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Research article

On analysing discrete sequential operators of fractional order and their monotonicity results

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Abstract: In this study, we consider the analysis of monotonicity for the Riemann-Liouville fractional differences of sequential type. The results are defined on the subsets of $(0, 1) \times (0, 1)$ with a certain restriction. By analysing the difference operator in the point-wise form into a delta form, we use the standard sequential formulas as stated in Theorems 2.1 and 2.2 to establish the positivity of the delta difference operator of the proposed the discrete sequential operators. Finally, some numerical experiments are conducted which confirm our theoretical monotonicity results.

Keywords: delta Riemann-Liouville fractional difference; sequential operators; monotonicity analysis
Mathematics Subject Classification: 39A12, 39B62, 33B10, 26A48, 26A51

1. Introduction

The decay of discrete fractional operators with respect to time scale has received much attention over the years. On the other hand, many results for alternative theories of discrete fractional operators have been obtained, see [1–4]. Also, these have some applications in many physical and mathematical situations, for example in astrophysics [5–7], meteorology [8–10], oceanography [11–13], geophysics [14, 15], engineering [16, 17], and contaminant transport [18–20].

The positivity and monotonicity analyses are nowadays an object of many studies and as far as it is concerned, many researchers investigated the problem of convection under the strict assumption of local discrete fractional operators. Mathematical models associated to phenomena governed by discrete fractional operators are based on suitable refinements of the singular and nonsingular kernels, such as standard, exponential and Mittag-Leffler function in kernels. Indeed, the traditional models are always more appropriate to describe certain positivity and monotonicity analysis since they involve a simple kernel. Also, these kinds of discrete fractional operators are much more flexible and provide in general a better fit of the observed discrete analysis, furthermore, they are examined with various delta and nabla difference operators in time space $\mathbb{N}_a$, see recent published works [21–25] and the references therein for more details.

Furthermore, monotonicity and positivity results are important examples of useful properties that discrete fractional operators may have. In recent years we also started to see a growing interest in the study of monotonicity and positivity analysis with discrete sequential fractional operators involving mixed orders analytically and numerically, see, for example, [26–30] to mention only a few works. Therefore, it is of interest to analyse a discrete sequential fractional operator of mixed order correctly, provided that they allow a development of the applications and theory based on them successfully.

This study aims to analyse some monotonicity results via the sequential fractional difference of mixed order

$$\left({}_{a+1-\beta}^{\ \ \mathrm{RL}}\Delta^{\alpha}\ {}_{a}^{\mathrm{RL}}\Delta^{\beta} f\right)(z) \quad \text{and} \quad \left({}_{a+1}^{\mathrm{RL}}\nabla^{\alpha}\ {}_{a}^{\mathrm{RL}}\nabla^{\beta} f\right)(z), \tag{1.1}$$

on the sets $\mathscr{D}_1 \cap \mathscr{D}_2$ and $\mathscr{D}_1 \setminus \mathscr{D}_2$ as shown in Figure 1, where

$$\mathscr{D}_1 := \Big\{(\alpha,\beta) \in (0,1)\times(0,1) \quad \text{with} \quad 1 < \alpha+\beta < 2\Big\}, \tag{1.2}$$

$$\mathscr{D}_2 := \Big\{(\alpha,\beta) \in (0,1)\times(0,1) \quad \text{with} \quad \frac{\beta}{2} < 1-\alpha\Big\}, \tag{1.3}$$

where ${}_{a}^{\mathrm{RL}}\Delta^{\beta}$ is the Riemann-Liouville fractional difference (see [22]), defined by

$$\left({}_{a}^{\mathrm{RL}}\Delta^{\beta} f\right)(z) = \frac{1}{\Gamma(-\beta)}\sum_{s=a}^{z+\beta}(z-1-s)^{\underline{-\beta-1}} f(s), \tag{1.4}$$

for $p-1 < \beta \le p$, $p \in \mathbb{N}_1$ and $z \in \mathbb{N}_{a+p-\beta} := \{a+p-\beta, a+p+1-\beta, \ldots\}$. Above it is used that

$$z^{\underline{\beta}} = \frac{\Gamma(z+1)}{\Gamma(z+1-\beta)}. \tag{1.5}$$

Moreover, ${}_{a}^{\mathrm{RL}}\nabla^{\beta}$ is the nabla Riemann-Liouville fractional difference (see [25, Lemma 2.1]), defined by

$$\left({}_{a}^{\mathrm{RL}}\nabla^{\beta} f\right)(z) = \frac{1}{\Gamma(-\beta)}\sum_{s=a+1}^{z}(z+1-s)^{\overline{-\beta-1}} f(s), \tag{1.6}$$

for $z \in \mathbb{N}_{a+p} := \{a+p, a+p+1, \ldots\}$, where $p-1 < \beta < p$, $p \in \mathbb{N}_1$. Above it is used that

$$z^{\overline{\beta}} = \frac{\Gamma(z+\beta)}{\Gamma(z)}, \tag{1.7}$$

such that it will tend to zero when $\Gamma(z)$ is undefined.

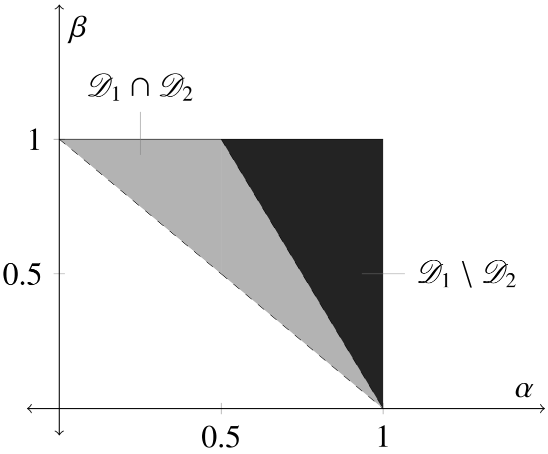

Figure 1. Graph of $\mathscr{D}_1$.

The design of this study is as follows: First, in Section 2, we recall an essential theorem of discrete fractional calculus and then we conclude a lemma which will be a base to get other main results. Then, in Section 3, we present a framework that clarifies the analysis of (1.1) and the monotonicity results on the set $\mathscr{D}_1 \setminus \mathscr{D}_2$ and $\mathscr{D}_1 \cap \mathscr{D}_2$. Finally, Section 5 provides the concluding notes and significance of the present article.

2. Preliminaries results

The basic and main results depend on the following theorem, proved by Holm in [31].

Theorem 2.1. *Let f be defined on $\mathbb{N}_a$ and $N_1 - 1 < \alpha \leq N_1, N_2 - 1 < \beta \leq N_2$, for $N_1, N_2 \in \mathbb{N}_1$. Then, we have*

$$\left({}_{a+1-\beta}^{\ \ \mathrm{RL}}\Delta^{\alpha}\ {}_{a}^{\mathrm{RL}}\Delta^{\beta} f\right)(z) = \left({}_{a}^{\mathrm{RL}}\Delta^{\alpha+\beta} f\right)(z) - \sum_{\kappa=0}^{N_2-1} \frac{(z+\beta-N_2-a)^{\underline{\kappa-\alpha-N_2}}}{\Gamma(\kappa+1-\alpha-N_2)} \left({}_{a}^{\mathrm{RL}}\Delta^{\kappa+\beta-N_2} f\right)(a+N_2-\beta), \quad (2.1)$$

for $z \in \mathbb{N}_{a+N_1+N_2-\alpha-\beta}$.

Based on the above theorem the following lemma can be deduced.

Lemma 2.1. *Let f be defined on $\mathbb{N}_a$ and $0 < \alpha,\ \beta \leq 1$. Then, we have*

$$\left({}_{a+1-\beta}^{\ \ \mathrm{RL}}\Delta^{\alpha}\ {}_{a}^{\mathrm{RL}}\Delta^{\beta} f\right)(z) = \left({}_{a}^{\mathrm{RL}}\Delta^{\alpha+\beta} f\right)(z) - \frac{(z+\beta-1-a)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)} f(a), \quad (2.2)$$

for $z \in \mathbb{N}_{a+2-\alpha-\beta}$.

Proof. For $N_1 = 1$ and $N_2 = 1$ in Theorem 2.1, we get

$$\begin{aligned}\left({}_{a+1-\beta}^{\ \ \mathrm{RL}}\Delta^{\alpha}\ {}_{a}^{\mathrm{RL}}\Delta^{\beta} f\right)(z) &= \left({}_{a}^{\mathrm{RL}}\Delta^{\alpha+\beta} f\right)(z) - \frac{(z+\beta-1-a)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)} \left({}_{a}^{\mathrm{RL}}\Delta^{-(1-\beta)} f\right)(a+1-\beta) \\ &= \left({}_{a}^{\mathrm{RL}}\Delta^{\alpha+\beta} f\right)(z) - \frac{(z+\beta-1-a)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)} f(a),\end{aligned}$$

where we have used

$$\left({}_{a}^{\mathrm{RL}}\Delta^{-(1-\beta)} f\right)(a+1-\beta) = \frac{1}{\Gamma(1-\beta)} \sum_{s=a}^{a+1-\beta-(1-\beta)} (a-\beta-s)^{\underline{-\beta}} f(s) = f(a).$$

Hence the proof is done. □

Next, we recall the following sequential result established by Mohammed et al. [32, Theorem 2.1].

Theorem 2.2. *For f be defined on $\mathbb{N}_a$, we have*

$$\left({}_{a+1}^{\mathrm{RL}}\nabla^{\alpha}\, {}_{a}^{\mathrm{RL}}\nabla^{\beta} f\right)(z) = \frac{1}{\Gamma(-\alpha-\beta)} \sum_{s=a+1}^{z} \frac{\Gamma(z-\alpha-\beta-s)}{\Gamma(z+1-s)} f(s) - \frac{\Gamma(z-a-\alpha-1)}{\Gamma(z-a)\Gamma(-\alpha)} f(a+1), \tag{2.3}$$

for $z \in \mathbb{N}_{a+3}$, $1 < \alpha \leq 2$ and $0 < \beta < 1$.

The above theorem enables us to have the following lemma.

Lemma 2.2. *For f be defined on $\mathbb{N}_a$ with $\left({}_{a+1}^{\mathrm{RL}}\nabla^{\alpha}\, {}_{a}^{\mathrm{RL}}\nabla^{\beta} f\right)(z) \geq 0$, for $z \in \mathbb{N}_{a+3}$, we have*

$$\begin{aligned}(\nabla f)(a+k) \geq{}& \left[\frac{\Gamma(k-a-\alpha-1)}{(k-1)!\Gamma(-\alpha)} - \frac{\Gamma(k-\alpha-\beta)}{(k-1)!\Gamma(1-\alpha-\beta)}\right] f(a+1) \\ &- \frac{1}{\Gamma(1-\alpha-\beta)} \sum_{\mathrm{j}=0}^{k-3} \frac{\Gamma(k-\mathrm{j}-1-\alpha-\beta)}{\Gamma(k-\mathrm{j}-1)} (\nabla f)(a+\mathrm{j}+2),\end{aligned} \tag{2.4}$$

for $k \in \mathbb{N}_2$, $0 < \alpha \leq 1$ and $0 < \beta < 1$.

Proof. Rewriting (2.3) as follows:

$$\begin{aligned}\left({}_{a+1}^{\mathrm{RL}}\nabla^{\alpha}\, {}_{a}^{\mathrm{RL}}\nabla^{\beta} f\right)(z) ={}& \left[\frac{\Gamma(z-a-\alpha-\beta)}{\Gamma(z-a)\Gamma(1-\alpha-\beta)} - \frac{\Gamma(z-a-\alpha-1)}{\Gamma(z-a)\Gamma(-\alpha)}\right] f(a+1) \\ &+ (\nabla f)(z) + \frac{1}{\Gamma(1-\alpha-\beta)} \sum_{s=a+2}^{z-1} \frac{\Gamma(z-s+1-\alpha-\beta)}{\Gamma(z-s+1)} (\nabla f)(s).\end{aligned}$$

Rearranging the above equality for $(\nabla f)(z)$ and by using the assumption $\left({}_{a+1}^{\mathrm{RL}}\nabla^{\alpha}\, {}_{a}^{\mathrm{RL}}\nabla^{\beta} f\right)(z) \geq 0$ together with the changing of the variable $z = a + k$, for $k \in \mathbb{N}_3$, the proof can be obtained. □

3. Monotonicity results

This section is divided in to two parts of delta and nabla monotonicity results.

3.1. Delta monotonicity results

We proceed by investigating a delta formula on the set $\mathscr{D}_1$ defined in (1.2), in the following lemma.

Theorem 3.1. *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given. Then*

(i) for each $(\alpha, \beta) \in \mathscr{D}_1 \setminus \mathscr{D}_2$, if

(1) $\left({}_{a+1-\beta}^{\quad\mathrm{RL}}\Delta^{\alpha}\ {}_{a}^{\mathrm{RL}}\Delta^{\beta}f\right)(z) \geq 0$*, for* $z \in \mathbb{N}_{a+2-\alpha-\beta}$*;*
(2) $(\Delta f)(a) \geq 0$*; and*
(3) $f(a) \geq 0$*;*

then f *is increasing on* $\mathbb{N}_a$*; and*

(ii) for each $(\alpha,\beta) \in \mathscr{D}_1 \cap \mathscr{D}_2$ *there exists a discrete function* f *defined on* $\mathbb{N}_a$ *which satisfies conditions (1)–(3) of part (i), but it is not monotone increasing on* $\mathbb{N}_{a+1}$*.*

Proof. Considering Lemma 2.1 and the identity (1.4), we see that

$$\begin{aligned}\left({}_{a+1-\beta}^{\quad\mathrm{RL}}\Delta^{\alpha}\ {}_{a}^{\mathrm{RL}}\Delta^{\beta}f\right)(z) &= \frac{1}{\Gamma(-\alpha-\beta)}\sum_{s=a}^{z+\alpha+\beta}(z-1-s)^{\underline{-\alpha-\beta-1}}f(s) - \frac{(z+\beta-1-a)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)}f(a)\\ &= \frac{1}{\Gamma(-\alpha-\beta+1)}\sum_{s=a}^{z+\alpha+\beta}\Delta_z(z-1-s)^{\underline{-\alpha-\beta}}f(s) - \frac{(z+\beta-1-a)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)}f(a)\\ &= \frac{1}{\Gamma(-\alpha-\beta+1)}\left[(z-a)^{\underline{-\alpha-\beta}}f(a) + \sum_{s=a}^{z+\alpha+\beta-1}(z-1-s)^{\underline{-\alpha-\beta}}(\Delta f)(s)\right]\\ &\quad - \frac{(z+\beta-1-a)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)}f(a), \end{aligned} \tag{3.1}$$

where we have used the notion that $(-\alpha-\beta-1)^{\underline{-\alpha-\beta}} = 0$. Now, using the assumption that $\left({}_{a+1-\beta}^{\quad\mathrm{RL}}\Delta^{\alpha}\ {}_{a}^{\mathrm{RL}}\Delta^{\beta}f\right)(z) \geq 0$, we can write

$$\begin{aligned}0 \leq \left({}_{a+1-\beta}^{\quad\mathrm{RL}}\Delta^{\alpha}\ {}_{a}^{\mathrm{RL}}\Delta^{\beta}f\right)(z) &= \frac{1}{\Gamma(-\alpha-\beta+1)}\left[(z-a)^{\underline{-\alpha-\beta}}f(a) + \sum_{s=a}^{z+\alpha+\beta-2}(z-1-s)^{\underline{-\alpha-\beta}}(\Delta f)(s)\right]\\ &\quad - \frac{(z+\beta-1-a)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)}f(a) + (\Delta f)(z+\alpha+\beta-1), \end{aligned} \tag{3.2}$$

for each $z \in \mathbb{N}_{a+2-\alpha-\beta}$. Whereupon solving (3.2) for $(\Delta f)(z+\alpha+\beta-1)$ yields the inequality

$$\begin{aligned}(\Delta f)(z+\alpha+\beta-1) &\geq -\frac{1}{\Gamma(-\alpha-\beta+1)}\sum_{s=a}^{z+\alpha+\beta-2}(z-1-s)^{\underline{-\alpha-\beta}}(\Delta f)(s)\\ &\quad + \left[\frac{(z+\beta-1-a)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)} - \frac{(z-a)^{\underline{-\alpha-\beta}}}{\Gamma(-\alpha-\beta+1)}\right]f(a). \end{aligned} \tag{3.3}$$

Now, we first observe that

$$-\frac{1}{\Gamma(-\alpha-\beta+1)}(z-1-s)^{\underline{-\alpha-\beta}} = -\frac{\Gamma(z-s)}{\underbrace{\Gamma(-\beta-\alpha+1)}_{<0}\Gamma(z-s+\alpha+\beta)} > 0,$$

for each $s \in \mathbb{N}_a^{z+\alpha+\beta-2}$ and $z \in \mathbb{N}_{a+2-\alpha-\beta}$. This means that if $(\Delta f)(s) \geq 0$ for $s \in \mathbb{N}_a^{z+\alpha+\beta-2}$, then it will follow that

$$-\frac{1}{\Gamma(-\alpha-\beta+1)}\sum_{s=a}^{z+\alpha+\beta-2}(z-1-s)^{\underline{-\alpha-\beta}}(\Delta f)(s) \geq 0.$$

At the same time, we notice that

$$\begin{aligned}&\frac{(z+\beta-1-a)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)}-\frac{(z-a)^{\underline{-\alpha-\beta}}}{\Gamma(-\alpha-\beta+1)}\\&=\frac{\Gamma(z+\beta-a)}{\Gamma(-\alpha)\Gamma(z+\alpha+\beta+1-a)}-\frac{\Gamma(z+1-a)}{\Gamma(-\alpha-\beta+1)\Gamma(z+1+\alpha+\beta-a)}.\end{aligned}\tag{3.4}$$

Next, we will show that

$$\frac{\Gamma(z+\beta-a)}{\Gamma(-\alpha)\Gamma(z+\alpha+\beta+1-a)}-\frac{\Gamma(z+1-a)}{\Gamma(-\alpha-\beta+1)\Gamma(z+1+\alpha+\beta-a)}\geq 0.\tag{3.5}$$

It is known that (3.5) holds iff

$$\frac{\Gamma(z+\beta-a)}{\Gamma(-\alpha)}\geq\frac{\Gamma(z+1-a)}{\Gamma(-\alpha-\beta+1)}.$$

Changing the variable $z := x-\alpha-\beta-1$, it follows that

$$\frac{\Gamma(x-\alpha-a-1)}{\Gamma(-\alpha)}\geq\frac{\Gamma(x-\alpha-\beta-a)}{\Gamma(-\alpha-\beta+1)},$$

for $x\in\mathbb{N}_{a+3}$, and the last inequality holds if and only if

$$\prod_{\ell=0}^{x-a-2}(-\alpha+\ell)\geq\prod_{\ell=0}^{x-a-2}(-\beta-\alpha+1+\ell).\tag{3.6}$$

We proceed the proof of (3.6) by using mathematical induction.

By using (3.6) with $x=a+3$, we have the base case as follows

$$(1-\alpha)(-\alpha)\geq(2-\beta-\alpha)(1-\beta-\alpha),$$

which can be simplified to the inequality

$$\beta^2+(2\alpha-3)\beta+(2-2\alpha)\leq 0.\tag{3.7}$$

Let us fix $\alpha\in(0,1)$. Then the left-hand side of (3.7) is equal to 0 if and only if

$$\beta=\begin{cases}\frac{(3-2\alpha)\pm(2\alpha-1)}{2}, & \alpha\geq 0.5\\ \frac{(3-2\alpha)\pm(1-2\alpha)}{2}, & \alpha<0.5\end{cases}.$$

Define the functions $\beta_{\alpha^-}\ :\ [0,1]\to\mathbb{R}$ and $\beta_{\alpha^+}\ :\ [0,1]\to\mathbb{R}$ by

$$\beta_{\alpha^-}(\alpha):=\begin{cases}1, & 0\leq\alpha<\frac{1}{2}\\ 2(1-\alpha), & \frac{1}{2}\leq\alpha\leq 1\end{cases}$$

and

$$\beta_{\alpha^+}(\alpha):=\begin{cases}2(1-\alpha), & 0\leq\alpha<\frac{1}{2}\\ 1, & \frac{1}{2}\leq\alpha\leq 1\end{cases}.$$

Now, it follows that inequality (3.7) will hold such that $\beta \in (\beta_{\alpha^-}(\alpha), \beta_{\alpha^+}(\alpha))$ because the resulting coefficient of the polynomial $\beta \mapsto \beta^2 + (2\alpha - 3)\beta + (2 - 2\alpha)$ is positive for fixed α. Further, this region is shaded dark grey in Figures 2 and 3 graphically.

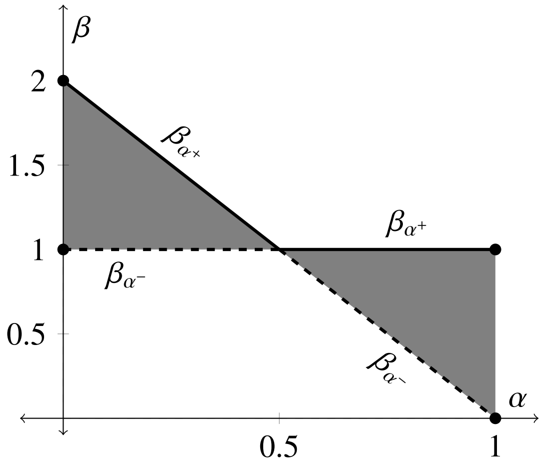

Figure 2. Plots of the functions β_{α^-} and β_{α^+}.

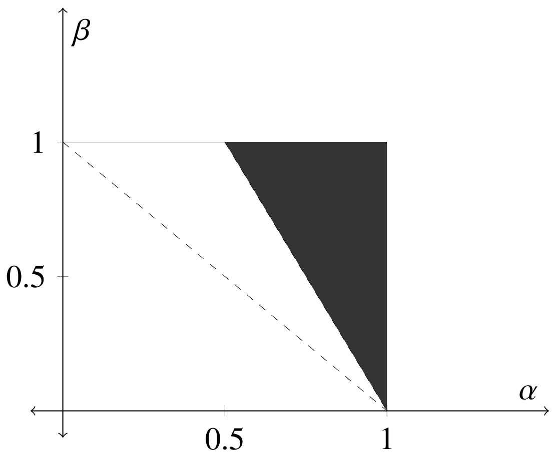

Figure 3. Plot of the region of $\mathscr{D}_1 \setminus \mathscr{D}_2$.

It worth mentioning that this admissible region is the set $\mathscr{D}_1 \setminus \mathscr{D}_2$ exactly. Therefore, the inequality (3.5) is true for $z = a + 2 - \alpha - \beta$ such that $(\alpha, \beta) \in \mathscr{D}_1 \setminus \mathscr{D}_2$ by assumption.

Next, observing that

$$p - \beta - \alpha \geq p - 1 - \alpha,$$

for each $p \in \mathbb{N}_3$. Then by making the use of

$$\max\{(1 - \alpha)(-\alpha), (2 - \beta - \alpha)(1 - \beta - \alpha)\} < 0,$$

it follows that if inequality (3.7) holds, then

$$(1 - \alpha)(-\alpha) \prod_{\ell=2}^{2+m} (\ell - \alpha) \geq (2 - \beta - \alpha)(1 - \beta - \alpha) \prod_{\ell=3}^{3+m} (\ell - \beta - \alpha),$$

for any $m \in \mathbb{N}_3$ by the same method used in the proof of [33, Theorem 2.5]. Therefore, by inducting on ℓ in (3.6) we can establish (3.5) for all $z \in \mathbb{N}_{a+2-\alpha-\beta}$ whereas $(\alpha,\beta) \in \mathscr{D}_1 \setminus \mathscr{D}_2$. Finally, we return to the inequality (3.3) and we see that $(\Delta f)(a+1) \geq 0$ such that $f(a)$, $(\Delta f)(a) \geq 0$, which are true by assumptions.

By iterating the inequality (3.3) by the same method as used in many recent papers inductively such as [33–37], we can deduce that $(\Delta f)(z) \geq 0$ for all $z \in \mathbb{N}_a$, as requested. Thus, the part (i) of Theorem 3.1 is proved.

To prove part (i) of Theorem 3.1, we will improve the methodological analysis that was introduced by Goodrich in [38]. Let $(\alpha,\beta) \in \mathscr{D}_1 \cap \mathscr{D}_2$ be fixed but otherwise arbitrary. Without loss of gentility, we will only prove part (ii) for $a = 0$; we will leave the other case when $a \neq 0$ which can be made by a simple modification of the proof in the first case $a = 0$. Now, let us define the function $f : \mathbb{N}_1 \to \mathbb{R}$ by

$$f(z) := \begin{cases} 1, & z \in \mathbb{N}_0^1, \\ 1-\rho, & z = 2, \\ 2^{z-2}, & z \in \mathbb{N}_3, \end{cases}$$

where ρ is a positive constant that will be defined later. It is clear that f satisfies conditions (2) and (3) since $f(0) \geq 0$ and $(\Delta f)(0) = 0 \geq 0$. However, $(\Delta f)(1) = -\rho < 0$, hence f is not increasing on $\mathbb{N}_1$.

Next, we demonstrate that $\left({}^{\mathrm{RL}}_{1-\beta}\Delta^{\alpha}\ {}^{\mathrm{RL}}_{0}\Delta^{\beta} f\right)(z) \geq 0$ for all $z \in \mathbb{N}_{2-\alpha-\beta}$ by fixing $(\alpha,\beta) \in \mathscr{D}_1 \cap \mathscr{D}_2$. To prove this, we proceed iteratively. First, we consider the base case $z = 2-\alpha-\beta$ and in this case, we use (3.1) to get

$$\begin{aligned} \left({}^{\mathrm{RL}}_{1-\beta}\Delta^{\alpha}\ {}^{\mathrm{RL}}_{0}\Delta^{\beta} f\right)(2-\alpha-\beta) &= \frac{1}{\Gamma(-\alpha-\beta)}\sum_{s=0}^{2}(1-\alpha-\beta-s)^{\underline{-\alpha-\beta-1}}f(s) - \frac{(1-\alpha)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)}f(0) \\ &= \left[\frac{\Gamma(2-\beta-\alpha)}{2\Gamma(-\beta-\alpha)}f(0) + \frac{\Gamma(1-\beta-\alpha)}{\Gamma(-\beta-\alpha)}f(1) + \frac{\Gamma(-\beta-\alpha)}{\Gamma(-\beta-\alpha)}f(2)\right] + \frac{1}{2}(\alpha)(1-\alpha) \\ &= \underbrace{\left[\frac{1}{2}\beta^2 + \left(\alpha - \frac{3}{2}\right)\beta + (-\alpha+1)\right]}_{=:\Upsilon(\alpha,\beta)} - \rho, \end{aligned} \tag{3.8}$$

where we have defined the function $\Upsilon : (0,1)\times(0,1) \to \mathbb{R}$ by

$$\Upsilon(\alpha,\beta) = \frac{1}{2}\beta^2 + \left(\alpha - \frac{3}{2}\right)\beta + (-\alpha+1).$$

For fixed α it follows that $\Upsilon(\alpha,\beta) = 0$ if and only if

$$\beta = \left(-\alpha + \frac{3}{2}\right) \pm \left|\alpha - \frac{1}{2}\right|.$$

Define the functions $\widehat{\beta}_{\alpha^-} : [0,1] \to \mathbb{R}$ and $\widehat{\beta}_{\alpha^+} : [0,1] \to \mathbb{R}$ by

$$\widehat{\beta}_{\alpha^-}(\alpha) := \left(-\alpha + \frac{3}{2}\right) - \left|\alpha - \frac{1}{2}\right|$$

and

$$\widehat{\beta}_{\alpha^+}(\alpha) := \left(-\alpha + \frac{3}{2}\right) + \left|\alpha - \frac{1}{2}\right|.$$

Then $\Upsilon(\alpha,\beta) \geq 0$ if and only if $\beta \in (-\infty, \widehat{\beta}_{\alpha^-}(\alpha)) \cup (\widehat{\beta}_{\alpha^+}(\alpha), +\infty)$. But since $(\alpha,\beta) \in \mathscr{D}_1 \cap \mathscr{D}_2$ implies that $\beta \in (-\infty, \widehat{\beta}_{\alpha^-}(\alpha)) \cup (\widehat{\beta}_{\alpha^+}(\alpha), +\infty)$, it follows that $\Upsilon(\alpha,\beta) \geq 0$ whenever $(\alpha,\beta) \in \mathscr{D}_1 \cap \mathscr{D}_2$. Therefore, we conclude from (3.8) that for each $(\alpha,\beta) \in \mathscr{D}_1 \cap \mathscr{D}_2$ we have that

$$\left({}^{\text{RL}}_{1-\beta}\Delta^{\alpha}\ {}^{\text{RL}}_{0}\Delta^{\beta} f\right)(2-\alpha-\beta) \geq 0,$$

which completes the proof of the initial case. Indeed, to prove $\left({}^{\text{RL}}_{1-\beta}\Delta^{\alpha}\ {}^{\text{RL}}_{0}\Delta^{\beta} f\right)(2-\alpha-\beta) \geq 0$, we must be able to fix $\rho := \rho(\alpha,\beta)$ in such a way that the inequality is allowed. Nevertheless, it can always be proved with the apparent choice of

$$\rho := \frac{1}{2}\Upsilon(\alpha,\beta).$$

Since for $(\alpha,\beta) \in \mathscr{D}_1 \cap \mathscr{D}_2$ it holds that $\Upsilon(\alpha,\beta) > 0$, then we will obtain $\rho > 0$ and $\Upsilon(\alpha,\beta) - \rho > 0$ correspondingly.

To proceed with the iteration, we first consider the case when $z = 3-\alpha-\beta$ and then we see that

$$\begin{aligned}
\left({}^{\text{RL}}_{1-\beta}\Delta^{\alpha}\ {}^{\text{RL}}_{0}\Delta^{\beta} f\right)(3-\alpha-\beta) &= \frac{1}{\Gamma(-\alpha-\beta)}\sum_{s=0}^{3}(2-\alpha-\beta-s)^{\underline{-\alpha-\beta-1}} f(s) \underbrace{- \frac{(2-\alpha)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)} f(0)}_{\geq 0} \\
&\geq f(3) + (-\beta-\alpha) f(2) \\
&\geq (-2)(1-\rho) + 2 \\
&= 2\rho \\
&\geq 0.
\end{aligned} \tag{3.9}$$

Similarly, we see that for any $z \in \mathbb{N}_{4-\alpha-\beta}$ we may compute

$$\begin{aligned}
\left({}^{\text{RL}}_{1-\beta}\Delta^{\alpha}\ {}^{\text{RL}}_{0}\Delta^{\beta} f\right)(z) &= \frac{1}{\Gamma(-\alpha-\beta)}\sum_{s=0}^{z+\alpha+\beta}(z-1-s)^{\underline{-\alpha-\beta-1}} f(s) \underbrace{- \frac{(z+\beta-1)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)} f(0)}_{\geq 0} \\
&\geq \frac{1}{\Gamma(-\alpha-\beta)}\sum_{s=0}^{z+\alpha+\beta}(z-1-s)^{\underline{-\alpha-\beta-1}} f(s) \\
&\geq f(z+\alpha+\beta) + (-\beta-\alpha) f(z+\alpha+\beta-1) \\
&> 2^{z+\alpha+\beta-2} - 2 \cdot 2^{z+\alpha+\beta-3} \\
&= 0.
\end{aligned} \tag{3.10}$$

And upon combining (3.9) and (3.10) we conclude that

$$\left({}^{\text{RL}}_{1-\beta}\Delta^{\alpha}\ {}^{\text{RL}}_{0}\Delta^{\beta} f\right)(z) \geq 0,$$

for each $z \in \mathbb{N}_{4-\alpha-\beta}$. And this completes the proof of part (ii) of the theorem. □

3.2. Nabla monotonicity results

We start with our first theorem considering the monotonicity on the set $\mathbb{N}_{a+2}$.

Theorem 3.2. *Let* $0<\alpha<1$ *and* $0<\beta<1$ *with* $2<\alpha+\beta<3$, *and* $f:\mathbb{N}_a\to\mathbb{R}$ *be given with the following conditions*

(i) $f(a+1)\geq 0$;
(ii) $(\nabla f)(a+2)\geq 0$;
(iii) $\left({}^{\mathrm{RL}}_{a+1}\nabla^{\alpha}\,{}^{\mathrm{RL}}_{a}\nabla^{\beta}f\right)(z)\geq 0$, *for* $z\in\mathbb{N}_{a+3}$;
(v) $(\alpha,\beta)\in\mathscr{D}_1\setminus\mathscr{D}_2$.

Then f *is monotone increasing on* $\mathbb{N}_{a+2}$.

Proof. Considering inequality (2.4), for $k=3$, we have that

$$\begin{aligned}&-\frac{1}{\Gamma(1-\alpha-\beta)}\sum_{\mathrm{j}=0}^{0}\frac{\Gamma(2-\mathrm{j}-\alpha-\beta)}{\Gamma(2-\mathrm{j})}(\nabla f)(a+\mathrm{j}+2)\\&=-\frac{\Gamma(2-\alpha-\beta)}{\Gamma(1-\alpha-\beta)}(\nabla f)(a+2)\\&=-\underbrace{(1-\alpha-\beta)}_{<0}\underbrace{(\nabla f)(a+2)}_{\geq 0\text{ by (ii)}}\geq 0.\end{aligned}$$

This process can be repeated for each $k\in\mathbb{N}_3$ until the following can be deduced

$$-\frac{1}{\Gamma(1-\alpha-\beta)}\sum_{\mathrm{j}=0}^{k-3}\frac{\Gamma(k-\mathrm{j}-1-\alpha-\beta)}{\Gamma(k-\mathrm{j}-1)}(\nabla f)(a+\mathrm{j}+2)\geq 0. \tag{3.11}$$

Now, we will try to show that

$$\frac{\Gamma(k-\alpha-1)}{\Gamma(-\alpha)(k-1)!}-\frac{\Gamma(k-\alpha-\beta)}{\Gamma(1-\alpha-\beta)(k-1)!}\geq 0. \tag{3.12}$$

By induction we can establish (3.12). At first, we consider the case when $k=3$ in which we need to show that

$$\frac{\Gamma(2-\alpha)}{\Gamma(-\alpha)2!}\geq\frac{\Gamma(3-\beta-\alpha)}{\Gamma(1-\beta-\alpha)2!}. \tag{3.13}$$

But (3.13) holds if and only if

$$(-\alpha)(1-\alpha)\geq(1-\beta-\alpha)(2-\beta-\alpha). \tag{3.14}$$

It follows from expanding both sides of (3.14) that

$$-\beta^2-2\beta\alpha+3\beta+2\alpha-2\geq 0. \tag{3.15}$$

It can be observed that (3.15) is trivially hold for $\beta=1$ as $2\alpha-2\alpha\geq 0$. Solving for α in (3.15), for fixed $\beta\in(0,1)$, to get

$$\alpha\geq\frac{\beta^2-3\beta+2}{2(1-\beta)}=1-\frac{\beta}{2}, \tag{3.16}$$

which is true by condition (v). Thus, the basic step for induction exists.

Now, we assume that inequality (3.14) holds in case $k = k_0$ for some integer $k_0 \geq 3$. Then we claim that

$$\frac{\Gamma(3-\alpha+k_0)}{\Gamma(-\alpha)} \geq \frac{\Gamma(4-\beta-\alpha+k_0)}{\Gamma(-\beta-\alpha+1)}. \tag{3.17}$$

First of all, we know that (3.17) is equivalent to the following inequality

$$(k_0-1-\alpha)(k_0-2-\alpha)\cdots(-\alpha) \geq (k_0-\beta-\alpha)(k_0-1-\beta-\alpha)\cdots(1-\beta-\alpha). \tag{3.18}$$

The induction hypothesis tells us that the following inequality holds

$$(k_0-2-\alpha)\cdots(-\alpha) \geq (k_0-1-\beta-\alpha)\cdots(1-\beta-\alpha). \tag{3.19}$$

It is important to observe that

$$\min\{k_0-1-\alpha, k_0-\beta-\alpha\} \geq 0,$$

and

$$\max\{\overbrace{(k_0-2-\alpha)}^{>0}\cdots\overbrace{(-\alpha)}^{<0}, \overbrace{(k_0-1-\beta-\alpha)}^{>0}\cdots\overbrace{(1-\beta-\alpha)}^{<0}\} < 0.$$

Now, we put

$$\begin{aligned} A_0 &:= (k_0-2-\alpha)\cdots(-\alpha) < 0, \\ B_0 &:= (k_0-1-\beta-\alpha)\cdots(1-\beta-\alpha) < 0, \\ z_M &:= k_0-\beta-\alpha > 0, \\ z_m &:= k_0-1-\alpha > 0. \end{aligned}$$

It is clear that $B_0 < A_0 < 0$ and $0 < z_m \leq z_M$ since $\beta \leq 1$ by assumption. Then it follows that $z_m A_0 \geq z_M B_0$. Thus, we can deduce from inequality (3.19) that (3.18) holds. Hence, (3.17) holds, and consequently the inequality (3.12) holds for each $k \in \mathbb{N}_3$ by induction.

Finally, we see from (2.4), for $k = 3$, that

$$\begin{aligned} (\nabla f)(a+3) &\geq \left[\frac{\Gamma(2-a-\alpha)}{2!\Gamma(-\alpha)} - \frac{\Gamma(3-\alpha-\beta)}{2!\Gamma(1-\alpha-\beta)}\right] f(a+1) \\ &\quad - \frac{1}{\Gamma(1-\alpha-\beta)} \sum_{\mathrm{j}=0}^{0} \frac{\Gamma(2-\mathrm{j}-\alpha-\beta)}{\Gamma(2-\mathrm{j})} (\nabla f)(a+\mathrm{j}+2) \\ &\geq 0, \end{aligned}$$

where conditions (i)–(v) from the statement of the theorem were in force. Therefore, by iterating (2.4) inductively together with $(\nabla f)(a+3) \geq 0$, we can deduce that $(\nabla f)(z) \geq 0$ for each $z \in \mathbb{N}_{a+2}$. And the proof is done. □

4. Test examples

In the final section, we give some examples to support our conclusions. These examples show that the case (i) of Theorem 3.1 can occur.

Example 4.1. Let $f : \mathbb{N}_0 \to \mathbb{R}$ be defined by

$$f(z) = z^{\underline{\alpha+\beta}}.$$

Then for $\alpha = 0.7, \beta = 0.8$, and $z = 2 - \alpha - \beta$, we have

$$\begin{aligned}\left({}^{\mathrm{RL}}_{1-\beta}\Delta^{\alpha}\ {}^{\mathrm{RL}}_{0}\Delta^{\beta} f\right)(2-\alpha-\beta) &= \frac{1}{\Gamma(-\alpha-\beta)}\sum_{s=0}^{2}(1-\alpha-\beta-s)^{\underline{-\alpha-\beta-1}} f(s) - \frac{(1-\alpha)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)} f(0) \\ &= 1.2751 \geq 0,\end{aligned}$$

and for $z = 3 - \alpha - \beta$, we have

$$\begin{aligned}\left({}^{\mathrm{RL}}_{1-\beta}\Delta^{\alpha}\ {}^{\mathrm{RL}}_{0}\Delta^{\beta} f\right)(3-\alpha-\beta) &= \frac{1}{\Gamma(-\alpha-\beta)}\sum_{s=0}^{3}(2-\alpha-\beta-s)^{\underline{-\alpha-\beta-1}} f(s) - \frac{(2-\alpha)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)} f(0) \\ &= 1.3095 \geq 0.\end{aligned}$$

Similarly, we can deduce that

$$\left({}^{\mathrm{RL}}_{1-\beta}\Delta^{\alpha}\ {}^{\mathrm{RL}}_{0}\Delta^{\beta} f\right)(z) \geq 0,$$

for each $\mathbb{N}_{a+2-\alpha-\beta}$. Furthermore, $f(0) \geq 0$ and $(\Delta f)(0) = 0.8463 \geq 0$. Hence $z^{\underline{\alpha+\beta}}$ is increasing on $\mathbb{N}_0$ according to Theorem 3.1.

In addition, we have drown the increasing of $z^{\underline{\alpha+\beta}}$ in Figure 4 graphically.

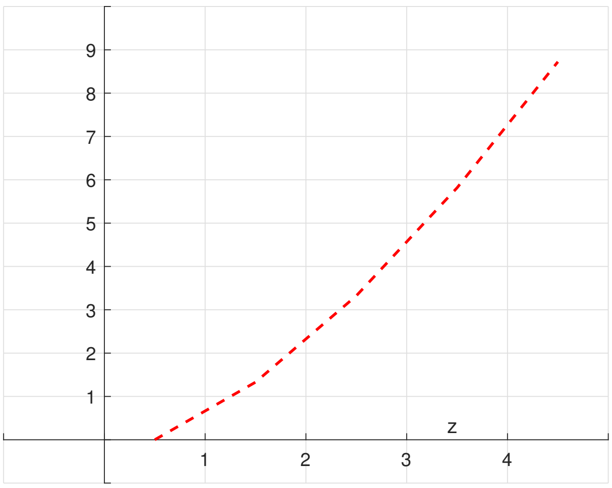

Figure 4. Graph of $f(z)$ in Example 4.1.

Example 4.2. Our last example considers the function $f : \mathbb{N}_0 \to \mathbb{R}$ given by

$$f(z) = \left(\frac{5}{4}\right)^{z-\alpha-\beta}.$$

Note that for $\alpha = \beta = 0.75$ and $z = 2 - \alpha - \beta$, one can have

$$\left({}^{\mathrm{RL}}_{1-\beta}\Delta^{\alpha}\ {}^{\mathrm{RL}}_{0}\Delta^{\beta} f\right)(2-\alpha-\beta) = \frac{1}{\Gamma(-\alpha-\beta)} \sum_{s=0}^{2} (1-\alpha-\beta-s)^{\underline{-\alpha-\beta-1}} f(s) - \frac{(1-\alpha)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)} f(0)$$
$$= 0.1398 \geq 0,$$

and for $z = 3 - \alpha - \beta$, we have

$$\left({}^{\mathrm{RL}}_{1-\beta}\Delta^{\alpha}\ {}^{\mathrm{RL}}_{0}\Delta^{\beta} f\right)(3-\alpha-\beta) = \frac{1}{\Gamma(-\alpha-\beta)} \sum_{s=0}^{3} (2-\alpha-\beta-s)^{\underline{-\alpha-\beta-1}} f(s) - \frac{(2-\alpha)^{\underline{-\alpha-1}}}{\Gamma(-\alpha)} f(0)$$
$$= 0.1607 \geq 0.$$

In similar manner, we can deduce that

$$\left({}^{\mathrm{RL}}_{1-\beta}\Delta^{\alpha}\ {}^{\mathrm{RL}}_{0}\Delta^{\beta} f\right)(z) \geq 0,$$

for each $\mathbb{N}_{a+2-\alpha-\beta}$. Also, $f(0) = 1 \geq 0$ and $(\Delta f)(0) = 0.2236 \geq 0$. Consequently $z^{\underline{\alpha+\beta}}$ will be increasing on $\mathbb{N}_0$ according to Theorem 3.1. Moreover, $\left(\frac{5}{4}\right)^{z-\alpha-\beta}$ is graphically drown in Figure 5.

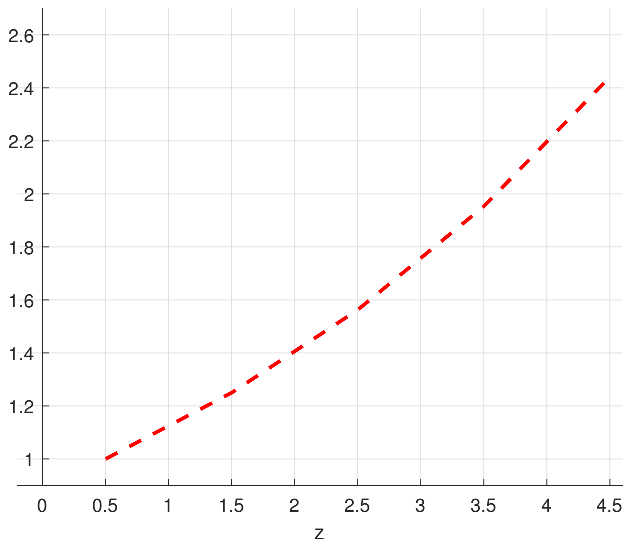

Figure 5. Graph of $f(z)$ in Example 4.2.

Example 4.3. Let $f : \mathbb{N}_0 \to \mathbb{R}$ be defined by $f(z) = z^{\overline{\alpha+\beta}}$. Then for $\alpha = 0.7, \beta = 0.8$, and $z = 3$, we have

$$\left({}^{\mathrm{RL}}_{1}\nabla^{\alpha}\ {}^{\mathrm{RL}}_{0}\nabla^{\beta} f\right)(3) = \frac{1}{\Gamma(-\alpha-\beta)} \sum_{s=1}^{3} (4-s)^{\overline{-\alpha-\beta-1}} f(s) - \frac{(3)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} f(1)$$
$$= 1.4689 \geq 0,$$

and for $z = 4$, we have

$$\left({}^{\mathrm{RL}}_{1}\nabla^{\alpha}\ {}^{\mathrm{RL}}_{0}\nabla^{\beta} f\right)(4) = \frac{1}{\Gamma(-\alpha-\beta)} \sum_{s=1}^{4} (5-s)^{\overline{-\alpha-\beta-1}} f(s) - \frac{(4)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} f(1)$$
$$= 1.3898 \geq 0.$$

Similarly, we can deduce that

$$\left({}^{\mathrm{RL}}_{1}\nabla^{\alpha}\ {}^{\mathrm{RL}}_{0}\nabla^{\beta} f\right)(z) \geq 0,$$

for each $\mathbb{N}_{a+2}$. Furthermore, $f(1) \geq 0$ and $(\nabla f)(2) = 1.9940 \geq 0$. Hence $z^{\overline{\alpha+\beta}}$ is increasing on $\mathbb{N}_2$ according to Theorem 3.2.

5. Conclusions

The Holm theorem (Theorem 2.1) was performed to analyze monotonicity of the discrete sequential operators $\left({}^{\mathrm{RL}}_{a+1-\beta}\Delta^{\alpha}\ {}^{\mathrm{RL}}_{a}\Delta^{\beta} f\right)(z)$ and $\left({}^{\mathrm{RL}}_{a+1}\nabla^{\alpha}\ {}^{\mathrm{RL}}_{a}\nabla^{\beta} f\right)(z)$. The major conclusions are as follows:

- The main idea of the study in (2.1) was recalled generally, and in (2.2) specifically.
- The sets $\mathscr{D}_1$ and $\mathscr{D}_2$ are defined in (1.2) and (1.3) and the discrete sequential operator was defined on these sets.
- The positivity of the difference operator Δ was deduced and conclusively the monotonicity of the function was pointed out.
- A strong connection between the monotonicity of f and the sign of $\left({}^{\mathrm{RL}}_{a+1-\beta}\Delta^{\alpha}\ {}^{\mathrm{RL}}_{a}\Delta^{\beta} f\right)(z)$ was depicted.
- The positivity of the difference operator ∇ was deduced and conclusively the monotonicity of the function was established.
- Again, a strong correlation between the monotonicity of f and the sign of $\left({}^{\mathrm{RL}}_{a+1}\nabla^{\alpha}\ {}^{\mathrm{RL}}_{a}\nabla^{\beta} f\right)(z)$ was depicted.
- Through the test examples, it can be found the effectiveness of our theoretical results.

Conflict of interest

The authors declare that they have no conflicts of interest.

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