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Research article

Closure properties for second-order λ-Hadamard product of a class of meromorphic Janowski function

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Abstract: In this paper, we define second-order λ-Hadamard product of a class of meromorphic Janowski function and study the closure properties of the above product.

Keywords: Meromorphic Janowski function; second-order λ-Hadamard product; closure property
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1. Introduction

Let $\Sigma(c, n)(c > 0, n \geq 2, n \in \mathbb{N})$ denote the class of meromorphic functions f which are analytic in the punctured open unit disk $\hat{\mathcal{U}} = \mathcal{U}\backslash\{0\} = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and has the Taylor series expansion of form

$$f(z) = \frac{c}{z} + \sum_{k=n}^{\infty} a_k z^k. \tag{1.1}$$

We define the following class $\Sigma S_n^*(c,\beta)$ of the generalized meromorphically starlike of order β [1,2],

$$\Sigma S_n^*(c,\beta) = \left\{ f(z) \in \Sigma(c, n) : -\mathrm{Re}\frac{zf'(z)}{f(z)} > \beta, \beta \in [0, 1) \right\}.$$

Let $f(z)$ and $g(z)$ be members of analytic function in $\mathcal{U}$, $f(z)$ is said to be subordinate to $g(z)$, if there exists a Schwarz function $\omega(z)$, analytic in $\mathcal{U}$ with

$$\omega(0) = 0 \;\; and \;\; |\omega| < 1,$$

such that

$$f(z) = g(\omega(z))(z \in \mathbb{U}).$$

In such a case, we write $f(z) \prec g(z)$. Furthermore if the function $g(z)$ is univalent in $\mathcal{U}$, then we have [3,4],

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathcal{U}) \subset g(\mathcal{U}).$$

In this paper, we study a class of meromorphic Janowski functions as follows. [5,6]

Definition 1.1. *A function $f(z) \in \Sigma(c, n)$ belongs to the class $\Sigma S_n^*(c, a, b)$, if and only if*

$$-\frac{zf'(z)}{f(z)} \prec \frac{1 + az}{1 + bz} \quad (z \in \hat{\mathcal{U}}).$$

It is quite clear that

$$\Sigma S_n^*(c, a, b) \subset \Sigma S_n^*(c, \frac{1 - a}{1 - b}) \ (-1 \leq b < a \leq 1).$$

Using the subordination relationship, $f(z) \in \Sigma S_n^*(c, a, b)$, if satisfies the following condition

$$-\frac{zf'(z)}{f(z)} = \frac{1 + aw(z)}{1 + bw(z)} \quad (z \in \hat{\mathcal{U}}),$$

where $\omega(z)$ is analytic in $\mathcal{U}$ with $|\omega(z)| < 1$ and $\omega(0) = 0$.

The above equation can also be equivalent to

$$\left|\frac{f(z) + zf'(z)}{af(z) + bzf'(z)}\right| < 1, \quad (z \in \hat{\mathcal{U}}).$$

Let $T(c, n)$ be a subclass of $\Sigma(c, n)$, and $f \in T(c, n)$ is defined as

$$f(z) = \frac{c}{z} - \sum_{k=n}^{\infty} |a_k| z^k. \tag{1.2}$$

In particular, taking

$$\overline{\Sigma S}_n^*(c, a, b) = T(c, n) \cap \Sigma S_n^*(c, a, b).$$

Next, we will introduce the second-order generalized λ-Hadamard product of the class $T(c, n)$.

Definition 1.2. *Let $f_i(z) = \frac{c}{z} - \sum_{k=n}^{\infty} |a_{k,i}| z^k \in T(c, n)$ $(i = 1, 2)$. The second-order generalized λ-Hadamard product $(f_1 \widetilde{\triangle} f_2)_\lambda(u, v, c; z)$ of the function f_1 and f_2 is defined by*

$$(f_1 \widetilde{\triangle} f_2)_\lambda(u, v, c; z) = (1 - \lambda) z (f_1 \triangle f_2)'(u, v, c; z) - \lambda z (z (f_1 \triangle f_2)')'(u, v, c; z) + \frac{2c^2}{z}, \tag{1.3}$$

where

$$(f_1 \triangle f_2)(u, v, c; z) = \frac{c^2}{z} - \sum_{k=n}^{\infty} |a_{k,1}|^u |a_{k,2}|^v z^k. \tag{1.4}$$

We can obtain, from (1.3) and (1.4), that

$$(f_1\widetilde{\triangle}f_2)_\lambda(u,v,c;z)=\frac{c^2}{z}-\sum_{k=n}^{\infty}[k-k(k+1)\lambda]|a_{k,1}|^u|a_{k,2}|^v z^k.$$

In particular, if $u=v=1$, then $(f_1\widetilde{\triangle}f_2)_\lambda(1,1,c;z)=(f_1\overline{*}f_2)_\lambda(z)$ is the second-order λ-Hadamard product as follows:

$$\begin{aligned}(f_1\overline{*}f_2)_\lambda(z)&=(1-\lambda)z(f_1\triangle f_2)'(1,1,c;z)-\lambda z(z(f_1\triangle f_2)')'(1,1,c;z)+\frac{2c^2}{z}\\ &=\frac{c^2}{z}-\sum_{k=n}^{\infty}[k-k(k+1)\lambda]|a_{k,1}||a_{k,2}|z^k.\end{aligned} \tag{1.5}$$

Note. By choosing different parameters c,λ,u and v, we can get special convolutions as below:

(1) For $\lambda=0,\ c=1,\ (f_1\triangle f_2)_0(u,v,1;z)=(f_1\triangle f_2)(u,v;z)$ is the product defined in [7,8].
(2) For $\lambda=0,\ u=v=1, c=1,\ (f_1\widetilde{\triangle}f_2)_0(1,1,1;z)=(f_1 * f_2)(z)$ is the famous Hadamard product defined in [1].

For the sake of simplicity, the parameters for the rest of the article are specified below

$$c>0, n\in\mathbb{N}, n\geq 2, a\in R, |a|\leq 1, b\in R, |b|\leq 1, a\neq b.$$

At the same time, let

$$\overline{\Sigma S}_n^*(c,a,b)=T(c,n)\cap\Sigma S_n^*(c,a,b).$$

In 1996, Choi et al. [7] studied the generalized λ-Hadamard product of univalent functions. In 2021, the authors [3] discuss the closure properties of the first-order λ-Hadamard product of the class $\overline{\Sigma S}_n^*(c,a,b)$.

In this paper, we will continue to discuss the closed problems of second-order generalized λ-Hadamard product of the class $\Sigma S_n^*(c,a,b)$ and obtain some new results.

2. Perliminaries

Lemma 2.1. *[3] If $f\in\Sigma(c,k)$ satisfies*

$$\sum_{k=n}^{\infty}\frac{[(k+1)+|a+kb|]|a_k|}{c|a-b|}\leq 1, \tag{2.1}$$

then $f(z)\in\Sigma S_n^*(c,a,b)$.

Lemma 2.2. *[3] Let the function $f\in T(c,k)$, then $f\in\overline{\Sigma S}_n^*(c,a,b)$ if and only if*

$$\begin{cases}\sum_{k=n}^{\infty}\frac{[k+1+(a+kb)]|a_k|}{c(a-b)}\leq 1, & 0\leq b<a\leq 1,\\ \sum_{k=n}^{\infty}\frac{[k+1-(a+kb)]|a_k|}{c(b-a)}\leq 1, & -1\leq a<b\leq 0.\end{cases} \tag{2.2}$$

3. Main results

First, we investigate the closure properties of the second-order λ-Hadamard product $(f_1 \overline{*} f_2)_\lambda(z)$.

Theorem 3.1. *If* $f_i(z) = \frac{c}{z} - \sum_{k=n}^{\infty} |a_{k,i}|z^k \in \overline{\Sigma S}^*_n(c, a, b)$ $(i = 1, 2)$ *satisfy one of the following conditions:*

(1) $0 \le b < a \le 1,\ 0 < \lambda < \frac{1}{n+1}$,
(2) $-1 \le a < b \le 0,\ 0 < \lambda < \frac{1}{n+1}$,

then the second-order λ*-Hadamard product* $\frac{1}{c}(f_1 \overline{*} f_2)_\lambda(z) \in \overline{\Sigma S}^*_n(c, a, b)$.

Proof. Let $f_1, f_2 \in \overline{\Sigma S}^*_n(c, a, b)$, which are given by (1.2).

(1) For $0 \le b < a \le 1$, we can get from (2.2),

$$\sum_{k=n}^{\infty} \frac{(k + 1 + (a + kb))|a_{k,1}|}{c(a - b)} \le 1 \tag{3.1}$$

and

$$\sum_{k=n}^{\infty} \frac{(k + 1 + (a + kb))|a_{k,2}|}{c(a - b)} \le 1. \tag{3.2}$$

Since

$$\frac{1}{c}(f_1 \overline{*} f_2)_\lambda(z) = \frac{1}{c}\left[(1 - \lambda)z(f_1 \triangle f_2)'(1, 1, c; z) - \lambda z(z(f_1 \triangle f_2)')'(1, 1, c; z) + \frac{2c^2}{z}\right]$$
$$= \frac{c}{z} - \sum_{k=n}^{\infty} \frac{1}{c}(k - k(k + 1)\lambda|a_{k,1}||a_{k,2}|z^k,$$

then, to obtain $\frac{1}{c}(f_1 \overline{*} f_2) \in \overline{\Sigma S}^*_n(c, a, b)$, we only need to verify the following condition is established.

$$\sum_{k=n}^{\infty} \frac{[(k - k(k + 1)]\lambda)(k + 1 + (a + kb))|a_{k,1}||a_{k,2}|}{c^2(a - b)} \le 1. \tag{3.3}$$

Using Cauchy-Schwarz inequality, it can be obtained from Eqs (3.1) and (3.2) that

$$\sum_{k=n}^{\infty} \frac{(k + 1 + (a + kb))\sqrt{|a_{k,1}||a_{k,2}|}}{c(a - b)} \le 1. \tag{3.4}$$

From (3.3) and (3.4), we just need to prove

$$\sqrt{|a_{k,1}||a_{k,2}|} \le \frac{c}{(k - k(k + 1)\lambda)}.$$

Since

$$\sqrt{|a_{k,1}||a_{k,2}|} \le \frac{c(a - b)}{k + 1 + (a + kb)}.$$

Thus, we just have to show

$$a - b \leq \frac{k + 1 + (a + kb)}{(k - k(k + 1)\lambda)}, n \geq k. \tag{3.5}$$

For $\lambda < \frac{1}{n+1}$, (3.5) can be simplified to $\lambda > \frac{k(a-b)-[k+1+(a+kb)]}{(a-b)k(k+1)}$.

Therefore, we conclude that if

$$\frac{1}{n+1} > \lambda > \max\left\{\frac{k(a-b) - [k+1+(a+kb)]}{(a-b)k(k+1)}, 0\right\} = 0,$$

then $\frac{1}{c}(f_1 \tilde{*} f_2)_\lambda(z) \in \overline{\Sigma S}^*_n(c, a, b)$.

(2) For $-1 \leq a < b \leq 0$. We can using the same method above, obtain that if

$$\frac{1}{n+1} > \lambda > \max\left\{\frac{k(b-a) - (k+1-a-kb)}{(b-a)k(k+1)}, 0\right\} = 0,$$

then $\frac{1}{c}(f_1 \tilde{*} f_2)_\lambda(z) \in \overline{\Sigma S}^*_n(c, a, b)$. This completes the proof of Theorem 3.1. □

Corollary 3.1. *Let $u > 1, f_i(z) \in \overline{\Sigma S}^*_n(c, -1, 0)$ $(i = 1, 2)$. If $0 < \lambda < \frac{1}{n+1}$, then*

$$(f_1 \widetilde{\triangle} f_2)_\lambda(1, 1, c; z) \in \overline{\Sigma S}^*_n(c, -1, 0).$$

Next, we consider the closure properties of the second-order generalized λ-Hadamard product.

Theorem 3.2. *Let $u > 1, f_i(z) = \frac{c}{z} - \sum_{k=n}^{\infty} |a_k| z^k \in \overline{\Sigma S}^*_n(c, a, b)$ $(i = 1, 2)$. If one of the following conditions,*

(i) $0 \leq \hat{b} < b < a \leq 1$, $\frac{2n+1}{(n+1)(3n+1)} < \lambda < \min(\frac{c(1+b)+2n(a-b)}{(3n^2+2n)(a-b)}, \frac{1}{n+1})$, *and* $0 \leq \hat{b} < \min(1, a - \frac{(a-b)[n+1+(a+nb)]n[1-(n+1)\lambda]}{c[n+1+(a+nb)]+(a-b)n^2[1-(n+1)\lambda]})$;

(ii) $-1 \leq a < b < \hat{b} \leq 0$, *and* $0 < \lambda < \frac{2n+1}{(n+1)(3n+1)}$, *holds true, then* $\frac{1}{c}(f_1 \widetilde{\triangle} f_2)_\lambda\left(\frac{1}{u}, \frac{u-1}{u}, c, z\right) \in \overline{\Sigma S}^*_n(c, a, \hat{b})$.

Proof. Let $f_i(z) \in \overline{\Sigma S}^*_n(c, a, b) (i = 1, 2)$.

(1) Let $0 \leq b < a \leq 1$, using (2.2), we have

$$\sum_{k=n}^{\infty} \frac{k + 1 + (a + kb)}{c(a - b)} |a_{k,i}| \leq \ (i = 1, 2).$$

So we get

$$\left\{\sum_{k=n}^{\infty} \frac{k + 1 + (a + kb)}{c(a - b)} |a_{k,1}|\right\}^{\frac{1}{u}} \leq 1, \tag{3.6}$$

and

$$\left\{\sum_{k=n}^{\infty} \frac{k + 1 + (a + kb)}{c(a - b)} |a_{k,2}|\right\}^{\frac{u-1}{u}} \leq 1. \tag{3.7}$$

Since

$$(f_1\widetilde{\triangle}f_2)_\lambda(u,v,c;z)=\frac{c^2}{z}-\sum_{k=n}^{\infty}[k-k(k+1)\lambda]|a_{k,1}|^u|a_{k,2}|^v z^k,$$

then, to get $\frac{1}{c}(f_1\widetilde{\triangle}f_2)(\frac{1}{u},\frac{u-1}{u},c;z)\in\overline{\Sigma S}_n^*(c,a,\widehat{b})$, we just only need to verify that the following condition:

$$\sum_{k=n}^{\infty}\frac{[k+1+(a+k\widehat{b})][(k-k(k+1)\lambda]}{c^2(a-\widehat{b})}|a_{n,1}|^{\frac{1}{u}}|a_{k,2}|^{\frac{u-1}{u}}\le 1 \tag{3.8}$$

holds true.

Applying the Hölder inequality, from (3.6) and (3.7), we get

$$\sum_{k=n}^{\infty}\frac{k+1+(a+kb)}{c(a-b)}|a_{k,1}|^{\frac{1}{u}}|a_{k,2}|^{\frac{u-1}{u}}\le 1.$$

In order to obtain inequality (3.8), we only need to prove

$$\frac{[k-k(k+1)\lambda][k+1+(a+k\widehat{b})]}{c^2(a-\widehat{b})}\le\frac{k+1+(a+kb)}{c(a-b)}, \tag{3.9}$$

that is,

$$\widehat{b}[\frac{c[k+1+(a+kb)]}{a-b}+[k^2-k^2(k+1)]\lambda)\le\frac{ca[k+1+(a+kb)]}{a-b}-(a+k+1)[k-k(k+1)\lambda].$$

Let

$$P_1(k)=\frac{c[k+1+(a+kb)]}{a-b}+[k^2-k^2(k-1)\lambda]$$

and

$$Q_1(k)=\frac{ca[k+1+(a+kb)]}{a-b}-(a+k+1)[(k-k(k+1)\lambda].$$

After simplification, we can get

$$\frac{Q_1(k)}{P_1(k)}=a-\frac{[k+1+(a+kb)](a-b)(k-k(k+1)\lambda)}{c[k+1+(a+kb)]+(a-b)[k^2-k^2(k+1)\lambda]}.$$

Suppose

$$\overline{P_1(k)}=[k+1+(a+kb)](a-b)(k-k(k+1)\lambda)$$

and

$$\overline{Q_1(k)}=c[k+1+(a+kb)]+(a-b)[k^2-k^2(k+1)\lambda].$$

To prove Theorem 3.2, we divide the procedure into two cases.

(a) Because $P_1(k)$ is increasing with respect to k and $P_1(n)=\frac{c[n+1+(a+nb)]}{a-b}+n^2[1-(n+1)\lambda]>0$, $P_1(k)>0$ is true for $k\ge n$. In order to prove (3.9), we only need to show the following inequality

$$\widehat{b}\le\frac{Q_1(k)}{P_1(k)}=a-\frac{(a-b)[k+1+(a+kb)](k-k(k+1)\lambda)}{c[k+1+(a+kb)]+(a-b)[k^2-k^2(k+1)\lambda]}. \tag{3.10}$$

It is easily to verify (3.10) holds true if $\lambda < \frac{(a-b)n^2+a[n+1+(a+nb)]}{(a-b)(n+1)n^2}$.

It is clear that $\overline{P_1(k)}$ is monotonically increasing, so we get $\lambda < \frac{c(1+b)+2k(a-b)}{(3k^2+2k)(a-b)}$.

Since $\overline{Q_1(k)}$ is monotonically decreasing with respect to k if $\lambda > \frac{2k+1}{(k+1)(3k+1)}$, then

$$\widehat{b} \leq \frac{\overline{P_1(k)}}{\overline{Q_1(k)}} = a - \frac{(n - n(n+1))\lambda(a-b)(n+1+(a+nb))}{c(n+1+(a+nb)) + (a-b)(n^2 - n^2(n+1)\lambda)},$$

and $\overline{Q_1(k)}$ is increasing with respect to k if $\lambda < \frac{2k+1}{(k+1)(3k+1)}$, then

$$\widehat{b} \leq \lim_{k\to+\infty} \frac{\overline{P(k)}}{\overline{Q(k)}} = -1.$$

Because of $\widehat{b} \geq 0$, this case is removed.

(b) Because $P_1(k)$ is decreasing with respect to k and $P_1(n) = \frac{c[n+1+(a+nb)]}{a-b} + n^2[1-(n+1)\lambda] < 0$, $P_1(k) < 0$ is true for $k \geq n$. (3.9) holds true if

$$\widehat{b} \geq \frac{Q_1(k)}{P_1(k)} = a - \frac{[k+1+(a+kb)](a-b)(k-k(k+1)\lambda)}{c[k+1+(a+kb)] + (a-b)[k^2 - k^2(k+1)\lambda]}. \tag{3.11}$$

Clearly, (3.11) holds true if $\lambda > \frac{(a-b)n^2+c[k+1+(a+kb)]}{(a-b)(k+1)k^2}$. Since $\lambda < \frac{1}{k+1}$, there is no solution for λ.

Since $P_1(k)$ and $\overline{P_1(k)}$ are decreasing functions, we have $\lambda > \frac{c(1+b)+2k(a-b)}{(3k^2+2k)(a-b)}$. $\overline{Q_1(k)}$ is monotonically decreasing with respect to k if $\lambda > \frac{2k+1}{(k+1)(3k+1)}$, we have

$$\widehat{b} \geq \frac{\overline{Q_1(k)}}{\overline{P_1(k)}} = a - \frac{(a-b)\lambda(n+1+(a+nb))(n-n(n+1))}{c(n+1+(a+nb)) + (a-b)n^2(1-n+1)\lambda)}.$$

And $\overline{Q_1(k)}$ is increasing if $\lambda < \frac{2k+1}{(k+1)(3k+1)}$. On the other hand, because $\lambda > \frac{c(1+b)+2k(a-b)}{(3k^2+2k)(a-b)}$, there is no solution for λ.

If the condition (i) is satisfied, then (3.9) is true. Therefore, $\frac{1}{c}(f_1 \widetilde{\triangle} f_2)_\lambda\left(\frac{1}{u}, \frac{u-1}{u}, c, z\right) \in \overline{\Sigma S}_n^*(c, a, \widehat{b})$.

(2) For $-1 \leq a < b \leq 0$, the proof is similar to (1), so we can obtain

$$\begin{aligned} &\widehat{b}[\frac{c[k+1-(a+kb)]}{b-a} + (k^2 - k^2(k+1)\lambda)] \\ \leq\ &\frac{ca[k+1-(a+kb)]}{b-a} + (-a+k+1)[k - k(k+1)\lambda]. \end{aligned} \tag{3.12}$$

Let

$$P_2(k) = \frac{c[k+1-(a+kb)]}{b-a} + [k^2 - k^2(k-1)\lambda]$$

and

$$Q_2(k) = \frac{ca[k+1-(a+kb)]}{b-a} + (-a+k+1)[k - k(k+1)\lambda].$$

After simple calculation, we can obtain

$$\frac{Q_2(k)}{P_2(k)} = a + \frac{(k+1-(a+kb))(b-a)k(1-(k+1)\lambda)}{c[k+1-(a+kb)] + (b-a)k^2[1-(k+1)\lambda]}.$$

Assume

$$\overline{P_2(k)} = (k+1-(a+kb))(b-a)(k-k(k+1)\lambda)$$

and

$$\overline{Q_2(k)} = c[k+1-(a+kb)] + (b-a)[k^2 - k^2(k+1)\lambda].$$

We divide the discussion into two cases as follows:

(c) Because $P_2(k)$ is increasing with respect to k and $P_2(n) = \frac{c[n+1+(a+nb)]}{b-a} + n^2[1-(n+1)\lambda] > 0$, $P_2(k) > 0$ is true for $k \geq n$. In order to prove (3.12), we only need the following inequality to be true

$$\widehat{b} \leq \frac{Q_2(k)}{P_2(k)} = a + \frac{(k+1-(a+kb))(b-a)(k-k(k+1)\lambda)}{c[k+1-(a+kb)] + (b-a)[k^2 - k^2(k+1)\lambda]}. \tag{3.13}$$

It is not hard to verify (3.13) holds true if $\lambda < \frac{(b-a)n^2 + c(n+1-(a+nb))}{(b-a)(n+1)n^2}$.

On the other hand, $\overline{P_2(k)}$ is increasing with respect to k, we can get $\lambda < \frac{c(1-b)+2k(b-a)}{(3k^2+2k)(b-a)}$.

It is clear that $\overline{Q_2(k)}$ is decreasing if $\lambda > \frac{2k+1}{(k+1)(3k+1)}$, then

$$\widehat{b} \leq \frac{\overline{P_2(k)}}{\overline{Q_2(k)}} = a + \frac{(n-n(n+1))\lambda(b-a)(n+1-(a+nb))}{c(n+1-(a+nb)) + n^2(b-a)(1-(n+1)\lambda)},$$

and $\overline{Q_2(k)}$ is increasing with respect to k if $\lambda < \frac{2k+1}{(k+1)(3k+1)}$, then

$$\widehat{b} \leq a + \lim_{k\to+\infty} \frac{\overline{P_2(k)}}{\overline{Q_2(k)}} = 1.$$

(d) Because $P_2(k)$ is decreasing with respect to k and $P_2(n) = n^2[1-(n+1)\lambda] + \frac{c[n+1-(a+nb)]}{b-a} < 0$, $P_2(k) < 0$ is true for $k \geq n$. In order to prove (3.12), we only need the following inequality to be true

$$\widehat{b} \geq \frac{Q_2(k)}{P_2(k)} = a + \frac{(k+1-(a+kb))(b-a)(k-k(k+1)\lambda)}{c[k+1-(a+kb)] + [k^2 - k^2(k+1)\lambda](b-a)}. \tag{3.14}$$

It is not hard to verify (3.14) holds true if $\lambda > \frac{(b-a)n^2 + c(n+1-(a+nb))}{(b-a)(n+1)n^2}$.

Also, $\overline{P_2(k)}$ is decreasing with respect to k, we can obtain $\lambda > \frac{c(1-b)+2k(b-a)}{(3k^2+2k)(b-a)}$. So, there is no solution for λ.

If the condition (ii) is satisfied, then (3.12) is true. Therefore, $\frac{1}{c}(f_1 \widetilde{\triangle} f_2)_\lambda\left(\frac{1}{u}, \frac{u-1}{u}, c, z\right) \in \overline{\Sigma S}^*_n(c, a, \hat{b})$. □

When $b = \frac{1}{2}$ and $a = 1$ in Theorem 3.2, we have the following corollary.

Corollary 3.2. *Let* $u > 1, f_i(z) = \frac{c}{z} - \sum_{k=n}^{\infty} |a_{k,i}|z^k (i = 1,2) \in \overline{\Sigma S}_n^*(c, 1, \frac{1}{2})$*. If*

$$\frac{2n+1}{(n+1)(3n+1)} < \lambda < \min(\frac{3c+2n}{3n^2+2n}, \frac{1}{n+1}) \text{ and } 1 - \frac{[n-n(n+1)\lambda](3n+4)}{c(6n+8)+2[n^2-n^2(n+1)\lambda]} > \hat{b} \geq 0,$$

then $\frac{1}{c}(f_1 \widetilde{\triangle} f_2)_\lambda \left(\frac{1}{u}, \frac{u-1}{u}, c, z\right) \in \overline{\Sigma S}_n^*(c, 1, \frac{1}{2})$.

By putting $b = -\frac{1}{2}$ and $a = -1$ in Theorem 3.2, we have

Corollary 3.3. *Let* $u > 1, f_i(z) = \frac{c}{z} - \sum_{k=n}^{\infty} |c_{k,i}|z^k (i = 1,2) \in \overline{\Sigma S}_n^*(c, -1, -\frac{1}{2})$.

If $-\frac{1}{2} < \hat{b} \leq 0,\ 0 < \lambda < \frac{2n+1}{(n+1)(3n+1)}$, *then* $\frac{1}{c}(f_1 \widetilde{\triangle} f_2)_\lambda \left(\frac{1}{u}, \frac{u-1}{u}, c, z\right) \in \overline{\Sigma S}_n^*(c, -1, -\frac{1}{2})$.

4. Conclusions

In this paper, we prove the closure properties of the second-order λ-Hadamard product and the second-order generalized λ-Hadamard product of the class $\overline{\Sigma S}_n^*(c, a, b)$. The results presented in this paper would find further applications for the λ-Hadamard product of the class of meromorphic Janowski function, which can enrich the research field of Hadamard procduct.

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Conflict of interest

The authors declare no conflict of interest.

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