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Research article

A weak form of soft α-open sets and its applications via soft topologies

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Abstract: In this work, we present some concepts that are considered unique ideas for topological structures generated by soft settings. We first define the concept of weakly soft α-open subsets and characterize it. It is demonstrated the relationships between this class of soft subsets and some generalizations of soft open sets with the help of some illustrative examples. Some interesting results and relationships are obtained under some stipulations like extended and hyperconnected soft topologies. Then, we introduce the interior and closure operators inspired by the classes of weakly soft α-open and weakly soft α-closed subsets. We establish their master features and derive some formulas that describe the relations among them. Finally, we study soft continuity with respect to this class of soft subsets and investigate its essential properties. In general, we discuss the systematic relations and results that are missing through the frame of our study. The line adopted in this study will create new roads in the branch of soft topology.

Keywords: weakly soft α-open set; extended soft topology; weakly soft α-interior and α-closure operators; weakly soft α-continuous function
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1. Introduction

To deal with vagueness and uncertainty existing in practical situations, e.g., social science, economics, engineering, and medical science, it has been proposed various mathematical instruments. The recent one, introduced by Molodtsov [33], is a soft set whose parameterizations are adequate to process uncertainties and are free from the inherent limitations of the previous instruments. The advantages of soft sets and their applications to different scopes were elaborated in the pioneering work of Molodtsov [33]. In 2002, Maji et al. [30] proposed a technique to handle decision-making problems using soft sets, which was then developed by many researchers. The basic ideas of soft set theory (operators and operations) were initiated by [31]. Then, the authors of [5] showed the awkwardness of these ideas and updated to be consistent with their analogs in the crisp set theory. Also, it was taken advantage of the parameterization families of soft sets to establish several types of operators and operations between soft sets as illustrated in [14, 35]. To raise the efficiency of soft set theory to cope with uncertain and complicated issues, it has been hybridized with other vague instruments like fuzzy and rough sets as debated in [13, 40, 41].

The year 2011 is the birth of topology induced from soft sets. It was defined by Shabir and Naz [42] and Çağman et al. [21] at the same time. Their approaches differ in the way of choosing the set of parameters as variable or constant. Herein, we follow the line of Shabir and Naz, which stipulate the necessity of a constant set of parameters for each element of the soft topology. Afterward, the field of soft topology attracted many researchers and intellectuals who studied the ideas of classical topology via soft topology. Min [32] described the shape of soft open and closed subsets of soft regular spaces and proved the systematic relation between soft T_2 and soft T_3-spaces. El-Shafei et al. [24] founded a strong family of soft T_i-spaces that preserves more features of classical separation axioms. El-Shafei and Al-shami [23] established another type of soft separation axioms and evinced their relationships with the previous types. Important adjustments for the previous studies of soft separation axioms have been conducted by some authors [8, 43]. Recently, Al-shami [11] has investigated how soft separation axioms are applied to select the optimal alternatives for tourism programs.

Aygünoğlu and Aygün [20] familiarized the concepts of compact and Lindelöf spaces. Hida [27] presented and described other kinds of compactness. The concepts of covering properties have been popularized with respect to soft regular closed [6], soft somewhat open sets [15] and soft somewhere dense set [17]. An interesting application of soft compactness to information systems was provided by Al-shami [9]. The correspondence between enriched and extended soft topologies was proved by Al-shami and Kočinac [18]. They also pointed out that many topological properties are transposable between this type of soft topology and their parametric topologies.

The notion of functions between soft topological spaces was defined by Kharal and Ahmad [28], which it refined using the crisp functions and soft points by Al-shami [10]. Specific sorts of soft functions such as soft continuous, open, and closed functions were discussed in [44]. The concepts of Menger spaces [29], maximal topologies [4] and expandable spaces [36] were integrated via soft settings as well. The authors of [26, 37–39] studied topological structures inspired by the hybridizations of soft sets with fuzzy sets and neutrosophic sets. To expand the topological ideas and relax topological conditions, generalizations of soft open sets have been probed. The main contributions to this topic were done by Chen [22], Akdag and Ozkan [1], Al-shami [7] and Al-Ghour [3] who respectively put forward the concepts of soft semi-open, α-open, somewhere dense and Q-sets. Al-shami et al. [16] defined the concept of weakly soft semi-open sets and debated its main characterizations. Al-shami [12] exploited soft somewhat open sets to discover to what extent the nutrition followed by individuals is convenient for the needs of their bodies.

The motivations for writing this article are, first, to suggest a new approach to generalizing soft topology inspired by its classical topologies. Second, to offer a new framework to produce soft topological concepts such as soft operators and continuity, which are achieved in this work. Of course, the researchers can explore other notions like soft covering properties and separation axioms via the proposed class of weakly soft α-open sets. Finally, to enhance the importance of the soft topological environment to create various analogs for each classical topological concept.

The content of this study is regulated as follows. In Section 2, we gather the most essential definitions and properties that are necessary to make this manuscript self-contained. Then, Section 3 provides a novel approach to introduce a new generalization of soft open sets, namely, weakly soft α-sets. It constructs an illustrative example to describe its main characteristics. Section 4 applies this generalization to establish the concepts of weakly soft α-interior, weakly soft α-closure, weakly soft α-boundary, and weakly soft α-limit soft points and explores the relationships between them. Section 5 discusses the idea of weakly soft α-continuity and elucidates that the equivalent conditions of soft continuity are invalid for this type of continuity. We close this work with some conclusions and proposed future work in Section 6.

2. Preliminaries

In this segment, we recapitulate the fundamentals that are required for the readers to become conscious of the manuscript's context.

Definition 2.1. *[33] A soft set over a nonempty (crisp) set Σ is a set-valued function F from nonempty set of parameters Δ to the power set 2^{Σ} of Σ; it is denoted by the ordered pair (F, Δ).*

That is, a soft set (F, Δ) over $\Sigma \neq \emptyset$ provides a parameterized collection of subsets of Σ; so it may represented as follows

$$(F, \Delta) = \{(\delta, F(\delta)) : \delta \in \Delta \text{ and } F(\delta) \in 2^{\Sigma}\},$$

where each $F(\delta)$ is termed a δ-component of (F, Δ). We denote the family of all soft sets over Σ with a set of parameters Δ by $2^{\Sigma_{\Delta}}$.

Through this manuscript, (F, Δ), (G, Δ) denote soft sets over Σ.

Definition 2.2. *[31, 34] A soft set (F, Δ) is called:*

(i) Absolute, symbolized by $\widetilde{\Sigma}$, if $F(\delta) = \Sigma$ for all $\delta \in \Delta$.

(ii) Null, symbolized by ϕ, if $F(\delta) = \emptyset$ for all $\delta \in \Delta$.

(iii) A soft point if there are $\delta \in \Delta$ and $\sigma \in \Sigma$ with $F(\delta) = \{\sigma\}$ and $F(a) = \emptyset$ for all $a \in \Delta - \{\delta\}$. A soft point is symbolized by σ_{δ}. We write $\sigma_{\delta} \in (F, \Delta)$ if $\sigma \in F(\delta)$.

(iv) Pseudo constant if $F(\delta) = \Sigma$ or $\emptyset$ for all $\delta \in \Delta$.

Definition 2.3. *[25] We call (F, Δ) a soft subset of (G, Δ) (or (G, Δ) a soft superset of (G, Δ)), symbolized by $(F, \Delta)\widetilde{\subseteq}(G, \Delta)$ if $F(\delta) \subseteq G(\delta)$ for each $\delta \in \Delta$.*

Definition 2.4. *[5] If $G(\delta) = \Sigma - F(\delta)$ for all $\delta \in \Delta$, then we call (G, Δ) a complement of (F, Δ). The complement of (F, Δ) is symbolized by $(F, \Delta)^c = (F^c, \Delta)$.*

Definition 2.5. *[14] Let (F, Δ) and (G, Δ) be soft sets. Then:*

(i) $(F, \Delta)\widetilde{\bigcup}(G, \Delta) = (H, \Delta)$, where $H(\delta) = F(\delta) \bigcup G(\delta)$ for all $\delta \in \Delta$.

(ii) $(F, \Delta)\widetilde{\bigcap}(G, \Delta) = (H, \Delta)$, where $H(\delta) = F(\delta) \bigcap G(\delta)$ for all $\delta \in \Delta$.

(iii) $(F,\Delta) \setminus (G,\Delta) = (H,\Delta)$, *where* $H(\delta) = F(\delta) \setminus G(\delta)$ *for all* $\delta \in \Delta$.

(iv) $(F,\Delta) \times (G,\Delta) = (H,\Delta)$, *where* $H(\delta_1,\delta_2) = F(\delta_1) \times G(\delta_2)$ *for all* $(\delta_1,\delta_2) \in \Delta \times \Delta$.

The adjusted version of the definition of soft functions is given in the following.

Definition 2.6. *[10] Let* $\mathrm{M}\colon \Sigma \to \Upsilon$ *and* $\mathrm{P}\colon \Delta \to \Omega$ *be crisp functions. A soft function* $\mathrm{M_P}$ *of* 2^{Σ_Δ} *into* 2^{Υ_Ω} *is a relation such that each* $\sigma_\delta \in 2^{\Sigma_\Delta}$ *is related to one and only one* $\epsilon_\omega \in 2^{\Upsilon_\Omega}$ *such that*

$$\mathrm{M_P}(\sigma_\delta) = \mathrm{M}(\sigma)_{\mathrm{P}(\delta)} \text{ for all } \sigma_\delta \in 2^{\Sigma_\Delta}.$$

In addition, $\mathrm{M_P^{-1}}(\epsilon_\omega) = \widetilde{\bigcup}_{\substack{\sigma\in \mathrm{M}^{-1}(\epsilon)\\ \delta\in \mathrm{P}^{-1}(\omega)}} \sigma_\delta$ *for each* $\epsilon_\omega \in 2^{\Upsilon_\Omega}$.

That is, the image of (F,Δ) and pre-image of (G,Ω) under a soft function $\mathrm{M_P}\colon 2^{\Sigma_\Delta} \to 2^{\Upsilon_\Omega}$ are respectively given by:

$$\mathrm{M_P}(F,\Delta) = \widetilde{\bigcup}_{\sigma_\delta\in(F,\Delta)} \mathrm{M_P}(\sigma_\delta,),$$

and

$$\mathrm{M_P^{-1}}(G,\Omega) = \widetilde{\bigcup}_{\epsilon_\omega\in(G,\Omega)} \mathrm{M_P^{-1}}(\epsilon_\omega).$$

A soft function is described as surjective (resp., injective, bijective) if its two crisp functions satisfy this description.

Proposition 2.7. *[28] Let* $\mathrm{M_P}\colon 2^{\Sigma_\Delta} \to 2^{\Upsilon_\Omega}$ *be a soft function and let* (F,Δ) *and* (G,Δ) *be soft subsets of* $\widetilde{\Sigma}$ *and* $\widetilde{\Upsilon}$*, respectively. Then*

(i) $(F,\Delta)\widetilde{\subseteq}\mathrm{M_P^{-1}}(\mathrm{M_P}(F,\Delta))$.

(ii) If $\mathrm{M_P}$ *is injective, then* $(F,\Delta) = \mathrm{M_P^{-1}}(\mathrm{M_P}(F,\Delta))$.

(iii) $\mathrm{M_P}(\mathrm{M_P^{-1}}(G,\Omega))\widetilde{\subseteq}(G,\Omega)$.

(iv) If $\mathrm{M_P}$ *is surjective, then* $\mathrm{M_P}(\mathrm{M_P^{-1}}(G,\Omega)) = (G,\Omega)$.

Definition 2.8. *[33] A subfamily* $\mathcal{T}$ *of* 2^{Σ_Δ} *is said to be a soft topology if the following terms are satisfied:*

(i) $\widetilde{\Sigma}$ *and* ϕ *are elements of* $\mathcal{T}$.

(ii) $\mathcal{T}$ *is closed under the arbitrary unions.*

(iii) $\mathcal{T}$ *is closed under the finite intersections.*

We will call the triplet $(\Sigma,\mathcal{T},\Delta)$ *a soft topological space (briefly, soft*$_{TS}$*). Each element in* $\mathcal{T}$ *is called soft open and its complement is called soft closed.*

Definition 2.9. *[33] For a soft subset* (F,Δ) *of a soft*$_{TS}$ $(\Sigma,\mathcal{T},\Delta)$*, the soft interior and soft closure of* (F,Δ)*, denoted respectively by* $int(F,\Delta)$ *and* $cl(F,\Delta)$*, are defined as follows:*

(i) $int(F,\Delta) = \widetilde{\bigcup}\{(G,\Delta) \in \mathcal{T} : (G,\Delta)\widetilde{\subseteq}(F,\Delta)\}$.

(ii) $cl(F,\Delta) = \widetilde{\bigcap}\{(H,\Delta) : (F,\Delta)\widetilde{\subseteq}(H,\Delta) \text{ and } (H^c,\Delta) \in \mathcal{T}\}$.

Definition 2.10. *[15] A soft$_{TS}$ $(\Sigma,\mathcal{T},\Delta)$ is called full if every non-null soft open set has no empty component.*

Proposition 2.11. *[33] Let $(\Sigma,\mathcal{T},\Delta)$ be a soft$_{TS}$. Then*

$$\mathcal{T}_\delta = \{F(\delta) : (F,\Delta) \in \mathcal{T}\}$$

is a topology on Σ for every $\delta \in \Delta$. We will call this topology a parametric topology.

Definition 2.12. *[33] Let (F,Δ) be a soft subset of a soft$_{TS}$ $(\Sigma,\mathcal{T},\Delta)$. Then $(int(F),\Delta)$ and $(cl(F),\Delta)$ are respectively defined by*

$$int(F)(\delta) = int(F(\delta)),$$

and

$$cl(F)(\delta) = cl(F(\delta)),$$

where $int(F(\delta))$ and $cl(F(\delta))$ are respectively the interior and closure of $F(\delta)$ in $(\Sigma,\mathcal{T}_\delta)$.

Definition 2.13. *[20,34] Let $(\Sigma,\mathcal{T},\Delta)$ be a soft$_{TS}$.*

(i) If all pseudo constant soft sets are elements of $\mathcal{T}$, then $\mathcal{T}$ is called an enriched soft topology.

(ii) $\mathcal{T}$ with the property "$(F,\Delta) \in \mathcal{T}$ iff $F(\delta) \in \mathcal{T}_\delta$ for each $\delta \in \Delta$" is called an extended soft topology.

A comprehensive investigation of the extended and enriched soft topologies was comported on [18]. The corresponding between these kinds of soft topologies was one of the valuable results attained in [18]. Henceforth, this sort of soft topology will be called an extended soft topology. Under this soft topology, it was elucidated several consequences that associated soft topology with its parametric topologies. De facto, the next result will be a key point in the proof of many results.

Theorem 2.14. *[18] A soft$_{TS}$ $(\Sigma,\mathcal{T},\Delta)$ is extended iff $(int(F),\Delta) = int(F,\Delta)$ and $(cl(F),\Delta) = cl(F,\Delta)$ for any soft subset (F,Δ).*

Definition 2.15. *A soft subset (F,Δ) of $(\Sigma,\mathcal{T},\Delta)$ is said to be:*

(i) soft α-open [1] if $(F,\Delta)\widetilde{\subseteq}int(cl(int(F,\Delta)))$.

(ii) soft semi-open [22] if $(F,\Delta)\widetilde{\subseteq}cl(int(F,\Delta))$.

(iii) soft β-open [2] if $(F,\Delta)\widetilde{\subseteq}cl(int(cl(F,\Delta)))$.

(iv) soft sw-open [19] if $(F,\Delta) = \phi$ or $int(F,\Delta) \neq \phi$.

Definition 2.16. *[44] A soft function $\mathrm{M_P}$: $(\Sigma,\mathcal{T}_\Sigma,\Delta) \to (\Upsilon,\mathcal{T}_\Upsilon,\Delta)$ is said to be soft continuous if $\mathrm{M_P^{-1}}(F,\Delta)$ is a soft open set where (F,Δ) is soft open.*

Theorem 2.17. *[18] If $\mathrm{M_P}$: $(\Sigma,\mathcal{T},\Delta) \to (\Upsilon,\mathcal{S},\Omega)$ is soft continuous, then h: $(\Sigma,\mathcal{T}_\delta) \to (\Upsilon,\mathcal{S}_{\mathrm{P}(\delta)})$ is continuous for each $\delta \in \Delta$.*

3. Weakly soft α-open sets and their basic properties

We introduce the main idea of this manuscript called "weakly soft α-open sets" in this section. We show that this class of soft subset is a novel extension of soft open subsets and it lies between soft α-open and soft *sw*-open subsets of extended soft topology. Additionally, we construct some counterexamples to point out some divergences between this class and other extensions such as this class is not closed under soft unions. Among other obtained results, we investigate how this class behaves with respect to topological properties and the product of soft spaces.

Definition 3.1. *A soft subset* (F, Δ) *of* $(\Sigma, \mathcal{T}, \Delta)$ *is called weakly soft* α*-open if it is a null soft set or there is a component of it which is a nonempty* α*-open set. That is,* $F(\delta) = \emptyset$ *for all* $\delta \in \Delta$ *or*

$$\emptyset \neq F(\delta) \subseteq int(cl(int(F(\delta))))$$

for some $\delta \in \Delta$.

We call (F, Δ) *a weakly soft* α*-closed set if its complement is weakly soft* α*-open.*

Proposition 3.2. *A subset* (F, Δ) *of* $(\Sigma, \mathcal{T}, \Delta)$ *is weakly soft* α*-closed iff*

$$(F, \Delta) = \widetilde{\Sigma}$$

or

$$cl(int(cl(F(\delta)))) \subseteq F(\delta) \neq \Sigma$$

for some $\delta \in \Delta$.

Proof. " $\Rightarrow$ ": Let (F, Δ) be a weakly soft α-closed set. Then,

$$(F^c, \Delta) = \phi,$$

or

$$\emptyset \neq F^c(\delta) \subseteq int(cl(int(F^c(\delta)))),$$

for some $\delta \in \Delta$. This means that

$$(F, \Delta) = \widetilde{\Sigma},$$

or

$$cl(int(cl(F(\delta)))) \subseteq F(\delta) \neq \Sigma,$$

for some $\delta \in \Delta$, as required.

" $\Leftarrow$ ": Let (F, Δ) be a soft set such that

$$(F, \Delta) = \widetilde{\Sigma},$$

or

$$cl(int(cl(F(\delta)))) \subseteq F(\delta) \neq \Sigma,$$

for some $\delta \in \Delta$. Then,

$$(F^c, \Delta) = \phi,$$

or

$$\emptyset \neq F^c(\delta) \subseteq int(cl(int(F^c(\delta)))),$$

for some $\delta \in \Delta$. This implies that (F^c, Δ) is weakly soft α-open. Hence, (F, Δ) is weakly soft α-closed, as required.

The following example clarifies that the family of weakly soft α-open (weakly soft α-closed) subsets is *not* closed under soft union or soft intersection.

Example 3.3. *Let $\mathbb{R}$ be the set of real numbers and $\Delta = \{\delta_1, \delta_2\}$ be a set of parameters. Let $\mathcal{T}$ be the soft topology on $\mathbb{R}$ generated by*

$$\{(\delta_i, F(\delta_i)) : F(\delta_i) = (a_i, b_i);\ a_i, b_i \in \mathbb{R};\ a_i \leq\ b_i \text{ and } i = 1, 2\}.$$

Set

$$(F, \Delta) = \{(\delta_1, (0, 1)), (\delta_2, [0, 1])\},$$

and

$$(G, \Delta) = \{(\delta_1, [0, 1]), (\delta_2, (0, 1))\},$$

over $\mathbb{R}$. It is obvious that (F, Δ) and (G, Δ) are both weakly soft α-open and weakly soft α-closed. On the other hand, their soft union is not weakly soft α-open and their soft intersection is not weakly soft α-closed. Also,

$$(H, \Delta) = \{(\delta_1, (1, 6)), (\delta_2, [2, 3])\}$$

and

$$(K, \Delta) = \{(\delta_1, [2, 3]), (\delta_2, (1, 6))\}$$

are both weakly soft α-open and weakly soft α-closed sets over $\mathbb{R}$. But their soft intersection is not weakly soft α-open and their soft union is not weakly soft α-closed.

Proposition 3.4. *Let $(\Sigma, \mathcal{T}, \Delta)$ be a full soft$_{TS}$ with the property of soft hyperconnected. Then the soft intersection of soft α-open and weakly soft α-open subsets is weakly soft α-open.*

Proof. Assume that (F, Δ) and (G, Δ) are respectively soft α-open and weakly soft α-open sets. Then there exist a non-null soft open set (U, Δ) and $\delta \in \Delta$ such that $(U, \Delta)\widetilde{\subseteq}(F, \Delta)$ and $G(\delta)$ is a nonempty α-open subset of $(\Sigma, \mathcal{T}_\delta)$. So there exists a nonempty open subset V_δ of $G(\delta)$. This means that $\mathcal{T}$ contains a non-null soft open set (V, Δ) with $V(\delta) = V_\delta$. Since $\mathcal{T}$ is soft hyperconnected and full, we get $V_\delta \cap U(\delta) \neq \emptyset$. Therefore, $G(\delta)$ and $U(\delta)$ have a nonempty intersection. It follows from general topology that $G(\delta) \cap U(\delta)$ is a nonempty α-open subset of $(\Sigma, \mathcal{T}_\delta)$. Hence, $(F, \Delta)\widetilde{\cap}(G, \Delta)$ is a weakly soft α-open set. ∎

Corollary 3.5. *Let $(\Sigma, \mathcal{T}, \Delta)$ be a full soft$_{TS}$ with the property of soft hyperconnected. Then the soft intersection of soft open and weakly soft α-open subsets is weakly soft α-open.*

Remark 3.6. *Every pseudo constant soft subset (F, Δ) is a weakly soft α-subset because $F(\delta) = \emptyset$ for all $\delta \in \Delta$ or $int(F(\delta)) = \Sigma$ for some $\delta \in \Delta$.*

The next propositions are obvious.

Proposition 3.7. *Every soft open set is weakly soft α-open.*

Proposition 3.8. *Any soft subset (F, Δ) of $(\Sigma, \mathcal{T}, \Delta)$ with $F(\delta) = \Sigma$ (resp., $F(\delta) = \emptyset$) is weakly soft α-open (resp., weakly soft α-closed).*

In the next result we provide a condition that guarantees the relation between weakly soft α-open sets and some generalizations of soft open sets.

Proposition 3.9. *If $(\Sigma, \mathcal{T}, \Delta)$ is extended, then every soft α-open set is weakly soft α-open.*

Proof. Let (F, Δ) be a non-null soft α-open set. Then

$$(F, \Delta)\widetilde{\subseteq} int(cl(int(F, \Delta))).$$

Since $\mathcal{T}$ is an extended soft topology, we get

$$F(\delta) \subseteq int(cl(int(F(\delta))))$$

for each $\delta \in \Delta$. This implies that there is a component of (F, Δ) which is a nonempty α-open subset. Hence, (F, Δ) is weakly soft α-open.

Following similar arguments one can prove the other cases.

Proposition 3.10. *If $(\Sigma, \mathcal{T}, \Delta)$ is extended, then every weakly soft α-open set is soft sw-open.*

Proof. Let (F, Δ) be a non-null weakly soft α-open set. Then there is a component of (F, Δ) which is a nonempty α-open set. So $int(F(\delta)) \neq \emptyset$ for some $\delta \in \Delta$. Since $\mathcal{T}$ is extended, we get

$$int(F, \Delta) = (int(F), \Delta) \neq \phi.$$

This completes the proof.

The next example elaborates that a condition of "extended soft topology" furnished in Propositions 3.9 and 3.10 is indispensable.

Example 3.11. *Let $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ be unverse and $\Delta = \{\delta_1, \delta_2\}$ be a parameters set. Take the family $\mathcal{T}$ consisting of ϕ, $\widetilde{\Sigma}$ and the following soft subsets over Σ with Δ*

$$(F_1, \Delta) = \{(\delta_1, \{\sigma_1\}), (\delta_2, \emptyset)\},$$

$$(F_2, \Delta) = \{(\delta_1, \emptyset), (\delta_2, \{\sigma_1\})\},$$

$$(F_3, \Delta) = \{(\delta_1, \Sigma), (\delta_2, \{\sigma_1\})\},$$

$$(F_4, \Delta) = \{(\delta_1, \{\sigma_1\}), (\delta_2, \Sigma)\},$$

$$(F_5, \Delta) = \{(\delta_1, \{\sigma_1\}), (\delta_2, \{\sigma_1\})\},$$

$$(F_6, \Delta) = \{(\delta_1, \{\sigma_1\}), (\delta_2, \{\sigma_2, \sigma_3\})\},$$

and

$$(F_7, \Delta) = \{(\delta_1, \{\sigma_2, \sigma_3\}), (\delta_2, \{\sigma_1\})\}.$$

Then, $(\Sigma, \mathcal{T}, \Delta)$ is a $soft_{TS}$. Remark that a soft set

$$(H, \Delta) = \{(\delta_1, \{\sigma_1, \sigma_2\}), (\delta_2, \{\sigma_1, \sigma_2\})\}$$

is soft α-open because

$$int(cl(int(H, \Delta))) = \widetilde{\Sigma}.$$

But it is not a weakly soft α-open set because

$$int(cl(int(H(\delta_1)))) = int(cl(int(H(\delta_2)))) = \{\sigma_1\} \not\supseteq H(\delta_1) = H(\delta_2).$$

Also, a soft set

$$(G, \Delta) = \{(\delta_1, \{\sigma_2, \sigma_3\}), (\delta_2, \{\sigma_3\})\}$$

is a weakly soft α-open set because

$$int(cl(int(G(\delta_1)))) = G(\delta_1).$$

But it is not a soft sw-open set because $int(G, \Delta) = \phi$.

To demonstrate that the converse of Propositions 3.9 and 3.10 fail, the following example is shown.

Example 3.12. *Let* $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ *be unverse and* $\Delta = \{\delta_1, \delta_2\}$ *be a parameters set. Take the family* $\mathcal{T}$ *consisting of* ϕ, $\widetilde{\Sigma}$ *and the following soft subsets over* Σ *with* Δ

$$(F_1, \Delta) = \{(\delta_1, \{\sigma_1\}), (\delta_2, \{\sigma_2, \sigma_3\})\},$$

$$(F_2, \Delta) = \{(\delta_1, \{\sigma_2, \sigma_3\}), (\delta_2, \{\sigma_1\})\},$$

$$(F_3, \Delta) = \{(\delta_1, \{\sigma_1\}), (\delta_2, \emptyset)\},$$

$$(F_4, \Delta) = \{(\delta_1, \emptyset), (\delta_2, \{\sigma_1\})\},$$

$$(F_5, \Delta) = \{(\delta_1, \{\sigma_2, \sigma_3\}), (\delta_2, \emptyset)\},$$

$$(F_6, \Delta) = \{(\delta_1, \emptyset), (\delta_2, \{\sigma_2, \sigma_3\})\},$$

$$(F_7, \Delta) = \{(\delta_1, \Sigma), (\delta_2, \{\sigma_1\})\},$$

$$(F_8, \Delta) = \{(\delta_1, \{\sigma_1\}), (\delta_2, \Sigma)\},$$

$$(F_9, \Delta) = \{(\delta_1, \{\sigma_2, \sigma_3\}), (\delta_2, \Sigma)\},$$

$$(F_{10}, \Delta) = \{(\delta_1, \Sigma), (\delta_2, \{\sigma_2, \sigma_3\})\},$$

$$(F_{11}, \Delta) = \{(\delta_1, \{\sigma_1\}), (\delta_2, \{\sigma_1\})\},$$

$$(F_{12}, \Delta) = \{(\delta_1, \{\sigma_2, \sigma_3\}), (\delta_2, \{\sigma_2, \sigma_3\})\},$$

$$(F_{13}, \Delta) = \{(\delta_1, \Sigma), (\delta_2, \emptyset)\},$$

and

$$(F_{14}, \Delta) = \{(\delta_1, \emptyset), (\delta_2, \Sigma)\}.$$

Then, $(\Sigma, \mathcal{T}, \Delta)$ *is an extended* $soft_{TS}$. *Remark that a soft set*

$$(H, \Delta) = \{(\delta_1, \{\sigma_2, \sigma_3\}), (\delta_2, \{\sigma_3\})\}$$

is weakly soft α-open. But it is not a soft α-open set because

$$int(cl(int(H, \Delta))) = \{(\delta_1, \{\sigma_2, \sigma_3\}), (\delta_2, \emptyset)\} \not\supseteq (H, \Delta).$$

Also, a soft set

$$(G, \Delta) = \{(\delta_1, \{\sigma_1, \sigma_2\}), (\delta_2, \emptyset)\}$$

is soft sw-open because

$$int(G, \Delta) = \{(\delta_1, \{\sigma_1\}), (\delta_2, \emptyset)\} \neq \phi.$$

But it is not a weakly soft α-open set because

$$int(cl(int(G(\delta_1)))) = \{\sigma_1\} \not\supseteq G(\delta_1)$$

and $G(\delta_2)$ *is empty.*

Proposition 3.13. *The image and pre-image of weakly soft α-open set under a soft bi-continuous function (soft open and continuous) is weakly soft α-open.*

Proof. To show the case of image, let M_P: $(\Sigma, \mathcal{T}, \Delta) \to (\Upsilon, \mathcal{S}, \Omega)$ be a soft bi-continuous function and let (F, Δ) be a weakly soft α-subset of $(\Sigma, \mathcal{T}, \Delta)$. Suppose that there exists $\delta \in \Delta$ such that $F(\delta)$ is a nonempty α-open subset and let $P(\delta) = \omega$. According to Theorem 2.17, it follows from the soft bicontinuity of M_P that M: $(\Sigma, \mathcal{T}_\delta) \to (\Upsilon, \mathcal{S}_{P(\delta)=\omega})$ is a bicontinuous function.

It is well known that a continuity of M implies that $M(cl(V)) \subseteq cl(M(V))$, and an openness of M implies that $M(int(V)) \subseteq int(M(V))$ for each subset V of Σ. This implies that

$$M(F(\delta)) \widetilde{\subseteq} M(int(cl(int(F(\delta))))) \widetilde{\subseteq} int(cl(int(M(F(\delta))))).$$

According to Definition 2.16, we find that $M(F(\delta))$ is a nonempty α-open subset of $M_P(F, \Delta)$; hence, $M_P(F, \Delta)$ is a weakly soft α-open subset of $(\Sigma, \mathcal{T}, \Delta)$.

Corollary 3.14. *The property of being a weakly soft α-open set is a topological property.*

Proposition 3.15. *The product of two weakly soft α-open sets is weakly soft α-open.*

Proof. Suppose that (F, Δ) and (G, Δ) are weakly soft α-open subsets and let

$$(H, \Delta \times \Delta) = (F, \Delta) \times (G, \Delta).$$

Then there are $\delta_1, \delta_2 \in \Delta$ such that $F(\delta_1)$ and $G(\delta_2)$ are nonempty α-open subsets. Now, $(\delta_1, \delta_2) \in \Delta \times \Delta$ such that

$$H(\delta_1, \delta_2) = F(\delta_1) \times G(\delta_2).$$

As we know from the classical topology the product of two nonempty α-open subsets is still a nonempty α-open subset; therefore, $H(\delta_1, \delta_2)$ is a nonempty α-open subset. Hence, $(H, \Delta \times \Delta)$ is a weakly soft α-open subset.

4. Weakly α-interior and weakly α-closure operators

As an expected line of this type of study, we build the operators of interior, closure, boundary, and limit inspired by the class of weakly soft α-open and weakly soft α-closed sets. We elucidate their master properties and scrutinize the relationships among them. By some counterexamples, we illustrate that the weakly α-interior (resp., weakly α-closure) of soft subset need not be weakly α-open (resp., weakly α-closed) sets, in general.

Definition 4.1. *The weakly α-interior points of a subset (F,Δ) of $(\Sigma,\mathcal{T},\Delta)$, denoted by $int_{w\alpha}(F,\Delta)$, is defined as the union of all weakly soft α-open sets contained in (F,Δ).*

By Example 3.3 we remark that the weakly α-interior points of a subset need not be a weakly α-open set. That is,

$$int_{w\alpha}(F,\Delta) = (F,\Delta)$$

does not imply that (F,Δ) is a weakly α-open set.

One can easily prove the next propositions.

Proposition 4.2. *Let (F,Δ) be a subset of $(\Sigma,\mathcal{T},\Delta)$ and $\sigma_\delta \in \widetilde{\Sigma}$. Then $\sigma_\delta \in int_{w\alpha}(F,\Delta)$ iff there is a weakly soft α-open set (G,Δ) contains σ_δ such that $(G,\Delta)\widetilde{\subseteq}(F,\Delta)$.*

Proposition 4.3. *Let (F,Δ), (G,Δ) be soft subsets of $(\Sigma,\mathcal{T},\Delta)$. Then*

(i) $int_{w\alpha}(F,\Delta)\widetilde{\subseteq}(F,\Delta)$.

(ii) if $(F,\Delta)\widetilde{\subseteq}(G,\Delta)$, then $int_{w\alpha}(F,\Delta)\widetilde{\subseteq}int_{w\alpha}(G,\Delta)$.

Corollary 4.4. *For any two subsets (F,Δ), (G,Δ) of $(\Sigma,\mathcal{T},\Delta)$, we have the following results:*

(i) $int_{w\alpha}[(F,\Delta)\widetilde{\bigcap}(G,\Delta)]\widetilde{\subseteq}int_{w\alpha}(F,\Delta)\widetilde{\bigcap}int_{w\alpha}(G,\Delta)$.

(ii) $int_{w\alpha}(F,\Delta)\widetilde{\bigcup}int_{w\alpha}(G,\Delta)\widetilde{\subseteq}int_{w\alpha}[(F,\Delta)\widetilde{\bigcup}(G,\Delta)]$.

Proof. It automatically comes from the following:
(i) $(F,\Delta)\widetilde{\bigcap}(G,\Delta)\widetilde{\subseteq}(F,\Delta)$ and $(F,\Delta)\widetilde{\bigcap}(G,\Delta)\widetilde{\subseteq}(G,\Delta)$.
(ii) $(F,\Delta)\widetilde{\subseteq}[(F,\Delta)\widetilde{\bigcup}(G,\Delta)]$ and $(G,\Delta)\widetilde{\subseteq}[(F,\Delta)\widetilde{\bigcup}(G,\Delta)]$.

Let

$$(E,\Delta) = \{(\delta_1,\{5\}),(\delta_2,\{6,7\})\},$$
$$(F,\Delta) = \{(\delta_1,(1,2)),(\delta_2,[1,2])\},$$
$$(G,\Delta) = \{(\delta_1,(2,3)),(\delta_2,[2,3])\},$$
$$(H,\Delta) = \{(\delta_1,\emptyset),(\delta_2,(0,1])\},$$

and

$$(J,\Delta) = \{(\delta_1,(0,1]),(\delta_2,\emptyset)\}$$

be soft subsets of a soft$_{TS}$ given in Example 3.3. We remark the following properties:
(i) $(E,\Delta)\widetilde{\nsubseteq}int_{w\alpha}(E,\Delta) = \phi$.
(ii) $int_{w\alpha}(E,\Delta)\widetilde{\subseteq}int_{w\alpha}(H,\Delta)$, whereas $(E,\Delta)\widetilde{\nsubseteq}(H,\Delta)$.
(iii) $int_{w\alpha}[(F,\Delta)\widetilde{\bigcap}(G,\Delta)] = \phi$, whereas $int_{w\alpha}(F,\Delta)\widetilde{\bigcap}int_{w\alpha}(G,\Delta) = \{(\delta_1,\emptyset),(\delta_2,\{2\})\}$.
(iv)

$$int_{w\alpha}(H,\Delta)\widetilde{\bigcup}int_{w\alpha}(J,\Delta) = \{(\delta_1,(0,1)),(\delta_2,(0,1))\},$$

whereas

$$int_{w\alpha}[(H,\Delta)\widetilde{\bigcup}(J,\Delta)] = \{(\delta_1,(0,1]),(\delta_2,(0,1])\}.$$

Hence, the inclusion relations of Proposition 4.3 and Corollary 4.4 are proper.

Definition 4.5. *The weakly α-closure points of a subset (F,Δ) of $(\Sigma,\mathcal{T},\Delta)$, denoted by $cl_{w\alpha}(F,\Delta)$, is defined as the intersection of all weakly soft α-closed sets containing (F,Δ).*

By Example 3.3 we remark that the weakly α-closure points of a subset need not be a weakly α-closed set. That is, $cl_{w\alpha}(F,\Delta) = (F,\Delta)$ does not imply that (F,Δ) is a weakly α-closed set.

Proposition 4.6. *Let (F,Δ) be a subset of $(\Sigma,\mathcal{T},\Delta)$ and $\sigma_\delta \in \widetilde{\Sigma}$. Then $\sigma_\delta \in cl_{w\alpha}(F,\Delta)$ iff $(G,\Delta)\widetilde{\bigcap}(F,\Delta) \neq \phi$ for each weakly soft α-open set (G,Δ) contains σ_δ.*

Proof. [⇒] Let $\sigma_\delta \in cl_{w\alpha}(F,\Delta)$. Suppose that there is weakly soft α-open set (G,Δ) containing σ_δ with

$$(G,\Delta)\widetilde{\bigcap}(F,\Delta) = \phi.$$

Then

$$(F,\Delta)\widetilde{\subseteq}(G^c,\Delta).$$

Therefore,

$$cl_{w\alpha}(F,\Delta)\widetilde{\subseteq}(G^c,\Delta).$$

Thus

$$\sigma_\delta \notin cl_{w\alpha}(F,\Delta).$$

This is a contradiction, which means that

$$(G,\Delta)\widetilde{\bigcap}(F,\Delta) \neq \phi,$$

as required.
[⇐] Let

$$(G,\Delta)\widetilde{\bigcap}(F,\Delta) \neq \phi$$

for each weakly soft α-open set (G,Δ) contains σ_δ. Suppose that

$$\sigma_\delta \notin cl_{w\alpha}(F,\Delta).$$

Then there is a weakly soft α-closed set (H,Δ) containing (F,Δ) with $\sigma_\delta \notin (H,\Delta)$. So

$$\sigma_\delta \in (H^c,\Delta),$$

and

$$(H^c,\Delta)\widetilde{\bigcap}(F,\Delta) = \phi.$$

This is a contradiction. Hence, we obtain the desired result. □

Corollary 4.7. *If*

$$(F,\Delta)\widetilde{\bigcap}(G,\Delta) = \phi$$

such that (F,Δ) is a weakly soft α-open set and (G,Δ) is a soft set in $(\Sigma,\mathcal{T},\Delta)$, then

$$(F,\Delta)\widetilde{\bigcap}cl_{w\alpha}(G,\Delta) = \phi.$$

Proof. Obvious.

Proposition 4.8. *The following properties hold for a subset* (F, Δ) *of* $(\Sigma, \mathcal{T}, \Delta)$.

(i) $[int_{w\alpha}(F, \Delta)]^c = cl_{w\alpha}(F^c, \Delta)$.

(ii) $[cl_{w\alpha}(F, \Delta)]^c = int_{w\alpha}(F^c, \Delta)$.

Proof. (i) If

$$\sigma_\delta \notin [int_{w\alpha}(F, \Delta)]^c,$$

then there is a weakly soft α-open set (G, Δ) with

$$\sigma_\delta \in (G, \Delta)\widetilde{\subseteq}(F, \Delta).$$

Therefore,

$$(F^c, \Delta)\widetilde{\bigcap}(G, \Delta) = \phi,$$

and hence,

$$\sigma_\delta \notin cl_{w\alpha}(F^c, \Delta).$$

Conversely, if $\sigma_\delta \notin cl_{w\alpha}(F^c, \Delta)$ we can follow the previous steps to verify $\sigma_\delta \notin [int_{w\alpha}(F, \Delta)]^c$.
(ii) Following similar approach given in (i).

The next proposition is easy, so we omit its proof.

Proposition 4.9. *Let* (F, Δ)*,* (G, Δ) *be soft subsets of* $(\Sigma, \mathcal{T}, \Delta)$*. Then*

(i) $(F, \Delta)\widetilde{\subseteq}cl_{w\alpha}(F, \Delta)$.

(ii) if $(F, \Delta)\widetilde{\subseteq}(G, \Delta)$*, then* $cl_{w\alpha}(F, \Delta)\widetilde{\subseteq}cl_{w\alpha}(G, \Delta)$.

Corollary 4.10. *The following results hold for any subsets* (F, Δ)*,* (G, Δ) *of* $(\Sigma, \mathcal{T}, \Delta)$.

(i) $cl_{w\alpha}[(F, \Delta)\widetilde{\bigcap}(G, \Delta)]\widetilde{\subseteq}cl_{w\alpha}(F, \Delta)\widetilde{\bigcap}cl_{w\alpha}(G, \Delta)$.

(ii) $cl_{w\alpha}(F, \Delta)\widetilde{\bigcup}cl_{w\alpha}(G, \Delta)\widetilde{\subseteq}cl_{w\alpha}[(F, \Delta)\widetilde{\bigcup}(G, \Delta)]$.

Proof. It automatically comes from the following:
(i) $(F, \Delta)\widetilde{\bigcap}(G, \Delta)\widetilde{\subseteq}(F, \Delta)$ and $(F, \Delta)\widetilde{\bigcap}(G, \Delta)\widetilde{\subseteq}(G, \Delta)$.
(ii) $(F, \Delta)\widetilde{\subseteq}[(F, \Delta)\widetilde{\bigcup}(G, \Delta)]$ and $(G, \Delta)\widetilde{\subseteq}[(F, \Delta)\widetilde{\bigcup}(G, \Delta)]$.

Let

$$(E, \Delta) = \{(\delta_1, \mathbb{R}), (\delta_2, [0, 1))\},$$

$$(F, \Delta) = \{(\delta_1, (1, 2)), (\delta_2, [1, 2])\},$$

$$(G, \Delta) = \{(\delta_1, (2, 3)), (\delta_2, [2, 3])\},$$

and

$$(H, \Delta) = \{(\delta_1, \mathbb{R}), (\delta_2, (0, 1])\}$$

be soft subsets of a soft$_{TS}$ given in Example 3.3. We remark the following properties:

(i) $cl_{w\alpha}(E,\Delta) = \{(\delta_1,\mathbb{R}),(\delta_2,[0,1])\}\widetilde{\not\subset}(E,\Delta)$.

(ii) $cl_{w\alpha}(E,\Delta)\widetilde{\subseteq}cl_{w\alpha}(H,\Delta)$, whereas $(E,\Delta)\widetilde{\not\subseteq}(H,\Delta)$.

(iii) $cl_{w\alpha}[(F,\Delta)\widetilde{\bigcap}(G,\Delta)] = \{(\delta_1,\emptyset),(\delta_2,\{2\})\}$, whereas $cl_{w\alpha}(F,\Delta)\widetilde{\bigcap}cl_{w\alpha}(G,\Delta) = \{(\delta_1,\{2\}),(\delta_2,\{2\})\}$.

Hence, the inclusion relations of Proposition 4.9 and Corollary 4.10 are proper.

Definition 4.11. *A soft point σ_δ is said to be a weakly α-boundary point of a subset (F,Δ) of $(\Sigma,\mathcal{T},\Delta)$ if σ_δ belongs to the complement of $int_{w\alpha}(F,\Delta)\widetilde{\bigcup}int_{w\alpha}(F^c,\Delta)$.*

All α-boundary points of (F,Δ) is called a weakly α-boundary set, denoted by $b_{w\alpha}(F,\Delta)$.

Proposition 4.12.

$$b_{w\alpha}(F,\Delta) = cl_{w\alpha}(F,\Delta)\widetilde{\bigcap}cl_{w\alpha}(F^c,\Delta)$$

for every subset (F,Δ) of $(\Sigma,\mathcal{T},\Delta)$.

Proof.

$$\begin{aligned} b_{w\alpha}(F,\Delta) &= [int_{w\alpha}(F,\Delta)\widetilde{\bigcup}int_{w\alpha}(F^c,\Delta)]^c \\ &= [int_{w\alpha}(F,\Delta)]^c\widetilde{\bigcap}[int_{w\alpha}(F^c,\Delta)]^c \quad (De\ \ Morgan's\ \ law) \\ &= cl_{w\alpha}(F^c,\Delta)\widetilde{\bigcap}cl_{w\alpha}(F,\Delta) \quad (Proposition\ \ 4.8(ii)). \end{aligned}$$

Corollary 4.13. *For every subset (F,Δ) of $(\Sigma,\mathcal{T},\Delta)$, the following properties hold.*

(i) $b_{w\alpha}(F,\Delta) = b_{w\alpha}(F^c,\Delta)$.

(ii) $b_{w\alpha}(F,\Delta) = cl_{w\alpha}(F,\Delta)\setminus int_{w\alpha}(F,\Delta)$.

(iii) $cl_{w\alpha}(F,\Delta) = int_{w\alpha}(F,\Delta)\widetilde{\bigcup}\, b_{w\alpha}(F,\Delta)$.

(iv) $int_{w\alpha}(F,\Delta) = (F,\Delta)\setminus b_{w\alpha}(F,\Delta)$.

Proof. (i) Obvious.

(ii) $b_{w\alpha}(F,\Delta) = cl_{w\alpha}(F,\Delta)\widetilde{\bigcap}cl_{w\alpha}(F^c,\Delta) = cl_{w\alpha}(F,\Delta)\setminus [cl_{w\alpha}(F^c,\Delta)]^c$. By (ii) of Proposition 4.8 we obtain the required relation.

(iii) $int_{w\alpha}(F,\Delta)\widetilde{\bigcup}\, b_{w\alpha}(F,\Delta) = int_{w\alpha}(F,\Delta)\widetilde{\bigcup}\,[cl_{w\alpha}(F,\Delta)\setminus int_{w\alpha}(F,\Delta)] = cl_{w\alpha}(F,\Delta)$.

(iv)

$$\begin{aligned} (F,\Delta)\setminus b_{w\alpha}(F,\Delta) &= (F,\Delta)\setminus [cl_{w\alpha}(F,\Delta)\setminus int_{w\alpha}(F,\Delta)] \\ &= (F,\Delta)\widetilde{\bigcap}[cl_{w\alpha}(F,\Delta)\widetilde{\bigcap}(int_{w\alpha}(F,\Delta))^c]^c \\ &= (F,\Delta)\widetilde{\bigcap}[(cl_{w\alpha}(F,\Delta))^c\widetilde{\bigcup}\, int_{w\alpha}(F,\Delta)] \\ &= [(F,\Delta)\widetilde{\bigcap}(cl_{w\alpha}(F,\Delta))^c]\widetilde{\bigcup}[(F,\Delta)\widetilde{\bigcap}\, int_{w\alpha}(F,\Delta)] \\ &= int_{w\alpha}(F,\Delta). \end{aligned}$$

Proposition 4.14. *Let $(F, \Delta), (G, \Delta)$ be subsets of $(\Sigma, \mathcal{T}, \Delta)$, the following properties hold.*

(i) $b_{w\alpha}(int_{w\alpha}(F, \Delta))\widetilde{\subseteq} b_{w\alpha}(F, \Delta)$.

(ii) $b_{w\alpha}(cl_{w\alpha}(F, \Delta))\widetilde{\subseteq} b_{w\alpha}(F, \Delta)$.

Proof. By substituting in the formula No. (iii) of Corollary 4.13, the proof follows.

Proposition 4.15. *Let (F, Δ) be a subset of $(\Sigma, \mathcal{T}, \Delta)$. Then*

(i) $(F, \Delta) = int_{w\alpha}(F, \Delta)$ *iff* $b_{w\alpha}(F, \Delta)\widetilde{\bigcap}(F, \Delta) = \phi$.

(ii) $(F, \Delta) = cl_{w\alpha}(F, \Delta)$ *iff* $b_{w\alpha}(F, \Delta)\widetilde{\subseteq}(F, \Delta)$.

Proof. (i) Suppose that

$$(F, \Delta) = int_{w\alpha}(F, \Delta).$$

Then by (iv) of Corollary 4.13,

$$(F, \Delta) = int_{w\alpha}(F, \Delta) = (F, \Delta) \setminus b_{w\alpha}(F, \Delta),$$

and hence,

$$b_{w\alpha}(F, \Delta)\widetilde{\bigcap}(F, \Delta) = \phi.$$

Conversely, let $\sigma_\delta \in (F, \Delta)$. Since $\sigma_\delta \notin b_{w\alpha}(F, \Delta)$ and $\sigma_\delta \in cl_{w\alpha}(F, \Delta)$, by (iii) of Corollary 4.13, $\sigma_\delta \in int_{w\alpha}(F, \Delta)$. Therefore,

$$int_{w\alpha}(F, \Delta) = (F, \Delta),$$

as required.
(ii) Assume that

$$(F, \Delta) = cl_{w\alpha}(F, \Delta).$$

Then

$$b_{w\alpha}(F, \Delta) = cl_{w\alpha}(F, \Delta)\widetilde{\bigcap} cl_{w\alpha}(F^c, \Delta)\widetilde{\subseteq} cl_{w\alpha}(F, \Delta) = (F, \Delta),$$

as required.

Conversely, if $b_{w\alpha}(F, \Delta)\widetilde{\subseteq}(F, \Delta)$, then by (iii) of Corollary 4.13,

$$cl_{w\alpha}(F, \Delta)\widetilde{\subseteq} int_{w\alpha}(F, \Delta)\widetilde{\bigcup}(F, \Delta) = (F, \Delta),$$

and hence,

$$cl_{w\alpha}(F, \Delta) = (F, \Delta),$$

as required.

Corollary 4.16. *Let (F, Δ) be a subset of $(\Sigma, \mathcal{T}, \Delta)$. Then*

$$int_{w\alpha}(F, \Delta) = (F, \Delta) = cl_{w\alpha}(F, \Delta),$$

iff

$$b_{w\alpha}(F, \Delta) = \phi.$$

Definition 4.17. *A soft point σ_δ is said to be a weakly α-limit point of a subset (F, Δ) of $(\Sigma, \mathcal{T}, \Delta)$ if*

$$[(G, \Delta) \backslash \sigma_\delta] \bigcap (F, \Delta) \neq \phi,$$

for each weakly soft α-open set (G, Δ) containing σ_δ.

All weakly α-limit points of (F, Δ) is called a weakly α-derived set and denoted by $l_{w\alpha}(F, \Delta)$.

Proposition 4.18. *Let (F, Δ) and (G, Δ) be subsets of $(\Sigma, \mathcal{T}, \Delta)$. If $(F, \Delta) \widetilde{\subseteq} (G, \Delta)$, then $l_{w\alpha}(F, \Delta) \widetilde{\subseteq} l_{w\alpha}(G, \Delta)$.*

Proof. Straightforward by Definition 4.17.

Corollary 4.19. *Consider (F, Δ) and (G, Δ) are subsets of $(\Sigma, \mathcal{T}, \Delta)$. Then:*

(i) $l_{w\alpha}[(F, \Delta) \widetilde{\bigcap} (G, \Delta)] \widetilde{\subseteq} l_{w\alpha}(F, \Delta) \widetilde{\bigcap} l_{w\alpha}(G, \Delta)$.

(ii) $l_{w\alpha}(F, \Delta) \widetilde{\bigcup} l_{w\alpha}(G, \Delta) \widetilde{\subseteq} l_{w\alpha}[(F, \Delta) \widetilde{\bigcup} (G, \Delta)]$.

Theorem 4.20. *Let (F, Δ) be a subset of $(\Sigma, \mathcal{T}, \Delta)$, then*

$$cl_{w\alpha}(F, \Delta) = (F, \Delta) \widetilde{\bigcup} l_{w\alpha}(F, \Delta).$$

Proof. The side

$$(F, \Delta) \widetilde{\bigcup} l_{w\alpha}(F, \Delta) \widetilde{\subseteq} cl_{w\alpha}(F, \Delta)$$

is obvious. To prove the other side let

$$\sigma_\delta \notin [(F, \Delta) \widetilde{\bigcup} l_{w\alpha}(F, \Delta)].$$

Then $\sigma_\delta \notin (F, \Delta)$ and $\sigma_\delta \notin l_{w\alpha}(F, \Delta)$. Therefore, there is weakly soft α-open (G, Δ) containing σ_δ with

$$(G, \Delta) \widetilde{\cap} (F, \Delta) = \phi.$$

Thus, $\sigma_\delta \notin cl_{w\alpha}(F, \Delta)$. Hence, we find that

$$cl_{w\alpha}(F, \Delta) = (F, \Delta) \widetilde{\bigcup} l_{w\alpha}(F, \Delta).$$

Corollary 4.21. *Let (F, Δ) be a weakly soft α-closed subset of $(\Sigma, \mathcal{T}, \Delta)$, then $l_{w\alpha}(F, \Delta) \widetilde{\subseteq} (F, \Delta)$.*

5. Continuity via weakly soft α-open sets

This is the last main section we dedicate to tackling the concept of soft continuity via weakly soft α-open sets. We establish its main characterizations and show that loss of the property says that "weakly α-interior of the soft subset is weakly soft α-open" leads to evaporating some descriptions of this type of soft continuity. An elucidative counterexample is supplied.

Definition 5.1. *A soft function $\mathrm{M_P}$: $(\Sigma, \mathcal{T}_\Sigma, \Delta) \to (\Upsilon, \mathcal{T}_\Upsilon, \Delta)$ is said to be weakly soft α-continuous if the inverse image of each soft open set is weakly soft α-open.*

It is straightforward to prove the next result, so omit its proof.

Proposition 5.2. *If* $\mathrm{M_P}$*:* $(\Sigma, \mathcal{T}_\Sigma, \Delta) \to (\Upsilon, \mathcal{T}_\Upsilon, \Delta)$ *is a weakly soft* α*-continuous function and* $\mathrm{N_K}$*:* $(\Upsilon, \mathcal{T}_\Upsilon, \Delta) \to (\Gamma, \mathcal{T}_\Gamma, \Delta)$ *is a soft continuous function, then* $\mathrm{N_K} \circ \mathrm{M_P}$ *is weakly soft* α*-continuous.*

Proposition 5.3. *Every soft continuous function is weakly soft* α*-continuous.*

Proof. It follows from Proposition 3.7.

Proposition 5.4. *Let* $\mathrm{M_P} : (\Sigma, \mathcal{T}_\Sigma, \Delta) \to (\Upsilon, \mathcal{T}_\Upsilon, \Delta)$ *be a soft function such that* $\mathcal{T}_\Sigma$ *is extended. Then*

(i) If $\mathrm{M_P}$ *is soft* α*-continuous, then* $\mathrm{M_P}$ *is weakly soft* α*-continuous.*

(ii) If $\mathrm{M_P}$ *is weakly soft* α*-continuous, then* $\mathrm{M_P}$ *is soft sw-continuous.*

Proof. It respectively follows from Propositions 3.9 and 3.10.

Proposition 5.5. *A soft function* $\mathrm{M_P}$*:* $(\Sigma, \mathcal{T}_\Sigma, \Delta) \to (\Upsilon, \mathcal{T}_\Upsilon, \Delta)$ *is weakly soft* α*-continuous iff the inverse image of every soft closed subset is weakly soft* α*-closed.*

Proof. Necessity: suppose that (F, Δ) is a soft closed subset of $(\Upsilon, \mathcal{T}_\Upsilon, \Delta)$. Then (F^c, Δ) is soft open. Therefore,

$$\mathrm{M_P^{-1}}(F^c, \Delta) = \widetilde{\Sigma} - \mathrm{M_P^{-1}}(F, \Delta)$$

is weakly soft α-open. Thus, $\mathrm{M_P^{-1}}(F, \Delta)$ is a weakly soft α-closed set.

Following similar argument one can prove the sufficient part.

Theorem 5.6. *If* $\mathrm{M_P}$*:* $(\Sigma, \mathcal{T}_\Sigma, \Delta) \to (\Upsilon, \mathcal{T}_\Upsilon, \Delta)$ *is weakly soft* α*-continuous, then the next properties are equivalent.*

(i) For each soft open subset (F, Δ) *of* $(\Upsilon, \mathcal{T}_\Upsilon, \Delta)$*, we have* $\mathrm{M_P^{-1}}(F, \Delta) = int_{w\alpha}(\mathrm{M_P^{-1}}(F, \Delta))$.

(ii) For each soft closed subset (F, Δ) *of* $(\Upsilon, \mathcal{T}_\Upsilon, \Delta)$*, we have* $\mathrm{M_P^{-1}}(F, \Delta) = cl_{w\alpha}(\mathrm{M_P^{-1}}(F, \Delta))$.

(iii) $cl_{w\alpha}(\mathrm{M_P^{-1}}(F, \Delta)) \widetilde{\subseteq} \mathrm{M_P^{-1}}(cl(F, \Delta))$ *for each* $(F, \Delta) \widetilde{\subseteq} \widetilde{\Upsilon}$.

(iv) $\mathrm{M_P}(cl_{w\alpha}(G, \Delta)) \widetilde{\subseteq} cl(\mathrm{M_P}(G, \Delta))$ *for each* $(G, \Delta) \widetilde{\subseteq} \widetilde{\Sigma}$.

(v) $\mathrm{M_P^{-1}}(int(F, \Delta)) \widetilde{\subseteq} int_{w\alpha}(\mathrm{M_P^{-1}}(F, \Delta))$ *for each* $(F, \Delta) \widetilde{\subseteq} \widetilde{\Upsilon}$.

Proof. $(i) \to (ii)$: Suppose that (F, Δ) is a soft closed subset of $(\Upsilon, \mathcal{T}_\Upsilon, \Delta)$. Then (F^c, Δ) is soft open. Therefore,

$$\mathrm{M_P^{-1}}(F^c, \Delta) = int_{w\alpha}(\mathrm{M_P^{-1}}(F^c, \Delta)).$$

According to Proposition 4.8, we obtain

$$\mathrm{M_P^{-1}}(F, \Delta) = cl_{w\alpha}(\mathrm{M_P^{-1}}(F, \Delta)).$$

$(ii) \to (iii)$: For any soft set $(F, \Delta) \widetilde{\subseteq} \widetilde{\Upsilon}$, we have

$$\mathrm{M_P^{-1}}(cl(F, \Delta)) = cl_{w\alpha}(\mathrm{M_P^{-1}}(cl(F, \Delta))).$$

Then

$$cl_{w\alpha}(\mathrm{M_P^{-1}}(F, \Delta)) \widetilde{\subseteq} cl_{w\alpha}(\mathrm{M_P^{-1}}(cl(F, \Delta))) = \mathrm{M_P^{-1}}(cl(F, \Delta)).$$

$(iii) \to (iv)$: It is obvious that

$$cl_{w\alpha}(G,\Delta)\widetilde{\subseteq}cl_{w\alpha}(M_P^{-1}(M_P(G,\Delta)))$$

for each $(G,\Delta)\widetilde{\subseteq}\widetilde{\Sigma}$. By (iii), we get

$$cl_{w\alpha}(M_P^{-1}(M_P(G,\Delta)))\widetilde{\subseteq}M_P^{-1}(cl(M_P(G,\Delta))).$$

Therefore,

$$M_P(cl_{w\alpha}(G,\Delta)\widetilde{\subseteq}M_P(M_P^{-1}(cl(M_P(G,\Delta))))\widetilde{\subseteq}cl(M_P(G,\Delta)).$$

$(iv) \to (v)$: Let (F,Δ) be an arbitrary soft set in $(\Upsilon,\mathcal{T}_\Upsilon,\Delta)$. Then

$$M_P(cl_{w\alpha}(M_P^{-1}(F^c,\Delta))\widetilde{\subseteq}cl(M_P(M_P^{-1}(F^c,\Delta)))\widetilde{\subseteq}cl(F^c,\Delta)$$

So that,

$$cl_{w\alpha}((M_P^{-1}(F,\Delta))^c)\widetilde{\subseteq}M_P^{-1}((int(F,\Delta))^c.$$

Hence,

$$M_P^{-1}(int(F,\Delta))\widetilde{\subseteq}int_{w\alpha}(M_P^{-1}(F,\Delta)).$$

$(v) \to (i)$: Suppose that (F,Δ) is a soft open subset in $(\Upsilon,\mathcal{T}_\Upsilon,\Delta)$. By (v), we obtain

$$M_P^{-1}(F,\Delta) = M_P^{-1}(int(F,\Delta))\widetilde{\subseteq}int_{w\alpha}(M_P^{-1}(F,\Delta)).$$

But

$$int_{w\alpha}(M_P^{-1}(F,\Delta))\widetilde{\subseteq}M_P^{-1}(F,\Delta),$$

so

$$M_P^{-1}(F,\Delta) = int_{w\alpha}(M_P^{-1}(F,\Delta)),$$

as required.

The converse of the above theorem fails. To demonstrate that the next example is furnished.

Example 5.7. *Let* $\Sigma = \{\sigma_1,\sigma_2,\sigma_3\}$ *and* $\Upsilon = \{\upsilon_1,\upsilon_2\}$ *with* $\Delta = \{\delta_1,\delta_2\}$. *Let*

$$\mathcal{T}_\Sigma = \{\phi,\widetilde{\Sigma},(F,\Delta),(G,\Delta)\}$$

and

$$\mathcal{T}_\Upsilon = \{\phi,\widetilde{\Upsilon},(H,\Delta)\}$$

be two soft topologies defined on Σ *and* Υ*, respectively, with the same set of parameters* Δ*, where*

$$(F,\Delta) = \{(\delta_1,\{\sigma_1\}),(\delta_2,\{\sigma_1\})\},$$

$$(G,\Delta) = \{(\delta_1,\{\sigma_2,\sigma_3\}),(\delta_2,\{\sigma_2,\sigma_3\})\},$$

and

$$(H,\Delta) = \{(\delta_1,\{\upsilon_1\}),(\delta_2,\{\upsilon_1\})\}.$$

Consider M_P*:* $(\Sigma,\mathcal{T}_\Sigma,\Delta) \to (\Upsilon,\mathcal{T}_\Upsilon,\Delta)$ *is a soft function, where* M*:* $\Sigma \to \Upsilon$ *is defined as follows*

$$M(\sigma_1) = M(\sigma_2) = \upsilon_1 \text{ and } M(\sigma_3) = \upsilon_2,$$

and $\mathrm{P}: \Delta \to \Delta$ *is the identity function.*

Now,

$$\mathrm{M_P}^{-1}(H,\Delta) = \{(\delta_1, \{\sigma_1, \sigma_2\}), (\delta_2, \{\sigma_1, \sigma_2\})\},$$

which is not a weakly soft α*-open subset because*

$$int(cl(int(\{\sigma_1, \sigma_2\}))) = \{\sigma_1\}.$$

Then $\mathrm{M_P}$ *is not weakly soft* α*-continuous. On the other hand,*

$$\mathrm{M_P}^{-1}(\phi) = int_{w\alpha}(\mathrm{M_P}^{-1}(\phi)),$$

$$\mathrm{M_P}^{-1}(\widetilde{\Upsilon}) = int_{w\alpha}(\mathrm{M_P}^{-1}(\widetilde{\Upsilon})),$$

and

$$\mathrm{M_P}^{-1}(H,\Delta) = int_{w\alpha}(\mathrm{M_P}^{-1}(H,\Delta)),$$

which means that the all properties given in Theorem 5.6 hold true.

Now, we introduce the concepts of weakly soft α-open, weakly soft α-closed and weakly soft α-homeomorphism functions.

Definition 5.8. *A soft function* $\mathrm{M_P}$*:* $(\Sigma, \mathcal{T}_\Sigma, \Delta) \to (\Upsilon, \mathcal{T}_\Upsilon, \Delta)$ *is called:*

(i) weakly soft α*-open provided that the image of each soft open set is weakly soft* α*-open.*

(ii) weakly soft α*-closed provided that the image of each soft closed set is weakly soft* α*-closed.*

Theorem 5.9. *Let* $\mathrm{M_P}$*:* $(\Sigma, \mathcal{T}_\Sigma, \Delta) \to (\Upsilon, \mathcal{T}_\Upsilon, \Delta)$ *be a soft function and* (F,Δ) *be any soft subset of* $\widetilde{\Sigma}$*. Then*

(i) If $\mathrm{M_P}$ *is weakly soft* α*-open, then* $\mathrm{M_P}(int(F,\Delta)) \widetilde{\subseteq} int_{w\alpha}(\mathrm{M_P}(F,\Delta))$*.*

(ii) If $\mathrm{M_P}$ *is weakly soft* α*-closed, then* $cl_{w\alpha}(\mathrm{M_P}(F,\Delta)) \widetilde{\subseteq} \mathrm{M_P}(cl(F,\Delta))$*.*

Proof. (i) Let (F,Δ) be a soft subset of $\widetilde{\Sigma}$. Then $\mathrm{M_P}(int(F,\Delta))$ is a weakly soft α-open subset of $(\Upsilon, \mathcal{T}_\Upsilon, \Delta)$ and so

$$\mathrm{M_P}(int(F,\Delta)) = int_{w\alpha}(\mathrm{M_P}(int(F,\Delta))) \widetilde{\subseteq} int_{w\alpha}(\mathrm{M_P}(F,\Delta)).$$

(ii) The proof is similar to that of (i).

Proposition 5.10. *A bijective soft function* $\mathrm{M_P}$*:* $(\Sigma, \mathcal{T}_\Sigma, \Delta) \to (\Upsilon, \mathcal{T}_\Upsilon, \Delta)$ *is weakly soft* α*-open iff it is weakly soft* α*-closed.*

Proof. Necessity: let (F,Δ) be a weakly soft α-closed subset of $(\Sigma, \mathcal{T}_\Sigma, \Delta)$. Since $\mathrm{M_P}$ is weakly soft α-open, $\mathrm{M_P}(F^c,\Delta)$ is weakly soft α-open. By bijectiveness of $\mathrm{M_P}$, we obtain

$$\mathrm{M_P}(F^c,\Delta) = (\mathrm{M_P}(F,\Delta))^c.$$

So that, $\mathrm{M_P}(F,\Delta)$ is a weakly soft α-closed set. Hence, $\mathrm{M_P}$ is weakly soft α-closed. To prove the sufficient, we follow similar approach.

Proposition 5.11. *Let* $\mathrm{M_P}$*:* $(\Sigma, \mathcal{T}_\Sigma, \Delta) \to (\Upsilon, \mathcal{T}_\Upsilon, \Delta)$ *be a weakly soft* α*-closed function and* $\widetilde{\Gamma}$ *be a soft closed subset of* $\widetilde{\Sigma}$*. Then* $\mathrm{M_P}\mid_\Gamma$*:* $(\Gamma, \mathcal{T}_\Gamma, \Delta) \to (\Upsilon, \mathcal{T}_\Upsilon, \Delta)$ *is weakly soft* α*-closed.*

Proof. Suppose that (F, Δ) is a soft closed subset of $(\Gamma, \mathcal{T}_\Gamma, \Delta)$. Then there is a soft closed subset (G, Δ) of $(\Sigma, \mathcal{T}_\Sigma, \Delta)$ with

$$(F, \Delta) = (G, \Delta) \widetilde{\bigcap} \widetilde{\Gamma}.$$

Since $\widetilde{\Gamma}$ is a soft closed subset of $(\Sigma, \mathcal{T}_\Sigma, \Delta)$, then (F, Δ) is also a soft closed subset of $(\Sigma, \mathcal{T}_\Sigma, \Delta)$. Since,

$$\mathrm{M_P}\mid_\Gamma (F, \Delta) = \mathrm{M_P}(F, \Delta),$$

then $\mathrm{M_P}\mid_\Gamma (F, \Delta)$ is a weakly soft α-closed set. Thus, $\mathrm{M_P}\mid_\Gamma$ is a weakly soft α-closed.

Proposition 5.12. *The next three statements hold for soft functions* $\mathrm{M_P}$*:* $(\Sigma, \mathcal{T}_\Sigma, \Delta) \to (\Upsilon, \mathcal{T}_\Upsilon, \Delta)$ *and* $\mathrm{N_K}$*:* $(\Upsilon, \mathcal{T}_\Upsilon, \Delta) \to (\Gamma, \mathcal{T}_\Gamma, \Delta)$*.*

(i) If $\mathrm{M_P}$ *is soft open and* $\mathrm{N_K}$ *is soft* α*-open such that* $\mathcal{T}_\Gamma$ *is extended, then* $\mathrm{N_K} \circ \mathrm{M_P}$ *is weakly soft* α*-open.*

(ii) If $\mathrm{N_K} \circ \mathrm{M_P}$ *is weakly soft* α*-open and* $\mathrm{M_P}$ *is surjective soft continuous, then* $\mathrm{N_K}$ *is weakly soft* α*-open.*

(iii) If $\mathrm{N_K} \circ \mathrm{M_P}$ *is soft open and* $\mathrm{N_K}$ *is injective weakly soft* α*-continuous, then* $\mathrm{M_P}$ *is weakly soft* α*-open.*

Proof. (i) Take

$$(F, \Delta) \neq \phi$$

as a soft open subset of $\widetilde{\Sigma}$. So

$$\mathrm{M_P}(F, \Delta) \neq \phi$$

is a soft open subset of $\widetilde{\Upsilon}$. Thus, $\mathrm{N_K}(\mathrm{M_P}(F, \Delta))$ is a soft α-open subset. According to Proposition 3.9, $\mathrm{N_K}(\mathrm{M_P}(F, \Delta))$ is a weakly soft α-open subset. Hence, $\mathrm{N_K} \circ \mathrm{M_P}$ is weakly soft α-open.

(ii) Suppose that $(F, \Delta) \neq \phi$ is a soft open subset of $\widetilde{\Upsilon}$. Then

$$\mathrm{M_P^{-1}}(F, \Delta) \neq \phi$$

is a soft open subset of $\widetilde{\Sigma}$. Therefore, $(\mathrm{N_K} \circ \mathrm{M_P})(\mathrm{M_P^{-1}}(F, \Delta))$ is a weakly soft α-open subset of $\widetilde{\Gamma}$. Since $\mathrm{M_P}$ is surjective, then

$$(\mathrm{N_K} \circ \mathrm{M_P})(\mathrm{M_P^{-1}}(F, \Delta)) = \mathrm{N_K}(\mathrm{M_P}(\mathrm{M_P^{-1}}(F, \Delta))) = \mathrm{N_K}(F, \Delta).$$

Thus, $\mathrm{N_K}$ is weakly soft α-open.

(iii) Let $(F, \Delta) \neq \phi$ be a soft open subset of $\widetilde{\Sigma}$. Then

$$(\mathrm{N_K} \circ \mathrm{M_P})(F, \Delta) \neq \phi$$

is a soft open subset of $\widetilde{\Gamma}$. Therefore, $\mathrm{N_K^{-1}}(\mathrm{N_K} \circ \mathrm{M_P}(F, \Delta))$ is a weakly soft α-open subset of $\widetilde{\Upsilon}$. Since $\mathrm{N_K}$ is injective,

$$\mathrm{N_K^{-1}}(\mathrm{N_K} \circ \mathrm{M_P}(F, \Delta)) = (\mathrm{N_K^{-1}}\mathrm{N_K})(\mathrm{M_P}(F, \Delta)) = \mathrm{M_P}(F, \Delta).$$

Thus, $\mathrm{M_P}$ is weakly soft α-open.

We cancel the proof of the next finding because it can be obtained following similar approach of the above proposition.

Proposition 5.13. *The next three statements hold for soft functions* $\mathrm{M_P}$*:* $(\Sigma, \mathcal{T}_\Sigma, \Delta) \to (\Upsilon, \mathcal{T}_\Upsilon, \Delta)$ *and* $\mathrm{N_K}$*:* $(\Upsilon, \mathcal{T}_\Upsilon, \Delta) \to (\Gamma, \mathcal{T}_\Gamma, \Delta)$.

(i) If $\mathrm{M_P}$ *is soft closed and* $\mathrm{N_K}$ *is soft* α*-closed, then* $\mathrm{N_K} \circ \mathrm{M_P}$ *is weakly soft* α*-closed.*

(ii) If $\mathrm{N_K} \circ \mathrm{M_P}$ *is weakly soft* α*-closed and* $\mathrm{M_P}$ *is surjective soft continuous, then* $\mathrm{N_K}$ *is weakly soft* α*-closed.*

(iii) If $\mathrm{N_K} \circ \mathrm{M_P}$ *is soft closed and* $\mathrm{N_K}$ *is injective weakly soft* α*-continuous, then* $\mathrm{M_P}$ *is weakly soft* α*-closed.*

Definition 5.14. *A bijective soft function* $\mathrm{M_P}$ *in which is weakly soft* α*-continuous and weakly soft* α*-open is called a weakly soft* α*-homeomorphism.*

6. Conclusions

It is well known that soft topology is defined as a family of soft sets fulfilling the basic axioms of general topology and creates a family of general topologies. it is crucial to examine the connections between these classical topologies and the soft topology that they generate. In this paper, we have benefited from the fruitful variety existing via soft topology to introduce a novel class of generalizations of soft open subsets called "weakly soft α-open sets", which we have constructed using its corresponding notion via parametric topologies.

First, we have studied the basic properties of this class and showed that this class lost the property of closing under arbitrary soft unions, which is satisfied by the previous famous generalizations. With respect to its relationship with the previous generalizations, we have demonstrated that it lies between soft α-open and soft *sw*-open subsets of extended (hyperconnected) soft topology. Then, we have introduced the concepts of interior, closure, boundary, and limit soft points via weakly soft α-open and weakly soft α-closed sets. We have scrutinized main their characterizations and inferred the formulas that connected each other. In the end, we have discussed the concepts of soft continuity, openness and closeness defined by weakly soft α-open and weakly soft α-closed sets.

Among the unique properties obtained in this study is that most descriptions of soft continuity have been evaporated for this type of continuity, which is due to the loss of the properties report that "a soft subset (F, Δ) is weakly soft α-open iff $int_{w\alpha}(F, \Delta) = (F, \Delta)$" and "a soft subset (F, Δ) is weakly soft α-closed iff $cl_{w\alpha}(F, \Delta) = (F, \Delta)$". To illustrate these divergences between this class and other generalizations, we have provided some counterexamples. To avoid this irregular behavior, we plan to produce another type of soft continuity inspired by weakly soft α-open sets. Moreover, we intend to discuss other topological ideas that can be formulated using this class of soft sets, e.g., covering property and separation axioms.

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Conflict of interest

The authors declare that they have no competing interests.

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