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Research article

Error estimates of mixed finite elements combined with Crank-Nicolson scheme for parabolic control problems

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Abstract: In this paper, a mixed finite element method combined with Crank-Nicolson scheme approximation of parabolic optimal control problems with control constraint is investigated. For the state and co-state, the order $m = 1$ Raviart-Thomas mixed finite element spaces and Crank-Nicolson scheme are used for space and time discretization, respectively. The variational discretization technique is used for the control variable. We derive optimal priori error estimates for the control, state and co-state. Some numerical examples are presented to demonstrate the theoretical results.

Keywords: parabolic optimal control problems; Crank-Nicolson scheme; mixed finite element method; variational discretization; a priori error estimates
Mathematics Subject Classification: 49J20, 65N22, 65N30

1. Introduction

There are numerous research on finite element methods (FEMs) solving elliptic optimal control problems (OCPs) with control constraint. A systematic introduction can be seen in [1–6]. Due to the low regularity of the control variable, it is usually approximated by piecewise constant functions. Then the optimal priori error estimate is $O(h)$ [7–9]. In order to improve the efficiency and accuracy of FEMs for solving such problems, many experts have considered its superconvergence [10–12], a posteriori error estimation [13–15], adaptive algorithm [16, 17] and variational discretization technique [18–20].

In the past two decades, many scholars have proposed different numerical methods for elliptic or parabolic OCPs, such as FEMs [21], space-time FEMs [22, 23], characteristic FEMs [24, 25], mixed finite element methods (MFEMs) [26,27], splitting positive definite mixed FEMs [28,29], finite volume methods [30, 31], spectral method [32–34], virtual element methods (VEMs) [35–37]. Unfortunately, almost of these numerical methods for parabolic OCPs use the backward Euler scheme (BES) for time discretization, and the error estimates of time variable is $O(k)$. In recent years, there are also a few works using the Crank-Nicolson scheme (CNS) for time discretization [38–41], which can improve the error convergence order of the time variable to $O(k^2)$.

To the best of our knowledge, all reported fully discrete MFEMs for parabolic OCPs use BES for time discretization and optimal error estimates of time variable is $O(k)$. The purpose of this paper is to develop a MFEM combined with CNS approximation of parabolic OCPs and establish optimal a priori error estimates $O(h^2 + k^2)$.

We are concerned with the following parabolic OCPs: Find $(y, \boldsymbol{p}, u)$ such that

$$\min_{u\in K} \frac{1}{2}\int_0^T \Big(\|\boldsymbol{p} - \boldsymbol{p}_d\|^2 + \|y - y_d\|^2 + \|u\|^2\Big)dt \tag{1.1}$$

$$y_t(x,t) + \mathrm{div}\boldsymbol{p}(x,t) = f(x,t) + u(x,t), \quad x \in \Omega, \quad t \in J, \tag{1.2}$$

$$\boldsymbol{p}(x,t) = -\nabla y(x,t), \quad x \in \Omega,\ t \in J, \tag{1.3}$$

$$y(x,t) = 0, \quad x \in \partial\Omega, \quad t \in J, \tag{1.4}$$

$$y(x,0) = y_0(x), \quad x \in \Omega, \tag{1.5}$$

where $\Omega \subset \mathbf{R}^2$ is a rectangle, $J = (0, T]$. Let $U = L^2(J; L^2(\Omega))$, $f, y_d \in U$, $\boldsymbol{p}_d \in U^2$ and $y_0 \in H^1(\Omega)$. K is a closed convex subset of U defined by

$$K = \{v \in U : a \le v(x,t) \le b,\ \text{a.e. in } \Omega \times J,\ a, b \in \mathbf{R}\}.$$

Throughout the paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p = \sum_{|\alpha|\le m} \|D^\alpha v\|_{L^p(\Omega)}^p$, a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. We denote by $L^s(J; W^{m,p}(\Omega))$ all L^s integrable functions from J into $W^{m,p}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,p}(\Omega))} = \Big(\int_0^T \|v\|_{W^{m,p}(\Omega)}^s dt\Big)^{1/s}$ for $s \in [1, \infty)$, and the standard modification for $s = \infty$. For ease of presentation, we denote $\|v\|_{L^s(J;W^{m,p}(\Omega))}$ by $\|v\|_{L^s(W^{m,p})}$. Similarly, one can define the spaces $H^l(W^{m,p})$. In addition C denotes a general positive constant.

The layout of this paper is as follows. In Section 2, we construct a MFEM combined with CNS approximation of the parabolic OCPs (1.1)–(1.5). In Section 3, we introduce some useful intermediate variables and important error estimates. In Section 4, we derive a priori error estimates for the control, state and co-state. In Section 5, we provide some numerical examples to illustrate our theoretical results.

2. MFEM combined with CNS approximation of parabolic OCPs

In this section, we shall consider a MFEM combined with CNS approximation of parabolic OCPs (1.1)–(1.5). For simplicity, we shall take the following state spaces $\boldsymbol{L} = H^1(J; \boldsymbol{V})$ and $Q = H^1(J; W)$, where $\boldsymbol{V}$ and W are defined as follows:

$$\boldsymbol{V} = H(\mathrm{div}; \Omega) = \Big\{\boldsymbol{v} \in (L^2(\Omega))^2, \mathrm{div}\boldsymbol{v} \in L^2(\Omega)\Big\},\ W = L^2(\Omega).$$

Furthermore, we define the space

$$K' = \{v \in W : a \le v(x) \le b, \ \text{a.e. in } \Omega\}.$$

Then OCPs (1.1)–(1.5) can be recast as the following weak form: Find $(y, \boldsymbol{p}, u) \in Q \times \boldsymbol{L} \times K$ such that

$$\min_{u\in K} \frac{1}{2} \int_0^T \left(\|\boldsymbol{p} - \boldsymbol{p}_d\|^2 + \|y - y_d\|^2 + \|u\|^2 \right) dt \tag{2.1}$$

$$(y_t, w) + (\mathrm{div}\boldsymbol{p}, w) = (f + u, w), \quad \forall\, w \in W, t \in J, \tag{2.2}$$

$$(\boldsymbol{p}, \boldsymbol{v}) - (y, \mathrm{div}\boldsymbol{v}) = 0, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, t \in J, \tag{2.3}$$

$$y(x, 0) = y_0(x), \quad \forall\, x \in \Omega. \tag{2.4}$$

It follows from [3] that OCPs (2.1)–(2.4) has a unique solution $(y, \boldsymbol{p}, u)$, and that a triplet $(y, \boldsymbol{p}, u) \in Q \times \boldsymbol{L} \times K$ is the solution of (2.1)–(2.4) if and only if there is a co-state $(z, \boldsymbol{q}) \in Q \times \boldsymbol{L}$ such that $(y, \boldsymbol{p}, z, \boldsymbol{q}, u)$ satisfies the following optimality conditions:

$$(y_t, w) + (\mathrm{div}\boldsymbol{p}, w) = (f + u, w), \quad \forall\, w \in W, t \in J, \tag{2.5}$$

$$(\boldsymbol{p}, \boldsymbol{v}) - (y, \mathrm{div}\boldsymbol{v}) = 0, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, t \in J, \tag{2.6}$$

$$y(x, 0) = y_0(x), \quad \forall\, x \in \Omega, \tag{2.7}$$

$$-(z_t, w) + (\mathrm{div}\boldsymbol{q}, w) = (y - y_d, w), \quad \forall\, w \in W, t \in J, \tag{2.8}$$

$$(\boldsymbol{q}, \boldsymbol{v}) - (z, \mathrm{div}\boldsymbol{v}) = -(\boldsymbol{p} - \boldsymbol{p}_d, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, t \in J, \tag{2.9}$$

$$z(x, T) = 0, \quad \forall\, x \in \Omega, \tag{2.10}$$

$$(u + z, \tilde{u} - u) \geq 0, \quad \forall\, \tilde{u} \in K', t \in J. \tag{2.11}$$

We introduce a pointwise projection $P_{[a,b]}$, which satisfies: For any $\varphi \in W$,

$$P_{[a,b]}\varphi(x) = \min\{b, \max\{a, -\varphi(x)\}\}, \quad \forall\, x \in \Omega.$$

Then the variational inequality (2.11) can be equivalently expressed as

$$u = P_{[a,b]}(z). \tag{2.12}$$

We use the Raviart-Thomas mixed finite element of the order $m = 1$ for space discretization. Let $\mathcal{T}_h$ be a regular triangulations of the domain Ω, h_e denotes the diameter of e and $h = \max_{e\in\mathcal{T}_h}\{h_e\}$. Let $P_m(e)$ indicates the space of polynomials of total degree no more than m on e and $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ denote Raviart-Thomas mixed finite element spaces [1,2] associated with the triangulations $\mathcal{T}_h$ of Ω, namely,

$$\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{V} : \boldsymbol{v}_h|_e \in (P_m(e))^2 + x \cdot P_m(e), \forall\, e \in \mathcal{T}_h, \},$$

$$W_h := \{w_h \in W : w_h|_e \in P_m(e), \forall\, e \in \mathcal{T}_h\}.$$

We shall use the CNS for time discretization. Let N be a positive integer, $k = T/N$ and $t_n = nk, n = 0, 1, \cdots, N$. Set $I_n = [t_n, t_{n+1}], n = 0, 1, \cdots, N - 1$. For any function φ, we define $\varphi^n = \varphi(x, t_n)$,

$$d_t\varphi^n = \left(\varphi^{n+1} - \varphi^n\right)/k,$$

$$\varphi^{n+\frac{1}{2}} = \left(\varphi^{n+1} + \varphi^n\right)/2,$$

and discrete time-dependent norms

$$\|\varphi\|_{l^p(W^{m,q})}=\left(\sum_{n=0}^{N-1}k\left\|\varphi^{n+\frac{1}{2}}\right\|_{W^{m,q}}^p\right)^{1/p}.$$

Then a MFEM combined with CNS approximation of (2.1)–(2.4) is as follows: Find $(y_h,\boldsymbol{p}_h,u_h)\in W_h\times\boldsymbol{V}_h\times K'$ such that

$$\min_{u_h^{n+\frac{1}{2}}\in K'}\frac{1}{2}\sum_{n=0}^{N-1}\left(\left\|\boldsymbol{p}_h^{n+\frac{1}{2}}-\boldsymbol{p}_d^{n+\frac{1}{2}}\right\|^2+\left\|y_h^{n+\frac{1}{2}}-y_d^{n+\frac{1}{2}}\right\|^2+\left\|u_h^{n+\frac{1}{2}}\right\|^2\right),\tag{2.13}$$

$$(d_ty_h^n,w_h)+\left(\mathrm{div}\boldsymbol{p}_h^{n+\frac{1}{2}},w_h\right)=\left(f^{n+\frac{1}{2}}+u_h^{n+\frac{1}{2}},w_h\right),\quad\forall\,w_h\in W_h,n=0,1,\cdots,N-1,\tag{2.14}$$

$$\left(\boldsymbol{p}_h^{n+\frac{1}{2}},\boldsymbol{v}_h\right)-\left(y_h^{n+\frac{1}{2}},\mathrm{div}\boldsymbol{v}_h\right)=0,\quad\forall\,\boldsymbol{v}_h\in\boldsymbol{V}_h,n=0,1,\cdots,N-1,\tag{2.15}$$

$$y_h^0(x)=R_hy_0(x),\quad\forall\,x\in\Omega,\tag{2.16}$$

where R_h is a L^2 projection operator, which will be specific later.

Like in [6], the OCPs (2.13)–(2.16) has a unique solution $\left(y_h^n,\boldsymbol{p}_h^n,u_h^n\right),n=0,1,\cdots,N$ and the triplet $\left(y_h^n,\boldsymbol{p}_h^n,u_h^n\right)\in W_h\times\boldsymbol{V}_h\times K',n=0,1,\cdots,N$ is the solution of (2.13)–(2.16) if and only if there is a co-state $\left(z_h^n,\boldsymbol{q}_h^n\right)\in W_h\times\boldsymbol{V}_h,(n=N,\cdots,1,0)$ such that $\left(y_h^n,\boldsymbol{p}_h^n,z_h,\boldsymbol{q}_h^n,u_h^n\right),(n=0,1,\cdots,N)$ satisfies the following optimality conditions:

$$(d_ty_h^n,w_h)+\left(\mathrm{div}\boldsymbol{p}_h^{n+\frac{1}{2}},w_h\right)=\left(f^{n+\frac{1}{2}}+u_h^{n+\frac{1}{2}},w_h\right),\quad\forall\,w_h\in W_h,\tag{2.17}$$

$$\left(\boldsymbol{p}_h^{n+\frac{1}{2}},\boldsymbol{v}_h\right)-\left(y_h^{n+\frac{1}{2}},\mathrm{div}\boldsymbol{v}_h\right)=0,\quad\forall\,\boldsymbol{v}_h\in\boldsymbol{V}_h,\tag{2.18}$$

$$y_h^0(x)=R_hy_0(x),\quad\forall\,x\in\Omega,\tag{2.19}$$

$$-(d_tz_h^n,w_h)+\left(\mathrm{div}\boldsymbol{q}_h^{n+\frac{1}{2}},w_h\right)=\left(y_h^{n+\frac{1}{2}}-y_d^{n+\frac{1}{2}},w_h\right),\quad\forall\,w_h\in W_h,\tag{2.20}$$

$$\left(\boldsymbol{q}_h^{n+\frac{1}{2}},\boldsymbol{v}_h\right)-\left(z_h^{n+\frac{1}{2}},\mathrm{div}\boldsymbol{v}_h\right)=-\left(\boldsymbol{p}_h^{n+\frac{1}{2}}-\boldsymbol{p}_d^{n+\frac{1}{2}},\boldsymbol{v}_h\right),\quad\forall\,\boldsymbol{v}_h\in\boldsymbol{V}_h,\tag{2.21}$$

$$z_h^N(x)=0,\quad\forall\,x\in\Omega,\tag{2.22}$$

$$\left(u_h^{n+\frac{1}{2}}+z_h^{n+\frac{1}{2}},\tilde{u}-u_h^{n+\frac{1}{2}}\right)\geq0,\quad\forall\,\tilde{u}\in K'.\tag{2.23}$$

Here, we use the variational discretization technique for the variational inequality. Similarly to (2.12), the variational inequality (2.23) can be equivalently rewritten as

$$u_h^{n+\frac{1}{2}}=P_{[a,b]}\left(z_h^{n+\frac{1}{2}}\right),\quad n=0,1,\cdots,N-1.\tag{2.24}$$

This means that, we can obtain $u_h^{n+\frac{1}{2}}$ from $z_h^{n+\frac{1}{2}}$ by using the relation (2.24).

The following projection operators are commonly used in the following error estimates of MFEMs approximation of OCPs. First, we define the standard $L^2(\Omega)$-projection [2] $R_h:\ W\to W_h$, which satisfies: For any $\phi\in W$,

$$(R_h\phi-\phi,w_h)=0,\quad\forall\,w_h\in W_h,\tag{2.25}$$

$$\|\phi - R_h\phi\|_{0,\rho} \le Ch^r\|\phi\|_{r,\rho}, \quad 0 \le \rho \le \infty, \quad \forall\, \phi \in W^{r,\rho}(\Omega), \quad 1 \le r \le 1+m. \tag{2.26}$$

Second, we define the Fortin projection [2] $\Pi_h : \boldsymbol{V} \to \boldsymbol{V}_h$, which satisfies: For any $\boldsymbol{q} \in \boldsymbol{V}$,

$$(\mathrm{div}(\Pi_h\boldsymbol{q} - \boldsymbol{q}), w_h) = 0, \quad \forall\, w_h \in W_h, \tag{2.27}$$
$$\|\boldsymbol{q} - \Pi_h\boldsymbol{q}\| \le Ch^r\|\boldsymbol{q}\|_r, \quad \forall\, \boldsymbol{q} \in (H^r(\Omega))^2, \quad 1 \le r \le 1+m, \tag{2.28}$$
$$\|\mathrm{div}(\boldsymbol{q} - \Pi_h\boldsymbol{q})\| \le Ch^r\|\mathrm{div}\boldsymbol{q}\|_r, \quad \forall\, \mathrm{div}\boldsymbol{q} \in H^r(\Omega), \quad 1 \le r \le 1+m. \tag{2.29}$$

3. Error estimates of intermediate variables

In this section, we will introduce some important intermediate variables and error estimates. For any $\tilde{u} \in K$, we define variables $(y_h^n(\tilde{u}), \boldsymbol{p}_h^n(\tilde{u}), z_h^n(\tilde{u}), \boldsymbol{q}_h^n(\tilde{u})), n = 0, 1, \cdots, N$, associated with $\tilde{u}$, which satisfies

$$(d_t y_h^n(\tilde{u}), w_h) + \left(\mathrm{div}\boldsymbol{p}_h^{n+\frac{1}{2}}(\tilde{u}), w_h\right) = \left(f^{n+\frac{1}{2}} + \tilde{u}^{n+\frac{1}{2}}, w_h\right), \quad \forall\, w_h \in W_h, \tag{3.1}$$
$$\left(\boldsymbol{p}_h^{n+\frac{1}{2}}(\tilde{u}), \boldsymbol{v}_h\right) - \left(y_h^{n+\frac{1}{2}}(\tilde{u}), \mathrm{div}\boldsymbol{v}_h\right) = 0, \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.2}$$
$$y_h^0(\tilde{u})(x) = R_h y_0(x), \quad \forall\, x \in \Omega, \tag{3.3}$$
$$-(d_t z_h^n(\tilde{u}), w_h) + \left(\mathrm{div}\boldsymbol{q}_h^{n+\frac{1}{2}}(\tilde{u}), w_h\right) = \left(y_h^{n+\frac{1}{2}}(\tilde{u}) - y_d^{n+\frac{1}{2}}, w_h\right), \quad \forall\, w_h \in W_h, \tag{3.4}$$
$$\left(\boldsymbol{q}_h^{n+\frac{1}{2}}(\tilde{u}), \boldsymbol{v}_h\right) - \left(z_h^{n+\frac{1}{2}}(\tilde{u}), \mathrm{div}\boldsymbol{v}_h\right) = -\left(\boldsymbol{p}_h^{n+\frac{1}{2}}(\tilde{u}) - \boldsymbol{p}_d^{n+\frac{1}{2}}, \boldsymbol{v}_h\right), \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.5}$$
$$z_h^N(\tilde{u})(x) = 0, \quad \forall\, x \in \Omega. \tag{3.6}$$

According to standard Raviart-Thomas mixed finite element approximation error analysis like in [20, 26], we can derive the following error estimates.

Lemma 3.1. *Let $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ and $(\boldsymbol{p}_h(u), y_h(u), \boldsymbol{q}_h(u), z_h(u))$ be the discrete solutions of (2.17)–(2.23) and (3.1)–(3.6) with $\tilde{u} = u$, respectively. Then we have*

$$|||y_h - y_h(u)|||_{l^\infty(L^2)} + |||\boldsymbol{p}_h - \boldsymbol{p}_h(u)|||_{l^2(L^2)} \le C|||u - u_h|||_{l^2(L^2)}, \tag{3.7}$$
$$|||z_h - z_h(u)|||_{l^\infty(L^2)} + |||\boldsymbol{q}_h - \boldsymbol{q}_h(u)|||_{l^2(L^2)} \le C|||u - u_h|||_{l^2(L^2)}. \tag{3.8}$$

Proof. Let $\alpha = y_h - y_h(u)$ and $\boldsymbol{\beta} = \boldsymbol{p}_h - \boldsymbol{p}_h(u)$. From (2.17), (2.18), (3.1) and (3.2) with $\tilde{u} = u$, we have the following error equations

$$(d_t\alpha^n, w_h) + \left(\mathrm{div}\boldsymbol{\beta}^{n+\frac{1}{2}}, w_h\right) = \left(u_h^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}, w_h\right), \quad \forall\, w_h \in W_h, \tag{3.9}$$
$$\left(\boldsymbol{\beta}^{n+\frac{1}{2}}, \boldsymbol{v}_h\right) - \left(\alpha^{n+\frac{1}{2}}, \mathrm{div}\boldsymbol{v}_h\right) = 0, \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{3.10}$$

Selecting $w_h = \alpha^{n+\frac{1}{2}}$ and $\boldsymbol{v}_h = \boldsymbol{\beta}^{n+\frac{1}{2}}$ in (3.9) and (3.10), respectively. Then add those equations, we get

$$\left(d_t\alpha^n, \alpha^{n+\frac{1}{2}}\right) + \left(\boldsymbol{\beta}^{n+\frac{1}{2}}, \boldsymbol{\beta}^{n+\frac{1}{2}}\right) = \left(u_h^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}, \alpha^{n+\frac{1}{2}}\right). \tag{3.11}$$

Note that $\left(d_t\alpha^n, \alpha^{n+\frac{1}{2}}\right) = \frac{\|\alpha^{n+1}\|^2-\|\alpha^n\|^2}{2k}$. From (3.11) and Young inequality, we obtain

$$\frac{\|\alpha^{n+1}\|^2 - \|\alpha^n\|^2}{2k} + \left\|\boldsymbol{\beta}^{n+\frac{1}{2}}\right\|^2 \le \varepsilon \left\|\alpha^{n+\frac{1}{2}}\right\|^2 + C(\varepsilon)\left\|u_h^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}\right\|^2. \tag{3.12}$$

Multiplying both sides of (3.12) by $2k$ and summing n from 0 to M $(0 \le M \le N-1)$, we have

$$\left\|\alpha^{M+1}\right\|^2 - \|\alpha^0\|^2 + 2\sum_{n=0}^{M} k\left\|\boldsymbol{\beta}^{n+\frac{1}{2}}\right\|^2 \le 2\varepsilon\sum_{n=0}^{M} k\left\|\alpha^{n+\frac{1}{2}}\right\|^2 + 2C(\varepsilon)\sum_{n=0}^{M} k\left\|u_h^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}\right\|^2. \tag{3.13}$$

According to $\alpha^0 = 0$, $|||\alpha|||_{l^2(L^2)} \le C|||\alpha|||_{l^\infty(L^2)}$ and (3.13), we get (3.7).

Set $\eta = z_h - z_h(u)$ and $\boldsymbol{\theta} = \boldsymbol{q}_h - \boldsymbol{q}_h(u)$. Subtract (3.4) and (3.5) from (2.20) and (2.21) to get the following error equations

$$-\left(d_t\eta^n, w_h\right) + \left(\mathrm{div}\boldsymbol{\theta}^{n+\frac{1}{2}}, w_h\right) = \left(\alpha^{n+\frac{1}{2}}, w_h\right), \quad \forall\, w_h \in W_h, \tag{3.14}$$

$$\left(\boldsymbol{\theta}^{n+\frac{1}{2}}, \boldsymbol{v}_h\right) - \left(\eta^{n+\frac{1}{2}}, \mathrm{div}\boldsymbol{v}_h\right) = -\left(\boldsymbol{\beta}^{n+\frac{1}{2}}, \boldsymbol{v}_h\right), \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{3.15}$$

Choosing $w_h = \eta^{n+\frac{1}{2}}$ and $\boldsymbol{v}_h = \boldsymbol{\theta}^{n+\frac{1}{2}}$ in (3.14) and (3.15), respectively. We derive

$$-\left(d_t\eta^n, \eta^{n+\frac{1}{2}}\right) + \left(\boldsymbol{\theta}^{n+\frac{1}{2}}, \boldsymbol{\theta}^{n+\frac{1}{2}}\right) = \left(\alpha^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}\right) - \left(\boldsymbol{\beta}^{n+\frac{1}{2}}, \boldsymbol{\theta}^{n+\frac{1}{2}}\right). \tag{3.16}$$

Note that $\eta^N = 0$ and $|||\eta|||_{l^2(L^2)} \le C|||\eta|||_{l^\infty(L^2)}$, similarly to (3.11)–(3.13), we can arrive at

$$|||\eta|||_{l^\infty(L^2)} + |||\boldsymbol{\theta}|||_{l^2(L^2)} \le |||\alpha|||_{l^2(L^2)} + |||\boldsymbol{\beta}|||_{l^2(L^2)}. \tag{3.17}$$

From (3.7) and (3.17), it is easy to get (3.8). □

For convenience, we use the following notations

$$\begin{aligned}
\rho_y &= y - y_h(u), & \boldsymbol{\varrho}_{\boldsymbol{p}} &= \boldsymbol{p} - \boldsymbol{p}_h(u),\\
\zeta_y &= y - R_h y, & \boldsymbol{\xi}_{\boldsymbol{p}} &= \boldsymbol{p} - \Pi_h\boldsymbol{p},\\
\upsilon_y &= R_h y - y_h(u), & \boldsymbol{\vartheta}_{\boldsymbol{p}} &= \Pi_h\boldsymbol{p} - \boldsymbol{p}_h(u).
\end{aligned}$$

Lemma 3.2. *Let $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ and $(\boldsymbol{p}_h(u), y_h(u), \boldsymbol{q}_h(u), z_h(u))$ be the solutions of (2.5)–(2.11) and (3.1)–(3.6) with $\tilde{u} = u$ respectively. Assume that $y, z \in L^2(H^2)$, $\boldsymbol{p}, \boldsymbol{q} \in L^2((H^2)^2)$ and $y_{ttt}, z_{ttt} \in L^2(L^2)$, then we have*

$$|||y - y_h(u)|||_{l^\infty(L^2)} + |||\boldsymbol{p} - \boldsymbol{p}_h(u)|||_{l^2(L^2)} \le C\left(h^2 + k^2\right), \tag{3.18}$$

$$|||z - z_h(u)|||_{l^\infty(L^2)} + |||\boldsymbol{q} - \boldsymbol{q}_h(u)|||_{l^2(L^2)} \le C\left(h^2 + k^2\right). \tag{3.19}$$

Proof. Set $t = \frac{t_{n+1}+t_n}{2}$ in (2.5) and (2.6) then subtract (3.1) and (3.2), we have

$$\left(d_t\rho_y^n, w_h\right) + \left(\mathrm{div}\boldsymbol{\varrho}_{\boldsymbol{p}}^{n+\frac{1}{2}}, w_h\right) = \left(d_t y^n - y_t^{n+\frac{1}{2}}, w_h\right), \quad \forall w_h \in W_h, \tag{3.20}$$

$$\left(\boldsymbol{\varrho}_{\boldsymbol{p}}^{n+\frac{1}{2}}, \boldsymbol{v}_h\right) - \left(\rho_y^{n+\frac{1}{2}}, \mathrm{div}\boldsymbol{v}_h\right) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{3.21}$$

Taking $w_h = \upsilon_y^{n+\frac{1}{2}}$ and $\boldsymbol{v}_h = \boldsymbol{\vartheta}_{\boldsymbol{p}}^{n+\frac{1}{2}}$ in (3.20) and (3.21), respectively. According to the definition of R_h and Π_h, we derive

$$\left(d_t\upsilon_y^n, \upsilon_y^{n+\frac{1}{2}}\right) + \left(\text{div}\boldsymbol{\vartheta}_{\boldsymbol{p}}^{n+\frac{1}{2}}, \upsilon_y^{n+\frac{1}{2}}\right) = \left(d_t y^n - y_t^{n+\frac{1}{2}}, \upsilon_y^{n+\frac{1}{2}}\right), \tag{3.22}$$

$$\left(\boldsymbol{\vartheta}_{\boldsymbol{p}}^{n+\frac{1}{2}}, \boldsymbol{\vartheta}_{\boldsymbol{p}}^{n+\frac{1}{2}}\right) - \left(\upsilon_y^{n+\frac{1}{2}}, \text{div}\boldsymbol{\vartheta}_{\boldsymbol{p}}^{n+\frac{1}{2}}\right) = -\left(\boldsymbol{\xi}_{\boldsymbol{p}}^{n+\frac{1}{2}}, \boldsymbol{\vartheta}_{\boldsymbol{p}}^{n+\frac{1}{2}}\right). \tag{3.23}$$

From (3.22)–(3.23), we get

$$\left(d_t\upsilon_y^n, \upsilon_y^{n+\frac{1}{2}}\right) + \left(\boldsymbol{\vartheta}_{\boldsymbol{p}}^{n+\frac{1}{2}}, \boldsymbol{\vartheta}_{\boldsymbol{p}}^{n+\frac{1}{2}}\right) = \left(d_t y^n - y_t^{n+\frac{1}{2}}, \upsilon_y^{n+\frac{1}{2}}\right) - \left(\boldsymbol{\xi}_{\boldsymbol{p}}^{n+\frac{1}{2}}, \boldsymbol{\vartheta}_{\boldsymbol{p}}^{n+\frac{1}{2}}\right). \tag{3.24}$$

By using (2.28), Cauchy-Schwarz inequality, Young inequality and interpolation theory, we have

$$\begin{aligned}\left(d_t y^n - y_t^{n+\frac{1}{2}}, \upsilon_y^{n+\frac{1}{2}}\right) &= \left(\frac{1}{k}\int_{t_n}^{t_{n+1}}\left(y_t - y_t^{n+\frac{1}{2}}\right)dt, \upsilon_y^{n+\frac{1}{2}}\right)\\ &\leq Ck^{\frac{3}{2}}\left(\int_{t_n}^{t_{n+1}}\left\|\frac{\partial^3 y}{\partial t^3}\right\|^2 dt\right)^{1/2}\left\|\upsilon_y^{n+\frac{1}{2}}\right\| \\ &\leq C(\varepsilon)k^3\left\|\frac{\partial^3 y}{\partial t^3}\right\|^2_{L^2(I_n;L^2)} + \varepsilon\left\|\upsilon_y^{n+\frac{1}{2}}\right\|^2\end{aligned} \tag{3.25}$$

and

$$\left(\boldsymbol{\xi}_{\boldsymbol{p}}^{n+\frac{1}{2}}, \boldsymbol{\vartheta}_{\boldsymbol{p}}^{n+\frac{1}{2}}\right) \leq C(\varepsilon)h^4\|\boldsymbol{p}\|_2^2 + \varepsilon\left\|\boldsymbol{\vartheta}_{\boldsymbol{p}}^{n+\frac{1}{2}}\right\|^2. \tag{3.26}$$

Combining (3.24)–(3.26), we obtain

$$\frac{\|\upsilon_y^{n+1}\|^2 - \|\upsilon_y^n\|^2}{2k} + \left\|\boldsymbol{\vartheta}_{\boldsymbol{p}}^{n+\frac{1}{2}}\right\|^2 \leq C(\varepsilon)k^3\left\|\frac{\partial^3 y}{\partial t^3}\right\|^2_{L^2(I_n;L^2)} + C(\varepsilon)h^4\|\boldsymbol{p}\|_2^2. \tag{3.27}$$

Multiplying both sides of (3.27) by $2k$ and summing n from 0 to M $(0 \leq M \leq N-1)$, we have

$$\left\|\upsilon_y^{M+1}\right\|^2 - \left\|\upsilon_y^0\right\|^2 + 2\sum_{n=0}^{M} k\left\|\boldsymbol{\vartheta}_{\boldsymbol{p}}^{n+\frac{1}{2}}\right\|^2 \leq 2C(\varepsilon)k^4\sum_{n=0}^{M}\left\|\frac{\partial^3 y}{\partial t^3}\right\|^2_{L^2(I_n;L^2)} + C(\varepsilon)h^4\sum_{n=0}^{M}k\|\boldsymbol{p}\|_2^2. \tag{3.28}$$

Noting that $\rho_y^0 = 0$. From (3.28), we arrive at

$$\|\upsilon_y\|_{l^\infty(L^2)} + \|\boldsymbol{\vartheta}_{\boldsymbol{p}}\|_{l^2(L^2)} \leq C\left(k^2 + h^2\right). \tag{3.29}$$

Then (3.18) follows from (2.26), (2.28), (3.29) and triangle inequality.

Analogously, we can define $\rho_z, \zeta_z, \upsilon_z$ and $\boldsymbol{\varrho}_{\boldsymbol{q}}, \boldsymbol{\xi}_{\boldsymbol{q}}, \boldsymbol{\vartheta}_{\boldsymbol{q}}$. Let $t = \frac{t_{n+1}+t_n}{2}$ in (2.8) and (2.9) then subtract (3.4) and (3.5), we get

$$-\left(d_t\rho_z^n, w_h\right) + \left(\text{div}\boldsymbol{\varrho}_{\boldsymbol{q}}^{n+\frac{1}{2}}, w_h\right) = \left(\rho_y^{n+\frac{1}{2}}, w_h\right) + \left(z_t^{n+\frac{1}{2}} - d_t z^n, w_h\right), \tag{3.30}$$

$$\left(\boldsymbol{\varrho}_{\boldsymbol{q}}^{n+\frac{1}{2}}, \boldsymbol{v}_h\right) - \left(\rho_z^{n+\frac{1}{2}}, \mathrm{div}\boldsymbol{v}_h\right) = -\left(\boldsymbol{\varrho}_{\boldsymbol{p}}^{n+\frac{1}{2}}, \boldsymbol{v}_h\right). \tag{3.31}$$

Choosing $w_h = \upsilon_z^{n+\frac{1}{2}}$ and $\boldsymbol{v}_h = \boldsymbol{\vartheta}_{\boldsymbol{q}}^{n+\frac{1}{2}}$ in (3.30) and (3.31), respectively. From the definition of R_h and Π_h, we arrive at

$$-\left(d_t\upsilon_z^n, \upsilon_z^{n+\frac{1}{2}}\right) + \left(\mathrm{div}\boldsymbol{\vartheta}_{\boldsymbol{q}}^{n+\frac{1}{2}}, \upsilon_z^{n+\frac{1}{2}}\right) = \left(\upsilon_y^{n+\frac{1}{2}}, \upsilon_z^{n+\frac{1}{2}}\right) + \left(z_t^{n+\frac{1}{2}} - d_t z^n, \upsilon_z^{n+\frac{1}{2}}\right), \tag{3.32}$$

$$\left(\boldsymbol{\vartheta}_{\boldsymbol{q}}^{n+\frac{1}{2}}, \boldsymbol{\vartheta}_{\boldsymbol{q}}^{n+\frac{1}{2}}\right) - \left(\upsilon_z^{n+\frac{1}{2}}, \mathrm{div}\boldsymbol{\vartheta}_{\boldsymbol{q}}^{n+\frac{1}{2}}\right) = -\left(\boldsymbol{\xi}_{\boldsymbol{q}}^{n+\frac{1}{2}}, \boldsymbol{\vartheta}_{\boldsymbol{q}}^{n+\frac{1}{2}}\right) - \left(\boldsymbol{\xi}_{\boldsymbol{p}}^{n+\frac{1}{2}}, \boldsymbol{\vartheta}_{\boldsymbol{q}}^{n+\frac{1}{2}}\right) - \left(\boldsymbol{\vartheta}_{\boldsymbol{p}}^{n+\frac{1}{2}}, \boldsymbol{\vartheta}_{\boldsymbol{q}}^{n+\frac{1}{2}}\right). \tag{3.33}$$

Similarly to (3.22)–(3.28), we can derive

$$\begin{aligned}\left\|\upsilon_z^{M+1}\right\|^2 - \left\|\upsilon_z^N\right\|^2 + 2\sum_{n=N-1}^{M} k\left\|\boldsymbol{\vartheta}_{\boldsymbol{q}}^{n+\frac{1}{2}}\right\|^2 \le{}& 2C(\varepsilon)h^4\sum_{n=N-1}^{M} k\left(\left\|y^{n+\frac{1}{2}}\right\|_2^2 + \left\|\boldsymbol{q}^{n+\frac{1}{2}}\right\|_2^2 + \left\|\boldsymbol{p}^{n+\frac{1}{2}}\right\|_2^2\right)\\ &+ 2C(\varepsilon)\sum_{n=N-1}^{M} k\left\|\boldsymbol{\vartheta}_{\boldsymbol{p}}^{n+\frac{1}{2}}\right\|^2 + 2C(\varepsilon)k^4\sum_{n=N-1}^{M}\left\|\frac{\partial^3 z}{\partial t^3}\right\|_{L^2(I_n;L^2)}^2.\end{aligned} \tag{3.34}$$

Since $\rho_z^N = 0$, (3.19) follows from (2.26), (2.28), (3.18), (3.34) and triangle inequality. □

4. A priori error estimates

In this section, we shall derive a priori error estimates of the MFEM combined with CNS approximation scheme (2.17)–(2.23).

Theorem 4.1. *Let $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ be the solution of (2.5)–(2.11) and $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ be the solution of (2.17)–(2.23). Assume that all the conditions in Lemmas 3.1 and 3.2 are valid. Then, we have*

$$|||u - u_h|||_{l^2(L^2)} \le C\left(h^2 + k^2\right). \tag{4.1}$$

Proof. From (2.11) and (2.23), we have

$$\left(u^{n+\frac{1}{2}} + z^{n+\frac{1}{2}}, u_h^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}\right) \ge 0, \tag{4.2}$$

and

$$\left(u_h^{n+\frac{1}{2}} + z_h^{n+\frac{1}{2}}, u^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}}\right) \ge 0. \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\begin{aligned}|||u - u_h|||_{l^2(L^2)}^2 &= \sum_{n=0}^{N-1} k\left(u^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}}, u^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}}\right)\\ &\le \sum_{n=0}^{N-1} k\left(z_h^{n+\frac{1}{2}} - z^{n+\frac{1}{2}}, u^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}}\right)\\ &= \sum_{n=0}^{N-1} k\left(z_h^{n+\frac{1}{2}} - z_h^{n+\frac{1}{2}}(u), u^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}}\right) + \sum_{n=0}^{N-1} k\left(z_h^{n+\frac{1}{2}}(u) - z^{n+\frac{1}{2}}, u^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}}\right).\end{aligned} \tag{4.4}$$

It follows from (2.17)–(2.22) and (3.1)–(3.6) that

$$\begin{aligned}\sum_{n=0}^{N-1} k\left(z_h^{n+\frac{1}{2}} - z_h^{n+\frac{1}{2}}(u), u^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}}\right) &= -\sum_{n=0}^{N-1} k\left(y_h^{n+\frac{1}{2}} - y_h^{n+\frac{1}{2}}(u), y_h^{n+\frac{1}{2}} - y_h^{n+\frac{1}{2}}(u)\right) \\ &\quad - \sum_{n=0}^{N-1} k\left(\boldsymbol{p}_h^{n+\frac{1}{2}} - \boldsymbol{p}_h^{n+\frac{1}{2}}(u), \boldsymbol{p}_h^{n+\frac{1}{2}} - \boldsymbol{p}_h^{n+\frac{1}{2}}(u)\right) \\ &\leq -\|y_h - y_h(u)\|_{l^2(L^2)}^2 - \|\boldsymbol{p}_h - \boldsymbol{p}_h(u)\|_{l^2(L^2)}^2 \leq 0.\end{aligned} \tag{4.5}$$

By using Hölder's inequality, Young's inequality and Lemma 3.2, we have

$$\begin{aligned}\sum_{n=0}^{N-1} k\left(z_h^{n+\frac{1}{2}}(u) - z^{n+\frac{1}{2}}, u^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}}\right) &\leq C(\varepsilon)\sum_{n=0}^{N-1} k\left\|z_h^{n+\frac{1}{2}}(u) - z^{n+\frac{1}{2}}\right\|^2 + \varepsilon\sum_{n=0}^{N-1} k\left\|u^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}}\right\|^2 \\ &\leq C(\varepsilon)\left(h^2 + k^2\right)^2 + \varepsilon\|u - u_h\|_{l^2(L^2)}^2.\end{aligned} \tag{4.6}$$

Combining (4.4)–(4.6), we obtain (4.1). □

Theorem 4.2. *Let* $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ *and* $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ *be the solution of (2.5)–(2.11) and the solution of (2.17)–(2.23), respectively. Assume that all the conditions in Theorem 4.1 are valid. Then we have*

$$|||y - y_h|||_{l^\infty(L^2)} + |||\boldsymbol{p} - \boldsymbol{p}_h|||_{l^2(L^2)} \leq C\left(h^2 + k^2\right), \tag{4.7}$$

$$|||z - z_h|||_{L^\infty(l^2)} + |||\boldsymbol{q} - \boldsymbol{q}_h|||_{l^2(L^2)} \leq C\left(h^2 + k^2\right). \tag{4.8}$$

Proof. By using the triangle inequality, Lemmas 3.1 and 3.2, Theorem 4.1, it is easy to get (4.7) and (4.8). □

5. Numerical experiments

In this section, we present two numerical examples to validate our theoretical results. The following parabolic OCPs were dealt numerically with codes developed based on AFEPack, which is a freely available software package and the details can be found in [42]. Their discretization schemes are described as (2.17)–(2.23) in Section 2. Let $\Omega = (0,1)\times(0,1)$ and $T = 1$.

Example 1. The data under testing are as follows:

$$\begin{aligned}&a = -0.5, b = 0.5,\\ &y(x,t) = t\sin(2\pi x_1)\sin(2\pi x_2),\\ &\boldsymbol{p}(x,t) = -\left(2\pi t\cos(2\pi x_1)\sin(2\pi x_2), 2\pi t\sin(2\pi x_1)\cos(2\pi x_2)\right),\\ &z(x,t) = (1-t)\sin(2\pi x_1)\sin(2\pi x_2),\\ &\boldsymbol{q}(x,t) = -\left(2\pi(1-t)\cos(2\pi x_1)\sin(2\pi x_2), 2\pi(1-t)\sin(2\pi x_1)\cos(2\pi x_2)\right),\\ &u(x,t) = \max\left\{a, \min\left\{b, -z(x,t)\right\}\right\},\\ &f(x,t) = y_t(x,t) + \mathrm{div}\boldsymbol{p}(x,t) - u(x,t),\\ &y_d(x,t) = z_t(x,t) - \mathrm{div}\boldsymbol{q}(x,t) + y(x,t),\\ &\boldsymbol{p}_d(x,t) = \boldsymbol{q}(x,t) + \nabla z(x,t) + \boldsymbol{p}(x,t).\end{aligned}$$

In Table 1, we list errors of $|||u-u_h|||_{l^2(L^2)}$, $|||y-y_h|||_{l^\infty(L^2)}$, $|||\boldsymbol{p}-\boldsymbol{p}_h|||_{l^2(L^2)}$, $|||z-z_h|||_{l^\infty(L^2)}$ and $|||\boldsymbol{q}-\boldsymbol{q}_h|||_{l^2(L^2)}$ based on a sequence of uniformly refined meshes. In Figure 1, we show the relationship between $\log_{10}(error)$ and $\log_{10}(node)$. It is easy to see that $|||u-u_h|||_{l^2(L^2)} = O(h^2+k^2)$, $|||y-y_h|||_{l^\infty(L^2)} + |||\boldsymbol{p}-\boldsymbol{p}_h|||_{l^2(L^2)} = O(h^2+k^2)$ and $|||z-z_h|||_{l^\infty(L^2)} + |||\boldsymbol{q}-\boldsymbol{q}_h|||_{l^2(L^2)} = O(h^2+k^2)$.

Table 1. Numerical results of Example 1.

$h=k$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{80}$
$\|\|\|u-u_h\|\|\|_{l^2(L^2)}$	4.1056e-02	1.0265e-02	2.5638e-03	6.4095e-04
$\|\|\|y-y_h\|\|\|_{l^\infty(L^2)}$	2.8315e-02	7.0790e-03	1.7697e-03	4.4242e-04
$\|\|\|\boldsymbol{p}-\boldsymbol{p}_h\|\|\|_{l^2(L^2)}$	6.4572e-02	1.6163e-02	4.0407e-03	1.0102e-03
$\|\|\|z-z_h\|\|\|_{l^\infty(L^2)}$	2.8854e-02	7.2135e-03	1.8036e-03	4.5090e-04
$\|\|\boldsymbol{q}-\boldsymbol{q}_h\|\|\|_{l^2(L^2)}$	6.6328e-02	1.6582e-02	4.1455e-03	1.0364e-03

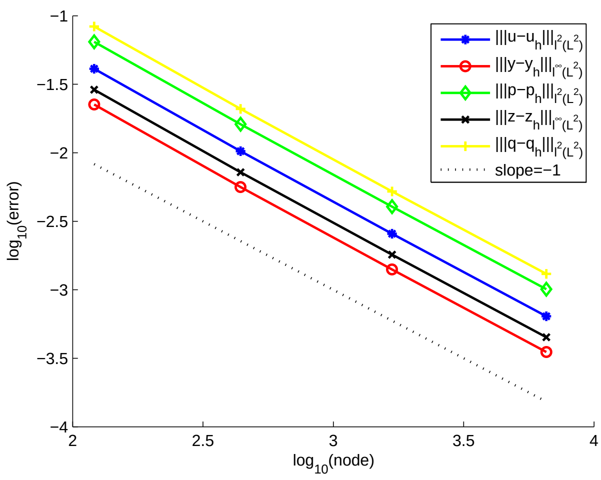

Figure 1. The convergence rate, Example 1.

Example 2. The data under testing are as follows:

$$
\begin{aligned}
&a = -0.25, b = 0.25,\\
&y(x,t) = t\left(x_1 - x_1^2\right)\left(x_2 - x_2^2\right),\\
&\boldsymbol{p}(x,t) = \left(t\,(2x_1 - 1)\left(x_2 - x_2^2\right), t\left(x_1 - x_1^2\right)(2x_2 - 1)\right),\\
&z(x,t) = (1-t)\left(x_1 - x_1^2\right)\left(x_2 - x_2^2\right),\\
&\boldsymbol{q}(x,t) = \left((1-t)\,(2x_1 - 1)\left(x_2 - x_2^2\right), (1-t)\left(x_1 - x_1^2\right)(2x_2 - 1)\right),\\
&u(x,t) = \max\left\{a, \min\left\{b, -z(x,t)\right\}\right\},\\
&f(x,t) = y_t(x,t) + \mathrm{div}\boldsymbol{p}(x,t) - u(x,t),\\
&y_d(x,t) = z_t(x,t) - \mathrm{div}\boldsymbol{q}(x,t) + y(x,t),\\
&\boldsymbol{p}_d(x,t) = \boldsymbol{q}(x,t) + \nabla z(x,t) + \boldsymbol{p}(x,t).
\end{aligned}
$$

The numerical results based on a sequence of uniformly refined meshes are reported in Table 2. We show the relationship between $\log_{10}(error)$ and $\log_{10}(node)$ in Figure 2. It is easy to see that errors of $|||u-u_h|||_{l^2(L^2)}$, $|||y-y_h|||_{l^\infty(L^2)}$, $|||\boldsymbol{p}-\boldsymbol{p}_h|||_{l^2(L^2)}$, $|||z-z_h|||_{l^\infty(L^2)}$ and $|||\boldsymbol{q}-\boldsymbol{q}_h|||_{l^2(L^2)}$ estimates are $O(h^2+k^2)$. It is consistent with the theoretical results in Section 4.

Table 2. Numerical results of Example 2.

$h = k$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{80}$
$\|\|\|u-u_h\|\|\|_{l^2(L^2)}$	2.8365e-02	7.0912e-03	1.7728e-03	4.4320e-04
$\|\|\|y-y_h\|\|\|_{l^\infty(L^2)}$	1.4674e-02	3.6685e-03	9.1712e-04	2.2928e-04
$\|\|\|\boldsymbol{p}-\boldsymbol{p}_h\|\|\|_{l^2(L^2)}$	4.1576e-02	1.0394e-02	2.5985e-03	6.4962e-04
$\|\|\|z-z_h\|\|\|_{l^\infty(L^2)}$	1.6554e-02	4.1389e-03	1.0347e-03	2.5867e-04
$\|\|\boldsymbol{q}-\boldsymbol{q}_h\|\|\|_{l^2(L^2)}$	4.2685e-02	1.0671e-02	2.6680e-03	6.6700e-04

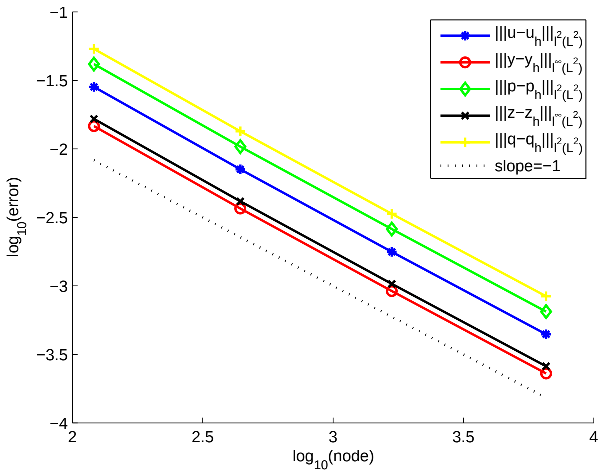

Figure 2. The convergence rate, Example 2.

6. Conclusions

In this paper, we investigate a MFEM combined with CNS approximation of constrained parabolic OCPs and obtain optimal priori error estimates, namely $|||u-u_h|||_{l^2(L^2)} = O(h^2+k^2)$, $|||y-y_h|||_{l^\infty(L^2)} + |||\boldsymbol{p}-\boldsymbol{p}_h|||_{l^2(L^2)} = O(h^2+k^2)$ and $|||z-z_h|||_{l^\infty(L^2)} + |||\boldsymbol{q}-\boldsymbol{q}_h|||_{l^2(L^2)} = O(h^2+k^2)$. Our results are new.

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Conflict of interest

The author declare no conflict of interest in this paper.

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