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Research article

Newly existence of solutions for pantograph a semipositone in Ψ-Caputo sense

Abdelkader Moumen[1,*], **Hamid Boulares**[2], **Tariq Alraqad**[1], **Hicham Saber**[1] **and Ekram E. Ali**[1]

[1] Department of Mathematics, Faculty of Science, University of Ha'il, Ha'il 55425, Saudi Arabia

[2] Laboratory of Analysis and Control of Differential Equations "ACED", Faculty MISM, Department of Mathematics, University of 8 May 1945 Guelma, P.O. Box 401, 24000 Guelma, Algeria

* **Correspondence:** Email: abdelkader.moumen@gmail.com.

Abstract: In the present manuscript, the BVP problem of a semipostone multipoint Ψ-Caputo fractional pantograph problem is addressed.

$$\mathcal{D}_r^{\gamma;\psi}\varkappa(\varsigma)+\mathcal{F}(\varsigma,\varkappa(\varsigma),\varkappa(r+\lambda\varsigma))=0,\ \varsigma \text{ in } (r,\mathfrak{I}),$$

$$\varkappa(r)=\vartheta_1,\ \varkappa(\mathfrak{I})=\sum_{i=1}^{m-2}\zeta_i\varkappa(\eta_i)+\vartheta_2,\ \vartheta_i\in\mathbb{R},\ i\in\{1,2\},$$

and λ in $\left(0,\frac{\mathfrak{I}-r}{\mathfrak{I}}\right)$. The seriousness of this research is to prove the existence of the solution of this problem by using Schauder's fixed point theorem (SFPT). We have developed our results in our research compared to some recent research in this field. We end our work by listing an example to demonstrate the result reached.

Keywords: Ψ-Caputo derivative; BVP; pantograph problem; changing sign nonlinearity; Schauder fixed point theorem
Mathematics Subject Classification: 34A08, 34B10

1. Introduction

Recently, fractional calculus methods became of great interest, because it is a powerful tool for calculating the derivation of multiples systems. These methods study real world phenomena in many areas of natural sciences including biomedical, radiography, biology, chemistry, and physics [1–7]. Abundant publications focus on the Caputo fractional derivative (CFD) and the Caputo-Hadamard derivative. Additionally, other generalization of the previous derivatives, such as Ψ-Caputo, study the existence of solutions to some FDEs (see [8–14]).

In general, an m-point fractional boundary problem involves a fractional differential equation with fractional boundary conditions that are specified at m different points on the boundary of a domain. The fractional derivative is defined using the Riemann-Liouville fractional derivative or the Caputo fractional derivative. Solving these types of problems can be challenging due to the non-local nature of fractional derivatives. However, there are various numerical and analytical methods available for solving such problems, including the spectral method, the finite difference method, the finite element method, and the homotopy analysis method. The applications of m-point fractional boundary problems can be found in various fields, including physics, engineering, finance, and biology. These problems are useful in modeling and analyzing phenomena that exhibit non-local behavior or involve memory effects (see [15–18]).

Pantograph equations are a set of differential equations that describe the motion of a pantograph, which is a mechanism used for copying and scaling drawings or diagrams. The equations are based on the assumption that the pantograph arms are rigid and do not deform during operation, we can simply say that see [19]. One important application of the pantograph equations is in the field of drafting and technical drawing. Before the advent of computer-aided design (CAD) software, pantographs were commonly used to produce scaled copies of drawings and diagrams. By adjusting the lengths of the arms and the position of the stylus, a pantograph can produce copies that are larger or smaller than the original [20], electrodynamics [21] and electrical pantograph of locomotive [22].

Many authors studied a huge number of positive solutions for nonlinear fractional BVP using fixed point theorems (FPTs) such as SFPT, Leggett-Williams and Guo-Krasnosel'skii (see [23, 24]). Some studies addressed the sign-changing of solution of BVPs [25–29].

In this work, we use Schauder's fixed point theorem (SFPT) to solve the semipostone multipoint Ψ-Caputo fractional pantograph problem

$$\mathcal{D}_r^{\nu;\psi}\varkappa(\varsigma) + \mathcal{F}(\varsigma, \varkappa(\varsigma), \varkappa(r+\lambda\varsigma)) = 0, \ \varsigma \text{ in } (r, \Im) \tag{1.1}$$

$$\varkappa(r) = \vartheta_1, \ \varkappa(\Im) = \sum_{i=1}^{m-2} \zeta_i \varkappa(\eta_i) + \vartheta_2, \ \vartheta_i \in \mathbb{R}, \ i \in \{1, 2\}, \tag{1.2}$$

where $\lambda \in \left(0, \frac{\Im - r}{\Im}\right), \mathcal{D}_r^{\nu;\psi}$ is Ψ-Caputo fractional derivative (Ψ-CFD) of order ν, $1 < \nu \le 2$, $\zeta_i \in \mathbb{R}^+ \ (1 \le i \le m-2)$ such that $0 < \Sigma_{i=1}^{m-2}\zeta_i < 1$, $\eta_i \in (r, \Im)$, and $\mathcal{F} : [r, \Im] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

The most important aspect of this research is to prove the existence of a positive solution of the above m-point FBVP. Note that in [30], the author considered a two-point BVP using Liouville-Caputo derivative.

The article is organized as follows. In the next section, we provide some basic definitions and arguments pertinent to fractional calculus (FC). Section 3 is devoted to proving the the main result and an illustrative example is given in Section 4.

2. Preliminaries

In the sequel, Ψ denotes an increasing map $\Psi : [r_1, r_2] \to \mathbb{R}$ via $\Psi'(\varsigma) \neq 0$, $\forall\ \varsigma$, and $[\alpha]$ indicates the integer part of the real number α.

Definition 2.1. *[4, 5] Suppose the continuous function $\varkappa : (0,\infty) \to \mathbb{R}$. We define (RLFD) the Riemann-Liouville fractional derivative of order $\alpha > 0, n = [\alpha] + 1$ by*

$$ {}^{RL}\mathcal{D}_{0+}^{\alpha}\varkappa(\varsigma) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{d\varsigma}\right)^{n}\int_{0}^{\varsigma}(\varsigma-\tau)^{n-\alpha-1}\varkappa(\tau)d\tau, $$

where $n-1 < \alpha < n$.

Definition 2.2. *[4, 5] The Ψ-Riemann-Liouville fractional integral (Ψ-RLFI) of order $\alpha > 0$ of a continuous function $\varkappa : [r,\Im] \to \mathbb{R}$ is defined by*

$$ \mathcal{I}_{r}^{\alpha;\Psi}\varkappa(\varsigma) = \int_{r}^{\varsigma}\frac{(\Psi(\varsigma)-\Psi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\Psi'(\tau)\varkappa(\tau)d\tau. $$

Definition 2.3. *[4, 5] The CFD of order $\alpha > 0$ of a function $\varkappa : [0,+\infty) \to \mathbb{R}$ is defined by*

$$ \mathcal{D}^{\alpha}\varkappa(\varsigma) = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{\varsigma}(\varsigma-\tau)^{n-\alpha-1}\varkappa^{(n)}(\tau)\,d\tau,\ \alpha\in(n-1,n),\ n\in\mathbb{N}. $$

Definition 2.4. *[4, 5] We define the Ψ-CFD of order $\alpha > 0$ of a continuous function $\varkappa : [r,\Im] \to \mathbb{R}$ by*

$$ \mathcal{D}_{r}^{\alpha;\Psi}\varkappa(\varsigma) = \int_{r}^{\varsigma}\frac{(\Psi(\varsigma)-\Psi(\tau))^{n-\alpha-1}}{\Gamma(n-\alpha)}\Psi'(\tau)\,\partial_{\Psi}^{n}\varkappa(\tau)d\tau,\ \varsigma > r,\ \alpha\in(n-1,n), $$

where $\partial_{\Psi}^{n} = \left(\frac{1}{\Psi'(\varsigma)}\frac{d}{d\varsigma}\right)^{n}$, $n\in\mathbb{N}$.

Lemma 2.1. *[4, 5] Suppose $q,\ell > 0$, and $\varkappa$ in $C([r,\Im],\mathbb{R})$. Then $\forall\varsigma\in[r,\Im]$ and by assuming $F_{r}(\varsigma) = \Psi(\varsigma)-\Psi(r)$, we have*

1) $\mathcal{I}_{r}^{q;\Psi}\mathcal{I}_{r}^{\ell;\Psi}\varkappa(\varsigma) = \mathcal{I}_{r}^{q+\ell;\Psi}\varkappa(\varsigma)$,
2) $\mathcal{D}_{r}^{q;\Psi}\mathcal{I}_{r}^{q;\Psi}\varkappa(\varsigma) = \varkappa(\varsigma)$,
3) $\mathcal{I}_{r}^{q;\Psi}(F_{r}(\varsigma))^{\ell-1} = \dfrac{\Gamma(\ell)}{\Gamma(\ell+q)}(F_{r}(\varsigma))^{\ell+q-1}$,
4) $\mathcal{D}_{r}^{q;\Psi}(F_{r}(\varsigma))^{\ell-1} = \dfrac{\Gamma(\ell)}{\Gamma(\ell-q)}(F_{r}(\varsigma))^{\ell-q-1}$,
5) $\mathcal{D}_{r}^{q;\Psi}(F_{r}(\varsigma))^{k} = 0,\ k = 0,\ldots,n-1,\ n\in\mathbb{N},\ q$ *in* $(n-1,n]$.

Lemma 2.2. *[4, 5] Let $n-1 < \alpha_1 \le n, \alpha_2 > 0,\ r > 0,\ \varkappa\in L(r,\Im),\ \mathcal{D}_{r}^{\alpha_1;\Psi}\varkappa\in L(r,\Im)$. Then the differential equation*

$$ \mathcal{D}_{r}^{\alpha_1;\Psi}\varkappa = 0 $$

has the unique solution

$$ \varkappa(\varsigma) = \mathcal{W}_0 + \mathcal{W}_1(\Psi(\varsigma)-\Psi(r)) + \mathcal{W}_2(\Psi(\varsigma)-\Psi(r))^2 + \cdots + \mathcal{W}_{n-1}(\Psi(\varsigma)-\Psi(r))^{n-1}, $$

and

$$ \mathcal{I}_{r}^{\alpha_1;\Psi}\mathcal{D}_{r}^{\alpha_1;\Psi}\varkappa(\varsigma) = \varkappa(\varsigma) + \mathcal{W}_0 + \mathcal{W}_1(\Psi(\varsigma)-\Psi(r)) + \mathcal{W}_2(\Psi(\varsigma)-\Psi(r))^2 + \cdots + \mathcal{W}_{n-1}(\Psi(\varsigma)-\Psi(r))^{n-1}, $$

with $\mathcal{W}_\ell \in \mathbb{R},\ \ell \in \{0,1,\ldots,n-1\}$.

Furthermore,

$$\mathcal{D}_r^{\alpha_1;\Psi}\mathcal{I}_r^{\alpha_1;\Psi}\varkappa(\varsigma)=\varkappa(\varsigma),$$

and

$$\mathcal{I}_r^{\alpha_1;\Psi}\mathcal{I}_r^{\alpha_2;\Psi}\varkappa(\varsigma)=\mathcal{I}_r^{\alpha_2;\Psi}\mathcal{I}_r^{\alpha_1;\Psi}\varkappa(\varsigma)=\mathcal{I}_r^{\alpha_1+\alpha_2;\Psi}\varkappa(\varsigma).$$

Here we will deal with the FDE solution of (1.1) and (1.2), by considering the solution of

$$-\mathcal{D}_r^{\nu;\psi}\varkappa(\varsigma)=h(\varsigma), \tag{2.1}$$

bounded by the condition (1.2). We set

$$\Delta := \Psi(\Im)-\Psi(r)-\Sigma_{i=1}^{m-2}\zeta_i\left(\Psi(\eta_i)-\Psi(r)\right).$$

Lemma 2.3. *Let* $\nu\in(1,2]$ *and* $\varsigma\in[r,\Im]$. *Then, the FBVP (2.1) and (1.2) have a solution* $\varkappa$ *of the form*

$$\varkappa(\varsigma)=\left[1+\frac{\Sigma_{i=1}^{m-2}\zeta_i-1}{\Delta}\left(\Psi(\varsigma)-\Psi(r)\right)\right]\vartheta_1+\frac{\Psi(\varsigma)-\Psi(r)}{\Delta}\vartheta_2+\int_r^{\Im}\varpi(\varsigma,\tau)\mathrm{h}(\tau)\Psi'(\tau)\,d\tau,$$

where

$$\varpi(\varsigma,\tau)=\frac{1}{\Gamma(\nu)}\begin{cases}\left[(\Psi(\Im)-\Psi(r))^{\nu-1}-\Sigma_{j=i}^{m-2}\zeta_j(\Psi\left(\eta_j\right)-\Psi(\tau))^{\nu-1}\right]\frac{\Psi(\varsigma)-\Psi(r)}{\Delta}\\ -(\Psi(\varsigma)-\Psi(\tau))^{\nu-1},\ \tau\le\varsigma,\ \eta_{i-1}<\tau\le\eta_i,\\ \left[(\Psi(\Im)-\Psi(\tau))^{\nu-1}-\Sigma_{j=i}^{m-2}\zeta_j(\Psi\left(\eta_j\right)-\Psi(\tau))^{\nu-1}\right]\frac{\Psi(\Im)-\Psi(r)}{\Delta},\\ \varsigma\le\tau,\ \eta_{i-1}<\tau\le\eta_i,\end{cases} \tag{2.2}$$

$i=1,2,...,m-2.$

Proof. According to the Lemma 2.2 the solution of $\mathcal{D}_r^{\nu;\psi}\varkappa(\varsigma)=-h(\varsigma)$ is given by

$$\varkappa(\varsigma)=-\frac{1}{\Gamma(\nu)}\int_r^{\varsigma}(\Psi(\varsigma)-\Psi(\tau))^{\nu-1}h(\tau)\Psi'(\tau)\,d\tau+c_0+c_1\left(\Psi(\varsigma)-\Psi(r)\right), \tag{2.3}$$

where $c_0,c_1\in\mathbb{R}$. Since $\varkappa(r)=\vartheta_1$ and $\varkappa(\Im)=\sum_{i=1}^{m-2}\zeta_i\varkappa(\eta_i)+\vartheta_2$, we get $c_0=\vartheta_1$ and

$$\begin{aligned}c_1 &= \frac{1}{\Delta}\Bigg(-\frac{1}{\Gamma(\nu)}\sum_{i=1}^{m-2}\zeta_i\int_r^{\eta_j}(\Psi(\eta_i)-\Psi(\tau))^{\nu-1}h(\tau)\Psi'(\tau)\,d\tau\\ &\quad+\frac{1}{\Gamma(\nu)}\int_r^{\Im}(\Psi(\Im)-\Psi(\tau))^{\nu-1}h(\tau)\Psi'(\tau)\,d\tau+\vartheta_1\left[\sum_{i=1}^{m-2}\zeta_i-1\right]+\vartheta_2\Bigg).\end{aligned}$$

By substituting c_0,c_1 into Eq (2.3) we find,

$$\begin{aligned}\varkappa(\varsigma) &= \left[1+\frac{\Sigma_{i=1}^{m-2}\zeta_i-1}{\Delta}\left(\Psi(\varsigma)-\Psi(r)\right)\right]\vartheta_1+\frac{(\Psi(\varsigma)-\Psi(r))}{\Delta}\vartheta_2\\ &\quad-\frac{1}{\Gamma(\nu)}\Bigg(\int_r^{\varsigma}(\Psi(\varsigma)-\Psi(\tau))^{\nu-1}h(\tau)\Psi'(\tau)\,d\tau\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\Psi(\varsigma)-\Psi(r))}{\Delta}\sum_{i=1}^{m-2}\zeta_i\int_r^{\eta_j}(\Psi(\eta_i)-\Psi(\tau))^{\nu-1}h(\tau)\Psi'(\tau)\,d\tau\\
&-\frac{\Psi(\varsigma)-\Psi(r)}{\Delta}\int_r^{\Im}(\Psi(\Im)-\Psi(\tau))^{\nu-1}h(\tau)\Psi'(\tau)\,d\tau\Big)\\
&=\left[1+\frac{\Sigma_{i=1}^{m-2}\zeta_i-1}{\Delta}(\Psi(\varsigma)-\Psi(r))\right]\vartheta_1+\frac{(\Psi(\varsigma)-\Psi(r))}{\Delta}\vartheta_2+\int_r^{\Im}\varpi(\varsigma,\tau)h(\tau)\Psi'(\tau)\,d\tau,
\end{aligned}
$$

where $\varpi(\varsigma,\tau)$ is given by (2.2). Hence the required result. □

Lemma 2.4. *If* $0<\sum_{i=1}^{m-2}\zeta_i<1$, *then*
i) $\Delta>0$,
ii) $(\Psi(\Im)-\Psi(\tau))^{\nu-1}-\sum_{j=i}^{m-2}\zeta_j(\Psi\left(\eta_j\right)-\Psi(\tau))^{\nu-1}>0$.

Proof. *i*) Since $\eta_i<\Im$, we have

$$\zeta_i(\Psi(\eta_i)-\Psi(r))<\zeta_i(\Psi(\Im)-\Psi(r)),$$

$$-\sum_{i=1}^{m-2}\zeta_i(\Psi(\eta_i)-\Psi(r))>-\sum_{i=1}^{m-2}\zeta_i(\Psi(\Im)-\Psi(r)),$$

$$
\begin{aligned}
\Psi(\Im)-\Psi(r)-\sum_{i=1}^{m-2}\zeta_i(\Psi(\eta_i)-\Psi(r))&>\Psi(\Im)-\Psi(r)-\sum_{i=1}^{m-2}\zeta_i(\Psi(\Im)-\Psi(r))\\
&=(\Psi(\Im)-\Psi(r))[1-\sum_{i=1}^{m-2}\zeta_i].
\end{aligned}
$$

If $1-\Sigma_{i=1}^{m-2}\zeta_i>0$, then $(\Psi(\Im)-\Psi(r))-\Sigma_{i=1}^{m-2}\zeta_i(\Psi(\eta_i)-\Psi(r))>0$. So we have $\Delta>0$.

ii) Since $0<\nu-1\le 1$, we have $(\Psi(\eta_i)-\Psi(\tau))^{\nu-1}<(\Psi(\Im)-\Psi(\tau))^{\nu-1}$. Then we obtain

$$
\begin{aligned}
\sum_{j=i}^{m-2}\zeta_j(\Psi\left(\eta_j\right)-\Psi(\tau))^{\nu-1}<\sum_{j=i}^{m-2}\zeta_j(\Psi(\Im)-\Psi(\tau))^{\nu-1}&\le(\Psi(\Im)-\Psi(\tau))^{\nu-1}\sum_{i=1}^{m-2}\zeta_i\\
&<(\Psi(\Im)-\Psi(\tau))^{\nu-1},
\end{aligned}
$$

and so

$$(\Psi(\Im)-\Psi(\tau))^{\nu-1}-\sum_{j=i}^{m-2}\zeta_j(\Psi\left(\eta_j\right)-\Psi(\tau))^{\nu-1}>0.$$

□

Remark 2.1. *Note that* $\int_r^{\Im}\varpi(\varsigma,\tau)\Psi'(\tau)\,d\tau$ *is bounded* $\forall\varsigma\in[r,\Im]$. *Indeed*

$$
\begin{aligned}
&\int_r^{\Im}|\varpi(\varsigma,\tau)|\,\Psi'(\tau)\,d\tau\\
&\le\frac{1}{\Gamma(\nu)}\int_r^{\varsigma}(\Psi(\varsigma)-\Psi(\tau))^{\nu-1}\Psi'(\tau)\,d\tau+\frac{\Psi(\varsigma)-\Psi(r)}{\Gamma(\nu)\Delta}\sum_{i=1}^{m-2}\zeta_i\int_r^{\eta_i}(\Psi\left(\eta_j\right)-\Psi(\tau))^{\nu-1}\Psi'(\tau)\,d\tau
\end{aligned}
$$

$$+\frac{\Psi(\varsigma)-\Psi(r)}{\Delta\Gamma(\nu)}\int_r^{\Im}(\Psi(\Im)-\Psi(\tau))^{\nu-1}\Psi'(\tau)\,d\tau$$

$$=\frac{(\Psi(\varsigma)-\Psi(r))^{\nu}}{\Gamma(\nu+1)}+\frac{\Psi(\varsigma)-\Psi(r)}{\Delta\Gamma(\nu+1)}\sum_{i=1}^{m-2}\zeta_i(\Psi(\eta_i)-\Psi(r))^{\nu}+\frac{\Psi(\varsigma)-\Psi(r)}{\Delta\Gamma(\nu+1)}(\Psi(\Im)-\Psi(r))^{\nu}$$

$$\leq\frac{(\Psi(\Im)-\Psi(r))^{\nu}}{\Gamma(\nu+1)}+\frac{\Psi(\Im)-\Psi(r)}{\Delta\Gamma(\nu+1)}\sum_{i=1}^{m-2}\zeta_i(\Psi(\eta_i)-\Psi(r))^{\nu}+\frac{(\Psi(\Im)-\Psi(r))^{\nu+1}}{\Delta\Gamma(\nu+1)}=M. \tag{2.4}$$

Remark 2.2. *Suppose $\Upsilon(\varsigma)\in L^1[r,\Im]$, and $w(\varsigma)$ verify*

$$\begin{cases} \mathcal{D}_r^{\nu;\psi}w(\varsigma)+\Upsilon(\varsigma)=0, \\ w(r)=0,\ w(\Im)=\Sigma_{i=1}^{m-2}\zeta_i w(\eta_i), \end{cases} \tag{2.5}$$

then $w(\varsigma)=\int_r^{\Im}\varpi(\varsigma,\tau)\Upsilon(\tau)\Psi'(\tau)\,d\tau$.

Next we recall the Schauder fixed point theorem.

Theorem 2.1. *[23] [SFPT] Consider the Banach space Ω. Assume $\aleph$ bounded, convex, closed subset in Ω. If $F:\aleph\to\aleph$ is compact, then it has a fixed point in $\aleph$.*

3. Existence result

We start this section by listing two conditions which will be used in the sequel.

- ($\Sigma1$) There exists a nonnegative function $\Upsilon\in L^1[r,\Im]$ such that $\int_r^{\Im}\Upsilon(\varsigma)d\varsigma>0$ and $\mathcal{F}(\varsigma,\varkappa,v)\geq-\Upsilon(\varsigma)$ for all $(\varsigma,\varkappa,v)\in[r,\Im]\times\mathbb{R}\times\mathbb{R}$.
- ($\Sigma2$) $\mathcal{G}(\varsigma,\varkappa,v)\neq0$, for $(\varsigma,\varkappa,v)\in[r,\Im]\times\mathbb{R}\times\mathbb{R}$.

Let $\aleph=C([r,\Im],\mathbb{R})$ the Banach space of CFs (continuous functions) with the following norm

$$\|\varkappa\|=\sup\{|\varkappa(\varsigma)|:\varsigma\in[r,\Im]\}.$$

First of all, it seems that the FDE below is valid

$$\mathcal{D}_r^{\nu;\psi}\varkappa(\varsigma)+\mathcal{G}(\varsigma,\varkappa^*(\varsigma),\varkappa^*(r+\lambda\varsigma))=0,\ \varsigma\in[r,\Im]. \tag{3.1}$$

Here the existence of solution satisfying the condition (1.2), such that $\mathcal{G}:[r,\Im]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$

$$\mathcal{G}(\varsigma,z_1,z_2)=\begin{cases} \mathcal{F}(\varsigma,z_1,z_2)+\Upsilon(\varsigma),\ z_1,z_2\geq0, \\ \mathcal{F}(\varsigma,0,0)+\Upsilon(\varsigma),\ z_1\leq0 \text{ or } z_2\leq0, \end{cases} \tag{3.2}$$

and $\varkappa^*(\varsigma)=\max\{(\varkappa-w)(\varsigma),0\}$, hence the problem (2.5) has w as unique solution. The mapping $Q:\aleph\to\aleph$ accompanied with the (3.1) and (1.2) defined as

$$(Q\varkappa)(\varsigma)=\left[1+\frac{\Sigma_{i=1}^{m-2}\zeta_i-1}{\Delta}(\Psi(\varsigma)-\Psi(r))\right]\vartheta_1+\frac{\Psi(\varsigma)-\Psi(r)}{\Delta}\vartheta_2$$

$$+\int_r^{\Im} \varpi(\varsigma,\tau)\mathcal{G}(\varsigma,\varkappa^*(\tau),\varkappa^*(r+\lambda\tau))\Psi'(\tau)\,d\tau, \tag{3.3}$$

where the relation (2.2) define $\varpi(\varsigma,\tau)$. The existence of solution of the problems (3.1) and (1.2) give the existence of a fixed point for Q.

Theorem 3.1. *Suppose the conditions* $(\Sigma 1)$ *and* $(\Sigma 2)$ *hold. If there exists* $\rho > 0$ *such that*

$$\left[1+\frac{\Sigma_{i=1}^{m-2}\zeta_i - 1}{\Delta}(\Psi(\Im)-\Psi(r))\right]\vartheta_1 + \frac{\Psi(\Im)-\Psi(r)}{\Delta}\vartheta_2 + LM \le \rho,$$

where $L \ge \max\{|\mathcal{G}(\varsigma,\varkappa,v)| : \varsigma \in [r,\Im], |\varkappa|, |v| \le \rho\}$ *and* M *is defined in (2.4), then, the problems (3.1) and (3.2) have a solution* $\varkappa(\varsigma)$.

Proof. Since $P := \{\varkappa \in \aleph : \|\varkappa\| \le \rho\}$ is a convex, closed and bounded subset of B described in the Eq (3.3), the SFPT is applicable to P. Define $Q : P \to \aleph$ by (3.3). Clearly Q is continuous mapping. We claim that range of Q is subset of P. Suppose $\varkappa \in P$ and let $\varkappa^*(\varsigma) \le \varkappa(\varsigma) \le \rho$, $\forall \varsigma \in [r,\Im]$. So

$$\begin{aligned}|Q\varkappa(\varsigma)| &= \Bigg|\left[1+\frac{\Sigma_{i=1}^{m-2}\zeta_i - 1}{\Delta}(\Psi(\varsigma)-\Psi(r))\right]\vartheta_1 + \frac{\Psi(\varsigma)-\Psi(r)}{\Delta}\vartheta_2 \\ &\quad + \int_r^{\Im} \varpi(\varsigma,\tau)\mathcal{G}(\tau,\varkappa^*(\tau),\varkappa^*(r+\lambda\tau))\Psi'(\tau)\,d\tau\Bigg| \\ &\le \left[1+\frac{\Sigma_{i=1}^{m-2}\zeta_i - 1}{\Delta}(\Psi(\Im)-\Psi(r))\right]\vartheta_1 + \frac{\Psi(\Im)-\Psi(r)}{\Delta}\vartheta_2 + LM \le \rho,\end{aligned}$$

for all $\varsigma \in [r,\Im]$. This indicates that $\|Q\varkappa\| \le \rho$, which proves our claim. Thus, by using the Arzela-Ascoli theorem, $Q : \aleph \to \aleph$ is compact. As a result of SFPT, Q has a fixed point $\varkappa$ in P. Hence, the problems (3.1) and (1.2) has $\varkappa$ as solution. □

Lemma 3.1. $\varkappa^*(\varsigma)$ *is a solution of the FBVP (1.1), (1.2) and* $\varkappa(\varsigma) > w(\varsigma)$ *for every* $\varsigma \in [r,\Im]$ *iff the positive solution of FBVP (3.1) and (1.2) is* $\varkappa = \varkappa^* + w$.

Proof. Let $\varkappa(\varsigma)$ be a solution of FBVP (3.1) and (1.2). Then

$$\begin{aligned}\varkappa(\varsigma) &= \left[1+\frac{\Sigma_{i=1}^{m-2}\zeta_i - 1}{\Delta}(\Psi(\varsigma)-\Psi(r))\right]\vartheta_1 + \frac{(\Psi(\varsigma)-\Psi(r))}{\Delta}\vartheta_2 \\ &\quad + \frac{1}{\Gamma(\nu)}\int_r^{\Im} \varpi(\varsigma,\tau)\mathcal{G}(\tau,\varkappa^*(\tau),\varkappa^*(r+\lambda\tau))\Psi'(\tau)\,d\tau \\ &= \left[1+\frac{\Sigma_{i=1}^{m-2}\zeta_i - 1}{\Delta}(\Psi(\varsigma)-\Psi(r))\right]\vartheta_1 + \frac{\Psi(\varsigma)-\Psi(r)}{\Delta}\vartheta_2 \\ &\quad + \frac{1}{\Gamma(\nu)}\int_r^{\Im} \varpi(\varsigma,\tau)\left(\mathcal{F}(\tau,\varkappa^*(\tau),\varkappa^*(r+\lambda\tau)) + p(\tau)\right)\Psi'(\tau)\,d\tau \\ &= \left[1+\frac{\Sigma_{i=1}^{m-2}\zeta_i - 1}{\Delta}(\Psi(\varsigma)-\Psi(r))\right]\vartheta_1 + \frac{\Psi(\varsigma)-\Psi(r)}{\Delta}\vartheta_2 \\ &\quad + \frac{1}{\Gamma(\nu)}\int_r^{\Im} \varpi(\varsigma,\tau)\mathcal{F}(\tau,(\varkappa-w)(\tau),(\varkappa-w)(r+\lambda\tau))\Psi'(\tau)\,d\tau\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\nu)}\int_r^{\Im}\varpi(\varsigma,\tau)p(\tau)\Psi'(\tau)\,d\tau\\
&=\left[1+\frac{\Sigma_{i=1}^{m-2}\zeta_i-1}{\Delta}\left(\Psi(\varsigma)-\Psi(r)\right)\right]\vartheta_1+\frac{\Psi(\varsigma)-\Psi(r)}{\Delta}\vartheta_2\\
&+\frac{1}{\Gamma(\nu)}\int_r^{\Im}\varpi(\varsigma,\tau)\mathcal{G}(\tau,(\varkappa-w)(\tau),(\varkappa-w)(r+\lambda\tau))\Psi'(\tau)\,d\tau+w(\varsigma).
\end{aligned}
$$

So,

$$
\begin{aligned}
\varkappa(\varsigma)-w(\varsigma)=&\left[1+\frac{\Sigma_{i=1}^{m-2}\zeta_i-1}{\Delta}\left(\Psi(\varsigma)-\Psi(r)\right)\right]\vartheta_1+\frac{\Psi(\varsigma)-\Psi(r)}{\Delta}\vartheta_2\\
&+\frac{1}{\Gamma(\nu)}\int_r^{\Im}\varpi(\varsigma,\tau)\mathcal{F}(\tau,(\varkappa-w)(\tau),(\varkappa-w)(r+\lambda\tau))\Psi'(\tau)\,d\tau.
\end{aligned}
$$

Then we get the existence of the solution with the condition

$$
\begin{aligned}
\varkappa^*(\varsigma)=&\left[1+\frac{\Sigma_{i=1}^{m-2}\zeta_i-1}{\Delta}\left(\Psi(\varsigma)-\Psi(r)\right)\right]\vartheta_1+\frac{\Psi(\varsigma)-\Psi(r)}{\Delta}\vartheta_2\\
&+\frac{1}{\Gamma(\nu)}\int_r^{\Im}\varpi(\varsigma,\tau)\mathcal{F}(\tau,\varkappa^*(\tau),\varkappa^*(r+\lambda\tau))\Psi'(\tau)\,d\tau.
\end{aligned}
$$

For the converse, if $\varkappa^*$ is a solution of the FBVP (1.1) and (1.2), we get

$$
\begin{aligned}
\mathcal{D}_r^{\nu;\psi}(\varkappa^*(\varsigma)+w(\varsigma))&=\mathcal{D}_r^{\nu;\psi}\varkappa^*(\varsigma)+\mathcal{D}_r^{\nu;\psi}w(\varsigma)=-\mathcal{F}(\varsigma,\varkappa^*(\varsigma),\varkappa^*(r+\lambda\varsigma))-p(\varsigma)\\
&=-\left[\mathcal{F}(\varsigma,\varkappa^*(\varsigma),\varkappa^*(r+\lambda\varsigma))+p(\varsigma)\right]=-\mathcal{G}(\varsigma,\varkappa^*(\varsigma),\varkappa^*(r+\lambda\varsigma)),
\end{aligned}
$$

which leads to

$$
\mathcal{D}_r^{\nu;\psi}\varkappa(\varsigma)=-\mathcal{G}(\varsigma,\varkappa^*(\varsigma),\varkappa^*(r+\lambda\varsigma)).
$$

We easily see that

$$
\varkappa^*(r)=\varkappa(r)-w(r)=\varkappa(r)-0=\vartheta_1,
$$

i.e., $\varkappa(r)=\vartheta_1$ and

$$
\varkappa^*(\Im)=\sum_{i=1}^{m-2}\zeta_i\varkappa^*(\eta_i)+\vartheta_2,
$$

$$
\varkappa(\Im)-w(\Im)=\sum_{i=1}^{m-2}\zeta_i\varkappa(\eta_i)-\sum_{i=1}^{m-2}\zeta_j w(\eta_i)+\vartheta_2=\sum_{i=1}^{m-2}\zeta_i(\varkappa(\eta_i)-w(\eta_i))+\vartheta_2.
$$

So,

$$
\varkappa(\Im)=\sum_{i=1}^{m-2}\zeta_i\varkappa(\eta_i)+\vartheta_2.
$$

Thus $\varkappa(\varsigma)$ is solution of the problem FBVP (3.1) and (3.2). □

4. Example

We propose the given FBVP as follows

$$\mathcal{D}^{\frac{7}{5}}\varkappa(\varsigma) + \mathcal{F}(\varsigma, \varkappa(\varsigma), \varkappa(1 + 0.5\varsigma)) = 0,\ \varsigma \in (1, e), \tag{4.1}$$

$$\varkappa(1) = 1,\ \varkappa(e) = \frac{1}{7}\varkappa(\frac{5}{2}) + \frac{1}{5}\varkappa(\frac{7}{4}) + \frac{1}{9}\varkappa(\frac{11}{5}) - 1. \tag{4.2}$$

Let $\Psi(\varsigma) = \log \varsigma$, where $\mathcal{F}(\varsigma, \varkappa(\varsigma), \varkappa(1 + \frac{1}{2}\varsigma)) = \frac{\varsigma}{1+\varsigma} \arctan(\varkappa(\varsigma) + \varkappa(1 + \frac{1}{2}\varsigma))$.

Taking $\Upsilon(\varsigma) = \varsigma$ we get $\int_1^e \varsigma d\varsigma = \frac{e^2-1}{2} > 0$, then the hypotheses ($\Sigma$1) and ($\Sigma$2) hold. Evaluate $\Delta \cong 0.366$, $M \cong 3.25$ we also get $|\mathcal{G}(\varsigma, \varkappa, v)| < \pi + e = L$ such that $|\varkappa| \leq \rho$, $\rho = 17$, we could just confirm that

$$\left[1 + \frac{\Sigma_{i=1}^{m-2}\zeta_i - 1}{\Delta}\left(\Psi(\mathfrak{I}) - \Psi(r)\right)\right]\vartheta_1 + \frac{\Psi(\mathfrak{I}) - \Psi(r)}{\Delta}\vartheta_2 + LM \cong 16.35 \leq 17. \tag{4.3}$$

By applying the Theorem 3.1 there exit a solution $\varkappa(\varsigma)$ of the problem (4.1) and (4.2).

5. Conclusions

In this paper, we have provided the proof of BVP solutions to a nonlinear Ψ-Caputo fractional pantograph problem or for a semi-positone multi-point of (1.1) and(1.2). What's new here is that even using the generalized Ψ-Caputo fractional derivative, we were able to explicitly prove that there is one solution to this problem, and that in our findings, we utilize the SFPT. The results obtained in our work are significantly generalized and the exclusive result concern the semi-positone multi-point Ψ-Caputo fractional differential pantograph problem (1.1) and (1.2).

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Conflict of interest

The authors declare no conflict of interest.

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