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Research article

Probabilistic linguistic multi-attribute decision making approach based upon novel GMSM operators

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Abstract: Probabilistic linguistic terms set (PLTS), a new tool for expressing uncertain decision information, is composed of all possible linguistic terms (LTs) and their related probabilities. It also increases the corresponding probability of LTs in hesitant fuzzy linguistic term set (HFLTS). On the other hand, aggregation operator is an important information fusion tool, the Maclaurin symmetric mean (MSM) operator can provide more flexibility and robustness in information fusion, and make it more suitable for solving MADM problems with independent attributes. This current study adopts the merits of PLTS and MSM operator, and then a novel probabilistic linguistic decision making approach are targeted. Firstly, the operations of two PLTSs are redefined based upon Archimedean t-norm (ATN) and Archimedean t-conorm (ATC); Secondly, the probabilistic linguistic generalized MSM operator (PLGMSM) is proposed based on ATN and ATC, some properties of PLGMSM are investigated, then some special PLGMSM operators have been studied in detail when the parameters take different values and the generator of ATN takes different functions. Thirdly, the weighted probabilistic linguistic generalized MSM operator (WPLGMSM) is studied along with some properties of PLGMSM, some special WPLGMSM operators have been also investigated in detail when the parameters take different values and the generator of ATN takes different functions. Finally, on the basis of our proposed aggregation operators, the aggregated-based decision making approach is designed and an example is supplied to manifest the effectiveness of the proposed approach. Furthermore, some comparison analyses with extant decision approaches are carried out to illustrate the validity and feasibility of the proposed approach.

Keywords: probabilistic linguistic term set; Archimedean t-norm; Archimedean t-conorm; (weighted) probabilistic linguistic GMSM operators; MADM
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1. Introduction

Generally speaking, multi-criteria evaluation refers to the evaluation conducted under multiple criteria that cannot be replaced by each other. In the specific evaluation process, information is often missing because of the wide range of evaluation criteria. Some information is available but very inaccurate; Some can only give a rough range based on empirical judgment. Therefore, in the process of scheme evaluation, quantitative calculation shall be carried out for those criteria that can be accurately quantified; For those criteria that are difficult to accurately quantify or cannot be quantified, it is necessary to make a rough estimation or invite relevant experts to conduct qualitative analysis and hierarchical semi quantitative description. On account of the complexity and uncertainty of objective things, as well as the fuzziness brought by human cognitive level and thinking mode, it is difficult for experts to give accurate and quantitative information in evaluation process. Therefore, how to realize mutual transformation between qualitative and quantitative as well as reflect the soft reasoning ability in linguistic expression has always been a research hot-spot in uncertain system evaluation and decision-making.

For example, the fuzzy set [1] shows the relationship between scheme and criterion in a quantitative way, which has been recognized by many scholars. Since then, quantitative decision tools, that is, various extensions of fuzzy set [2–5], have been emerged to show decision information. However, with the ceaseless advancement and change of decision-making environment, it is difficult for decision-makers to use a set of quantitative and specific values to describe the decision-making information of a scheme under a certain criterion. To solve this deficiency, Zadeh proposed linguistic variable (LV) [6], which qualitatively displays the decision information of decision-makers in a non-numerical way for the first time. Then, In line with LV, decision tools such as uncertain LV [7–9], hesitant fuzzy linguistic term set (HFLTS) [10, 11], terms with weakening modifiers [12] appeared to help decision-makers give qualitative decision information. In the face of complex MADM problems, owing to the influence of complex information as well as uncertain factors of group cognition, people sometimes use a single linguistic term (LT) to describe attribute evaluation information, but sometimes need to use several LTs to express decision information at the same time [13, 14]. For example, when students evaluate the quality of class teaching, they may use both “good” and “very good” to evaluate it at the same time. Inspired by hesitant fuzzy sets [15] and linguistic term set (LTS) [6], Rodriguez et al [10] defined HFLTS in 2012, which allows decision-makers to use several possible LTs to evaluate attributes simultaneously.

Although HFLTSs can meet the needs of decision-makers to express information by multiple LTs, the HFLTSs are assumed that the weight of all possible LTs are equal. Obviously, this assumption is too idealistic and inconsistent with actual situation. Because although decision-makers shilly-shally about several possible LTs, they may tend to use some of them under certain circumstances. Therefore, different LTs should possess different weights. In line with this reality, Pang et al. (2016) Proposed probabilistic linguistic term sets (PLTSs), which is composed of possible LTs and associated with their probabilities [16]. On the one hand, probability linguistic contains several LTs to show decision-makers view in decision-making, which retains the good nature of HFLTS. On the other hand, it reflects the corresponding weights of several LTs. This way of displaying decision information by combining qualitative and quantitative information well reflects decision-makers decision information, which will not lose linguistic evaluation information, so they can make decision evaluation more in line with the reality. Although it is difficult to give a definite decision-making view on the problems that needs to decide, it will give the weight of corresponding view and give relatively clear decision-making information as much as possible to help experts solve decision-making problems. Briefly, the merit of PLTSs is that it can express information more completely and accurately. Hence, PLTSs could be utilized to solve practical decision-making problems.

Aggregation operator (AO) is an important tool for information fusion. Most AOs are built on the special triangle t-norm. Archimedeans t-norm (ATN) and Archimedeans t-conorm (ATC) are composed of t-norm (TN) and t-conorm(TC) families. They can deduce some basic algorithms of fuzzy sets. Linguistic scaling functions can define different semantics for LTs in different linguistic environments. At the same time, the significant advantage of Muirhead operator is that it can reflect the relationship between any parameters. Liu et al. [17] defined the algorithm for PLTS based on ATN and linguistic scaling function, and then combined the Muirhead average operator with the PLTSs to propose the Archimedeans Muirhead average operator and Archimedeans weighted Muirhead average operator of probabilistic linguistic, Archimedean dual Muirhead average operator of probability linguistic and Archimedean weighted dual Muirhead average operator of probability linguistic. After that, more and more attention has been paid to various aggregation operators [18–20]. In real MADM, it is rare that the evaluation attributes of various alternatives are independent with each other. For example, there is a positive correlation between teaching quality and lesson design, that is, the better the curriculum design, the higher the quality of teaching. For capturing these dependencies, Maclaurin [21] initially proposed Maclaurin symmetric mean (MSM) operator, which can consider the relationship between multiple attribute values at the same time, and MSM has an adjusting parameter k. The MSM operator is a flexible operator that can consider the relationship between several attribute values. Therefore, it is essential to develop some MSM operators [22–34] in different polymerization environment; Besides, the operation laws are essential in the process of aggregation, and they can generate many operation laws based on certain ATTs and ATCs. Although MSM operators have attracted a lot of attentions since it's appearance, MSM operator has some disadvantages. The main disadvantage of MSM operator is that it only focus on the overall relationship, ignoring heterogeneity among individuals. To address this handicap, Detemple and Robertson [35] proposed generalized Maclaurin symmetric mean (GMSM), which is considered as a new generalization of MSM. GMSM can not only reflect the relationship of the whole, but also consider the importance level of individuals. Besides, compared with MSM, GMSM can avoid information loss. Because the polymerization process increased equality constraints. Therefore, GMSM is extensively employed in information fusion.

Since PLTSs are introduced by integrating the LTSs and the HFSs, PLTSs can successfully express random and fuzzy information. Therefore, it is necessary to develop a novel important probabilistic linguistic information fusion tool (that is, PLGMSM operator) which can not only combine the merits of PLTSs and MSM, but also reflect the relationship of the whole and consider the importance level of individuals. These considerations lead us to lock the main targets that follow from this work:

(1) To introduce new probabilistic linguistic GMSM operators along with investigate some properties as well as some special situations;

(2) To construct an MADM algorithm based upon the proposed PLGMSM operators;

(3) To manifest an example based on probabilistic linguistic information to prove the availability of the proposed MADM approach;

(4) To analyze the sensitivities of parameters in the proposed aggregation operators.

To achieve the above objective, some probabilistic linguistic GMSM operators are introduced for PLTSs based on ATN and ATC in current work. The structure of this work is arranged as: In Sect.2, some related basic concepts are presented, for instance PLTS, MSM operators, ATN and ATC, etc. In Sect.3, the PLGMSM are introduced based upon the ATN and ATC, some properties of the PLGMSM operators and special situations of PLGMSM operators are also given. In Sect.4, the weighted PLGMSM (WPLGMSM) is introduced based upon the ATN and ATC, some properties of the PLGMSM and special situations of WPLGMSM are also listed in this section. Section. 5 constructs a MADM method for evaluating quality of classroom teaching. Some comparisons are carried out in Sect.6. and a conclusion is made in Sect.7.

2. Related knowledge

Some basic concepts will be reviewed in this part, including linguistic term set (LTS), probabilistic linguistic term set (PLTS), ATN and ATC.

2.1. PLTS

Definition 2.1. *[36] Suppose $S = \{s_v | v = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a LTS, where s_v expresses a possible value of a LV, and τ is a positive integer. For any two LVs $s_\alpha, s_\beta \in S$, it satisfies: if $\alpha > \beta$, then $s_\alpha > s_\beta$.*

Definition 2.2. *[16] Suppose $S = \{s_v | v = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a LTS, a PLTS is defined as following:*

$$\ell(p) = \left\{ \ell^{(i)} \left(p^{(i)} \right) \middle| \ell^{(i)} \in S, p^{(i)} \geqslant 0, i = 1, \ldots, \#\ell(p), \sum_{i=1}^{\#\ell(p)} p^{(i)} \leqslant 1 \right\} \tag{2.1}$$

where $\ell^{(i)} \left(p^{(i)} \right)$ denotes the i-th LV ℓ with probability $p^{(i)}$, and $\#\ell(p)$ represents the number of all different elements in $\ell(p)$.

In line with Definition 2.2, Eq (2.1) can be transformed

$$\ell(\widetilde{p}) = \left\{ \ell^{(i)} \left(\widetilde{p}^{(i)} \right) \middle| \ell^{(i)} \in S, \widetilde{p}^{(i)} \geq 0, i = 1, \ldots, \#\ell(p), \sum_{i=1}^{\#\ell(p)} \widetilde{p}^{(i)} = 1 \right\} \tag{2.2}$$

where $\widetilde{p}^{(i)} = p^{(i)} / \sum_{i=1}^{\#\ell(p)} p^{(i)}$.

The score of $\ell(p)$ can be calculated as

$$E(\ell(p)) = s_r, \tag{2.3}$$

where $r = \sum_{i=1}^{\#\ell(p)} r^{(i)} p^{(i)} / \sum_{i=1}^{\#\ell(p)} p^{(i)}$, $r^{(k)}$ is subscript of $\ell^{(k)}$.

The deviation degree of $\ell(p)$ is

$$\sigma(\ell(p)) = \left(\sum_{i=1}^{\#\ell p} \left((r^{(i)} - r)p^{(i)}\right)^2\right)^{1/2} / \sum_{i=1}^{\#\ell(p)} p^{(i)}. \tag{2.4}$$

For any two LPTSs $\ell_1(p), \ell_2(p)$,
(1) if $E(\ell_1(p)) > E(\ell_2(p))$, the $\ell_1(p) > \ell_2(p)$;
(2) if $E(\ell_1(p)) = E(\ell_2(p))$, when $\sigma(\ell_1(p)) > \sigma(\ell_2(p))$, then $\ell_1(p) < \ell_2(p)$;
(3) if $E(\ell_1(p)) = E(\ell_2(p))$, when $\sigma(\ell_1(p)) = \sigma(\ell_2(p))$, then $\ell_1(p) = \ell_2(p)$.

In order to calculate the PLTSs more conveniently, the transformation function g was introduced by Gou et al. [37]. Suppose there is a LTS S and a PLTS $\ell(p)$, the g and g^{-1} are defined as:

$$g : [-\tau, \tau] \to [0, 1], g(\ell_v(p)) = \left\{\left(\frac{\nu + \tau}{2\tau}\right)\left(p^{(i)}\right)\right\} = \left\{(\gamma)\left(p^{(i)}\right)\right\}, \gamma \in [0, 1], \tag{2.5}$$

and

$$g^{-1} : [0, 1] \to [-\tau, \tau], g^{-1}(g(\ell(p))) = \left\{S_{(2\gamma-1)\tau}\left(p^{(i)}\right) | \gamma \in [0, 1]\right\} = \ell(p). \tag{2.6}$$

2.2. ATN and ATC

AOs are very important tools for information fusion in some decision-making problems. However, most of AOs are defined based on TNs and TCs. So it is essentially to review TNs and TCs before the operations of PLTS are given.

Definition 2.3. *[37] If the function $\varsigma : [0, 1]^2 \to [0, 1]$ meets the following four requirements for all $\alpha, \beta, \delta \in [0, 1]$, it was named as a TN:*
(1) $\varsigma(\alpha, \beta) = \varsigma(\beta, \alpha)$;
(2) $\varsigma(\alpha, \varsigma(\beta, \delta)) = \varsigma(\varsigma(\alpha, \beta), \delta)$;
(3) $\varsigma(\alpha, \beta) \le \varsigma(\alpha, \delta)$, if $\beta \le \delta$;
(4) $\varsigma(1, \alpha) = \alpha$.

A TC ς^* [38] is a mapping from $[0, 1]^2$ to $[0, 1]$, if ς^* meets the following four requirements for all $\alpha, \beta, \delta \in [0, 1]$:
(1) $\varsigma^*(\alpha, \beta) = \varsigma^*(\beta, \alpha)$;
(2) $\varsigma^*(\alpha, \varsigma^*(\beta, \delta)) = \varsigma^*(\varsigma^*(\alpha, \beta), \delta)$;
(3) $\varsigma^*(\alpha, \beta) \le \varsigma^*(\alpha, \delta)$, if $\beta \le \delta$;
(4) $\varsigma^*(0, \alpha) = \alpha$.

The TN ς and TC ς^* are dual, that is, $\varsigma^*(\alpha, \beta) = 1 - \varsigma(1 - \alpha, 1 - \beta)$.

A TN ς is Archimedean t-norm (ATN), if there exists an integral n, such that $\varsigma(\underbrace{a, \cdots, a}_{n \ times}) < b$ for any $(a, b) \in [0, 1]^2$. A TC ς^* is Archimedean t-conorm (ATC), if there is an integral n, such that $\varsigma^*(\underbrace{a, \cdots, a}_{n \ times}) > b$ for any $(a, b) \in [0, 1]^2$. Specially, if ς and ς^* satisfy the three given requirements:
(1) ς and ς^* are continuous;
(2) ς and ς^* and are strictly increasing;

(3) for all $\alpha \in [0, 1]$, and $\varsigma^*(\alpha, \alpha) > \alpha$,
then ς and ς^* are strict ATN and strict ATC respectively.

Assuming there is an additive generator $J : [0, 1] \to [0, \infty)$. A strict ATN $\varsigma(\alpha,\beta)$ can be defined by:

$$\varsigma(\alpha,\beta) = J^{-1}(J(\alpha) + J(\beta)), \tag{2.7}$$

where J^{-1} is the inverse of J. Similarly, its ATC $\varsigma^*(\alpha,\beta)$ also can be generated by its additive generator J^*:

$$\varsigma^*(\alpha,\beta) = (J^*)^{-1}(J^*(\alpha) + J^*(\beta)), \tag{2.8}$$

where $J^*(\alpha) = J(1-\alpha)$, $(J^*)^{-1}(\alpha) = 1 - J^{-1}(\alpha)$ and $(J^*)^{-1}$ is the inverse of J^*.

Moreover, we can also derive $\varsigma^*(\alpha,\beta)$ as:

$$\varsigma^*(\alpha,\beta) = 1 - J^{-1}(J(1-\alpha) + J(1-\beta)). \tag{2.9}$$

2.3. Generalized MSM operators

MSM operator [21] was originally proposed by Maclaurin and then further generalized by Detemple [35], it's merit is that it can reflect the relationship between multiple input parameters. The MSM is defined as follows:

Definition 2.4. *[21] Let $\xi_1, \xi_2, \cdots, \xi_n$ be n nonnegative real numbers, and $m = 1, \ldots, n$. A MSM operator will be expressed as*

$$MSM^{(m)}(\xi_1, \xi_2 \ldots, \xi_n) = \left(\frac{\sum\limits_{1 \le i_1 < \cdots < i_m \le n} \prod\limits_{j=1}^{m} \xi_{i_j}}{C_n^m} \right)^{\frac{1}{m}}, \tag{2.10}$$

where $(i_1, i_2, \ldots, i_n)$ is a permutation of $(1, 2, \ldots, n)$.

Definition 2.5. *[35] Let $\xi_1, \xi_2, \cdots, \xi_n$ be n nonnegative real numbers, and $\mu_j \ge 0$. A GMSM operator can be expressed as*

$$GMSM^{(m,u_1,\ldots,u_m)}(\xi_1, \xi_2, \ldots, \xi_n) = \left(\frac{\sum\limits_{1 \le i_1 < \cdots < i_m \le n} \prod\limits_{j=1}^{m} \xi_{i_j}^{u_j}}{C_n^m} \right)^{\frac{1}{(u_1+u_2+\cdots+u_m)}}, \tag{2.11}$$

where $(i_1, i_2, \ldots, i_n)$ is a permutation of $(1, 2, \ldots, n)$, $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$.

The properties of $\text{GMSM}^{(m,u_1,\ldots,u_m)}$ are given as follows:
(a) Idempotency. $GMSM^{(m,u_1,\ldots,u_m)}(\xi, \ldots, \xi) = \xi$, if $\xi_i = \xi$ for all i;
(b) Monotonicity. $GMSM^{(m,u_1,\ldots,u_m)}(\xi_1, \ldots, \xi_n) \le GMSM^{(m,u_1,\ldots,u_m)}(\eta_1, \ldots, \eta_n)$, if $\xi_i \le \eta_i$ for all i ;
(c) Boundedness. $\min_i(\xi_i) \le GMSM^{(m,u_1,\ldots,u_m)}(\xi_1, \xi_2, \ldots, \xi_n) \le \max_i(\xi_i)$.

Under some special situations, GMSM$^{(m,u_1,\dots,u_m)}$ can reduce to some concrete operators when m takes different values:

(1) When $m = 2$, the GMSM$^{(m,u_1,\dots,u_m)}$ will reduce to BM operator:

$$GMSM^{(m,u_1,u_2)}(\xi_1,\xi_2,\dots,\xi_n) = \left(\frac{\sum\limits_{1\le i<j\le n} \xi_i^{u_1}\xi_j^{u_2}}{n(n-1)}\right)^{\frac{1}{(u_1+u_2)}}. \tag{2.12}$$

(2) When $m = 3$, the GMSM$^{(m,u_1,\dots,u_m)}$ will reduce to generalized BM operator:

$$GMSM^{(m,u_1,u_2,u_3)}(\xi_1,\xi_2,\dots,\xi_n) = \left(\frac{\sum\limits_{1\le i,j,k\le n, i\ne j\ne k} \xi_i^{u_1}\xi_j^{u_2}\xi_k^{u_3}}{n(n-1)(n-2)}\right)^{\frac{1}{(u_1+u_2+u_3)}}. \tag{2.13}$$

(3) When u_1=u_2=$\cdots$=u_m=1, the GMSM$^{(m,u_1,\dots,u_m)}$ will reduced to MSM operator:

$$GMSM^{(m,u_1,\dots,u_m)}(\xi_1,\xi_2,\dots,\xi_n) = \left(\frac{\sum\limits_{1\le i_1<\cdots<i_m\le n} \prod\limits_{j=1}^{m} \xi_{i_j}}{C_n^m}\right)^{\frac{1}{m}}. \tag{2.14}$$

3. PLGMSM operators based on ATN and ATC

3.1. The operation laws of PLTSs based on ATN and ATC

In terms of ATN and ATC, a series of operation laws of PLTS can be defined as follows:

Definition 3.1. *Let $\ell_1(p), \ell_2(p)$ be two PLTSs, then*
(1)

$$\begin{aligned}
\ell_1(p) \oplus \ell_2(p) &= g^{-1}\left(\bigcup_{\eta_i^{(t)}\in g(\ell_i(p))(i=1,2)} \left\{\varsigma^*\left(\eta_1^{(t)},\eta_2^{(t)}\right)\left(p_1^{(i)}p_2^{(i)}\right)\right\}\right) \\
&= g^{-1}\left(\bigcup_{\eta_i^{(t)}\in g(\ell_i(p))(i=1,2)} \left\{\left(1-J^{-1}\left(J\left(1-\eta_1^{(i)}\right)+J\left(1-\eta_2^{(i)}\right)\right)\right)\left(p_1^{(i)}p_2^{(i)}\right)\right\}\right);
\end{aligned}$$

(2)

$$\begin{aligned}
\ell_1(p) \otimes \ell_2(p) &= g^{-1}\left(\bigcup_{\eta_i^{(t)}\in g(\ell_i(p))(i=1,2)} \left\{\varsigma\left(\eta_1^{(i)},\eta_2^{(i)}\right)\left(p_1^{(i)}p_2^{(i)}\right)\right\}\right) \\
&= g^{-1}\left(\bigcup_{\eta_i^{(t)}\in g(\ell_i(p))(i=1,2)} \left\{J^{-1}\left(J\left(\eta_1^{(i)}\right)+J\left(\eta_2^{(i)}\right)\right)\left(p_1^{(i)}p_2^{(i)}\right)\right\}\right);
\end{aligned}$$

(3)

$$\lambda\ell_1(p) = g^{-1}\left(\bigcup_{\eta_1^{(t)}\in g(\ell_1(p))} \left\{\left(1-J^{-1}\left(\lambda J\left(1-\eta_1^{(i)}\right)\right)\right)\left(p_1^{(i)}\right)\right\}\right), for\ all \quad \lambda\in R;$$

(4)

$$\ell_1(p)^{\lambda} = g^{-1}\left(\bigcup_{\eta_1^{(t)} \in g(\ell_1(p))} \left\{J^{-1}\left(\lambda J\left(\eta_1^{(i)}\right)\right)\left(p_1^{(i)}\right)\right\}\right), for\ all \quad \lambda \in R.$$

Remark 3.1. *In Definition 3.1, J is a generator of ATN, when J takes different function which satisfies the condition of generators, we can obtain different operations of two PLTSs. Therefore, Definition 3.1 can be regarded as a unified expression of some existing operations of PLTSs.*

3.2. PLGMSM operators based on ATN and ATC

In what follows, $\ell_i(p) = \left\{\ell_i^{(t)}\left(p_i^{(t)}\right) | t = 1, 2, \ldots, \#\ell_i(p)\right\}$ if not specifically stated. Based upon operational laws of PLTSs defined in Defintion 3.1, PLGMSM operator can be proposed and listed as follows.

Definition 3.2. *Let $\ell_1(p), \cdots, \ell_n(p)$ be a group of PLTSs, the probabilistic linguistic generalized MSM operator (PLGMSM) based on ATN and ATC is a function $PLGMSM : \Omega^n \to \Omega$ and*

$$PLGMSM^{(\mathrm{m,u_1,\ldots,u_m})}\left(\ell_1(p), \ldots, \ell_n(p)\right) = \left(\frac{\bigoplus\limits_{1 \le i_1 < \cdots < i_m \le n} \left(\bigotimes\limits_{j=1}^{m} \ell(p)_{i_j}{}^{u_j}\right)}{C_n^m}\right)^{\frac{1}{u_1+u_2+\cdots+u_m}}, \tag{3.1}$$

where Ω is the set of all PLTSs.

According to Definition 3.1 and Definition 3.2, the following result can be derived.

Theorem 3.1. *Let $\ell_1(p), \cdots, \ell_n(p)$ be a group of PLTSs, then*

$$\begin{aligned} &PLGMSM^{(\mathrm{m,u_1,u_2,\ldots,u_m})}\left(\ell_1(p), \cdots, \ell_n(p)\right) \\ &= g^{-1}\left(\bigcup_{\eta_i^{(t)} \in g(\ell_i(p))} \left\{J^{-1}\left(\frac{1}{\sum\limits_{k=1}^{m} u_k} \cdot J\left(1 - J^{-1}\left(\frac{1}{C_n^m}\left(\sum_{\substack{1 \le i_1 < \\ \cdots < i_m \le n}} J\left(1 - J^{-1}\left(\sum_{j=1}^{m} u_j \cdot J\left(\eta_{i_j}^{(t)}\right)\right)\right)\right)\right)\right)\right)\left(\prod_{\substack{1 \le i_1 < \\ \cdots < i_m \le n}} \prod_{j=1}^{m} p_{i_j}^{(t)}\right)\right\}\right) \end{aligned} \tag{3.2}$$

Proof. According to Definition 3.1, we have

$$\left(\ell(p)_{i_j}\right)^{u_j} = g^{-1}\left(\bigcup_{\eta_i^{(t)} \in g(\ell_i(p))} \left\{J^{-1}\left(u_j \cdot J\left(\eta_{i_j}^{(t)}\right)\right)\left(p_{i_j}^{(t)}\right)\right\}\right),$$

and

$$\begin{aligned} \bigotimes_{j=1}^{m}\left(\ell(p)_{i_j}\right)^{u_j} &= g^{-1}\left(\bigcup_{\eta_i^{(t)} \in g(\ell_i(p))} \left\{J^{-1}\left(\sum_{j=1}^{m}\left(J\left(J^{-1}\left(u_j \cdot J\left(\eta_{i_j}^{(t)}\right)\right)\right)\right)\right)\left(\prod_{j=1}^{m} p_{i_j}^{(t)}\right)\right\}\right) \\ &= g^{-1}\left(\bigcup_{\eta_i^{(t)} \in g(\ell_i(p))} \left\{J^{-1}\left(\sum_{j=1}^{m}\left(u_j \cdot J\left(\eta_{i_j}^{(t)}\right)\right)\right)\left(\prod_{j=1}^{m} p_{i_j}^{(t)}\right)\right\}\right), \end{aligned}$$

therefore

$$\begin{aligned}
&\underset{1\le i_1<\cdots<i_m\le n}{\oplus}\left(\underset{j=1}{\overset{m}{\otimes}}\left(\ell(p)_{i_j}\right)^{u_j}\right)\\
=\ & g^{-1}\left(\underset{\eta_i^{(t)}\in g(\ell_i(p))}{\cup}\left\{1-J^{-1}\left(\sum_{j=1}^{m}J\left(1-J^{-1}\left(\sum_{j=1}^{m}u_j\cdot J\left(\eta_{i_j}^{(t)}\right)\right)\right)\right)\left(\prod_{1\le i_1<\cdots<i_m\le n}\prod_{j=1}^{m}p_{i_j}^{(t)}\right)\right\}\right).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\frac{\underset{1\le i_1<\cdots<i_m\le n}{\oplus}\left(\underset{j=1}{\overset{m}{\otimes}}\ \ell(p)_{i_j}{}^{u_j}\right)}{C_n^m}\\
=\ & g^{-1}\left(\underset{\eta_i^{(t)}\in g(\ell_i(p))}{\cup}\left\{1-J^{-1}\left(\frac{1}{C_n^m}J\left(1-\left(1-J^{-1}\left(\underset{1\le i_1<\cdots<i_m\le n}{\Sigma}J\left(1-J^{-1}\left(\sum_{j=1}^{m}u_j\cdot J\left(\eta_{i_j}^{(t)}\right)\right)\right)\right)\right)\right)\right)\left(\prod_{1\le i_1<\cdots<i_m\le n}\prod_{j=1}^{m}p_{i_j}^{(t)}\right)\right\}\right).
\end{aligned}$$

So

$$\begin{aligned}
&\left(\frac{\underset{1\le i_1<\cdots<i_m\le n}{\oplus}\left(\underset{j=1}{\overset{m}{\otimes}}\ \ell(p)_{i_j}{}^{u_j}\right)}{C_n^m}\right)^{\frac{1}{u_1+u_2+\cdots+u_m}}\\
=\ & g^{-1}\left(\underset{\eta_i^{(t)}\in g(\ell_i(p))}{\cup}\left\{J^{-1}\left(\frac{1}{\sum\limits_{k=1}^{m}u_k}\cdot J\left(1-J^{-1}\left(\frac{1}{C_n^m}\left(\underset{1\le i_1<\cdots<i_m\le n}{\Sigma}J\left(1-J^{-1}\left(\sum_{j=1}^{m}u_j\cdot J\left(\eta_{i_j}^{(t)}\right)\right)\right)\right)\right)\right)\right)\left(\prod_{1\le i_1<\cdots<i_m\le n}\prod_{j=1}^{m}p_{i_j}^{(t)}\right)\right\}\right).
\end{aligned}$$

Proved. □

Theorem 3.2. *Let $\ell_1(p),\cdots,\ell_n(p)$ be a group of PLTSs. If $\left(\ell_1'(p)(p),\ell_2{}'(p),\cdots,\ell_n{}'(p)\right)$ is a permutation of $(\ell_1(p),\cdots,\ell_n(p))$, then*

$$PLGMSM^{(m,u_1,u_2,\ldots,u_m)}\left(\ell_1(p),\ldots,\ell_n(p)\right)=PLGMSM^{(m,u_1,u_2,\ldots,u_m)}\left(\ell_1'(p),\cdots,\ell_n{}'(p)\right). \tag{3.3}$$

Proof. The proofs is similar to Property 3 in [24]. So, the details are omitted. □

3.3. Some special PLGMSM operators based on different generators

In this section, some special PLGMSMS operators will be investigated when parameters take different values and the generator takes different functions.

3.3.1. When parameters takes different values

(a) When $m=1$, the PLGMSM operator based on ATN and ATC will reduce to

$$\begin{aligned}
&PLMSM\left(\ell_1(p),\ldots,\ell_n(p)\right)\\
=\ & g^{-1}\left(\underset{\eta_i^{(t)}}{\cup}\left\{J^{-1}\left(\frac{1}{u_1}\cdot J\left(1-J^{-1}\left(\frac{1}{n}\left(\sum_{1\le i\le n}J\left(1-J^{-1}\left(u_1J\left(\eta_i^{(t)}\right)\right)\right)\right)\right)\right)\right)\left(\prod_{1\le i\le n}p_i^{(t)}\right)\right\}\right).
\end{aligned} \tag{3.4}$$

(b) When $m = 2$, the PLGMSM operator based on ATN and ATC will reduce to

$$\begin{aligned} & PLGMSM^{(2,u_1,u_2,)}\left(\ell_1(p),\ldots,\ell_n(p)\right) \\ = \; & g^{-1}\left(\bigcup_{\eta_i^{(t)}}\left\{J^{-1}\left(\frac{1}{u_1+u_2}\cdot J\left(1-J^{-1}\left(\frac{1}{n(n-1)}\left(\sum_{1\le i_1<i_2\le n} J\left(1-J^{-1}\left(\sum_{j=1}^{2} u_j\cdot J\left(\eta_{i_j}^{(t)}\right)\right)\right)\right)\right)\right)\right)\left(\prod_{1\le i_1<i_2\le n}\prod_{j=1}^{2} p_{i_j}^{(t)}\right)\right\}\right). \end{aligned} \tag{3.5}$$

(c) When $u_1{=}u_2{=}\cdots{=}u_m{=}1$, the PLGMSM operator based on ATN and ATC will reduce to

$$\begin{aligned} & PLGMSM^{(m,u_1,u_2,\ldots,u_m)}\left(\ell_1(p),\ldots,\ell_n(p)\right) \\ = \; & g^{-1}\left(\bigcup_{\eta_i^{(t)}}\left\{J^{-1}\left(\frac{1}{m}\cdot J\left(1-J^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\leqslant i_1<\cdots<i_m\leqslant n} J\left(1-J^{-1}\left(\sum_{j=1}^{m} u_j\cdot J\left(\eta_{i_j}^{(t)}\right)\right)\right)\right)\right)\right)\right)\left(\prod_{1\leqslant i_1<\cdots<i_m\leqslant n}\prod_{j=1}^{m} p_{i_j}^{(t)}\right)\right\}\right). \end{aligned} \tag{3.6}$$

3.3.2. When generator takes different functions

(a) If $J(x) = -\ln x$, it has $J^{-1}(x) = e^{-x}$. Then we get probabilistic linguistic Archimedean Algebraic GMSM (PLAAGMSM) [16] operators as follows:

$$\begin{aligned} & PLAAGMSM^{(m,u_1,u_2,\ldots,u_m)}\left(\ell_1(p),\ldots,\ell_n(p)\right) \\ = \; & g^{-1}\left(\bigcup_{\eta_i^{(t)}}\left(1-\left(\prod_{1\leqslant i_1<\cdots<i_m\leqslant n}\left(1-\prod_{j=1}^{m}\left(\eta_{i_j}^{(t)}\right)^{u_j}\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{\sum_{k=1}^{m}u_k}}\right)\left(\prod_{1\leqslant i_1<\cdots<i_m\leqslant n}\prod_{j=1}^{m} p_{i_j}^{(t)}\right)\right). \end{aligned} \tag{3.7}$$

In this situation, GPLGMSM reduce to the PLAAMSM.

(b) If $J(x) = \ln\frac{2-x}{x}$, it gains $J^{-1}(x) = \frac{2}{e^x+1}$. We get probabilistic linguistic Archimedean Einstein GMSM (PLAEGMSM) operators as follows:

$$\begin{aligned} & PLAEGMSM^{(m,u_1,u_2,\ldots,u_m)}\left(\ell_1(p),\ldots,\ell_n(p)\right) \\ = \; & g^{-1}\left(\bigcup_{\eta_I^{(t)}}\left(\frac{2(A-1)^{\frac{1}{\sum_{k=1}^{m}u_k}}}{(2A+3)^{\frac{1}{\sum_{k=1}^{m}u_k}}+(A-1)^{\frac{1}{\sum_{k=1}^{m}u_k}}}\right)\left(\prod_{1\leqslant i_1<\cdots<i_m\leqslant n}\prod_{j=1}^{m} p_{i_j}^{(t)}\right)\right), \end{aligned} \tag{3.8}$$

in which,

$$A = \prod_{1\le i_1<\cdots<i_m\le n}\left(\frac{\prod_{j=1}^{m}\left(2-\eta_{i_j}^{(t)}\right)^{u_j}+3\prod_{j=1}^{m}\left(\eta_{i_j}^{(t)}\right)^{u_j}}{\prod_{j=1}^{m}\left(2-\eta_{i_j}^{(t)}\right)^{u_j}-\prod_{j=1}^{m}\left(\eta_{i_j}^{(t)}\right)^{u_j}}\right)^{\frac{1}{C_n^m}}.$$

(c) If $J(x) = \ln\frac{\varepsilon+(1-\varepsilon)x}{x}$ $(\varepsilon > 0)$, then it has $J^{-1}(x) = \frac{\varepsilon}{e^x+\varepsilon-1}$. We can get probabilistic linguistic Archimedean Hamacher GMSM (PLAHGMSM) operators as follows.

$$
\begin{aligned}
&PLAHGMSM^{(\mathrm{m},\mathrm{u}_1,\mathrm{u}_2,\ldots,\mathrm{u}_\mathrm{m})}\left(\ell_1(p),\ldots,\ell_n(p)\right)\\
=\ & g^{-1}\left(\bigcup_{\eta_i^{(t)}}\left(\frac{\varepsilon}{\left(\frac{\varepsilon^2}{b-1}+1\right)^{\frac{1}{\sum_{k=1}^{m}u_k}}+\varepsilon-1}\right)\left(\prod_{1\leqslant i_1<\cdots<i_m\leqslant n}\prod_{j=1}^{m}p_{i_j}^{(t)}\right)\right),
\end{aligned}\tag{3.9}
$$

in which,

$$
b=\prod_{1\leq i_1<\cdots<i_m\leq n}\left(\frac{\varepsilon^2}{a-1}+1\right)^{\frac{1}{C_n^m}},
$$

and where

$$
a=\prod_{j=1}^{m}\left(\frac{\varepsilon+(1-\varepsilon)\,\eta_{i_j}^{(t)}}{\eta_{i_j}^{(t)}}\right)^{u_j}.
$$

In this case, when $\varepsilon=1$, PLAHGMSM reduces to the GPLAAMSM.

Example 3.1. *Suppose $\ell_1=\{\ell_{-1}(1)\}$, $\ell_2=\{s_{-2}(1)\}$, $\ell_3=\{s_0(0.3),s_1(0.7)\}$ be three PLTSs, applying the function g to convert ℓ_i $(i=1,2,3)$ into*

$$
g(\ell_1)=\{0.25(1)\},g(\ell_2)=\{0(1)\},g(\ell_3)=\{0.5(0.3),0.75(0.7)\}.
$$

Besides, previous different operators could be used to aggregate ℓ_i $(i=1,2,3)$. Here, set $m=2$, $u_1=1,u_2=2$. In line with the above formula, we get

$$
\begin{aligned}
&PLAAGMSM^{(2,1,2)}\left(\ell_1(p),\ell_2(p),\ell_3(p)\right)\\
=\ & g^{-1}\left(\bigcup_{\eta_i^{(t)}\in g(\ell_i(p)),i=1,3}\left(1-\left(\prod_{1\leqslant i_1<i_2\leqslant 3}\left(1-\prod_{j=1}^{2}\left(\eta_{i_j}^{(t)}\right)^{u_j}\right)^{\frac{1}{C_3^2}}\right)^{\frac{1}{1+2}}\right)\left(\prod_{1\leqslant i_1<i_2\leqslant 3}\prod_{j=1}^{2}p_{i_j}^{(t)}\right)\right)\\
=\ & \{s_{-1.34}(0.3),s_{-0.8}(0.7)\}.
\end{aligned}
$$

4. Weighted probabilistic linguistic generalized MSM operators based upon ATN and ATC

Due to each individual's different background knowledge and preference, the importance should be different. Hence, it is essential to consider the individual weight information to make the decision results are more reasonable and scientific. In this section, the weighted probabilistic linguistic generalized MSM operators based on ATN and ATC will be introduced.

Definition 4.1. *Let $\ell_1(p),\cdots,\ell_n(p)$ be n PLTSs and w_i be the weight of $\ell_i(p)$ with $w_i\in[0,1]$, $\sum_{i=1}^{n}w_i=1$. The WPLGMSM is a function $WPLGMSM^{(m,u_1,u_2,\ldots,u_m)}:\Omega^n\rightarrow\Omega$, if*

$$
WPLGMSM^{(m,u_1,u_2,\ldots,u_m)}\left(\ell_1(p),\ldots,\ell_n(p)\right)=\left(\frac{\bigoplus_{1\leqslant i_1<\cdots<i_m\leqslant n}\left(\bigotimes_{j=1}^{m}\left(\left(nw_{i_j}\right)\otimes\ell(p)_{i_j}\right)^{u_j}\right)}{C_n^m}\right)^{\frac{1}{u_1+u_2+\cdots+u_m}},\tag{4.1}
$$

where Ω is the set of all PLTSs.

In the light of Definition 4.1, the following result can be derived.

Theorem 4.1. *Let $\ell_1(p), \cdots, \ell_n(p)$ be n PLTSs and w_i be the weight of $\ell_i(p)$ with $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$. Then*

$$WPLGMSM^{(m,u_1,u_2,\cdots,u_m)}(\ell_1(p), \ldots, \ell_n(p))$$
$$= g^{-1}\left(\bigcup_{\eta_i^{(t)} \in g(\ell_i(p))} \left\{J^{-1}\left(\frac{1}{\sum_{k=1}^{m} u_k} J\left(1 - J^{-1}\left(\frac{1}{C_n^m} \sum_{1\leqslant i_1<\cdots<i_m\leqslant n} J\left(1 - J^{-1}\left(\sum_{j=1}^{m} u_j J\left(1 - J^{-1}\left(nw_{i_j} J\left(1 - \eta_{i_j}^{(t)}\right)\right)\right)\right)\right)\right)\right)\right)\right\}\left(\prod_{1\leqslant i_1<\cdots<i_m\leqslant n} \prod_{j=1}^{m} p_{i_j}{}^{(t)}\right)\right).$$

Proof. In line with the operational formula, we have

$$\left(nw_{i_j} \otimes \ell(p)_{i_j}\right) = g^{-1}\left(\bigcup_{\eta_{i_j}^{(t)} \in g\left(\ell(p)_{i_j}\right)} \left\{1 - J^{-1}\left(nw_{i_j} J\left(1 - \eta_{i_j}^{(t)}\right)\right)\right\}\left(p_{i_j}^{(t)}\right)\right),$$

$$\left(nw_{i_j} \otimes \ell(p)_{i_j}\right)^{u_j} = g^{-1}\left(\bigcup_{\eta_{i_j}^{(t)} \in g\left(\ell(p)_{i_j}\right)} \left\{\left\{J^{-1}\left[u_j J\left(1 - J^{-1}\left(nw_{i_j} J\left(1 - \eta_{i_j}^{(t)}\right)\right)\right)\right]\right\}\right\}\left(p_{i_j}^{(t)}\right)\right).$$

Therefore,

$$\bigotimes_{j=1}^{m}\left(nw_{i_j} \otimes \ell(p)_{i_j}\right)^{u_j} = g^{-1}\left(\bigcup_{\eta_{i_j}^{(t)} \in g\left(\ell(p)_{i_j}\right)} \left\{J^{-1}\left(\sum_{j=1}^{m} J\left(J^{-1}\left[u_j J\left(1 - J^{-1}\left(nw_{i_j} J\left(1 - \eta_{i_j}^{(t)}\right)\right)\right)\right]\right)\right)\right\}\left(\prod_{j=1}^{m} p_{i_j}^{(t)}\right)\right)$$
$$= g^{-1}\left(\bigcup_{\eta_{i_j}^{(t)} \in g\left(\ell(p)_{i_j}\right)} \left\{J^{-1}\left(\sum_{j=1}^{m}\left(u_j J\left(1 - J^{-1}\left(nw_{i_j} J\left(1 - \eta_{i_j}^{(t)}\right)\right)\right)\right)\right)\right\}\left(\prod_{j=1}^{m} p_{i_j}^{(t)}\right)\right)$$

and so

$$\bigotimes_{j=1}^{m}\left(nw_{i_j} \otimes \ell(p)_{i_j}\right)^{u_j} = g^{-1}\left(\bigcup_{\eta_{i_j}^{(t)} \in g\left(\ell(p)_{i_j}\right)} \left\{J^{-1}\left(\sum_{j=1}^{m} J\left(J^{-1}\left[u_j J\left(1 - J^{-1}\left(nw_{i_j} J\left(1 - \eta_{i_j}^{(t)}\right)\right)\right)\right]\right)\right)\right\}\left(\prod_{j=1}^{m} p_{i_j}^{(t)}\right)\right)$$
$$= g^{-1}\left(\bigcup_{\eta_{i_j}^{(t)} \in g\left(\ell(p)_{i_j}\right)} \left\{J^{-1}\left(\sum_{j=1}^{m}\left(u_j J\left(1 - J^{-1}\left(nw_{i_j} J\left(1 - \eta_{i_j}^{(t)}\right)\right)\right)\right)\right)\right\}\left(\prod_{j=1}^{m} p_{i_j}^{(t)}\right)\right),$$

$$\frac{1}{C_n^m} \bigoplus_{1\leqslant i_1<\cdots<i_m\leqslant n}\left(\bigotimes_{j=1}^{m}\left(nw_{i_j} \otimes \ell(p)_{i_j}\right)^{u_j}\right)$$
$$= g^{-1}\left(\bigcup_{\eta_{i_j}^{(t)} \in g\left(\ell(p)_{i_j}\right)} \left\{1 - J^{-1}\left(\frac{1}{C_n^m} J\left(J^{-1}\left[\sum_{1\leqslant i_1<\cdots<i_m\leqslant n} J\left(1 - J^{-1}\left(\sum_{j=1}^{m} u_j J\left(1 - J^{-1}\left(nw_{i_j} J\left(1 - \eta_{i_j}^{(t)}\right)\right)\right)\right)\right)\right]\right)\right)\right\}\left(\prod_{1\leqslant i_1<\cdots<i_m\leqslant n} \prod_{j=1}^{m} p_{i_j}{}^{(t)}\right)\right)$$
$$= g^{-1}\left(\bigcup_{\eta_{i_j}^{(t)} \in g\left(\ell(p)_{i_j}\right)} \left\{1 - J^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\leqslant i_1<\cdots<i_m\leqslant n} J\left(1 - J^{-1}\left(\sum_{j=1}^{m} u_j J\left(1 - J^{-1}\left(nw_{i_j} J\left(1 - \eta_{i_j}^{(t)}\right)\right)\right)\right)\right)\right)\right)\right\}\left(\prod_{1\leqslant i_1<\cdots<i_m\leqslant n} \prod_{j=1}^{m} p_{i_j}{}^{(t)}\right)\right).$$

Hence,

$$\left(\frac{1}{C_n^m}\underset{1\leqslant i_1<\cdots<i_m\leqslant n}{\oplus}\left(\overset{m}{\underset{j=1}{\otimes}}\left(\left(nw_{i_j}\right)\otimes\ell(p)_{i_j}\right)^{u_j}\right)\right)^{\frac{1}{u_1+u_2+\cdots+u_m}}$$

$$=g^{-1}\left(\bigcup_{\eta_{i_j}^{(t)}\in g\left(\ell(p)_{i_j}\right)}\left\{J^{-1}\left(\frac{1}{\sum_{k=1}^{m}u_k}J\left(1-J^{-1}\left(\frac{1}{C_n^m}\sum_{1\leqslant i_1<\cdots<i_m\leqslant n}J\left(1-J^{-1}\left(\sum_{j=1}^{m}u_jJ\left(1-J^{-1}\left(nw_{i_j}J\left(1-\eta_{i_j}^{(t)}\right)\right)\right)\right)\right)\right)\right)\right)\right\}\left(\prod_{1\leqslant i_1<\cdots<i_m\leqslant n}\prod_{j=1}^{m}p_{i_j}{}^{(t)}\right)\right).$$

Proved. □

Theorem 4.2. *Let $\ell_1(p),\cdots,\ell_n(p)$ be a group of PLTSs. If $\left(\ell_1'(p),\cdots,\ell_n'(p)\right)$ is a permutation of $(\ell_1(p),\cdots,\ell_n(p))$, then*

$$WPLGMSM^{(m,u_1,u_2,\ldots,u_m)}\left(\ell_1(p),\ldots,\ell_n(p)\right)=WPLGMSM^{(m,u_1,u_2,\ldots,u_m)}\left(\ell_1'(p),\cdots,\ell_n'(p)\right).\quad(4.2)$$

Proof. The proofs of this theorem is similar to Property 3 in [24]. So, the details are omitted. □

4.1. Some special WPLGMSM operators based on different generators

In this section, some special PLGMSMS operators will be investigated when the parameters take different values and the generator takes different function.

4.1.1. When parameters take different values

(a) When $m=1$, the WPLGMSM operator based on ATN and ATC will reduce to

$$\begin{aligned}&WPLMSM\left(\ell_1(p),\ldots,\ell_n(p)\right)\\=\;&g^{-1}\left(\bigcup_{\eta_i^{(t)}\in g(\ell_i(p))}\left\{J^{-1}\left(\frac{1}{u_1}J\left(1-J^{-1}\left(\frac{1}{n}\sum_{1\leqslant i_1\leqslant n}J\left(1-J^{-1}\left(u_jJ\left(1-J^{-1}\left(nw_{i_j}J\left(1-\eta_{i_j}^{(t)}\right)\right)\right)\right)\right)\right)\right)\right)\right\}\left(\prod_{1\leqslant i_1\leqslant n}p_{i_1}{}^{(t)}\right)\right).\end{aligned}$$

(b) When $m=2$, the PLGMSM operator based on ATN and ATC will reduce to

$$\begin{aligned}&PLGMSM^{(2,u_1,u_2,)}\left(\ell_1(p),\ldots,\ell_n(p)\right)=\\&g^{-1}\left(\bigcup_{\eta_i^{(t)}\in g(\ell_i(p)),}\left\{J^{-1}\left(\frac{1}{u_1+u_2}\cdot J\left(1-J^{-1}\left(\frac{1}{n(n-1)}\left(\sum_{1\leqslant i_1<i_2\leqslant n}J\left(1-J^{-1}\left(\sum_{j=1}^{2}u_jJ\left(1-J^{-1}\left(nw_{i_j}J\left(1-\eta_{i_j}^{(t)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\left(\prod_{1\leqslant i_1<i_2\leqslant n}\prod_{j=1}^{2}p_{i_j}^{(t)}\right)\right\}\right).\end{aligned}$$

(c) When $u_1=u_2=\cdots=u_m=1$, the PLGMSM operator based on ATN and ATC will reduce to

$$\begin{aligned}&PLGMSM^{(\mathrm{m},\mathrm{u}_1,\mathrm{u}_2,\ldots,\mathrm{u}_\mathrm{m})}\left(\ell_1(p),\ldots,\ell_n(p)\right)\\&=g^{-1}\left(\bigcup_{\eta_{i_j}^{(t)}\in g\left(\ell(p)_{i_j}\right)}\left\{J^{-1}\left(\frac{J}{m}\left(1-J^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\leqslant i_1<\cdots<i_m\leqslant n}J\left(1-J^{-1}\left(\sum_{j=1}^{m}u_jJ\left(1-J^{-1}\left(nw_{i_j}J\left(1-\eta_{i_j}^{(t)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\left(\prod_{1\leqslant i_1<\cdots<i_m\leqslant n}\prod_{j=1}^{m}p_{i_j}^{(t)}\right)\right\}\right).\end{aligned}$$

4.2. When generator takes different functions

In the what follows, the special situations of the WPLGMSM based on ATN and ATC will be discussed.

(1) If $J(x)=-\ln x$, then $J^{-1}(x)=e^{-x}$. The weighted probabilistic linguistic Archimedean Algebraic GMSM (WPLAAGMSM) operators will be obtained as follows:

$$
\begin{aligned}
&WPLAAGMSM^{(\mathrm{m,u_1,u_2,\ldots,u_m})}\left(\ell_1(p),\ldots,\ell_n(p)\right)\\
&= g^{-1}\left(\bigcup_{\eta_{i_j}^{(t)}\in g\left(\ell(p)_{i_j}\right)}\left(1-\left(\prod_{1\leqslant i_1<\cdots<i_m\leqslant n}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\eta_{i_j}^{(t)}\right)^{nw_{i_j}}\right)^{u_j}\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{u_1+u_2+\cdots+u_m}}\right)\left(\prod_{1\leqslant i_1<\cdots<i_m\leqslant n}\prod_{j=1}^{m}p_{i_j}^{(t)}\right)\right). \qquad (4.3)
\end{aligned}
$$

(2) If $J(x)=\ln\frac{2-x}{x}$, it has $J^{-1}(x)=\frac{2}{\mathrm{e}^x+1}$. Then the weighted probabilistic linguistic Archimedean Einstein GMSM (WPLAEGMSM) operators will be obtained as follows:

$$
\begin{aligned}
&WPLAEGMSM^{(\mathrm{m,u_1,u_2,\ldots,u_m})}\left(\ell_1(p),\ldots,\ell_n(p)\right))\\
&= g^{-1}\left(\bigcup_{\eta_{i_j}^{(t)}\in g\left(\ell(p)_{i_j}\right)}\left(\frac{2\left(A^{\frac{1}{C_n^m}}-1\right)^{\frac{1}{u_1+u_2+\cdots+u_m}}}{\left(A^{\frac{1}{C_n^m}}-1\right)^{\frac{1}{u_1+u_2+\cdots+u_m}}+\left(A^{\frac{1}{C_n^m}}+1\right)^{\frac{1}{u_1+u_2+\cdots+u_m}}}\right)\left(\prod_{1\leqslant i_1<\cdots<i_m\leqslant n}\prod_{j=1}^{m}p_{i_j}^{(t)}\right)\right), \qquad (4.4)
\end{aligned}
$$

where,

$$
A=\prod_{1\leqslant i_1<\cdots<i_m\leqslant n}\left(\left(\prod_{j=1}^{m}\left(\frac{\left(2-\eta_{i_j}^{(t)}\right)^{nw_{i_j}}+\left(\eta_{i_j}^{(t)}\right)^{nw_{i_j}}}{\left(2-\eta_{i_j}^{(t)}\right)^{nw_{i_j}}-\left(\eta_{i_j}^{(t)}\right)^{nw_{i_j}}}\right)^{u_j}+1\right)\Bigg/\left(\prod_{j=1}^{m}\left(\frac{\left(2-\eta_{i_j}^{(t)}\right)^{nw_{i_j}}+\left(\eta_{i_j}^{(t)}\right)^{nw_{i_j}}}{\left(2-\eta_{i_j}^{(t)}\right)^{nw_{i_j}}-\left(\eta_{i_j}^{(t)}\right)^{nw_{i_j}}}\right)^{u_j}-1\right)\right).
$$

(3) If $J(x)=\ln\frac{\varepsilon+(1-\varepsilon)x}{x}$ $(\varepsilon>0)$, then $J^{-1}(x)=\frac{\varepsilon}{\mathrm{e}^x+\varepsilon-1}$, the weighted probabilistic linguistic Archimedean Hamacher GMSM (WPLAHGMSM) operators will be obtained as follows:

$$
\begin{aligned}
&WPLAHGMSM^{(\mathrm{m,u_1,u_2,\ldots,u_m})}\left(\ell_1(p),\ldots,\ell_n(p)\right)\\
&= g^{-1}\left(\bigcup_{\eta_{i_j}^{(t)}\in g\left(\ell(p)_{i_j}\right)}\left\{\frac{\varepsilon\left(C_{i_j}\right)^{\frac{1}{u_1+u_2+\cdots+u_m}}}{\left(\varepsilon+(1-\varepsilon)C_{i_j}\right)^{\frac{1}{u_1+u_2+\cdots+u_m}}-(1-\varepsilon)\left(C_{i_j}\right)^{\frac{1}{u_1+u_2+\cdots+u_m}}}\right\}\left(\prod_{1\leqslant i_1<\cdots<i_m\leqslant n}\prod_{j=1}^{m}p_{i_j}^{(t)}\right)\right), \qquad (4.5)
\end{aligned}
$$

where,

$$
\begin{aligned}
C_{i_j} &= \left(\left(\prod_{1\leqslant i_1<\cdots<i_m\leqslant n}\frac{\varepsilon+(1-\varepsilon)B_{i_j}}{B_{i_j}}\right)^{\frac{1}{C_n^m}}-1\right)\Bigg/\left(\left(\prod_{1\leqslant i_1<\cdots<i_m\leqslant n}\frac{\varepsilon+(1-\varepsilon)B_{i_j}}{B_{i_j}}\right)^{\frac{1}{C_n^m}}-(1-\varepsilon)\right),\\
B_{i_j}^{(t)} &= \frac{\left(\varepsilon+(1-\varepsilon)A_{i_j}\right)^{u_j}-A_{i_j}{}^{u_j}}{\left(\varepsilon+(1-\varepsilon)A_{i_j}\right)^{u_j}-(1-\varepsilon)A_{i_j}{}^{u_j}},\\
A_{i_j}^{(t)} &= \frac{\left(\varepsilon+(1-\varepsilon)\left(1-\eta^{(t)}{}_{i_j}\right)\right)^{nw_{i_j}}-\left(\eta^{(t)}{}_{i_j}\right)^{nw_{i_j}}}{\left(\varepsilon+(1-\varepsilon)\left(1-\eta^{(t)}{}_{i_j}\right)\right)^{nw_{i_j}}-(1-\varepsilon)\left(\eta^{(t)}{}_{i_j}\right)^{nw_{i_j}}}.
\end{aligned}
$$

In the following, we will continue to give some examples to testify different aggregation operators, and discuss some special situations for diverse parameters.

Example 4.1. *Let* $\ell_1 = \{s_{-1}(0.3), s_1(0.7)\}$, $\ell_2 = \{s_1(1)\}$, $\ell_3 = \{s_0(0.25), s_2(0.75)\}$ *be three PLTSs. Suppose* $w = (0.35, 0.25, 0.4)$ *is the weight vector of* ℓ_1, ℓ_2, ℓ_3*. Set* $\tau = 3$*, then, with the function* g*,* ℓ_1, ℓ_2, ℓ_3 *will converted into* $g(\ell_1(p)) = \{0.33(0.3), 0.67(0.7)\}$*,* $g(\ell_2(p)) = \{0.67(1)\}$*,* $g(\ell_3(p)) = \{0.5(0.25), 0.83(0.75)\}$*, respectively.*

Then will use the WPLGMSM operator based on ATN and ATC to fuse ℓ_1, ℓ_2, ℓ_3*. set* $m = 2, u_1 = 1, u_2 = 2$*, with different additive generators, the aggregated results are obtained as follows:*

(1) If $J(x) = -ln\ x$*:*

$$
\begin{aligned}
&WPLGMSM^{(2,1,2)}(\ell_1(P), \ell_2(P), \ell_3(P)) \\
&= \left(\frac{\underset{1 \leqslant i_1 < i_2 \leqslant 3}{\oplus} \left(\overset{2}{\underset{j=1}{\otimes}} \left(\left(3w_{i_j}\right) \otimes \ell(p)_{i_j} \right)^{u_j} \right)}{C_3^2} \right)^{\frac{1}{1+2}} \\
&= \left\{ \begin{array}{llll}
S_{0.068}(0.006) & S_{0.329}(0.013) & S_{0.436}(0.017) & S_{1.024}(0.039) \\
S_{0.329}(0.013) & S_{0.546}(0.031) & S_{0.637}(0.039) & S_{1.153}(0.092) \\
S_{0.708}(0.017) & S_{0.873}(0.039) & S_{0.943}(0.051) & S_{1.362}(0.118) \\
S_{0.873}(0.039) & S_{1.108}(0.092) & S_{1.081}(0.118) & S_{1.460}(0.276)
\end{array} \right\} \\
&= \left\{ \begin{array}{llll}
S_{0.068}(0.006) & S_{0.329}(0.026) & S_{0.436}(0.017) & S_{1.024}(0.039) \\
S_{0.546}(0.031) & S_{0.637}(0.039) & S_{1.153}(0.092) & S_{0.708}(0.017) \\
S_{0.873}(0.078) & S_{0.943}(0.051) & S_{1.362}(0.118) & S_{1.108}(0.092) \\
 & S_{1.081}(0.118) & S_{1.460}(0.276) &
\end{array} \right\}.
\end{aligned}
$$

(2) If $J(x) = ln\ \frac{2-x}{x}$*:*

$$
\begin{aligned}
&WPLAEGMSM^{(2,1,2)}(\ell_1(P), \ell_2(P), \ell_3(P)) \\
&= \left(\frac{\underset{1 \leqslant i_1 < i_2 \leqslant 3}{\oplus} \left(\overset{2}{\underset{j=1}{\otimes}} \left(\left(3w_{i_j}\right) \otimes \ell(p)_{i_j} \right)^{u_j} \right)}{C_3^2} \right)^{\frac{1}{1+2}} \\
&= \left\{ \begin{array}{llll}
s_{0.226}(0.006) & s_{0.469}(0.013) & s_{0.638}(0.017) & s_{1.136}(0.039) \\
s_{1.008}(0.013) & s_{1.136}(0.031) & s_{1.233}(0.039) & s_{1.558}(0.092) \\
s_{0.437}(0.017) & s_{0.641}(0.039) & s_{0.787}(0.051) & s_{1.235}(0.118) \\
s_{1.118}(0.039) & s_{1.235}(0.092) & s_{1.325}(0.118) & s_{1.628}(0.276)
\end{array} \right\}.
\end{aligned}
$$

(3) If $J(x) = ln\ \frac{\varepsilon + (1-\varepsilon)x}{x}$ $(\varepsilon = 2)$*:*

$$
\begin{aligned}
&WPLAHGMSM^{(2,1,2)}(\ell_1(P), \ell_2(P), \ell_3(P)) \\
&= \left(\frac{\underset{1 \leqslant i_1 < i_2 \leqslant 3}{\oplus} \left(\overset{2}{\underset{j=1}{\otimes}} \left(\left(3w_{i_j}\right) \otimes \ell(p)_{i_j} \right)^{u_j} \right)}{C_3^2} \right)^{\frac{1}{1+2}}
\end{aligned}
$$

$$= \left\{ \begin{array}{llll} s_{0.857}\,(0.006) & s_{0.782}\,(0.013) & s_{0.737}\,(0.017) & s_{0.568}\,(0.039) \\ s_{0.567}\,(0.013) & s_{0.529}\,(0.031) & s_{0.505}\,(0.039) & s_{0.406}\,(0.092) \\ s_{0.721}\,(0.017) & s_{0.666}\,(0.039) & s_{0.632}\,(0.051) & s_{0.497}\,(0.118) \\ s_{0.496}\,(0.039) & s_{0.465}\,(0.092) & s_{0.445}\,(0.118) & s_{0.361}\,(0.276) \end{array} \right\}$$

5. Decision-making approach based upon WPLGMSM and application in teaching quality evaluation

5.1. Aggregation-based decision-making approach

Before giving the decision-making approach, a formal description of a MADM problem with probabilistic linguistic information will be given. Suppose $A = \{A_1, \ldots, A_k\}$ be a set of diverse alternatives, $CR = \{CR_1, CR_2, \ldots, CR_l\}$ be the set of different attributes, and $w = \{w_1, w_2, \ldots, w_l\}$ be the weight vector of attributes CR_i with w_i and $\sum_{i=1}^{l} w_i = 1$. A probabilistic linguistic decision matrix can be expressed as $M = \left(\ell(p)_{ij}\right)_{k\times l}$, where $\ell(p)_{ij}$= $\left\{\ell_{ij}^{(t)}\left(p_{ij}^{(t)}\right)|t = 1, 2, \ldots, \#\ell(p)_{ij}\right\}$ is a PLTS, and $\ell(p)_{ij}$ expresses the evaluation value of alternatives $A_j\,(j = 1, 2, \ldots, k)$ for the attributes $CR_i\,(i = 1, \ldots, l)$.

In line with the given above-mentioned description of MADM problem, the proposed aggregated operators will adopted to address some actual issues and find an ideal alternative. Some main procedures are listed as follows:

Step 1. Standardize the attribute values by the following ways:

If the attribute is a benefit type, then

$$\ell(p)_{ij} = \left\{\ell_{ij}^{(t)}\left(p_{ij}^{(t)}\right)|t = 1, 2, \ldots, \#\ell(p)_{ij}\right\}, \tag{5.1}$$

If the attribute is a cost type, then

$$\ell(p)_{ij} = g^{-1}\left(\bigcup_{\eta_{ij}^{(t)}\in g\left(LS(p)_{\mathrm{ij}}\right)} \left\{\left(1-\eta_{ij}^{(t)}\right)\left(p_{ij}^{(t)}\right)\right\}\right). \tag{5.2}$$

Step 2. Transform all attributes values $\ell(p)_{ij}$ of each alternative to probabilistic hesitant fuzzy element $r(p)_{ij}$.

Step 3. Aggregate all attributes values $r(p)_{ij}$ of each alternative to the comprehensive values $r(p)_j$.

Step 4. Transform $r(p)_j$ into PLTS $\ell(p)_j$.

Step 5. Calculate the score function and the deviation degree of $A_j\,(j = 1, \ldots, k)$ by Eq (2.3) and Eq (2.4).

Step 6. Rank all alternatives and then choose the desirable one.

5.2. Teaching quality evaluation in universities

This section will discuss the decision making option based upon the given WPLGMSM with experimental cases.

Example 5.1. *At present, continuous improvement of education quality has been placed at an important position in colleges and universities, and the evaluation of teaching quality is the baton for the healthy development of education, as well as an essential part of education mechanism. Exploring the evaluation index system of teaching quality, building a scientific evaluation model, and forming* *a reasonable education evaluation system will help to improve the teaching quality and promote the high-quality development of education. In this study four main factors will be used as teaching quality evaluation indexes, they are CR_1: teaching content, CR_2: teaching method, CR_3: teaching effect and CR_4: teaching attitude. It is assumed that the weight of four indexes are $w = (0.2, 0.3, 0.4, 0.1)$, and four teachers' $(A_1 - A_4)$ teaching course Bayesian formula and its application in Probability Theory and Mathematical Statistics will be evaluated. At the same time, through the questionnaire survey on teaching experience of some graduated students and teachers, they are required to evaluate with the following linguistic variables through their own experience:*

$$\{s_{-3} = \textit{extremely bad}, s_{-2} = \textit{very bad},\ s_{-1} = \textit{bad},\ s_0 = \textit{medium},\\ s_1 = \textit{good},\ s_2 = \textit{very good},\ s_3 = \textit{extremely good}\}.$$

After collecting data, relevant decision-making information is obtained as in Table 1.

Table 1. Original decision making matrix.

	CR_1	CR_2	CR_3	CR_4
A_1	$\{s_0(0.45), s_1(0.55)\}$	$\{s_2(1)\}$	$\{s_0(0.3), s_2(0.7)\}$	$\{s_0(1)\}$
A_2	$\{s_0(1)\}$	$\{s_0(0.25), s_2(0.75)\}$	$\{s_1(1)\}$	$\{s_0(1)\}$
A_3	$\{s_0(0.45), s_1(0.55)\}$	$\{s_2(1)\}$	$\{s_0(0.3), s_2(0.7)\}$	$\{s_0(1)\}$
A_4	$\{s_0(0.45), s_1(0.55)\}$	$\{s_2(1)\}$	$\{s_0(0.3), s_2(0.7)\}$	$\{s_0(1)\}$

The following task is to make decision by using the procedure in Section 5.1:

Step 1. Standardize the attribute values $\ell(p)_{ij}$. As all criteria are benefit-type, so it is not necessary to standardize.

Step 2. Transformed all attributes values of each alternative to probabilistic hesitant fuzzy element $r(p)_{ij}$. We set $\tau = 3$ and use the function of g. Probabilistic linguistic information will transformed into probabilistic hesitant fuzzy element and listed $r(p)_{ij}$ in Table 2.

Table 2. Probabilistic hesitant fuzzy element decision information matrix.

	CR_1	CR_2	CR_3	CR_4
$g(A_1(p))$	$\left\{\frac{1}{2}(0.45), \frac{2}{3}(0.55)\right\}$	$\left\{\frac{5}{6}(1)\right\}$	$\left\{\frac{1}{2}(0.3), \frac{5}{6}(0.7)\right\}$	$\left\{\frac{1}{2}(1)\right\}$
$g(A_2(p))$	$\left\{\frac{1}{2}(1)\right\}$	$\left\{\frac{1}{2}(0.25), \frac{5}{6}(0.75)\right\}$	$\left\{\frac{2}{3}(1)\right\}$	$\left\{\frac{1}{2}(1)\right\}$
$g(A_3(p))$	$\left\{\frac{1}{2}(0.5), \frac{2}{3}(0.5)\right\}$	$\left\{\frac{2}{3}(1)\right\}$	$\left\{\frac{1}{3}(0.3), \frac{5}{6}(0.7)\right\}$	$\left\{\frac{1}{3}(1)\right\}$
$g(A_4(p))$	$\left\{\frac{5}{6}(1)\right\}$	$\left\{\frac{1}{2}(0.35), \frac{5}{6}(0.65)\right\}$	$\left\{\frac{2}{3}(1)\right\}$	$\left\{\frac{2}{3}(1)\right\}$

Step 3.-Step 4. We choose the aggregation operator based on algebraic generator to fuse decision information. As the vast numbers, the results not listed here.

Step 5. Calculate the expected value and listed as follows:

$$E(A_1) = 1.1233,\ E(A_2) = 1.1145,\ E(A_3) = 0.6608,\ E(A_4) = 1.4618.$$

Step 6. Determine the desirable alternative according to the expected values. From the calculated results of Step 5, we have $A_4 \succ A_1 \succ A_2 \succ A_3$. Therefore, A_4 is the desirable one.

Meanwhile, we use other three proposed aggregation operators to fuse above decision information, the results are listed in Table 3.

Table 3. The ranking based three proposed aggregation operators when $u_1 = 1$, $u_2 = 2$.

Aggregation operators	Expected values	Ranking
WPLAAGMSM	$E(A_1) = 1.1233, E(A_2) = 1.1145, E(A_3) = 0.6608, E(A_4) = 1.4618$	$A_4 > A_1 > A_2 > A_3$
WPLAEGMSM	$E(A_1) = 1.2210, E(A_2) = 0.8550, E(A_3) = 0.8217, E(A_4) = 1.2546$	$A_4 > A_1 > A_2 > A_3$
WPLAHGMSM	$E(A_1) = -0.3405, E(A_2) = -0.3837, E(A_3) = -0.3964, E(A_4) = -0.2681$	$A_4 > A_1 > A_2 > A_3$

It is obviously the ranking results are consistent with different aggregated operators. Meanwhile, we feed back the ranking results of this paper to some evaluators. Most of them state that the results are in line with their selection order, which also demonstrates the rationalities and effectiveness of the proposed method in Section 5.1.

6. Comparative analyses

To further justify the validity and robustness of our proposed decision-making method, more comparisons will be carried out in this section.

6.1. Comparison with the method of possibility degree matrix

B. Fang, et al. [40] proposed an improved possibility degree formula to assess the education and teaching quality in military academies with probabilistic linguistic MCDM method, some main concepts are reviewed as follows.

Definition 6.1. *[40] Assume $S = \{s_\upsilon | \upsilon = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a LTS, for any two PLTSs $\ell_1(p)$ and $\ell_2(p)$, the possibility degree of $\ell_1(p) \geqslant \ell_2(p)$ could be defined as follows:*

$$P(\ell_1 \geqslant \ell_2) = 0.5 + \sum_{t=1}^{\#\ell_1} \left(\frac{\tau + r_1^{(t)}}{2\tau} - 0.5 \left(\frac{\tau + r_1^{(t)}}{2\tau} \right)^2 \right) * p_1^{(t)} - \sum_{t=1}^{\#\ell_2} \left(\frac{\tau + r_2^{(t)}}{2\tau} - 0.5 \left(\frac{\tau + r_2^{(t)}}{2\tau} \right)^2 \right) * p_2^{(t)} \quad (6.1)$$

in which, $r_1^{(t)}$ and $r_2^{(t)}$ are the subscript of $\ell_1^{(t)}$ and $\ell_2^{(t)}$, $p_1^{(t)}$ and $p_2^{(t)}$ are the corresponding probability, respectively.

Step 1. Calculate comprehensive possibility degree matrix. For each attribute CR_j, w_j is the weight of CR_j and $\sum_{j=1}^{n} w_j = 1$. If $P_{ik}^j = P\left(\ell_{ij} \geq \ell_{kj}\right)$, then the possibility degree matrix of will be defined as follows:

$$P_j = \begin{pmatrix} P_{11}^j & P_{12}^j & \cdots & P_{1n}^j \\ P_{21}^j & P_{22}^j & \cdots & P_{2n}^j \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1}^j & P_{2n}^j & \cdots & P_{nn}^j \end{pmatrix}.$$

Step 2. Then with the method of weighted arithmetic average, the comprehensive possibility degree matrix will be calculated by

$$P=\sum_{j=1}^{n} w_j P_j.$$

Step 3. Calculate the ranking results of alternatives Set $\delta=(\delta_1,\delta_2,\cdots,\delta_m)^T$ is the ordering vector of matrix P, and $0\le\delta_i\le 1$ with $\sum\limits_{i=1}^{m}\delta_i=1$. Then the alternatives can be ranked with the values of δ_i. The higher value of δ_i, the better of the alternative, in which,

$$\delta_i=\frac{1}{m}\left(\sum_{k=1}^{m}P_{ik}+1\right)-0.5,$$

with this method, the teaching quality in Example 5.1 could be ranked as follows.

Firstly, calculating the possibility degree matrix with Step 1.

$$P_1=\begin{pmatrix} 0.5 & 0.5382 & 0.5035 & 0.4271\\ 0.4618 & 0.5 & 0.4653 & 0.3889\\ 0.4965 & 0.5347 & 0.5 & 0.4236\\ 0.5729 & 0.6111 & 0.5764 & 0.5 \end{pmatrix}, P_2=\begin{pmatrix} 0.5 & 0.5278 & 0.5417 & 0.5389\\ 0.4722 & 0.5 & 0.5139 & 0.5111\\ 0.4583 & 0.4861 & 0.5 & 0.4972\\ 0.4611 & 0.4889 & 0.5028 & 0.5 \end{pmatrix},$$

$$P_3=\begin{pmatrix} 0.5 & 0.5083 & 0.5292 & 0.5083\\ 0.4917 & 0.5 & 0.5208 & 0.5\\ 0.4708 & 0.4792 & 0.5 & 0.4792\\ 0.4917 & 0.5 & 0.5208 & 0.5 \end{pmatrix}, P_4=\begin{pmatrix} 0.5 & 0.5 & 0.5972 & 0.4306\\ 0.5 & 0.5 & 0.5972 & 0.4036\\ 0.4028 & 0.4028 & 0.5 & 0.3333\\ 0.5694 & 0.5694 & 0.6667 & 0.5 \end{pmatrix}.$$

Secondly, calculating the compressive possibility degree matrix with weight $w=(0.2,0.3,0.4,0.1)$ as follows:

$$P=\sum_{i=1}^{4} w_i P_i=\begin{pmatrix} 0.5 & 0.5193 & 0.5346 & 0.4935\\ 0.4807 & 0.5 & 0.5153 & 0.4742\\ 0.4654 & 0.4847 & 0.5 & 0.4589\\ 0.5065 & 0.5258 & 0.5411 & 0.5 \end{pmatrix}$$

Lastly, calculating the ranking results:

$$\delta_1=0.2618,\delta_2=0.2425,\delta_3=0.2273,\delta_4=0.2684.$$

According to the above results, the ranking order is

$$A_4\succ A_1\succ A_2\succ A_3,$$

which is the same as our proposed method.

6.2. Comparison with Zhao's method

We take the example of Zhao [39]. To evaluate four cities (Λ_1: Nanchang, Λ_2: Ganzhou, Λ_3: Jiujiang, Λ_4: Jingdezhen) intelligent transportation system. Taking four factors into consideration (CR_1: Traffic data collection, CR_2: Convenient transportation, CR_3: Accident emergency handling capacity, CR_4: Traffic signal equipment) to improve the rationality of evaluation. There we set $w = (0.2, 0.35, 0.25, 0.2)^T$ are the weight of $CR_i\,(i = 1, 3, 4)$, and the LTS is:

$$S = \{s_{-2}=\text{bad}, s_{-1}=\text{slightly bad}, s_0=\text{medium}, s_1=\text{slightly good}, s_2=\text{good}\},$$

using the data of the decision-making matrix without considering the hesitance, after normalizing the probability and listed in the following Table 4.

Table 4. Decision matrix.

	CR_1	CR_2	CR_3	CR_4
Λ_1	$\{s_{-1}(0.4), s_0(0.6)\}$	$\{s_0(0.6)\}$	$\{s_{-2}(0.5), s_{-1}(0.4)\}$	$\{s_{-1}(0.5)\}$
Λ_2	$\{s_{-1}(0.5), s_0(0.3)\}$	$\{s_0(0.4)\}$	$\{s_{-1}(0.7)\}$	$\{s_{-2}(0.7)\}$
Λ_3	$\{s_{-2}(0.4), s_0(0.2)\}$	$\{s_1(0.4)\}$	$\{s_{-2}(0.3), s_{-1}(0.2)\}$	$\{s_0(0.6)\}$
Λ_4	$\{s_{-2}(0.7)\}$	$\{s_{-1}(0.7)\}$	$\{s_{-1}(0.4)\}$	$\{s_{-2}(0.5), s_{-1}(0.4)\}$

After normalizing the probability and using the function of g, the probabilistic hesitant fuzzy matrix will be obtained and listed in Table 5.

Table 5. Normalized decision matrix.

	CR_1	CR_2	CR_3	CR_4
$g(\Lambda_1)$	$\left\{\frac{1}{4}(0.4), \frac{1}{2}(0.6)\right\}$	$\left\{\frac{1}{2}(1)\right\}$	$\left\{0(0.56), \frac{1}{4}(0.44)\right\}$	$\left\{\frac{1}{4}(1)\right\}$
$g(\Lambda_2)$	$\left\{\frac{1}{4}(0.625), \frac{1}{2}(0.375)\right\}$	$\left\{\frac{1}{4}(1)\right\}$	$\left\{\frac{1}{4}(1)\right\}$	$\{0(1)\}$
$g(\Lambda_3)$	$\left\{0(0.67), \frac{1}{2}(0.33)\right\}$	$\left\{\frac{1}{4}(3)\right\}$	$\left\{0(0.6), \frac{1}{4}(0.4)\right\}$	$\left\{\frac{1}{2}(1)\right\}$
$g(\Lambda_4)$	$\{0(1)\}$	$\left\{\frac{1}{4}(1)\right\}$	$\left\{\frac{1}{4}(1)\right\}$	$\left\{0(0.56), \frac{1}{4}(0.44)\right\}$

Based on the operator of $J(x) = -\ln x$, the following expected values and ranking order are obtained:

$$E(\Lambda_1) = -0.9548,\ E(\Lambda_2) = -1.0790,\ E(\Lambda_3) = -0.6757,\ E(\Lambda_4) = -1.9367.$$

It has $\Lambda_3 \succ \Lambda_1 \succ \Lambda_2 \succ \Lambda_4$. Hence, the optimal intelligent transportation system is Λ_3 (Jiujiang), which is the same as the answer in Zhao [39].

6.3. Comparison with Liu's method

Peide Liu et al. [17] proposed probabilistic linguistic Archimedean Muirhead mean operators to rank the alternatives. For the case of maximization profit problems, we make a comparison between our methods and Archimedean Muirhead Mean operators' methods.

For four potential projects $\Lambda_i\,(i = 1, 2, 3, 4)$, directors need to choose the desirable one through four attributes (Λ_1: Financial perspective, Λ_2: Customers satisfaction, Λ_3: Internal business process, Λ_4:

Learning and growth) to improve the rationality of evaluation. Suppose the weight of attributes are $w = (0.2, 0.3, 0.3, 0.2)$, and the LTS is:

$$S = \{s_{-2}= \text{low}, s_{-1}= \text{little low}, s_0= \text{medium}, s_1= \text{little high}, s_2= \text{high}\}.$$

The original decision-making matrix with PLTSs can be normalized and listed in Table 6.

Table 6. Original decision matrix.

	CR_1	CR_2	CR_3	CR_4
Λ_1	$\{s_0(1)\}$	$\{s_{-1}(0.6)\}$	$\{s_1(0.4), s_2(0.4)\}$	$\{s_1(0.8)\}$
Λ_2	$\{s_0(0.8)\}$	$\{s_{-1}(0.8)\}$	$\{s_{-2}(0.6), s_{-1}(0.2)\}$	$\{s_0(0.6)\}$
Λ_3	$\{s_{-1}(0.4)\}$	$\{s_1(0.6)\}$	$\{s_0(0.8), s_1(0.2)\}$	$\{s_{-2}(0.5)\}$
Λ_4	$\{s_{-1}(0.8)\}$	$\{s_{-1}(0.6)\}$	$\{s_1(0.5), s_2(0.5)\}$	$\{s_0(1)\}$

To assure the calculated results more accurate, we do not normalize the probabilities. Using the function of g, we get the probabilistic hesitant fuzzy matrix and listed in Table 7.

Table 7. Probabilistic hesitant fuzzy decision matrix.

	CR_1	CR_2	CR_3	CR_4
$g(\Lambda_1)$	$\left\{\frac{1}{2}(1)\right\}$	$\left\{\frac{1}{4}(0.6)\right\}$	$\left\{\frac{3}{4}(0.4), 1(0.4)\right\}$	$\left\{\frac{3}{4}(0.8)\right\}$
$g(\Lambda_2)$	$\left\{\frac{1}{2}(0.8)\right\}$	$\left\{\frac{1}{4}(0.8)\right\}$	$\left\{0(0.6)\,\frac{1}{4}(0.2)\right\}$	$\left\{\frac{1}{2}(0.6)\right\}$
$g(\Lambda_3)$	$\left\{\frac{1}{4}(0.4)\right\}$	$\left\{\frac{3}{4}(0.6)\right\}$	$\left\{\frac{1}{2}(0.8), \frac{3}{4}(0.2)\right\}$	$\{0(0.5)\}$
$g(\Lambda_4)$	$\left\{\frac{1}{4}(0.8)\right\}$	$\left\{\frac{1}{4}(0.6)\right\}$	$\left\{1(0.5)\,\frac{3}{4}(0.5)\right\}$	$\left\{\frac{1}{2}(1)\right\}$

Using different decision approaches in [16,17] and our proposed approaches to address this decision problem and the results are listed in Table 8.

Table 8. The results obtained by different decision approaches.

Methods	Aggregated results	Ranking
PLWA [16]	$E(\Lambda_1)=1.48, E(\Lambda_2)=0.83, E(\Lambda_3)=0.95, E(\Lambda_4)=1.27$	$\Lambda_1 \succ \Lambda_4 \succ \Lambda_3 \succ \Lambda_2$
PLWG [16]	$E(\Lambda_1)=1.59, E(\Lambda_2)=0.61, E(\Lambda_3)=0, E(\Lambda_4)=1.38$	$\Lambda_1 \succ \Lambda_4 \succ \Lambda_2 \succ \Lambda_3$
HPLAWMM [17]	$E(\Lambda_1)=3.20, E(\Lambda_2)=1.28, E(\Lambda_3)=2.05, E(\Lambda_4)=3.01$ (suppose $P = (1, 0, 0, 0)$ and $\delta = 1$)	$\Lambda_1 \succ \Lambda_4 \succ \Lambda_3 \succ \Lambda_2$
HPLADWMM [17]	$E(\Lambda_1)=2.08, E(\Lambda_2)=0.33, E(\Lambda_3)=0, E(\Lambda_4)=1.67$ (suppose $P = (1, 0, 0, 0)$ and $\delta = 1$)	$\Lambda_1 \succ \Lambda_4 \succ \Lambda_2 \succ \Lambda_3$
Proposed PLAAGMSM	$E(\Lambda_1)=0.516, E(\Lambda_2)=-1.278, E(\Lambda_3)=-0.258,$ $E(\Lambda_4)=-0.159$ (suppose $u_1 = 1\ \ u_1 = 2,\ \ n = 3$)	$\Lambda_1 \succ \Lambda_4 \succ \Lambda_3 \succ \Lambda_2$
Proposed PLAEGMSM	$E(\Lambda_1)=0.5433, E(\Lambda_2)=-0.7559, E(\Lambda_3)=-0.3227,$ $E(\Lambda_4)=-0.1177$ (suppose $u_1 = 1,\ \ u_1 = 2,\ \ n = 3$)	$\Lambda_1 \succ \Lambda_3 \succ \Lambda_4 \succ \Lambda_2$
Proposed PLAHGMSM	$E(\Lambda_1)=-0.2653, E(\Lambda_2)=-0.3507, E(\Lambda_3)=-0.3076,$ $E(\Lambda_4)=-0.3087$ (suppose $u_1 = 1,\ \ u_1 = 2,\ \ n = 3$)	$\Lambda_1 \succ \Lambda_3 \succ \Lambda_4 \succ \Lambda_2$

According to the ranking order, the optimal project is Λ_1, which is the same obtained by other extant decision making approach. This also shows the reliability and effectiveness of the method.

Although the proposed decision-making approach integrated the advantages of PLTSs and GMSM operators, there is a limitation of the proposed approach, that is, as the number of elements increases, the complexity of calculation will increase, corresponding.

7. Conclusions

Classical MSM operators have been attracted great attention of many scholars and widely been used in the field of information fusion due to their biggest merit that they can reflect the relationship between multiple input arguments. On this basis, Wang generalized the traditional MSM operators and introduced the generalized MSM operators. On the other hand, PLTS, a new tool for describing uncertain decision information, can better reflect the actual decision-making problems such as the hesitation of decision-makers, the relative importance of linguistic variables. Combining the merits of PTLS and GMSM operators, PLGMSM and WPLGMSM based on ATN and ATC are proposed and their properties are also investigated. Meanwhile, some special situations are discussed when parameters take different values and the generators of ATN take different function. Besides, this proposed method is applied in teaching quality evaluation in universities, to evaluate an ideal classroom teaching in four alternatives. At last, several comparison analysis are adopted to ensure the validity of the decision-making results, which further verify the reasonability of our proposed decision-making approach.

In future studies, we will continue the current work in expanding and applying the current operators into other contexts. Also some novel MADM approaches will be developed to address some decision-making problems with probabilistic linguistic information. The proposed MADM problem could also be used to other complicated issues.

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Conflict of interest

The authors declare no conflict of interest.

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