline{D}_\mu^{q,t}(x, \Theta) = 1 \quad \text{for} \quad \mathscr{P}_{*,\mu}^{q,t} \text{ almost every } \ x \in E.$$

For $\alpha < 1$, we set

$$G_k = \left\{ x \in E,\ r \leq 1/k \Rightarrow \mathscr{P}_{*,\mu}^{q,t}(E \cap B(x, r)) \geq \alpha \mu(B(x, r))^q \nu(B(x, r))^t \right\}$$

and let $G'_k = E \backslash G_k$. Therefore,

$$\lim_k \mathscr{P}^{q,t}_{*,\mu}(G_k) = \mathscr{P}^{q,t}_{*,\mu}(E), \quad \lim_k w^{q,h}_{*,\mu}(G_k) = w^{q,t}_{*,\mu}(E)$$

and

$$\lim_k \mathscr{P}^{q,t}_{*,\mu}(G'_k) = 0 = \lim_k w^{q,t}_{*,\mu}(G'_k).$$

Let Δ be a gauge satisfying $\Delta(x) < 1/k$. Then for any Δ-fine weak-relative-packing π of G_k, we have

$$\begin{aligned} & \sum_{(x,r)\in\pi} \alpha\mu(B(x,\ r)^q \nu(B(x,r))^t \\ \leq & \sum_{(x,r)\in\pi} \mathscr{P}^{q,t}_{*,\mu}(E \ \cap \ B(x,r)) \\ \leq & \sum_{(x,r)\in\pi} \mathscr{P}^{q,t}_{*,\mu}(G_k \ \cap \ B(x,r)) + \sum_{(x,r)\in\pi} \mathscr{P}^{q,t}_{*,\mu}(G'_k \ \cap \ B(x,r)). \end{aligned}$$

As π is a Δ-fine weak-relative-packing of G_k, the $(G_k \cap B)$'s are disjoint, and so

$$\sum_{(x,r)\in\pi} \mathscr{P}^{q,t}_{*,\mu}(G_k \ \cap \ B(x,r)) \leq \mathscr{P}^{q,t}_{*,\mu}(G_k).$$

Since $\mathscr{X}$ is amenable to packing, we may distribute the constituents $(x_i, r_i)_i$ into K sequences $\pi_i = \Big\{(x_{ik}, r_{ik}),\ k \in \mathbb{N}\Big\} \subseteq \pi$, $1 \leq i \leq K$ such that each π_i is a Δ-fine packing of G_k. Therefore, we have

$$\sum_{(x,r)\in\pi} \mathscr{P}^{q,t}_{*,\mu}(G'_k \ \cap \ B) \leq K\mathscr{P}^{q,t}_{*,\mu}(G'_k)$$

and so

$$\alpha w^{q,t}_{*,\mu}(G_k) \leq \mathscr{P}^{q,t}_{*,\mu}(G_k) + K\mathscr{P}^{q,t}_{*,\mu}(G'_k).$$

Letting $k \to \infty$ we get

$$\alpha w^{q,t}_{*,\mu}(E) \leq \mathscr{P}^{q,t}_{*,\mu}(E).$$

Since $\alpha < 1$ was arbitrary, the proof is complete.

□

6. Conclusions

In real-line and higher-dimensional Euclidean spaces, the properties of various fractal and multifractal measures and dimensions have been extensively studied. It is frequently hard to create non-trivial self-similar or self-conformal sets, etc., in non-Euclidean spaces. In this study, we discuss how to formulate the definitions for use in general metric spaces. We look into the relative Hausdorff measures and packing measures defined in a separable metric space. We present a few product inequalities that follow from a revised formulation of the density theorems for these measures. We also demonstrate that the Henstock-Thomson variation measures can be stated in terms of one another. In this situation, the weak-Vitali property becomes relevant.

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Conflict of interest

The authors declare that there is no conflicts of interest in this paper.

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