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Research article

The existence and stability results of multi-order boundary value problems involving Riemann-Liouville fractional operators

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Abstract: In this paper, a general framework for the fractional boundary value problems is presented. The problem is created by Riemann-Liouville type two-term fractional differential equations with a fractional bi-order setup. Moreover, the boundary conditions of the suggested system are considered as mixed Riemann-Liouville integro-derivative conditions with four different orders, which it cover a variety of specific instances previously researched. Further, the provided problem's Hyers-Ulam stability and the possibility of a fixed-point approach solution are both investigated. Finally, to support our theoretical findings, an example is developed.

Keywords: Riemann-Liouville fractional derivative; coupled fractional boundary value problem; fixed point; Hyers-Ulam stability
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1. Introduction and building system

The classical derivatives are local in nature, i.e., using classical derivatives we can describe changes in the neighborhood of a point, but using fractional derivatives we can describe changes in an interval.

Namely, a fractional derivative is nonlocal in nature. This property makes these derivatives suitable to simulate more physical phenomena such as earthquake vibrations, polymers, etc.

On the other hand, in the case of the heat conduction equation, the fractional order parameter α means the level of thermal conductivity. If $\alpha = 1$, the medium's thermal conductivity is normal; if $\alpha < 1$, the medium has weak conductivity; and if $\alpha \geq 1$, the medium has strong conductivity.

Further, in modeling various memory phenomena, it is observed that a memory process usually consists of two stages. One is short with permanent retention, and the other is governed by a simple model of fractional derivative. With the numerical least squares method, the fractional model perfectly fits the test data of memory phenomena in different disciplines, not only in mechanics but also in biology and psychology. Based on this model, it is found that the physical meaning of the fractional order is an index of memory. For more details, see [1, 2].

Fractional calculus and its applications have acquired a lot of interest in several disciplines of engineering and science such as biology, chemistry, physics, economics, control theory, signal and image processing, etc, see [3–5] and the references therein. Variant definitions for the fractional derivative have emerged over the years. The most famous ones are the Riemann-Liouville and Caputo fractional derivatives. In recent years, many nonlinear phenomena in numerous fields have been modeled by fractional differential equations. Due to the evolution of fractional calculus, these equations have emerged as a new branch of applied mathematics. Several works on the existence and multiplicity of solutions to fractional boundary value problems (FBVPs) have appeared in view of the qualitative properties of fractional differential equations.

Among the used methods to solve a FBVP, there are the variational methods used by Fix and Roop in [6] and Erwin and Roop in [7]. Also, some fixed point techniques have been applied successfully to ensure the existence of solutions of some FBVPs. Here, we may cite the works of Agarwal et al. [8], Benchohra et al. [9], Zhang [10], Ahmad and Nieto [11], etc. Going in the same direction, the critical point theory has been used to investigate the solutions for some FBVPs. For instance, see the works Jiao and Zhou [12] and Tang and Wu [13]. On the other hand, stability analysis of fractional differential equations with different types of initial and boundary conditions have attracted many researchers who discussed the analysis of stability in the setting of Ulam-Hyers (UH) and generalized UH theory. For more details, see [14–30].

In 2016, the boundary value problem (BVP) with 4-order Riemann-Liouville fractional (RLF) derivatives is studied by Niyom et al. [31]:

$$\begin{cases} \upsilon D^{\rho}(z(\tau)) + (1-\upsilon)D^{\theta}(z(\tau)) = \Xi(\tau, z(\tau)),\ \tau \in [0, G],\ \rho \in [1, 2), \\ z(0) = 0,\ \ \varrho_1 D^{\eta_1} z(G) + (1-\varrho_1)D^{\eta_2} z(G) = \xi_1, \end{cases} \tag{1.1}$$

under appropriate conditions. Also, Niyom et al. [32], modified the above problem under multiple orders of fractional integrals and derivatives as follows:

$$\begin{cases} \upsilon D^{\rho}(z(\tau)) + (1-\upsilon)D^{\theta}(z(\tau)) = \Xi(\tau, z(\tau)),\ \tau \in [0, G],\ \rho \in [1, 2), \\ z(0) = 0,\ \ \varrho_2 I^{s_1} z(G) + (1-\varrho_2)I^{s_2} z(G) = \xi_2. \end{cases} \tag{1.2}$$

In 2018, Xu et al. [33] examined the existence of solutions and UH stability for the FDEs

$$\begin{cases} \upsilon D^{\rho}(z(\tau)) + D^{\theta}(z(\tau)) = \Xi(\tau, z(\tau)),\ \tau \in [0, G],\ \rho \in [1, 2), \\ z(0) = 0,\ \ \varrho_1 D^{\eta_1} z(G) + I^{s_2} z(G) = \xi_2. \end{cases} \tag{1.3}$$

They focused on the RLF derivative and integral issues of the two-term class of three-point BVPs, where the notions and parameters in (1.1) and (1.2) are defined below the system (1.4).

Now, utilizing the concepts from the works described above and combining them, we investigate a new category of coupled boundary value problems (CBVPs) that includes a multi-order RLF equation plus various linear integro-derivative boundary stipulations as follows:

$$\begin{cases} \upsilon D^{\rho}(z(\tau)) + (1-\upsilon)D^{\theta}(z(\tau)) = \Xi(\tau, z(\tau), r(\tau)),\ \tau \in [0,G],\ \rho \in [2,3), \\ \upsilon^{*} D^{\rho^{*}}(r(\tau)) + (1-\upsilon^{*})D^{\theta^{*}}(r(\tau)) = \Xi^{*}(\tau, r(\tau), z(\tau)),\ \tau \in [0,G],\ \rho^{*} \in [2,3), \\ z(0)=0,\ \varrho_1 D^{\eta_1} z(G) + (1-\varrho_1)D^{\eta_2} z(G) = \xi_1, \\ r(0)=0,\ \varrho_1^{*} D^{\eta_1^{*}} r(G) + (1-\varrho_{*1})D^{\eta_2^{*}} r(G) = \xi_1^{*}, \\ \varrho_2 I^{s_1} z(G) + (1-\varrho_2) I^{s_2} z(G) = \xi_2,\ \varrho_2^{*} I^{s_1^{*}} r(G) + (1-\varrho_2^{*}) I^{s_2^{*}} r(G) = \xi_2^{*}, \end{cases} \tag{1.4}$$

where $2 < \theta < \rho$, $2 < \theta^{*} < \rho^{*}$, $\upsilon, \upsilon^{*}, \varrho_1, \varrho_2, \varrho_1^{*}, \varrho_2^{*} \in (0,1]$, $0 \le \eta_1, \eta_2 < \rho - \theta$, $0 \le \eta_1^{*}, \eta_2^{*} < \rho^{*} - \theta^{*}$, $s_1, s_2, s_1^{*}, s_2^{*} \in \mathbb{R}^{+}$, D^{q} is the RLF derivative of order $q \in \{\rho, \theta, \rho^{*}, \theta^{*}, \eta_1, \eta_2, \eta_1^{*}, \eta_2^{*}\}$, I^{m} is the RLF integral of order $m \in \{s_1, s_2, s_1^{*}, s_2^{*}\}$ and $\Xi, \Xi^{*} : [0,G] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

As many scholars are interested in exploring the idea of stability for various CBVPs, this can serve as inspiration for us to research the stability of complicated systems with added broad boundary stipulations. Consequently, to be more precise, the main objective of the current manuscript is to find some existing criteria for the solutions to a new general CBVP that includes a two-term fractional differential equation (FDE) (1.4) and multi-order RLF derivatives and integrals. The well-known standard fixed point (FP) theorems are employed in order to achieve this goal. Furthermore, in the follow-up, we examine the HU stability of the suggested problem (1.4) in the unique scenario when $\varrho_1 = \varrho_2 = 1$ and $\varrho_1^{*} = \varrho_2^{*} = 1$. Ultimately, to demonstrate the applicability of our theoretical results, two examples are provided. We think that the BVP that has been proposed is a generic one that incorporates a lot of fractional dynamical systems as special examples in the fields of physics and other applied disciplines.

2. Basic concepts

Let $G > 0$ and $U = [0,G]$. Assume that the piecewise continuous function space $PC(U, \mathbb{R}_{+})$ equipped with the norms $\|z\| = \max\{|z(\upsilon)| : \upsilon \in U\}$ and $\|r\| = \max\{|r(\upsilon)| : \upsilon \in U\}$ is a Banach space (BS), then the products of these norms are also a BS under the norm $\|(z,r)\| = \|z\| + \|r\|$.

Assume also $\Im_1$ and $\Im_2$ represent the piecewise continuous function spaces described as

$$\Im_1 = PC_1(U,\mathbb{R}) = \{z : U \to \mathbb{R}\} \text{ and } \Im_2 = PC_2(U,\mathbb{R}) = \{r : U \to \mathbb{R}\},$$

with norms

$$\|z\|_{\Im_1} = \sup\{|z(\upsilon)|,\ \upsilon \in U\} \text{ and } \|r\|_{\Im_2} = \sup\{|r(\upsilon)|,\ \upsilon \in U\},$$

respectively. Clearly, the product $\Im = \Im_1 \times \Im_2$ is a BS endowed with $\|(z,r)\|_{\Im} = \|z\|_{\Im_1} + \|r\|_{\Im_2}$.

Definition 2.1. *[34]* For a real valued function $z : (0,\infty) \to \mathbb{R}$, the RLF integral operator of order ρ is described as

$$I^{\rho} z(\tau) = \frac{1}{\Gamma(\rho)} \int_0^{\tau} (\tau - \hbar)^{\rho - 1} z(\hbar) d\hbar,$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2.2. *[34] The* RLF derivative of order ρ of a function $z:(0,\infty)\to\mathbb{R}$ takes the form

$$D^{\rho}z(\tau)=\frac{1}{\Gamma(n-\rho)}\left(\frac{d}{d\tau}\right)^{n}\int\limits_{0}^{\tau}(\tau-\hbar)^{n-\rho-1}z(\hbar)d\hbar,\ n=[\rho]+1.$$

where $[\rho]$ refers to the integer part of real number ρ.

Lemma 2.1. *[34, 35] Assume that $\rho>0$ and $z\in C(0,1)$. Then the FDE $D^{\rho}z(\tau)=0$ owns a general solution $z(\tau)=\sum\limits_{j=1}^{n}O_j\tau^{\rho-j}$, where $j-1<\rho\leq j$ and the constants $O_1,O_2,...,O_n\in\mathbb{R}$.*

Lemma 2.2. *[34] Assume that $\rho>0$ and $z\in C(0,1)$. Then, we have*

$$I^{\rho}D^{\rho}z(\tau)=z(\tau)+\sum_{j=1}^{n}O_j\tau^{\rho-j},$$

where $j-1<\rho\leq j$ and the constants $O_1,O_2,...O_j\in\mathbb{R}$.

Lemma 2.3. *[4] Assume that $\rho,\theta>0$ with $\rho>\theta$, then $I^{\rho}_{0^+}D^{\theta}_{0^+}=I^{\rho-\theta}_{0^+}$.*

The auxiliary theorems that follows is also required.

Theorem 2.1. *(Krasnoselskii's FP theorem [36]) Assume that S is a non-empty, closed, bounded and convex subset of a BS $\Im$. Let $\Omega,\Omega^*:S\to S$ be operators such that*

(1) $\Omega(z)+\Omega^*(r)\in S$, where $z,r\in S$;
(2) Ω^* is a contraction mapping;
(3) Ω is completely continuous.

Then there exists $z\in S$ so that $z=\Omega(z)+\Omega^*(z)$.

Theorem 2.2. *(Banach FP theorem [37]) Every contraction self-mapping defined on a complete metric space admits a unique FP.*

3. Existence results

We begin this section with the lemma below.

Lemma 3.1. *The mappings z_0,r_0 are a solution for CBVP (1.4) if z_0,r_0 are solutions to the following integral equations:*

$$\begin{aligned}
&z(\tau)\\
=\ &\frac{\upsilon-1}{\upsilon\Gamma(\rho-\theta)}\int_0^{\tau}(\tau-\hbar)^{\rho-\theta-1}z(\hbar)d\hbar+\frac{1}{\upsilon\Gamma(\rho)}\int_0^{\tau}(\tau-\hbar)^{\rho-1}\Xi(\hbar,z(\hbar),r(\hbar))d\hbar\\
&+\frac{\tau^{\rho-1}}{\Phi}\Bigg(\frac{\varrho_1\nabla_4(\upsilon-1)}{\upsilon}I^{\rho-\theta-\eta_1}z(G)-\frac{\varrho_2\nabla_2(\upsilon-1)}{\upsilon}I^{\rho-\theta-s_1}z(G)\\
&+\frac{\nabla_4(1-\varrho_1)(\upsilon-1)}{\upsilon}I^{\rho-\theta-\eta_2}z(G)-\frac{\nabla_2(1-\varrho_2)(\upsilon-1)}{\upsilon}I^{\rho-\theta-s_2}z(G)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\varrho_1\nabla_4}{\upsilon}I^{\rho-\eta_1}\Xi(G,z(G),r(G))-\frac{\varrho_2\nabla_2}{\upsilon}I^{\rho+s_1}\Xi(G,z(G),r(G))+\nabla_2\xi_2-\nabla_4\xi_1\\
&+\frac{(1-\varrho_1)\nabla_4}{\upsilon}I^{\rho-\eta_2}\Xi(G,z(G),r(G))-\frac{(1-\varrho_2)\nabla_2}{\upsilon}I^{\rho+s_2}\Xi(G,z(G),r(G))\Bigg)\\
&-\frac{\tau^{\rho-2}}{\Phi}\Bigg(\frac{\varrho_1\nabla_3(\upsilon-1)}{\upsilon}I^{\rho-\theta-\eta_1}z(G)-\frac{\varrho_2\nabla_1(\upsilon-1)}{\upsilon}I^{\rho-\theta-s_1}z(G)\\
&+\frac{\nabla_3(1-\varrho_1)(\upsilon-1)}{\upsilon}I^{\rho-\theta-\eta_2}z(G)-\frac{\nabla_1(1-\varrho_2)(\upsilon-1)}{\upsilon}I^{\rho-\theta-s_2}z(G)\\
&+\frac{\varrho_1\nabla_3}{\upsilon}I^{\rho-\eta_1}\Xi(G,z(G),r(G))-\frac{\varrho_2\nabla_1}{\upsilon}I^{\rho+s_1}\Xi(G,z(G),r(G))+\nabla_1\xi_2-\nabla_3\xi_1\\
&+\frac{(1-\varrho_1)\nabla_3}{\upsilon}I^{\rho-\eta_2}\Xi(G,z(G),r(G))-\frac{(1-\varrho_2)\nabla_1}{\upsilon}I^{\rho+s_2}\Xi(G,z(G),r(G))\Bigg),
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
r(\tau) \;=\; &\frac{\upsilon^*-1}{\upsilon^*\Gamma(\rho^*-\theta^*)}\int_0^\tau(\tau-\hbar)^{\rho^*-\theta^*-1}r(\hbar)d\hbar+\frac{1}{\upsilon^*\Gamma(\rho^*)}\int_0^\tau(\tau-\hbar)^{\rho^*-1}\Xi^*(\hbar,r(\hbar),z(\hbar))d\hbar\\
&+\frac{\tau^{\rho^*-1}}{\Phi^*}\Bigg(\frac{\varrho_1^*\nabla_4^*(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*-\eta_1^*}r(G)-\frac{\varrho_2^*\nabla_2^*(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*+s_1^*}r(G)\\
&+\frac{\nabla_4^*(1-\varrho_1^*)(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*-\eta_2^*}r(G)-\frac{\nabla_2^*(1-\varrho_2^*)(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*+s_2^*}r(G)\\
&+\frac{\varrho_1^*\nabla_4^*}{\upsilon^*}I^{\rho^*-\eta_1^*}\Xi^*(G,r(G),z(G))-\frac{\varrho_2^*\nabla_2^*}{\upsilon^*}I^{\rho^*+s_1^*}\Xi^*(G,r(G),z(G))+\nabla_2^*\xi_2^*-\nabla_4^*\xi_1^*\\
&+\frac{(1-\varrho_1^*)\nabla_4^*}{\upsilon^*}I^{\rho^*-\eta_2^*}\Xi^*(G,r(G),z(G))-\frac{\left(1-\varrho_2^*\right)\nabla_2^*}{\upsilon^*}I^{\rho^*+s_2^*}\Xi^*(G,r(G),z(G))\Bigg)\\
&-\frac{\tau^{\rho^*-2}}{\Phi^*}\Bigg(\frac{\varrho_1^*\nabla_3^*(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*-\eta_1^*}r(G)-\frac{\varrho_2^*\nabla_1^*(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*+s_1^*}r(G)\\
&+\frac{\nabla_3^*(1-\varrho_1^*)(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*-\eta_2^*}r(G)-\frac{\nabla_1^*(1-\varrho_2^*)(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*+s_2^*}r(G)\\
&\frac{\varrho_1^*\nabla_3^*}{\upsilon^*}I^{\rho^*-\eta_1^*}\Xi^*(G,r(G),z(G))-\frac{\varrho_2^*\nabla_1^*}{\upsilon^*}I^{\rho^*+s_1^*}\Xi^*(G,r(G),z(G))+\nabla_1^*\xi_2^*-\nabla_3^*\xi_1^*\\
&+\frac{(1-\varrho_1^*)\nabla_3^*}{\upsilon^*}I^{\rho^*-\eta_2^*}\Xi^*(G,r(G),z(G))-\frac{\left(1-\varrho_2^*\right)\nabla_1^*}{\upsilon^*}I^{\rho^*+s_2^*}\Xi^*(G,r(G),z(G))\Bigg),
\end{aligned}
\tag{3.2}
$$

where

$$
\begin{aligned}
&\nabla_1=\tfrac{\varrho_1\Gamma(\rho)}{\Gamma(\rho-\eta_1)}G^{\rho-\eta_1-1}+\tfrac{(1-\varrho_1)\Gamma(\rho)}{\Gamma(\rho-\eta_2)}G^{\rho-\eta_2-1}, &&\nabla_1^*=\tfrac{\varrho_1^*\Gamma(\rho^*)}{\Gamma(\rho^*-\eta_1^*)}G^{\rho^*-\eta_1^*-1}-\tfrac{(1-\varrho_1^*)\Gamma(\rho^*)}{\Gamma(\rho^*-\eta_2^*)}G^{\rho^*-\eta_2^*-1},\\
&\nabla_2=\tfrac{\varrho_1\Gamma(\rho-1)}{\Gamma(\rho-\eta_1-1)}G^{\rho-\eta_1-2}+\tfrac{(1-\varrho_1)\Gamma(\rho-1)}{\Gamma(\rho-\eta_2-1)}G^{\rho-\eta_2-2}, &&\nabla_2^*=\tfrac{\varrho_1^*\Gamma(\rho^*-1)}{\Gamma(\rho^*-\eta_1^*-1)}G^{\rho^*-\eta_1^*-2}-\tfrac{(1-\varrho_1^*)\Gamma(\rho^*-1)}{\Gamma(\rho^*-\eta_2^*-1)}G^{\rho^*-\eta_2^*-2},\\
&\nabla_3=\tfrac{\varrho_2\Gamma(\rho)}{\Gamma(\rho+s_1)}G^{\rho+s_1-1}+\tfrac{(1-\varrho_2)\Gamma(\rho)}{\Gamma(\rho+s_2)}G^{\rho+s_2-1}, &&\nabla_3^*=\tfrac{\varrho_2^*\Gamma(\rho^*)}{\Gamma(\rho^*+s_1^*)}G^{\rho^*+s_1^*-1}-\tfrac{(1-\varrho_2^*)\Gamma(\rho^*)}{\Gamma(\rho^*+s_2^*)}G^{\rho^*+s_2^*-1},\\
&\nabla_4=\tfrac{\varrho_2\Gamma(\rho-1)}{\Gamma(\rho+s_1-1)}G^{\rho+s_1-2}+\tfrac{(1-\varrho_2)\Gamma(\rho-1)}{\Gamma(\rho+s_2-1)}G^{\rho+s_2-2}, &&\nabla_4^*=\tfrac{\varrho_2^*\Gamma(\rho^*-1)}{\Gamma(\rho^*+s_1^*-1)}G^{\rho^*+s_1^*-2}-\tfrac{(1-\varrho_2^*)\Gamma(\rho^*-1)}{\Gamma(\rho^*+s_2^*-1)}G^{\rho^*+s_2^*-2},\\
&\Phi=\nabla_3\nabla_2-\nabla_1\nabla_4, &&\Phi^*=\nabla_3^*\nabla_2^*-\nabla_1^*\nabla_4^*.
\end{aligned}
\tag{3.3}
$$

Proof. Let (z_0, r_0) be a solution for the Eq (1.4), then, we get

$$\begin{cases} D^{\rho}z_0(\tau) = \frac{(\upsilon-1)}{\upsilon}D^{\theta}z_0(\tau) + \frac{1}{\upsilon}\Xi(\tau, z_0(\tau), r_0(\tau)), \\ D^{\rho^*}r_0(\tau) = \frac{(\upsilon^*-1)}{\upsilon^*}D^{\theta^*}z_0(\tau) + \frac{1}{\upsilon^*}\Xi(\tau, r_0(\tau), z_0(\tau)). \end{cases} \tag{3.4}$$

Taking the RLF integration of order ρ from both sides of the first equation in (3.4), we have

$$\begin{aligned} z_0(\tau) &= \frac{\upsilon-1}{\upsilon\Gamma(\rho-\theta)}\int_0^{\tau}(\tau-\hbar)^{\rho-\theta-1}z_0(\hbar)d\hbar + \frac{1}{\upsilon\Gamma(\rho)}\int_0^{\tau}(\tau-\hbar)^{\rho-1}\Xi(\tau, z_0(\tau), r_0(\tau))d\hbar \\ &\quad + O_1\tau^{\rho-1} + O_2\tau^{\rho-2} + O_3\tau^{\rho-3}, \end{aligned}$$

where O_1, O_2 and O_3 are real constants. From the first boundary stipulation of (1.4), for $\rho \in (2, 3)$, we have $O_3 = 0$. By Lemma 2.3, we can write

$$z_0(\tau) = \frac{\upsilon-1}{\upsilon}I^{\rho-\theta}z_0(\tau) + \frac{1}{\upsilon}I^{\rho}\Xi(\tau, z_0(\tau), r_0(\tau)) + O_1\tau^{\rho-1} + O_2\tau^{\rho-2}. \tag{3.5}$$

Using the RLF integral and derivative of order η and s, respectively with $\eta \in \{\eta_1, \eta_2\}$, $s \in \{s_1, s_2\}$, $0 < \eta < \rho - \theta$ and $2 < \theta < \rho$, we obtain

$$\begin{aligned} D^{\rho}z_0(\tau) &= \frac{\upsilon-1}{\upsilon\Gamma(\rho-\theta-\eta)}\int_0^{\tau}(\tau-\hbar)^{\rho-\theta-\eta-1}z_0(\hbar)d\hbar + O_1\frac{\Gamma(\rho)}{\Gamma(\rho-\eta)}\tau^{\rho-\eta-1} \\ &\quad + \frac{1}{\upsilon\Gamma(\rho-\eta)}\int_0^{\tau}(\tau-\hbar)^{\rho-\eta-1}\Xi(\hbar, z_0(\hbar), r_0(\hbar))d\hbar + O_2\frac{\Gamma(\rho-1)}{\Gamma(\rho-\eta-1)}\tau^{\rho-\eta-2}. \end{aligned}$$

and

$$\begin{aligned} I^{\rho}z_0(\tau) &= \frac{\upsilon-1}{\upsilon\Gamma(\rho-\theta+s)}\int_0^{\tau}(\tau-\hbar)^{\rho-\theta+s-1}z_0(\hbar)d\hbar + O_1\frac{\Gamma(\rho)}{\Gamma(\rho+s)}\tau^{\rho+s-1} \\ &\quad + \frac{1}{\upsilon\Gamma(\rho+s)}\int_0^{\tau}(\tau-\hbar)^{\rho+s-1}\Xi(\hbar, z_0(\hbar), r_0(\hbar))d\hbar + O_2\frac{\Gamma(\rho-1)}{\Gamma(\rho+s-1)}\tau^{\rho+s-2}. \end{aligned}$$

Replacing $\eta = \eta_1$, $\eta = \eta_2$, $s = s_1$, $s = s_2$ and using the boundary stipulations $\varrho_1 D^{\eta_1}z(G) + (1 - \varrho_1)D^{\eta_2}z(G) = \xi_1$ and $\varrho_2 I^{s_1}z(G) + (1 - \varrho_2)I^{s_2}z(G) = \xi_2$, we can write

$$\begin{aligned} \xi_1 &= \frac{\varrho_1(\upsilon-1)}{\upsilon\Gamma(\rho-\theta-\eta_1)}\int_0^{G}(G-\hbar)^{\rho-\theta-\eta_1-1}z_0(\hbar)d\hbar + \frac{(1-\varrho_1)(\upsilon-1)}{\upsilon\Gamma(\rho-\theta-\eta_2)}\int_0^{G}(G-\hbar)^{\rho-\theta-\eta_2-1}z_0(\hbar)d\hbar \\ &\quad + \frac{\varrho_1}{\upsilon\Gamma(\rho-\eta_1)}\int_0^{G}(G-\hbar)^{\rho-\eta_1-1}\Xi(\hbar, z_0(\hbar), r_0(\hbar))d\hbar + O_1\nabla_1 \\ &\quad + \frac{(1-\varrho_1)}{\upsilon\Gamma(\rho-\eta_2)}\int_0^{G}(G-\hbar)^{\rho-\eta_2-1}\Xi(\hbar, z_0(\hbar), r_0(\hbar))d\hbar + O_2\nabla_2, \end{aligned}$$

and

$$\begin{aligned} \xi_2 &= \frac{\varrho_2(\upsilon-1)}{\upsilon\Gamma(\rho-\theta+s_1)}\int_0^{G}(G-\hbar)^{\rho-\theta+s_1-1}z_0(\hbar)d\hbar + \frac{(1-\varrho_s)(\upsilon-1)}{\upsilon\Gamma(\rho-\theta+s_2)}\int_0^{G}(G-\hbar)^{\rho-\theta+s_2-1}z_0(\hbar)d\hbar \\ &\quad + \frac{\varrho_2}{\upsilon\Gamma(\rho+s_1)}\int_0^{G}(G-\hbar)^{\rho+s_1-1}\Xi(\hbar, z_0(\hbar), r_0(\hbar))d\hbar + O_1\nabla_3 \end{aligned}$$

$$+\frac{(1-\varrho_2)}{\upsilon\Gamma(\rho+s_2)}\int_0^G (G-\hbar)^{\rho+s_2-1}\Xi(\hbar, z_0(\hbar), r_0(\hbar))d\hbar + O_2\nabla_4,$$

which yields that

$$\begin{aligned}
O_1 = {} & \frac{\varrho_1\nabla_4(\upsilon-1)}{\upsilon}I^{\rho-\theta-\eta_1}z_0(G) - \frac{\varrho_2\nabla_2(\upsilon-1)}{\upsilon}I^{\rho-\theta+s_1}z_0(G) \\
& +\frac{\nabla_4(1-\varrho_1)(\upsilon-1)}{\upsilon}I^{\rho-\theta-\eta_2}z_0(G) - \frac{\nabla_2(1-\varrho_2)(\upsilon-1)}{\upsilon}I^{\rho-\theta+s_2}z_0(G) \\
& +\frac{\varrho_1\nabla_4}{\upsilon}I^{\rho-\eta_1}\Xi(G, z_0(G), r_0(G)) - \frac{\varrho_2\nabla_2}{\upsilon}I^{\rho+s_1}\Xi(G, z_0(G), r_0(G)) + \nabla_2\xi_2 - \nabla_4\xi_1 \\
& +\frac{(1-\varrho_1)\nabla_3}{\upsilon}I^{\rho-\eta_2}\Xi(G, z_0(G), r_0(G)) - \frac{(1-\varrho_2)\nabla_1}{\upsilon}I^{\rho+s_2}\Xi(G, z_0(G), r_0(G)).
\end{aligned}$$

and

$$\begin{aligned}
O_2 = {} & \frac{\varrho_1\nabla_3(\upsilon-1)}{\upsilon}I^{\rho-\theta-\eta_1}z_0(G) - \frac{\varrho_2\nabla_1(\upsilon-1)}{\upsilon}I^{\rho-\theta+s_1}z_0(G) \\
& +\frac{\nabla_3(1-\varrho_1)(\upsilon-1)}{\upsilon}I^{\rho-\theta-\eta_2}z_0(G) - \frac{\nabla_1(1-\varrho_2)(\upsilon-1)}{\upsilon}I^{\rho-\theta+s_2}z_0(G) \\
& +\frac{\varrho_1\nabla_3}{\upsilon}I^{\rho-\eta_1}\Xi(G, z_0(G), r_0(G)) - \frac{\varrho_2\nabla_1}{\upsilon}I^{\rho+s_1}\Xi(G, z_0(G), r_0(G)) + \nabla_1\xi_2 - \nabla_3\xi_1 \\
& +\frac{(1-\varrho_1)\nabla_3}{\upsilon}I^{\rho-\eta_2}\Xi(G, z_0(G), r_0(G)) - \frac{(1-\varrho_2)\nabla_1}{\upsilon}I^{\rho+s_2}\Xi(G, z_0(G), r_0(G))\Bigg),
\end{aligned}$$

Substituting O_1 and O_2 in (1.4), we have the first part of the solution (3.1). With the same scenario followed above, the second part of the solution (3.2) can easily be obtained.

Now, we convert the problem to the FP problem. Based on Lemma 3.1, define an operator $\Omega : \mathfrak{I} \to \mathfrak{I}$ by

$$\Omega(z, r) = (\Omega_1(z, r), \Omega_2(z, r)),$$

where

$$\begin{aligned}
& \Omega_1(z, r) \\
= {} & \frac{\upsilon-1}{\upsilon\Gamma(\rho-\theta)}\int_0^\tau (\tau-\hbar)^{\rho-\theta-1}z(\hbar)d\hbar + \frac{1}{\upsilon\Gamma(\rho)}\int_0^\tau (\tau-\hbar)^{\rho-1}\Xi(\hbar, z(\hbar), r(\hbar))d\hbar \\
& +\frac{\tau^{\rho-1}}{\Phi}\Bigg(\frac{\varrho_1\nabla_4(\upsilon-1)}{\upsilon}I^{\rho-\theta-\eta_1}z(G) - \frac{\varrho_2\nabla_2(\upsilon-1)}{\upsilon}I^{\rho-\theta+s_1}z(G) \\
& +\frac{\nabla_4(1-\varrho_1)(\upsilon-1)}{\upsilon}I^{\rho-\theta-\eta_2}z(G) - \frac{\nabla_2(1-\varrho_2)(\upsilon-1)}{\upsilon}I^{\rho-\theta+s_2}z(G) \\
& +\frac{\varrho_1\nabla_4}{\upsilon}I^{\rho-\eta_1}\Xi(G, z(G), r(G)) - \frac{\varrho_2\nabla_2}{\upsilon}I^{\rho+s_1}\Xi(G, z(G), r(G)) + \nabla_2\xi_2 - \nabla_4\xi_1 \\
& +\frac{(1-\varrho_1)\nabla_4}{\upsilon}I^{\rho-\eta_2}\Xi(G, z(G), r(G)) - \frac{(1-\varrho_2)\nabla_2}{\upsilon}I^{\rho+s_2}\Xi(G, z(G), r(G))\Bigg) \\
& -\frac{\tau^{\rho-2}}{\Phi}\Bigg(\frac{\varrho_1\nabla_3(\upsilon-1)}{\upsilon}I^{\rho-\theta-\eta_1}z(G) - \frac{\varrho_2\nabla_1(\upsilon-1)}{\upsilon}I^{\rho-\theta+s_1}z(G)
\end{aligned}$$

$$+\frac{\nabla_3(1-\varrho_1)(\upsilon-1)}{\upsilon}I^{\rho-\theta-\eta_2}z(G)-\frac{\nabla_1(1-\varrho_2)(\upsilon-1)}{\upsilon}I^{\rho-\theta+s_2}z(G)$$
$$+\frac{\varrho_1\nabla_3}{\upsilon}I^{\rho-\eta_1}\Xi(G,z(G),r(G))-\frac{\varrho_2\nabla_1}{\upsilon}I^{\rho+s_1}\Xi(G,z(G),r(G))+\nabla_1\xi_2-\nabla_3\xi_1$$
$$+\frac{(1-\varrho_1)\nabla_3}{\upsilon}I^{\rho-\eta_2}\Xi(G,z(G),r(G))-\frac{(1-\varrho_2)\nabla_1}{\upsilon}I^{\rho+s_2}\Xi(G,z(G),r(G))\Bigg), \tag{3.6}$$

and

$$\begin{aligned}
\Omega_2(z,r) &= \frac{\upsilon^*-1}{\upsilon^*\Gamma(\rho^*-\theta^*)}\int_0^\tau(\tau-\hbar)^{\rho^*-\theta^*-1}r(\hbar)d\hbar+\frac{1}{\upsilon^*\Gamma(\rho^*)}\int_0^\tau(\tau-\hbar)^{\rho^*-1}\Xi^*(\hbar,r(\hbar),z(\hbar))d\hbar\\
&+\frac{\tau^{\rho^*-1}}{\Phi^*}\Bigg(\frac{\varrho_1^*\nabla_4^*(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*-\eta_1^*}r(G)-\frac{\varrho_2^*\nabla_2^*(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*-s_1^*}r(G)\\
&+\frac{\nabla_4^*(1-\varrho_1^*)(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*-\eta_2^*}r(G)-\frac{\nabla_2^*(1-\varrho_2^*)(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*-s_2^*}r(G)\\
&+\frac{\varrho_1^*\nabla_4^*}{\upsilon^*}I^{\rho^*-\eta_1^*}\Xi^*(G,r(G),z(G))-\frac{\varrho_2^*\nabla_2^*}{\upsilon^*}I^{\rho^*+s_1^*}\Xi^*(G,r(G),z(G))+\nabla_2^*\xi_2^*-\nabla_4^*\xi_1^*\\
&+\frac{(1-\varrho_1^*)\nabla_4^*}{\upsilon^*}I^{\rho^*-\eta_2^*}\Xi^*(G,r(G),z(G))-\frac{\left(1-\varrho_2^*\right)\nabla_2^*}{\upsilon^*}I^{\rho^*+s_2^*}\Xi^*(G,r(G),z(G))\Bigg)\\
&-\frac{\tau^{\rho^*-2}}{\Phi^*}\Bigg(\frac{\varrho_1^*\nabla_3^*(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*-\eta_1^*}r(G)-\frac{\varrho_2^*\nabla_1^*(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*-s_1^*}r(G)\\
&+\frac{\nabla_3^*(1-\varrho_1^*)(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*-\eta_2^*}r(G)-\frac{\nabla_1^*(1-\varrho_2^*)(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*-s_2^*}r(G)\\
&\frac{\varrho_1^*\nabla_3^*}{\upsilon^*}I^{\rho^*-\eta_1^*}\Xi^*(G,r(G),z(G))-\frac{\varrho_2^*\nabla_1^*}{\upsilon^*}I^{\rho^*+s_1^*}\Xi^*(G,r(G),z(G))+\nabla_1^*\xi_2^*-\nabla_3^*\xi_1^*\\
&+\frac{(1-\varrho_1^*)\nabla_3^*}{\upsilon^*}I^{\rho^*-\eta_2^*}\Xi^*(G,r(G),z(G))-\frac{\left(1-\varrho_2^*\right)\nabla_1^*}{\upsilon^*}I^{\rho^*+s_2^*}\Xi^*(G,r(G),z(G))\Bigg).
\end{aligned} \tag{3.7}$$

Remember that the solution to CBVP (1.4) is (z_0, r_0) iff (z_0, r_0) is a FP of Ω. We employ the following notation to streamline calculations:

$$\begin{aligned}
\Lambda_1 &= \frac{|\upsilon-1|\left(\nabla_4+\nabla_3G^{-1}\right)}{|\Phi|}\left(\frac{\varrho_1G^{2\rho-\theta-\eta_1-1}}{\upsilon\Gamma(\rho-\theta-\eta_1+1)}+\frac{(1-\varrho_1)G^{2\rho-\theta-\eta_2-1}}{\upsilon\Gamma(\rho-\theta-\eta_2+1)}\right)\\
&+\frac{|\upsilon-1|\left(\nabla_2+\nabla_1G^{-1}\right)}{|\Phi|}\left(\frac{\varrho_2G^{2\rho-\theta+s_1-1}}{\upsilon\Gamma(\rho-\theta+s_1+1)}+\frac{(1-\varrho_2)G^{2\rho-\theta+s_2-1}}{\upsilon\Gamma(\rho-\theta+s_2+1)}\right)\\
&+\frac{|\upsilon-1|\,G^{\rho-\theta}}{\upsilon\Gamma(\rho-\theta+1)}.
\end{aligned} \tag{3.8}$$

$$\Lambda_1^* = \frac{|\upsilon^*-1|\left(\nabla_4^*+\nabla_3^*G^{-1}\right)}{|\Phi^*|}\left(\frac{\varrho_1^*G^{2\rho^*-\theta^*-\eta_1^*-1}}{\upsilon^*\Gamma(\rho^*-\theta^*-\eta_1^*+1)}+\frac{(1-\varrho_1^*)G^{2\rho^*-\theta^*-\eta_2^*-1}}{\upsilon^*\Gamma(\rho^*-\theta^*-\eta_2^*+1)}\right)$$

$$+\frac{|\upsilon^*-1|\left(\nabla_4^*+\nabla_3^*G^{-1}\right)}{|\Phi^*|}\left(\frac{\varrho_2^*G^{2\rho^*-\theta^*+s_1^*-1}}{\upsilon^*\Gamma(\rho^*-\theta^*+s_1^*+1)}+\frac{(1-\varrho_2^*)G^{2\rho^*-\theta^*+s_2^*-1}}{\upsilon^*\Gamma(\rho^*-\theta^*+s_2^*+1)}\right)$$
$$+\frac{|\upsilon^*-1|\,G^{\rho^*-\theta^*}}{\upsilon^*\Gamma(\rho^*-\theta^*+1)}. \tag{3.9}$$

$$\Lambda_2 = \frac{G^{\rho}}{\upsilon\Gamma(\rho+1)}+\frac{\nabla_4+\nabla_3G^{-1}}{|\Phi|}\left(\frac{\varrho_1G^{2\rho-\eta_1-1}}{\upsilon\Gamma(\rho-\eta_1+1)}+\frac{(1-\varrho_1)G^{2\rho-\eta_2-1}}{\upsilon\Gamma(\rho-\eta_2+1)}\right)$$
$$+\frac{\nabla_2+\nabla_1G^{-1}}{|\Phi|}\left(\frac{\varrho_2G^{2\rho+s_1-1}}{\upsilon\Gamma(\rho+s_1+1)}+\frac{(1-\varrho_2)G^{2\rho+s_2-1}}{\upsilon\Gamma(\rho+s_2+1)}\right). \tag{3.10}$$

$$\Lambda_2^* = \frac{G^{\rho^*}}{\upsilon^*\Gamma(\rho^*+1)}+\frac{\left(\nabla_4^*+\nabla_3^*G^{-1}\right)}{|\Phi^*|}\left(\frac{\varrho_1^*G^{2\rho^*-\eta_1^*-1}}{\upsilon^*\Gamma(\rho^*-\eta_1^*+1)}+\frac{(1-\varrho_1^*)G^{2\rho^*-\eta_2^*-1}}{\upsilon^*\Gamma(\rho^*-\eta_2^*+1)}\right)$$
$$+\frac{\nabla_4^*+\nabla_3^*G^{-1}}{|\Phi^*|}\left(\frac{\varrho_2^*G^{2\rho^*+s_1^*-1}}{\upsilon^*\Gamma(\rho^*+s_1^*+1)}+\frac{(1-\varrho_2^*)G^{2\rho^*+s_2^*-1}}{\upsilon^*\Gamma(\rho^*+s_2^*+1)}\right). \tag{3.11}$$

Now, our main theorem is as follows:

Theorem 3.1. *Assume that the mappings $\Xi, \Xi^* : U\times\mathbb{R}^2\to\mathbb{R}$ are continuous and there are constants $T_\Xi, \widetilde{T}_\Xi, T_{\Xi^*}, \widetilde{T}_{\Xi^*} > 0$ so that*

$$|\Xi(\tau, z_1(\tau), z_2(\tau)) - \Xi(\tau, \widetilde{z}_1(\tau), \widetilde{z}_2(\tau))| \le T_\Xi\,|z_1-\widetilde{z}_1| + \widetilde{T}_\Xi\,|z_2-\widetilde{z}_2|,$$

and

$$|\Xi^*(\tau, r_1(\tau), r_2(\tau)) - \Xi^*(\tau, \widetilde{r}_1(\tau), \widetilde{r}_2(\tau))| \le T_{\Xi^*}\,|r_1-\widetilde{r}_1| + \widetilde{T}_{\Xi^*}\,|r_2-\widetilde{r}_2|,$$

*for all $\tau\in U$ and $z_1, z_2, \widetilde{z}_1, \widetilde{z}_2, r_1, r_2, \widetilde{r}_1, \widetilde{r}_2\in\mathbb{R}$. If $\widehat{T}\Lambda_4+\Lambda_3<1$, then the considered problem (1.4) has a unique solution (**US**), where $\widehat{T}=\max\{T, T^*\}$, $T=\max\{T_\Xi, \widetilde{T}_\Xi\}$, $T^*=\max\{T_{\Xi^*}, \widetilde{T}_{\Xi^*}\}$, $\Lambda_1+\Lambda_1^*=\Lambda_3$ and $\Lambda_2+\Lambda_2^*=\Lambda_4$ and Λ_1, Λ_1^*, Λ_2, Λ_2^* are described as (3.8)–(3.11), respectively.*

Proof. Set $\sup_{\upsilon\in U}\Xi(\tau,0,0)=N<\infty$, $\sup_{\upsilon\in U}\Xi^*(\tau,0,0)=N^*<\infty$ and choose

$$y \ge \frac{\widehat{N}\Lambda_4}{1-\widehat{T}\Lambda_4-\Lambda_3}+\frac{(|\Phi|+|\Phi^*|)\,G^{\rho-1}\left(|\nabla_2\xi_1|+|\nabla_4\xi_2|+\left|\nabla_2^*\xi_1^*\right|+\left|\nabla_4^*\xi_2^*\right|\right)}{|\Phi|\,|\Phi^*|\,(1-\widehat{T}\Lambda_4-\Lambda_3)}$$
$$+\frac{G^{\rho-2}\left(|\nabla_2\xi_1|+|\nabla_4\xi_2|+\left|\nabla_2^*\xi_1^*\right|+\left|\nabla_4^*\xi_2^*\right|\right)}{|\Phi|\,|\Phi^*|\,(1-\widehat{T}\Lambda_4-\Lambda_3)}, \tag{3.12}$$

where ∇_i and ∇_i^*, $i\in\{1,2,3,4\}$, Φ and Φ^* are defined by (3.3) and $\widehat{N}=\max\{N, N^*\}$. As a first step, we show that $\Omega Q_y\subset Q_y$, where $Q_y=\{(z,r)\in\mathfrak{I} : \|(z,r)\|\le y\}$. For any $(z,r)\in Q_y$, we have

$$\|\Omega(z,r)\|_{\mathfrak{I}} = \|\Omega_1(z,r)\|_{\mathfrak{I}_1}+\|\Omega_2(z,r)\|_{\mathfrak{I}_2} \tag{3.13}$$

From (3.6) and (3.7), we get

$$|\Omega_1(z,r)| \le \frac{|\upsilon-1|}{\upsilon\Gamma(\rho-\theta)}\int_0^\tau(\tau-\hbar)^{\rho-\theta-1}\,|z(\hbar)|\,d\hbar$$

$$
\begin{aligned}
&+\frac{1}{\upsilon\Gamma(\rho)}\int_0^\tau (\tau-\hbar)^{\rho-1}\left(|\Xi(\hbar,z(\hbar),r(\hbar))-\Xi(\hbar,0,0)|+|\Xi(\hbar,0,0)|\right)d\hbar\\
&+\frac{G^{\rho-1}}{|\Phi|}\left(\frac{\varrho_1\nabla_4|\upsilon-1|}{\upsilon}I^{\rho-\theta-\eta_1}|z(G)|-\frac{\varrho_2\nabla_2|\upsilon-1|}{\upsilon}I^{\rho-\theta+s_1}|z(G)|\right.\\
&+\frac{\nabla_4|1-\varrho_1|(|\upsilon-1|)}{\upsilon}I^{\rho-\theta-\eta_2}|z(G)|-\frac{\nabla_2|1-\varrho_2||\upsilon-1|}{\upsilon}I^{\rho-\theta+s_2}|z(G)|\\
&+\frac{\varrho_1\nabla_4}{\upsilon}I^{\rho-\eta_1}\left[|\Xi(G,z(G),r(G))-\Xi(G,0,0)|+|\Xi(G,0,0)|\right]\\
&-\frac{\varrho_2\nabla_2}{\upsilon}I^{\rho+s_1}\left[|\Xi(G,z(G),r(G))-\Xi(G,0,0)|+|\Xi(G,0,0)|\right]+|\nabla_2\xi_2|+|\nabla_4\xi_1|\\
&+\frac{(1-\varrho_1)\nabla_4}{\upsilon}I^{\rho-\eta_2}\left[|\Xi(G,z(G),r(G))-\Xi(G,0,0)|+|\Xi(G,0,0)|\right]\\
&\left.-\frac{(1-\varrho_2)\nabla_2}{\upsilon}I^{\rho+s_2}\left[|\Xi(G,z(G),r(G))-\Xi(G,0,0)|+|\Xi(G,0,0)|\right]\right)\\
&-\frac{G^{\rho-2}}{\Phi}\left(\frac{\varrho_1\nabla_3|\upsilon-1|}{\upsilon}I^{\rho-\theta-\eta_1}|z(G)|-\frac{\varrho_2\nabla_1|\upsilon-1|}{\upsilon}I^{\rho-\theta+s_1}|z(G)|\right.\\
&+\frac{\nabla_3|1-\varrho_1|(|\upsilon-1|)}{\upsilon}I^{\rho-\theta-\eta_2}|z(G)|-\frac{\nabla_1|1-\varrho_2||\upsilon-1|}{\upsilon}I^{\rho-\theta+s_2}|z(G)|\\
&+\frac{\varrho_1\nabla_3}{\upsilon}I^{\rho-\eta_1}\left[|\Xi(G,z(G),r(G))-\Xi(G,0,0)|+|\Xi(G,0,0)|\right]\\
&-\frac{\varrho_2\nabla_2}{\upsilon}I^{\rho+s_1}\left[|\Xi(G,z(G),r(G))-\Xi(G,0,0)|+|\Xi(G,0,0)|\right]+|\nabla_1\xi_2|+|\nabla_3\xi_1|\\
&+\frac{(1-\varrho_1)\nabla_3}{\upsilon}I^{\rho-\eta_2}\left[|\Xi(G,z(G),r(G))-\Xi(G,0,0)|+|\Xi(G,0,0)|\right]\\
&\left.-\frac{(1-\varrho_2)\nabla_1}{\upsilon}I^{\rho+s_2}\left[|\Xi(G,z(G),r(G))-\Xi(G,0,0)|+|\Xi(G,0,0)|\right]\right),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\|\Omega_1(z,r)\|_{\mathfrak{I}_1}\\
\leq\;& (T\|(z,r)\|+N)\Lambda_2+\|(z,r)\|\Lambda_1+\frac{1}{|\Phi|}\left[G^{\rho-1}(|\nabla_2\xi_2|+|\nabla_4\xi_1|)+G^{\rho-2}(|\nabla_1\xi_2|+|\nabla_3\xi_1|)\right]\\
=\;& (T\Lambda_2+\Lambda_1)\|(z,r)\|+N\Lambda_2+\frac{1}{|\Phi|}\left[G^{\rho-1}(|\nabla_2\xi_2|+|\nabla_4\xi_1|)+G^{\rho-2}(|\nabla_1\xi_2|+|\nabla_3\xi_1|)\right]. \qquad (3.14)
\end{aligned}
$$

In the same scenario, we can write

$$
\begin{aligned}
&\|\Omega_2(z,r)\|_{\mathfrak{I}_2} \qquad (3.15)\\
\leq\;& (T^*\Lambda_2^*+\Lambda_1^*)\|(z,r)\|+N^*\Lambda_2^*+\frac{1}{|\Phi^*|}\left[G^{\rho-1}\left(\left|\nabla_2^*\xi_2^*\right|+\left|\nabla_4^*\xi_1^*\right|\right)+G^{\rho-2}\left(\left|\nabla_1^*\xi_2^*\right|+\left|\nabla_3^*\xi_1^*\right|\right)\right].
\end{aligned}
$$

Applying (3.14) and (3.15) in (3.13) and using (3.12), we have

$$
\begin{aligned}
\|\Omega(z,r)\|_{\mathfrak{I}} &= \|\Omega_1(z,r)\|_{\mathfrak{I}_1}+\|\Omega_2(z,r)\|_{\mathfrak{I}_2}\\
&= (T\Lambda_2+\Lambda_1+T^*\Lambda_2^*+\Lambda_1^*)\|(z,r)\|+N\Lambda_2+N^*\Lambda_2^*
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{|\Phi|}\left[G^{\rho-1}\left(|\nabla_2\xi_2|+|\nabla_4\xi_1|\right)+G^{\rho-2}\left(|\nabla_1\xi_2|+|\nabla_3\xi_1|\right)\right]\\
&+\frac{1}{|\Phi^*|}\left[G^{\rho-1}\left(\left|\nabla_2^*\xi_2^*\right|+\left|\nabla_4^*\xi_1^*\right|\right)+G^{\rho-2}\left(\left|\nabla_1^*\xi_2^*\right|+\left|\nabla_3^*\xi_1^*\right|\right)\right]\\
\leq\;& \left(T\Lambda_2+\Lambda_1+T^*\Lambda_2^*+\Lambda_1^*\right)y+N\Lambda_2+N^*\Lambda_2^*\\
&+\frac{1}{|\Phi|}\left[G^{\rho-1}\left(|\nabla_2\xi_2|+|\nabla_4\xi_1|\right)+G^{\rho-2}\left(|\nabla_1\xi_2|+|\nabla_3\xi_1|\right)\right]\\
&+\frac{1}{|\Phi^*|}\left[G^{\rho-1}\left(\left|\nabla_2^*\xi_2^*\right|+\left|\nabla_4^*\xi_1^*\right|\right)+G^{\rho-2}\left(\left|\nabla_1^*\xi_2^*\right|+\left|\nabla_3^*\xi_1^*\right|\right)\right]\\
\leq\;& \left(\widehat{T}\Lambda_4+\Lambda_3\right)y+\widehat{N}\Lambda_4\\
&+\frac{1}{|\Phi|}\left[G^{\rho-1}\left(|\nabla_2\xi_2|+|\nabla_4\xi_1|\right)+G^{\rho-2}\left(|\nabla_1\xi_2|+|\nabla_3\xi_1|\right)\right]\\
&+\frac{1}{|\Phi^*|}\left[G^{\rho-1}\left(\left|\nabla_2^*\xi_2^*\right|+\left|\nabla_4^*\xi_1^*\right|\right)+G^{\rho-2}\left(\left|\nabla_1^*\xi_2^*\right|+\left|\nabla_3^*\xi_1^*\right|\right)\right]\\
\leq\;& y+
\end{aligned}
$$

Hence, $\|\Omega(z,r)\|_{\mathfrak{I}} \leq y$ and so $\Omega Q_y \subset Q_y$. For each $\upsilon \in U$ and for $z, r, \widetilde{z}, \widetilde{r} \in \mathfrak{I}$, we get

$$
\begin{aligned}
& \left|\Omega_1(z,r)(\tau)-\Omega_1(\widetilde{z},\widetilde{r})(\upsilon)\right|\\
\leq\;& \frac{|\upsilon-1|}{\upsilon\Gamma(\rho-\theta)}\int_0^\tau(\tau-\hbar)^{\rho-\theta-1}\left|z(\hbar)-\widetilde{z}(\hbar)\right|d\hbar\\
&+\frac{1}{\upsilon\Gamma(\rho)}\int_0^\tau(\tau-\hbar)^{\rho-1}\left|\Xi(\hbar,z(\hbar),r(\hbar))-\Xi(\hbar,\widetilde{z}(\hbar),\widetilde{r}(\hbar))\right|d\hbar\\
&+\frac{G^{\rho-1}}{|\Phi|}\Bigg(\frac{\varrho_1\nabla_4|\upsilon-1|}{\upsilon}I^{\rho-\theta-\eta_1}\left|z(G)-\widetilde{z}(G)\right|-\frac{\varrho_2\nabla_2|\upsilon-1|}{\upsilon}I^{\rho-\theta+s_1}\left|z(G)-\widetilde{z}(G)\right|\\
&+\frac{\nabla_4|1-\varrho_1|\left(|\upsilon-1|\right)}{\upsilon}I^{\rho-\theta-\eta_2}\left|z(G)-\widetilde{z}(G)\right|-\frac{\nabla_2|1-\varrho_2||\upsilon-1|}{\upsilon}I^{\rho-\theta+s_2}\left|z(G)-\widetilde{z}(G)\right|\\
&+\frac{\varrho_1\nabla_4}{\upsilon}I^{\rho-\eta_1}\left|\Xi(G,z(G),r(G))-\Xi(G,\widetilde{z}(G),\widetilde{r}(G))\right|\\
&-\frac{\varrho_2\nabla_2}{\upsilon}I^{\rho+s_1}\left|\Xi(G,z(G),r(G))-\Xi(G,\widetilde{z}(G),\widetilde{r}(G))\right|\\
&+\frac{(1-\varrho_1)\nabla_4}{\upsilon}I^{\rho-\eta_2}\left|\Xi(G,z(G),r(G))-\Xi(G,\widetilde{z}(G),\widetilde{r}(G))\right|\\
&-\frac{(1-\varrho_2)\nabla_2}{\upsilon}I^{\rho+s_2}\left|\Xi(G,z(G),r(G))-\Xi(G,\widetilde{z}(G),\widetilde{r}(G))\right|\Bigg)\\
&-\frac{G^{\rho-2}}{\Phi}\Bigg(\frac{\varrho_1\nabla_3|\upsilon-1|}{\upsilon}I^{\rho-\theta-\eta_1}|z(G)|-\frac{\varrho_2\nabla_1|\upsilon-1|}{\upsilon}I^{\rho-\theta+s_1}|z(G)|\\
&+\frac{\nabla_3|1-\varrho_1|\left(|\upsilon-1|\right)}{\upsilon}I^{\rho-\theta-\eta_2}\left|z(G)-\widetilde{z}(G)\right|-\frac{\nabla_1|1-\varrho_2||\upsilon-1|}{\upsilon}I^{\rho-\theta+s_2}\left|z(G)-\widetilde{z}(G)\right|\\
&+\frac{\varrho_1\nabla_3}{\upsilon}I^{\rho-\eta_1}\left|\Xi(G,z(G),r(G))-\Xi(G,\widetilde{z}(G),\widetilde{r}(G))\right|\\
&-\frac{\varrho_2\nabla_2}{\upsilon}I^{\rho+s_1}\left|\Xi(G,z(G),r(G))-\Xi(G,\widetilde{z}(G),\widetilde{r}(G))\right|
\end{aligned}
$$

$$+\frac{(1-\varrho_1)\nabla_3}{\upsilon} I^{\rho-\eta_2} \left|\Xi(G,z(G),r(G)) - \Xi(G,\widetilde{z}(G),\widetilde{r}(G))\right|$$
$$\left. -\frac{(1-\varrho_2)\nabla_1}{\upsilon} I^{\rho+s_2} \left|\Xi(G,z(G),r(G)) - \Xi(G,\widetilde{z}(G),\widetilde{r}(G))\right| \right),$$

which leads to

$$\|\Omega_1(z,r) - \Omega_1(\widetilde{z},\widetilde{r})\|_{\mathfrak{I}_1} \leq T\Lambda_2 \left(\|z-\widetilde{z}\| + \|r-\widetilde{r}\|\right) + \|z-\widetilde{z}\|\,\Lambda_1.$$

Similarly, one can obtain

$$\|\Omega_2(z,r) - \Omega_2(\widetilde{z},\widetilde{r})\|_{\mathfrak{I}_2} \leq T^*\Lambda_2^* \left(\|z-\widetilde{z}\| + \|r-\widetilde{r}\|\right) + \|r-\widetilde{r}\|\,\Lambda_1^*.$$

Hence,

$$\begin{aligned}
\|\Omega(z,r) - \Omega(\widetilde{z},\widetilde{r})\|_{\mathfrak{I}} &\leq \|\Omega_1(z,r) - \Omega_1(\widetilde{z},\widetilde{r})\|_{\mathfrak{I}_1} + \|\Omega_2(z,r) - \Omega_2(\widetilde{z},\widetilde{r})\|_{\mathfrak{I}_2} \\
&= T\Lambda_2 \left(\|z-\widetilde{z}\| + \|r-\widetilde{r}\|\right) + \|z-\widetilde{z}\|\,\Lambda_1 \\
&\quad + T^*\Lambda_2^* \left(\|z-\widetilde{z}\| + \|r-\widetilde{r}\|\right) + \|r-\widetilde{r}\|\,\Lambda_1^* \\
&= \left(\Lambda_1 + T\Lambda_2 + T^*\Lambda_2^*\right)\|z-\widetilde{z}\| + \left(\Lambda_1^* + T^*\Lambda_2^* + T\Lambda_2\right)\|r-\widetilde{r}\| \\
&\leq \left(\Lambda_1 + \widehat{T}\Lambda_4\right)\|z-\widetilde{z}\| + \left(\Lambda_1^* + \widehat{T}\Lambda_4\right)\|r-\widetilde{r}\| \\
&\leq \left(\widehat{T}\Lambda_4 + \Lambda_3\right)\|(z,r) - (\widetilde{z},\widetilde{r})\|.
\end{aligned}$$

Since $\widehat{T}\Lambda_4 + \Lambda_3 < 1$, then Ω is a contraction mapping. Using the contraction principle, Ω has a unique FP, which is the US for the CBVP (1.4).

Now, we present an existence result by applying Krasnoselskii's FP theorem.

Theorem 3.2. *Suppose that the mappings $\Xi, \Xi^* : U \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and there are positive constants $T_\Xi, \widetilde{T}_\Xi, T_{\Xi^*}, \widetilde{T}_{\Xi^*}$ so that*

$$|\Xi(\tau, z_1(\tau), z_2(\tau)) - \Xi(\tau, \widetilde{z}_1(\tau), \widetilde{z}_2(\tau))| \leq T_\Xi\,|z_1 - \widetilde{z}_1| + \widetilde{T}_\Xi\,|z_2 - \widetilde{z}_2|,$$

and

$$|\Xi^*(\tau, r_1(\tau), r_2(\tau)) - \Xi^*(\tau, \widetilde{r}_1(\tau), \widetilde{r}_2(\tau))| \leq T_{\Xi^*}\,|r_1 - \widetilde{r}_1| + \widetilde{T}_{\Xi^*}\,|r_2 - \widetilde{r}_2|,$$

for all $\tau \in U$ and $z_1, z_2, \widetilde{z}_1, \widetilde{z}_2, r_1, r_2, \widetilde{r}_1, \widetilde{r}_2 \in \mathbb{R}$. If there are $V(\tau), V^*(\tau) \in C(U, \mathbb{R}_+)$ so that

$$\Xi(\tau, z(\tau), r(\tau)) \leq V(\tau) \text{ and } \Xi^*(\tau, z(\tau), r(\tau)) \leq V^*(\tau),$$

for all $(\tau, z, r) \in U \times \mathbb{R} \times \mathbb{R}$ and $\Lambda_3 < 1$, then, the CBVP (1.4) has at least one solution.

Proof. Consider $\sup_{\tau\in U} |V(\tau)| = \|V\|$, $\sup_{tau\in U} |V^*(\tau)| = \|V^*\|$ and the set $Q_x = \{(z,r) \in \mathfrak{I} : \|(z,r)\| \leq x\}$, where

$$\begin{aligned}
x \geq\ & \frac{\widehat{V}\Lambda_3}{1-\Lambda_4} + \frac{(|\Phi| + |\Phi^*|)\,G^{\rho-1}\left(|\nabla_2\xi_2| + |\nabla_4\xi_1| + \left|\nabla_2^*\xi_2^*\right| + \left|\nabla_4^*\xi_1^*\right|\right)}{|\Phi|\,|\Phi^*|\,(1-\Lambda_4)} \\
& + \frac{(|\Phi| + |\Phi^*|)\,G^{\rho-2}\left(|\nabla_1\xi_2| + |\nabla_1\xi_1| + \left|\nabla_1^*\xi_2^*\right| + \left|\nabla_3^*\xi_1^*\right|\right)}{|\Phi|\,|\Phi^*|\,(1-\Lambda_4)},
\end{aligned}$$

and ∇_i, ∇_i^*, $i \in \{1,2,3,4\}$, Φ and Φ^* are defined by (3.3), $\widehat{V} = \max\{V, V^*\}$ and $\Lambda_3 = \Lambda_1 + \Lambda_1^*$. For any $(z,r) \in Q_x$, define the operators $\Omega, \Omega^* : \mathfrak{I} \to \mathfrak{I}$ by

$$\Omega(z,r) = \widetilde{\Omega}_1(z,r) + \widetilde{\Omega}_2(z,r) \text{ and } \Omega^*(z,r) = \widehat{\Omega}_1(z,r) + \widehat{\Omega}_2(z,r),$$

where

$$\begin{aligned} \widetilde{\Omega}_1(z,r) &= \frac{\upsilon-1}{\upsilon\Gamma(\rho-\theta)}\int_0^\tau (\tau-\hbar)^{\rho-\theta-1} z(\hbar)d\hbar + \frac{\tau^{\rho-1}}{\Phi}\times \\ &\left(\frac{\varrho_1\nabla_4(\upsilon-1)}{\upsilon} I^{\rho-\theta-\eta_1} z(G) - \frac{\varrho_2\nabla_2(\upsilon-1)}{\upsilon} I^{\rho-\theta+s_1} z(G)\right. \\ &\left.+\frac{\nabla_4(1-\varrho_1)(\upsilon-1)}{\upsilon} I^{\rho-\theta-\eta_2} z(G) - \frac{\nabla_2(1-\varrho_2)(\upsilon-1)}{\upsilon} I^{\rho-\theta+s_2} z(G)\right) \\ &\frac{\tau^{\rho-2}}{\Phi}\left(\frac{\varrho_1\nabla_3(\upsilon-1)}{\upsilon} I^{\rho-\theta-\eta_1} z(G) - \frac{\varrho_2\nabla_1(\upsilon-1)}{\upsilon} I^{\rho-\theta+s_1} z(G)\right. \\ &\left.+\frac{\nabla_3(1-\varrho_1)(\upsilon-1)}{\upsilon} I^{\rho-\theta-\eta_2} z(G) - \frac{\nabla_1(1-\varrho_2)(\upsilon-1)}{\upsilon} I^{\rho-\theta+s_2} z(G)\right), \end{aligned} \tag{3.16}$$

$$\begin{aligned} \widehat{\Omega}_1(z,r) &= \frac{1}{\upsilon\Gamma(\rho)}\int_0^\tau (\tau-\hbar)^{\rho-1}\Xi(\hbar, z(\hbar), r(\hbar))d\hbar + \frac{\tau^{\rho-1}}{\Phi}\times \\ &\left(\frac{\varrho_1\nabla_4}{\upsilon} I^{\rho-\eta_1}\Xi(G,z(G),r(G)) - \frac{\varrho_2\nabla_2}{\upsilon} I^{\rho+s_1}\Xi(G,z(G),r(G)) + \nabla_2\xi_2 - \nabla_4\xi_1\right. \\ &\left.+\frac{(1-\varrho_1)\nabla_4}{\upsilon} I^{\rho-\eta_2}\Xi(G,z(G),r(G)) - \frac{(1-\varrho_2)\nabla_2}{\upsilon} I^{\rho+s_2}\Xi(G,z(G),r(G))\right) \\ &-\frac{\tau^{\rho-2}}{\Phi}\left(\frac{\varrho_1\nabla_3}{\upsilon} I^{\rho-\eta_1}\Xi(G,z(G),r(G)) - \frac{\varrho_2\nabla_1}{\upsilon} I^{\rho+s_1}\Xi(G,z(G),r(G))\right. \\ &+\frac{(1-\varrho_1)\nabla_3}{\upsilon} I^{\rho-\eta_2}\Xi(G,z(G),r(G)) - \frac{(1-\varrho_2)\nabla_1}{\upsilon} I^{\rho+s_2}\Xi(G,z(G),r(G)) \\ &\left.+\nabla_1\xi_2 - \nabla_3\xi_1\right), \end{aligned} \tag{3.17}$$

$$\begin{aligned} \widetilde{\Omega}_2(z,r) &= \frac{\upsilon^*-1}{\upsilon^*\Gamma(\rho^*-\theta^*)}\int_0^\tau (\tau-\hbar)^{\rho^*-\theta^*-1} r(\hbar)d\hbar + \frac{\tau^{\rho^*-1}}{\Phi^*}\times \\ &\left(\frac{\varrho_1^*\nabla_4^*(\upsilon^*-1)}{\upsilon^*} I^{\rho^*-\theta^*-\eta_1^*} r(G) - \frac{\varrho_2^*\nabla_2^*(\upsilon^*-1)}{\upsilon^*} I^{\rho^*-\theta^*+s_1^*} r(G)\right. \\ &\left.\frac{\nabla_4^*(1-\varrho_1^*)(\upsilon^*-1)}{\upsilon^*} I^{\rho^*-\theta^*-\eta_2^*} r(G) - \frac{\nabla_2^*(1-\varrho_2^*)(\upsilon^*-1)}{\upsilon^*} I^{\rho^*-\theta^*+s_2^*} r(G)\right) \\ &-\frac{\tau^{\rho^*-2}}{\Phi^*}\left(\frac{\varrho_1^*\nabla_3^*(\upsilon^*-1)}{\upsilon^*} I^{\rho^*-\theta^*-\eta_1^*} r(G) - \frac{\varrho_2^*\nabla_1^*(\upsilon^*-1)}{\upsilon^*} I^{\rho^*-\theta^*+s_1^*} r(G)\right. \\ &\left.+\frac{\nabla_3^*(1-\varrho_1^*)(\upsilon^*-1)}{\upsilon^*} I^{\rho^*-\theta^*-\eta_2^*} r(G) - \frac{\nabla_1^*(1-\varrho_2^*)(\upsilon^*-1)}{\upsilon^*} I^{\rho^*-\theta^*+s_2^*} r(G)\right), \end{aligned} \tag{3.18}$$

and

$$\widehat{\Omega}_2(z,r) = \frac{1}{\upsilon^*\Gamma(\rho^*)}\int_0^\tau (\tau-\hbar)^{\rho^*-1}\Xi^*(\hbar, r(\hbar), z(\hbar))d\hbar + \frac{\tau^{\rho^*-1}}{\Phi^*}\times$$

$$\Bigg(+\frac{\varrho_1^*\nabla_4^*}{\upsilon^*}I^{\rho^*-\eta_1^*}\Xi^*(G,r(G),z(G))-\frac{\varrho_2^*\nabla_2^*}{\upsilon^*}I^{\rho^*+s_1^*}\Xi^*(G,r(G),z(G))+\nabla_2^*\xi_2^*-\nabla_4^*\xi_1^*$$
$$+\frac{(1-\varrho_1^*)\nabla_4^*}{\upsilon^*}I^{\rho^*-\eta_2^*}\Xi^*(G,r(G),z(G))-\frac{\left(1-\varrho_2^*\right)\nabla_2^*}{\upsilon^*}I^{\rho^*+s_2^*}\Xi^*(G,r(G),z(G))\Bigg)$$
$$-\frac{\tau^{\rho^*-2}}{\Phi^*}\Bigg(\frac{\varrho_1^*\nabla_3^*}{\upsilon^*}I^{\rho^*-\eta_1^*}\Xi^*(G,r(G),z(G))-\frac{\varrho_2^*\nabla_1^*}{\upsilon^*}I^{\rho^*+s_1^*}\Xi^*(G,r(G),z(G))+$$
$$+\frac{(1-\varrho_1^*)\nabla_3^*}{\upsilon^*}I^{\rho^*-\eta_2^*}\Xi^*(G,r(G),z(G))-\frac{\left(1-\varrho_2^*\right)\nabla_1^*}{\upsilon^*}I^{\rho^*+s_2^*}\Xi^*(G,r(G),z(G))$$
$$+\nabla_1^*\xi_2^*-\nabla_3^*\xi_1^*\Bigg). \tag{3.19}$$

We shall show that $\Omega(z,r)+\Omega^*(z,r)\in Q_x$, for all $(z,r)\in Q_x$. From (3.16) and (3.17), we have

$$\begin{aligned}
&\left|\widetilde{\Omega}_1(z,r)(\tau)+\widehat{\Omega}_1(z,r)(\tau)\right| \\
\le\ & \|V\|\Bigg[\frac{G^{\rho}}{\upsilon\Gamma(\rho+1)}+\frac{\nabla_4+\nabla_3G^{-1}}{|\Phi|}\left(\frac{\varrho_1G^{2\rho-\eta_1-1}}{\upsilon\Gamma(\rho-\eta_1+1)}+\frac{(1-\varrho_1)G^{2\rho-\eta_2-1}}{\upsilon\Gamma(\rho-\eta_2+1)}\right) \\
&+\frac{\nabla_2+\nabla_1G^{-1}}{|\Phi|}\left(\frac{\varrho_2G^{2\rho+s_1-1}}{\upsilon\Gamma(\rho+s_1+1)}+\frac{(1-\varrho_2)G^{2\rho+s_2-1}}{\upsilon\Gamma(\rho+s_2+1)}\right)\Bigg] \\
&+\|(z,r)\|\Bigg[\frac{|\upsilon-1|\,G^{\rho-\theta}}{\upsilon\Gamma(\rho-\theta+1)}+\frac{|\upsilon-1|\left(\nabla_4+\nabla_3G^{-1}\right)}{|\Phi|}\left(\frac{\varrho_1G^{2\rho-\theta-\eta_1-1}}{\upsilon\Gamma(\rho-\theta-\eta_1+1)}+\frac{(1-\varrho_1)G^{2\rho-\theta-\eta_2-1}}{\upsilon\Gamma(\rho-\theta-\eta_2+1)}\right) \\
&+\frac{|\upsilon-1|\left(\nabla_2+\nabla_1G^{-1}\right)}{|\Phi|}\left(\frac{\varrho_2G^{2\rho-\theta+s_1-1}}{\upsilon\Gamma(\rho-\theta+s_1+1)}+\frac{(1-\varrho_2)G^{2\rho-\theta+s_2-1}}{\upsilon\Gamma(\rho-\theta+s_2+1)}\right) \\
&+\frac{1}{|\Phi|}\left[G^{\rho-1}\left(|\nabla_2\xi_2|+|\nabla_4\xi_1|\right)+G^{\rho-2}\left(|\nabla_1\xi_2|+|\nabla_3\xi_1|\right)\right] \\
\le\ & \|V\|\,\Lambda_1+y\Lambda_2+\frac{1}{|\Phi|}\left[G^{\rho-1}\left(|\nabla_2\xi_2|+|\nabla_4\xi_1|\right)+G^{\rho-2}\left(|\nabla_1\xi_2|+|\nabla_3\xi_1|\right)\right].
\end{aligned} \tag{3.20}$$

Analogously, using (3.18) and (3.19), we get

$$\begin{aligned}
&\left|\widetilde{\Omega}_2(z,r)(\tau)+\widehat{\Omega}_2(z,r)(\tau)\right| \\
\le\ & \|V^*\|\,\Lambda_1^*+y\Lambda_2^*+\frac{1}{|\Phi^*|}\left[G^{\rho-1}\left(\left|\nabla_2^*\xi_2^*\right|+\left|\nabla_4^*\xi_1^*\right|\right)+G^{\rho-2}\left(\left|\nabla_1^*\xi_2^*\right|+\left|\nabla_3^*\xi_1^*\right|\right)\right].
\end{aligned} \tag{3.21}$$

Combining (3.20) and (3.21), we obtain that

$$\begin{aligned}
|\Omega(z,r)+\Omega^*(z,r)| &\le \left|\widetilde{\Omega}_1(z,r)(\tau)+\widehat{\Omega}_1(z,r)(\tau)\right|+\left|\widetilde{\Omega}_2(z,r)(\tau)+\widehat{\Omega}_2(z,r)(\tau)\right| \\
&= \|V\|\,\Lambda_1+y\Lambda_2+\frac{1}{|\Phi|}\left[G^{\rho-1}\left(|\nabla_2\xi_2|+|\nabla_4\xi_1|\right)+G^{\rho-2}\left(|\nabla_1\xi_2|+|\nabla_3\xi_1|\right)\right] \\
&\quad+\|V^*\|\,\Lambda_1^*+y\Lambda_2^*+\frac{1}{|\Phi^*|}\left[G^{\rho-1}\left(\left|\nabla_2^*\xi_2^*\right|+\left|\nabla_4^*\xi_1^*\right|\right)+G^{\rho-2}\left(\left|\nabla_1^*\xi_2^*\right|+\left|\nabla_3^*\xi_1^*\right|\right)\right] \\
&= \widehat{V}\Lambda_3+\Lambda_4y+\frac{1}{|\Phi|}\left[G^{\rho-1}\left(|\nabla_2\xi_2|+|\nabla_4\xi_1|\right)+G^{\rho-2}\left(|\nabla_1\xi_2|+|\nabla_3\xi_1|\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|\Phi^*|}\left[G^{\rho-1}\left(\left|\nabla_2^*\xi_2^*\right|+\left|\nabla_4^*\xi_1^*\right|\right)+G^{\rho-2}\left(\left|\nabla_1^*\xi_2^*\right|+\left|\nabla_3^*\xi_1^*\right|\right)\right] \\
\leq\ & x.
\end{aligned}
$$

Thus, $\Omega(z,r)+\Omega^*(z,r)\in Q_x$. Hence the condition (1) of Theorem 2.1 is true. Next, we prove that $\Omega(z,r)$ is a contraction mapping. Let $(z,r),(\widetilde{z},\widetilde{r})\in Q_x$, then by (3.16), one has

$$
\begin{aligned}
&\left|\widetilde{\Omega}_1(z,r)(\tau)-\widetilde{\Omega}_1(\widetilde{z},\widetilde{r})(\tau)\right| \\
\leq\ & \frac{|\upsilon-1|}{\upsilon\Gamma(\rho-\theta)}\int_0^\tau(\tau-\hbar)^{\rho-\theta-1}\left|z(\hbar)-\widetilde{z}(\hbar)\right|d\hbar+\frac{G^{\rho-1}}{\Phi}\times \\
&+\left(\frac{\varrho_1\nabla_4|\upsilon-1|}{\upsilon}I^{\rho-\theta-\eta_1}\left|z(G)-\widetilde{z}(G)\right|-\frac{\varrho_2\nabla_2|\upsilon-1|}{\upsilon}I^{\rho-\theta+s_1}\left|z(G)-\widetilde{z}(G)\right|\right. \\
&\left.+\frac{\nabla_4(1-\varrho_1)|\upsilon-1|}{\upsilon}I^{\rho-\theta-\eta_2}\left|z(G)-\widetilde{z}(G)\right|-\frac{\nabla_2(1-\varrho_2)|\upsilon-1|}{\upsilon}I^{\rho-\theta+s_2}\left|z(G)-\widetilde{z}(G)\right|\right) \\
&+\frac{G^{\rho-2}}{\Phi}\left(\frac{\varrho_1\nabla_3|\upsilon-1|}{\upsilon}I^{\rho-\theta-\eta_1}\left|z(G)-\widetilde{z}(G)\right|-\frac{\varrho_2\nabla_1|\upsilon-1|}{\upsilon}I^{\rho-\theta+s_1}\left|z(G)-\widetilde{z}(G)\right|\right. \\
&\left.+\frac{\nabla_3(1-\varrho_1)|\upsilon-1|}{\upsilon}I^{\rho-\theta-\eta_2}\left|z(G)-\widetilde{z}(G)\right|-\frac{\nabla_1(1-\varrho_2)|\upsilon-1|}{\upsilon}I^{\rho-\theta+s_2}\left|z(G)-\widetilde{z}(G)\right|\right) \\
\leq\ & \Lambda_1\left\|z-z\right\|.
\end{aligned}
$$

Similarly, we can write

$$
\left|\widetilde{\Omega}_2(z,r)(\tau)-\widetilde{\Omega}_2(\widetilde{z},\widetilde{r})(\tau)\right|\leq\Lambda_1^*\left\|r-r\right\|.
$$

It follows that

$$
\begin{aligned}
\left\|\Omega(z,r)-\Omega(\widetilde{z},\widetilde{r})\right\|_{\mathfrak{I}} &\leq \left\|\widetilde{\Omega}_1(z,r)-\widetilde{\Omega}_1(\widetilde{z},\widetilde{r})\right\|_{\mathfrak{I}_1}+\left\|\widetilde{\Omega}_2(z,r)-\widetilde{\Omega}_2(\widetilde{z},\widetilde{r})\right\|_{\mathfrak{I}_2} \\
&= \Lambda_3\left\|(z,\widetilde{z})-(z,\widetilde{z})\right\|.
\end{aligned}
$$

Since $\Lambda_3<1$, then Ω_1 is a contraction mapping. Hence the condition (2) of Theorem 2.1 holds. The continuity of Ξ and Ξ^* lead to the continuity of Ω^*. If $(z,r)\in Q_x$, then

$$
\begin{aligned}
\left\|\widehat{\Omega}_1(z,r)\right\|_{\mathfrak{I}_1} &\leq \|V\|\left[\frac{G^\rho}{\upsilon\Gamma(\rho+1)}+\frac{\nabla_4+\nabla_3G^{-1}}{|\Phi|}\left(\frac{\varrho_1G^{2\rho-\eta_1-1}}{\upsilon\Gamma(\rho-\eta_1+1)}+\frac{(1-\varrho_1)G^{2\rho-\eta_2-1}}{\upsilon\Gamma(\rho-\eta_2+1)}\right)\right. \\
&\quad\left.+\frac{\nabla_2+\nabla_1G^{-1}}{|\Phi|}\left(\frac{\varrho_2G^{2\rho+s_1-1}}{\upsilon\Gamma(\rho+s_1+1)}+\frac{(1-\varrho_2)G^{2\rho+s_2-1}}{\upsilon\Gamma(\rho+s_2+1)}\right)\right. \\
&= \Lambda_2\left\|(z,r)\right\|.
\end{aligned}
$$

Similarly, we have

$$
\left\|\widehat{\Omega}_2(z,r)\right\|_{\mathfrak{I}_2}\leq\Lambda_2^*\left\|(z,r)\right\|.
$$

Hence,

$$
\left\|\Omega^*(z,r)\right\|_{\mathfrak{I}}\leq\left\|\widehat{\Omega}_1(z,r)\right\|_{\mathfrak{I}_1}+\left\|\widehat{\Omega}_1(z,r)\right\|_{\mathfrak{I}_2}\leq\Lambda_4\left\|(z,r)\right\|,\ \text{where } \Lambda_4=\Lambda_2+\Lambda_2^*.
$$

This means that Ω^* is a uniformly bounded operator on Q_x. Finally, we prove that the operator Ω^* is completely continuous. Set for $(z, r) \in Q_x$, $\sup_{\tau\in U} \Xi(\tau, z(\tau), r(\tau)) = R$, and $\sup_{\tau\in U} \Xi^*(\tau, z(\tau), r(\tau)) = R^*$. Then, for each $\tau_1, \tau_2 \in U$ with $\tau_1 < \tau_2$, we get

$$
\begin{aligned}
& \left|\widehat{\Omega}_1(z,r)(\tau_2) - \widehat{\Omega}_1(z,r)(\tau_1)\right| \\
& \left|\frac{1}{\upsilon\Gamma(\rho)}\int_0^{\tau_2}(\tau_2-\hbar)^{\rho-1}\Xi(\hbar,z(\hbar),r(\hbar))d\hbar - \frac{1}{\upsilon\Gamma(\rho)}\int_0^{\tau_1}(\tau_1-\hbar)^{\rho-1}\Xi(\hbar,z(\hbar),r(\hbar))d\hbar\right. \\
& \frac{\tau_2^{\rho-1}-\tau_1^{\rho-1}}{\Phi}\left(\frac{\varrho_1\nabla_4}{\upsilon}I^{\rho-\eta_1}\Xi(G,z(G),r(G)) - \frac{\varrho_2\nabla_2}{\upsilon}I^{\rho+s_1}\Xi(G,z(G),r(G)) + \nabla_2\xi_2 - \nabla_4\xi_1\right. \\
& \left.+\frac{(1-\varrho_1)\nabla_4}{\upsilon}I^{\rho-\eta_2}\Xi(G,z(G),r(G)) - \frac{(1-\varrho_2)\nabla_2}{\upsilon}I^{\rho+s_2}\Xi(G,z(G),r(G))\right) \\
& -\frac{\tau_2^{\rho-1}-\tau_1^{\rho-1}}{\Phi}\left(\frac{\varrho_1\nabla_3}{\upsilon}I^{\rho-\eta_1}\Xi(G,z(G),r(G)) - \frac{\varrho_2\nabla_1}{\upsilon}I^{\rho+s_1}\Xi(G,z(G),r(G))\right. \\
& \left.\left.\frac{(1-\varrho_1)\nabla_3}{\upsilon}I^{\rho-\eta_2}\Xi(G,z(G),r(G)) - \frac{(1-\varrho_2)\nabla_1}{\upsilon}I^{\rho+s_2}\Xi(G,z(G),r(G)) + \nabla_1\xi_2 - \nabla_3\xi_1\right)\right| \\
\leq\; & \frac{R\left[2(\tau_2-\tau_1)^\rho + \left|\tau_2^\rho - \tau_1^\rho\right|\right]}{\upsilon\Gamma(\rho+1)} + \frac{\tau_2^{\rho-1}-\tau_1^{\rho-1}}{|\Phi|}\left[\frac{R\varrho_1\nabla_4 I^{\rho-\eta_1}}{\upsilon\Gamma(\rho-\eta_1+1)} + \frac{R\varrho_2\nabla_2 I^{\rho+s_1}}{\upsilon\Gamma(\rho+s_1+1)} + |\nabla_2\xi_2| + |\nabla_4\xi_1|\right. \\
& \left.+\frac{R(1-\varrho_1)\nabla_4 I^{\rho-\eta_2}}{\upsilon\Gamma(\rho-\eta_2+1)} + \frac{R(1-\varrho_2)\nabla_2 I^{\rho+s_2}}{\upsilon\Gamma(\rho+s_2+1)}\right] + \frac{\tau_2^{\rho-2}-\tau_1^{\rho-2}}{|\Phi|}\left[\frac{R\varrho_1\nabla_3 I^{\rho-\eta_1}}{\upsilon\Gamma(\rho-\eta_1+1)} + \frac{R\varrho_2\nabla_1 I^{\rho+s_1}}{\upsilon\Gamma(\rho+s_1+1)}\right. \\
& \left.+\frac{R(1-\varrho_1)\nabla_3 I^{\rho-\eta_2}}{\upsilon\Gamma(\rho-\eta_2+1)} + \frac{R(1-\varrho_2)\nabla_1 I^{\rho+s_2}}{\upsilon\Gamma(\rho+s_2+1)} + |\nabla_1\xi_2| + |\nabla_3\xi_1|\right],
\end{aligned}
$$

which implies that

$$\left|\widehat{\Omega}_1(z,r)(\tau_2) - \widehat{\Omega}_1(z,r)(\tau_1)\right| \to 0 \text{ as } \tau_1 \to \tau_2.$$

Similarly

$$\left|\widehat{\Omega}_2(z,r)(\tau_2) - \widehat{\Omega}_2(z,r)(\tau_1)\right| \to 0 \text{ as } \tau_1 \to \tau_2.$$

Hence

$$
\begin{aligned}
|\Omega^*(z,r)(\tau_2) - \Omega^*(z,r)(\tau_1)| &\leq \left|\widehat{\Omega}_1(z,r)(\tau_2) - \widehat{\Omega}_1(z,r)(\tau_1)\right| + \left|\widehat{\Omega}_2(z,r)(\tau_2) - \widehat{\Omega}_2(z,r)(\tau_1)\right| \\
&\to 0 \text{ as } \tau_1 \to \tau_2,
\end{aligned}
$$

which yields that Ω^* is equicontinuous, and so Ω^* is relatively compact on Q_x. Since every compact operator is completely continuous, then by the Arzela-Ascoli theorem, Ω^* is completely continuous. Thus, condition (3) of Theorem 2.1 is satisfied. Hence, all conditions of Theorem 2.1 are satisfied. Consequently, the CBVP (1.4) has at least one solution.

4. Hyers-Ulam stability

In this part, we discuss the Hyers–Ulam stability of the CBVP

$$\begin{cases} \upsilon D^{\rho}(z(\tau)) + (1-\upsilon)D^{\theta}(z(\tau)) = \Xi(\tau, z(\tau), r(\tau)), \\ \upsilon^* D^{\rho^*}(r(\tau)) + (1-\upsilon^*)D^{\theta^*}(r(\tau)) = \Xi^*(\tau, r(\tau), z(\tau)), \\ z(0) = 0,\ r(0) = 0,\ D^{\eta_1} z(G) = \xi_1,\ D^{\eta_1^*} r(G) = \xi_1^*, \\ I^{s_1} z(G) = \xi_2,\ \ I^{s_1^*} r(G) = \xi_2^*, \end{cases} \tag{4.1}$$

for each $\tau \in [0, G]$ and $\rho \in [2, 3)$. The CBVP (4.1) is a special case of (1.4) when we take $\varrho_1 = \varrho_2 = 1$ and $\varrho_1^* = \varrho_2^* = 1$.

Definition 4.1. The CBVP (4.1) is called HU stable if there is a positive constant $\widehat{\Delta} > 0$ so that, for each $\epsilon, \epsilon^* > 0$ and $(z, r) \in \mathfrak{I}$ as a solution to the inequalities

$$\begin{cases} \left|\upsilon D^{\rho}(z(\tau)) + (\upsilon - 1)D^{\theta}(z(\tau)) - \Xi(\tau, z(\tau), r(\tau))\right| \le \epsilon, \\ \left|\upsilon^* D^{\rho^*}(r(\tau)) + (\upsilon^* - 1)D^{\theta^*}(r(\tau)) - \Xi^*(\tau, r(\tau), z(\tau))\right| \le \epsilon^*, \end{cases}$$

there is a US $(\widetilde{z}, \widetilde{r}) \in \mathfrak{I}$ with

$$\|(z, r) - (\widetilde{z}, \widetilde{r})\|_{\mathfrak{I}} \le \widehat{\Delta\epsilon}, \text{ for all } \upsilon \in U,$$

where $\widehat{\epsilon} = \max\{\epsilon, \epsilon^*\}$.

Theorem 4.1. *Assume that $\Xi, \Xi^* : U \times \mathbb{R}^2 \to \mathbb{R}$ are continuous maps and there are constants $T_\Xi, \widetilde{T}_\Xi, T_{\Xi^*}, \widetilde{T}_{\Xi^*} > 0$ so that*

$$|\Xi(\tau, z_1(\tau), z_2(\tau)) - \Xi(\tau, \widetilde{z}_1(\tau), \widetilde{z}_2(\tau))| \le T_\Xi |z_1 - \widetilde{z}_1| + \widetilde{T}_\Xi |z_2 - \widetilde{z}_2|,$$

and

$$|\Xi^*(\tau, r_1(\tau), r_2(\tau)) - \Xi^*(\tau, \widetilde{r}_1(\tau), \widetilde{r}_2(\tau))| \le T_{\Xi^*} |r_1 - \widetilde{r}_1| + \widetilde{T}_{\Xi^*} |r_2 - \widetilde{r}_2|,$$

for all $\tau \in U$ and $z_1, z_2, \widetilde{z}_1, \widetilde{z}_2, r_1, r_2, \widetilde{r}_1, \widetilde{r}_2 \in \mathbb{R}$. Then, the CBVP (4.1) is HU stable provided that $\beth = 1 - \frac{\eth\eth^*}{(1-\wp)(1-\wp^*)} > 0$.

Proof. Let $\epsilon, \epsilon^* > 0$ and $(z, r) \in \mathfrak{I}$ be so that

$$\begin{cases} \left|\upsilon D^{\rho}(z(\tau)) + (\upsilon - 1)D^{\theta}(z(\tau)) - \Xi(\tau, z(\tau), r(\tau))\right| \le \epsilon, \\ \left|\upsilon^* D^{\rho^*}(r(\tau)) + (\upsilon^* - 1)D^{\theta^*}(r(\tau)) - \Xi^*(\tau, r(\tau), z(\tau))\right| \le \epsilon^*. \end{cases}$$

Choose the functions ζ and ζ^* satisfying

$$\begin{cases} \upsilon D^{\rho}(z(\tau)) + (\upsilon - 1)D^{\theta}(z(\tau)) = \Xi(\tau, z(\tau), r(\tau)) + \zeta(\tau), \\ \upsilon^* D^{\rho^*}(r(\tau)) + (\upsilon^* - 1)D^{\theta^*}(r(\tau)) = \Xi^*(\tau, r(\tau), z(\tau)) + \zeta^*(\tau), \end{cases}$$

such that $|\zeta(\tau)| \le \epsilon$ and $|\zeta^*(\tau)| \le \epsilon^*$ for all $\tau \in U$. Then, we get

$$\begin{aligned} z(\tau) \;\;=\;\; & \frac{\upsilon - 1}{\upsilon} I^{\rho-\theta} z(\tau) + \frac{1}{\upsilon} I^{\rho} \Xi(\tau, z(\tau), r(\tau)) + \frac{1}{\upsilon} I^{\rho} \zeta(\tau) + \frac{\tau^{\rho-1}}{\Phi} \Bigg(\frac{\nabla_4(\upsilon - 1)}{\upsilon} I^{\rho-\theta-\eta_1} z(G) \\ & - \frac{\nabla_2(\upsilon - 1)}{\upsilon} I^{\rho-\theta+s_1} z(G) + \frac{\nabla_4}{\upsilon} I^{\rho-\eta_1} \Xi(G, z(G), r(G)) - \frac{\nabla_2}{\upsilon} I^{\rho+s_1} \Xi(G, z(G), r(G)) \end{aligned}$$

$$
+\frac{\nabla_4}{\upsilon}I^{\rho-\eta_1}\zeta(G)-\frac{\nabla_2}{\upsilon}I^{\rho+s_1}\zeta(G)+\nabla_2\xi_2-\nabla_4\xi_1\bigg)+\frac{\tau^{\rho-2}}{\Phi}\bigg(\frac{\nabla_3(\upsilon-1)}{\upsilon}I^{\rho-\theta-\eta_1}z(G)
$$

$$
-\frac{\nabla_1(\upsilon-1)}{\upsilon}I^{\rho-\theta+s_1}z(G)+\frac{\nabla_3}{\upsilon}I^{\rho-\eta_1}\Xi(G,z(G),r(G))-\frac{\nabla_1}{\upsilon}I^{\rho+s_1}\Xi(G,z(G),r(G))
$$

$$
+\frac{\nabla_3}{\upsilon}I^{\rho-\eta_1}\zeta(G)-\frac{\nabla_1}{\upsilon}I^{\rho+s_1}\zeta(G)+\nabla_1\xi_2-\nabla_3\xi_1\bigg),
$$

and

$$
\begin{aligned}
r(\tau) \;=\;& \frac{\upsilon^*-1}{\upsilon^*}I^{\rho^*-\theta^*}r(\tau)+\frac{1}{\upsilon^*}I^{\rho^*}\Xi^*(\tau,r(\tau),z(\tau))+\frac{1}{\upsilon}I^{\rho^*}\zeta^*(\tau)+\frac{\tau^{\rho^*-1}}{\Phi^*}\bigg(\frac{\nabla_4^*(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*-\eta_1^*}r(G)\\
&-\frac{\nabla_2^*(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*+s_1^*}r(G)+\frac{\nabla_4^*}{\upsilon^*}I^{\rho^*-\eta_1^*}\Xi^*(G,r(G),z(G))-\frac{\nabla_2^*}{\upsilon^*}I^{\rho^*+s_1^*}\Xi^*(G,r(G),z(G))\\
&+\frac{\nabla_4^*}{\upsilon^*}I^{\rho^*-\eta_1^*}\zeta^*(G)-\frac{\nabla_2^*}{\upsilon^*}I^{\rho^*+s_1^*}\zeta^*(G)+\nabla_2^*\xi_2^*-\nabla_4^*\xi_1^*\bigg)-\frac{\tau^{\rho^*-2}}{\Phi^*}\bigg(\frac{\nabla_3^*(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*-\eta_1^*}r(G)\\
&-\frac{\nabla_1^*(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*+s_1^*}r(G)+\frac{\nabla_3^*}{\upsilon^*}I^{\rho^*-\eta_1^*}\Xi^*(G,r(G),z(G))-\frac{\nabla_1^*}{\upsilon^*}I^{\rho^*+s_1^*}\Xi^*(G,r(G),z(G))\\
&+\frac{\nabla_3^*}{\upsilon^*}I^{\rho^*-\eta_1^*}\zeta^*(G)-\frac{\nabla_1^*}{\upsilon^*}I^{\rho^*+s_1^*}\zeta^*(G)+\nabla_1^*\xi_2^*-\nabla_3^*\xi_1^*\bigg).
\end{aligned}
$$

Let $(\widetilde{z},\widetilde{r})$ be a US of the CBVP (4.1), then $\widetilde{z}(\tau)$ and $\widetilde{r}(\tau)$ are given by

$$
\begin{aligned}
\widetilde{z}(\tau) \;=\;& \frac{\upsilon-1}{\upsilon}I^{\rho-\theta}\widetilde{z}(\tau)+\frac{1}{\upsilon}I^{\rho}\Xi(\tau,\widetilde{z}(\tau),\widetilde{r}(\tau))+\frac{1}{\upsilon}I^{\rho}\zeta(\tau)+\frac{\tau^{\rho-1}}{\Phi}\bigg(\frac{\nabla_4(\upsilon-1)}{\upsilon}I^{\rho-\theta-\eta_1}\widetilde{z}(G)\\
&-\frac{\nabla_2(\upsilon-1)}{\upsilon}I^{\rho-\theta+s_1}\widetilde{z}(G)+\frac{\nabla_4}{\upsilon}I^{\rho-\eta_1}\Xi(\tau,\widetilde{z}(\tau),\widetilde{r}(\tau))-\frac{\nabla_2}{\upsilon}I^{\rho+s_1}\Xi(\tau,\widetilde{z}(\tau),\widetilde{r}(\tau))\\
&+\frac{\nabla_4}{\upsilon}I^{\rho-\eta_1}\zeta(G)-\frac{\nabla_2}{\upsilon}I^{\rho+s_1}\zeta(G)+\nabla_2\xi_2-\nabla_4\xi_1\bigg)+\frac{\tau^{\rho-2}}{\Phi}\bigg(\frac{\nabla_3(\upsilon-1)}{\upsilon}I^{\rho-\theta-\eta_1}\widetilde{z}(G)\\
&-\frac{\nabla_1(\upsilon-1)}{\upsilon}I^{\rho-\theta+s_1}\widetilde{z}(G)+\frac{\nabla_3}{\upsilon}I^{\rho-\eta_1}\Xi(\tau,\widetilde{z}(\tau),\widetilde{r}(\tau))-\frac{\nabla_1}{\upsilon}I^{\rho+s_1}\Xi(\tau,\widetilde{z}(\tau),\widetilde{r}(\tau))\\
&+\frac{\nabla_3}{\upsilon}I^{\rho-\eta_1}\zeta(G)-\frac{\nabla_1}{\upsilon}I^{\rho+s_1}\zeta(G)+\nabla_1\xi_2-\nabla_3\xi_1\bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
&\widetilde{r}(\tau)\\
=\;& \frac{\upsilon^*-1}{\upsilon^*}I^{\rho^*-\theta^*}\widetilde{r}(\tau)+\frac{1}{\upsilon^*}I^{\rho^*}\Xi^*(\tau,\widetilde{r}(\tau),\widetilde{z}(\tau))+\frac{1}{\upsilon}I^{\rho^*}\zeta^*(\tau)+\frac{\tau^{\rho^*-1}}{\Phi^*}\bigg(\frac{\nabla_4^*(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*-\eta_1^*}\widetilde{r}(G)\\
&-\frac{\nabla_2^*(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*+s_1^*}\widetilde{r}(G)+\frac{\nabla_4^*}{\upsilon^*}I^{\rho^*-\eta_1^*}\Xi^*(\tau,\widetilde{r}(\tau),\widetilde{z}(\tau))-\frac{\nabla_2^*}{\upsilon^*}I^{\rho^*+s_1^*}\Xi^*(\tau,\widetilde{r}(\tau),\widetilde{z}(\tau))\\
&+\frac{\nabla_4^*}{\upsilon^*}I^{\rho^*-\eta_1^*}\zeta^*(G)-\frac{\nabla_2^*}{\upsilon^*}I^{\rho^*+s_1^*}\zeta^*(G)+\nabla_2^*\xi_2^*-\nabla_4^*\xi_1^*\bigg)-\frac{\tau^{\rho^*-2}}{\Phi^*}\bigg(\frac{\nabla_3^*(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*-\eta_1^*}\widetilde{r}(G)\\
&-\frac{\nabla_1^*(\upsilon^*-1)}{\upsilon^*}I^{\rho^*-\theta^*+s_1^*}\widetilde{r}(G)+\frac{\nabla_3^*}{\upsilon^*}I^{\rho^*-\eta_1^*}\Xi^*(\tau,\widetilde{r}(\tau),\widetilde{z}(\tau))-\frac{\nabla_1^*}{\upsilon^*}I^{\rho^*+s_1^*}\Xi^*(\tau,\widetilde{r}(\tau),\widetilde{z}(\tau))
\end{aligned}
$$

$$+\frac{\nabla_3^*}{\upsilon^*}I^{\rho^*-\eta_1^*}\zeta^*(G)-\frac{\nabla_1^*}{\upsilon^*}I^{\rho^*+s_1^*}\zeta^*(G)+\nabla_1^*\xi_2^*-\nabla_3^*\xi_1^*\Bigg).$$

Hence,

$$\begin{aligned}
&|z(\tau)-\widetilde{z}(\tau)|\\
\leq\ & \frac{|\upsilon-1|}{\upsilon}I^{\rho-\theta}\left|z(\tau)-\widetilde{z}(\tau)\right|+\frac{1}{\upsilon}I^{\rho}\left|\Xi(\tau,z(\tau),r(\tau))-\Xi(\tau,\widetilde{z}(\tau),\widetilde{r}(\tau))\right|\\
&+\frac{G^{\rho-1}}{|\Phi|}\Bigg(\frac{\nabla_4(\upsilon-1)}{\upsilon}I^{\rho-\theta-\eta_1}\left|z(G)-\widetilde{z}(G)\right|+\frac{\nabla_2(\upsilon-1)}{\upsilon}I^{\rho-\theta+s_1}\left|z(G)-\widetilde{z}(G)\right|\\
&+\frac{\nabla_4}{\upsilon}I^{\rho-\eta_1}\left|\Xi(G,z(G),r(G))-\Xi(G,\widetilde{z}(G),\widetilde{r}(G))\right|+\frac{\nabla_2}{\upsilon}I^{\rho+s_1}\left|\Xi(G,z(G),r(G))-\Xi(G,\widetilde{z}(G),\widetilde{r}(G))\right|\Bigg)\\
&+\frac{G^{\rho-2}}{|\Phi|}\Bigg(\frac{\nabla_3(\upsilon-1)}{\upsilon}I^{\rho-\theta-\eta_1}\left|z(G)-\widetilde{z}(G)\right|+\frac{\nabla_1(\upsilon-1)}{\upsilon}I^{\rho-\theta+s_1}\left|z(G)-\widetilde{z}(G)\right|\\
&+\frac{\nabla_3}{\upsilon}I^{\rho-\eta_1}\left|\Xi(G,z(G),r(G))-\Xi(G,\widetilde{z}(G),\widetilde{r}(G))\right|\\
&+\frac{\nabla_1}{\upsilon}I^{\rho+s_1}\left|\Xi(G,z(G),r(G))-\Xi(G,\widetilde{z}(G),\widetilde{r}(G))\right|\Bigg)+\frac{1}{\upsilon}I^{\rho}\left|\zeta(\tau)\right|\\
&+\frac{G^{\rho-1}}{|\Phi|}\left(\frac{\nabla_4}{\upsilon}I^{\rho-\eta_1}\left|\zeta(G)\right|+\frac{\nabla_2}{\upsilon}I^{\rho+s_1}\left|\zeta(G)\right|\right)+\frac{G^{\rho-2}}{|\Phi|}\left(\frac{\nabla_3}{\upsilon}I^{\rho-\eta_1}\left|\zeta(G)\right|+\frac{\nabla_1}{\upsilon}I^{\rho+s_1}\left|\zeta(G)\right|\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
\|z(\tau)-\widetilde{z}(\tau)\| \leq\ & \Bigg(\frac{G^{\rho-\theta}|\upsilon-1|}{\upsilon\Gamma(\rho-\theta+1)}+\frac{TG^{\rho-1}}{\upsilon\Gamma(\rho)}+\frac{T}{\upsilon\Gamma(\rho+1)}+\frac{\nabla_4(\upsilon-1)G^{2\rho-\theta-\eta_1-1}}{\upsilon|\Phi|\Gamma(\rho-\theta-\eta_1+1)}\\
&+\frac{\nabla_2(\upsilon-1)G^{2\rho-\theta+s_1-1}}{\upsilon|\Phi|\Gamma(\rho-\theta+s_1+1)}+\frac{T\nabla_4G^{2\rho-\eta_1-1}}{\upsilon|\Phi|\Gamma(\rho-\eta_1+1)}+\frac{T\nabla_2G^{2\rho+s_1-1}}{\upsilon|\Phi|\Gamma(\rho+s_1+1)}\\
&+\frac{\nabla_3(\upsilon-1)G^{2\rho-\theta-\eta_1-2}}{\upsilon|\Phi|\Gamma(\rho-\theta-\eta_1+1)}+\frac{\nabla_1(\upsilon-1)G^{2\rho-\theta+s_1-2}}{\upsilon|\Phi|\Gamma(\rho-\theta+s_1+1)}+\frac{\nabla_3G^{2\rho-\theta-\eta_1-2}}{\upsilon|\Phi|\Gamma(\rho-\theta-\eta_1+1)}\\
&+\frac{\nabla_1G^{2\rho-\theta+s_1-2}}{\upsilon|\Phi|\Gamma(\rho-\theta+s_1+1)}\Bigg)\|z(\tau)-\widetilde{z}(\tau)\|\\
&+\Bigg(\frac{TG^{\rho-1}}{\upsilon\Gamma(\rho)}+\frac{T\nabla_4G^{2\rho-\eta_1-1}}{\upsilon|\Phi|\Gamma(\rho-\eta_1+1)}+\frac{T\nabla_2G^{2\rho+s_1-1}}{\upsilon|\Phi|\Gamma(\rho+s_1+1)}+\frac{\nabla_3G^{2\rho-\theta-\eta_1-2}}{\upsilon|\Phi|\Gamma(\rho-\theta-\eta_1+1)}\\
&\frac{\nabla_1G^{2\rho-\theta+s_1-2}}{\upsilon|\Phi|\Gamma(\rho-\theta+s_1+1)}\Bigg)\|r(\tau)-\widetilde{r}(\tau)\|+\\
&\frac{\varepsilon G^{\rho}}{\Gamma(\rho+1)}+\frac{\varepsilon G^{\rho-1}}{|\Phi|}\left(\frac{\nabla_4G^{\rho-\eta_1}}{\upsilon|\Phi|\Gamma(\rho-\eta_1+1)}+\frac{\nabla_2G^{\rho+s_1}}{\upsilon|\Phi|\Gamma(\rho+s_1+1)}\right)\\
&+\frac{\varepsilon G^{\rho-2}}{|\Phi|}\left(\frac{\nabla_3G^{\rho-\eta_1}}{\upsilon|\Phi|\Gamma(\rho-\eta_1+1)}+\frac{\nabla_1G^{\rho+s_1}}{\upsilon|\Phi|\Gamma(\rho+s_1+1)}\right),
\end{aligned}$$

where $T=\max\{T_\Xi,\widetilde{T}_\Xi\}$. For simplicity, we consider

$$\wp = \frac{G^{\rho-\theta}|\upsilon-1|}{\upsilon\Gamma(\rho-\theta+1)}+\frac{T}{\upsilon\Gamma(\rho+1)}+\frac{\nabla_4(\upsilon-1)G^{2\rho-\theta-\eta_1-1}}{\upsilon|\Phi|\Gamma(\rho-\theta-\eta_1+1)}+\frac{\nabla_2(\upsilon-1)G^{2\rho-\theta+s_1-1}}{\upsilon|\Phi|\Gamma(\rho-\theta+s_1+1)}$$

$$+\frac{\nabla_3(\upsilon-1)G^{2\rho-\theta-\eta_1-2}}{\upsilon\,|\Phi|\,\Gamma(\rho-\theta-\eta_1+1)}+\frac{\nabla_1(\upsilon-1)G^{2\rho-\theta+s_1-2}}{\upsilon\,|\Phi|\,\Gamma(\rho-\theta+s_1+1)}+\Game,$$

$$\begin{aligned}\Game &= \frac{TG^{\rho-1}}{\upsilon\Gamma(\rho)}+\frac{T\nabla_4G^{2\rho-\eta_1-1}}{\upsilon\,|\Phi|\,\Gamma(\rho-\eta_1+1)}+\frac{T\nabla_2G^{2\rho+s_1-1}}{\upsilon\,|\Phi|\,\Gamma(\rho+s_1+1)}\\ &\quad+\frac{\nabla_3G^{2\rho-\theta-\eta_1-2}}{\upsilon\,|\Phi|\,\Gamma(\rho-\theta-\eta_1+1)}+\frac{\nabla_1G^{2\rho-\theta+s_1-2}}{\upsilon\,|\Phi|\,\Gamma(\rho-\theta+s_1+1)},\end{aligned}$$

and

$$\begin{aligned}S &= \frac{G^{\rho}}{\Gamma(\rho+1)}+\frac{G^{\rho-1}}{\upsilon\,|\Phi|}\left(\frac{\nabla_4G^{\rho-\eta_1}}{\Gamma(\rho-\eta_1+1)}+\frac{\nabla_2G^{\rho+s_1}}{\Gamma(\rho+s_1+1)}\right)\\ &\quad+\frac{G^{\rho-2}}{\upsilon\,|\Phi|}\left(\frac{\nabla_3G^{\rho-\eta_1}}{\Gamma(\rho-\eta_1+1)}+\frac{\nabla_1G^{\rho+s_1}}{\Gamma(\rho+s_1+1)}\right).\end{aligned}$$

It follows that

$$\|z(\tau)-\widetilde{z}(\tau)\|_{\mathfrak{J}_1}-\frac{\Game}{1-\wp}\,\|r(\tau)-\widetilde{r}(\tau)\|_{\mathfrak{J}_2}\le\frac{S\,\varepsilon}{1-\wp}. \tag{4.2}$$

Similarly, one can obtain under $T^*=\max\{T_{\Xi^*},\widetilde{T}_{\Xi^*}\}$ and $|\zeta^*(\tau)|\le\epsilon^*$ that

$$\|r(\tau)-\widetilde{r}(\tau)\|_{\mathfrak{J}_2}-\frac{\Game^*}{1-\wp^*}\,\|z(\tau)-\widetilde{z}(\tau)\|_{\mathfrak{J}_1}\le\frac{S^*\varepsilon^*}{1-\wp^*}, \tag{4.3}$$

where

$$\begin{aligned}\wp^* &= \frac{G^{\rho^*-\theta^*}\,|\upsilon^*-1|}{\upsilon^*\Gamma(\rho^*-\theta^*+1)}+\frac{T^*}{\upsilon^*\Gamma(\rho^*+1)}+\frac{\nabla_4^*(\upsilon^*-1)G^{2\rho^*-\theta^*-\eta_1^*-1}}{\upsilon^*\,|\Phi^*|\,\Gamma(\rho^*-\theta^*-\eta_1^*+1)}\\ &\quad+\frac{\nabla_2^*(\upsilon^*-1)G^{2\rho^*-\theta^*+s_1^*-1}}{\upsilon^*\,|\Phi^*|\,\Gamma(\rho^*-\theta^*+s_1^*+1)}+\frac{\nabla_3^*(\upsilon^*-1)G^{2\rho^*-\theta^*-\eta_1^*-2}}{\upsilon^*\,|\Phi^*|\,\Gamma(\rho^*-\theta^*-\eta_1^*+1)}\\ &\quad+\frac{\nabla_1^*(\upsilon^*-1)G^{2\rho^*-\theta^*+s_1^*-2}}{\upsilon^*\,|\Phi^*|\,\Gamma(\rho^*-\theta^*+s_1^*+1)}+\Game^*,\end{aligned}$$

$$\begin{aligned}\Game^* &= \frac{T^*G^{\rho^*-1}}{\upsilon^*\Gamma(\rho^*)}+\frac{T^*\nabla_4^*G^{2\rho^*-\eta_1^*-1}}{\upsilon^*\,|\Phi^*|\,\Gamma(\rho^*-\eta_1^*+1)}+\frac{T^*\nabla_2^*G^{2\rho^*+s_1^*-1}}{\upsilon^*\,|\Phi^*|\,\Gamma(\rho^*+s_1^*+1)}\\ &\quad+\frac{\nabla_3^*G^{2\rho^*-\theta^*-\eta_1^*-2}}{\upsilon^*\,|\Phi^*|\,\Gamma(\rho^*-\theta^*-\eta_1^*+1)}+\frac{\nabla_3^*G^{2\rho^*-\theta^*-\eta_1^*-2}}{\upsilon^*\,|\Phi^*|\,\Gamma(\rho^*-\theta^*-\eta_1^*+1)},\end{aligned}$$

and

$$\begin{aligned}S^* &= \frac{G^{\rho^*}}{\upsilon^*\Gamma(\rho^*+1)}+\frac{G^{\rho^*-1}}{\upsilon^*\,|\Phi^*|}\left(\frac{\nabla_4^*G^{\rho^*-\eta_{*1}}}{\Gamma(\rho^*-\eta_1^*+1)}+\frac{\nabla_2^*G^{\rho^*+s_1^*}}{\Gamma(\rho^*+s_1^*+1)}\right)\\ &\quad+\frac{G^{\rho^*-2}}{\upsilon^*\,|\Phi^*|}\left(\frac{\nabla_3^*G^{\rho^*-\eta_1^*}}{\Gamma(\rho^*-\eta_1^*+1)}+\frac{\nabla_1G^{\rho^*+s_1^*}}{\Gamma(\rho^*+s_1^*+1)}\right).\end{aligned}$$

Inequalities (4.2) and (4.3) can be written as

$$\begin{bmatrix} 1 & -\frac{\eth}{1-\wp} \\ -\frac{\eth^*}{1-\wp^*} & 1 \end{bmatrix} \begin{bmatrix} \|z(\tau)-\widetilde{z}(\tau)\|_{\mathfrak{I}_1} \\ \|r(\tau)-\widetilde{r}(\tau)\|_{\mathfrak{I}_2} \end{bmatrix} \leq \begin{bmatrix} \frac{S\varepsilon}{1-\wp} \\ \frac{S^*\varepsilon^*}{1-\wp^*} \end{bmatrix}.$$

Hence

$$\begin{bmatrix} \|z(\tau)-\widetilde{z}(\tau)\|_{\mathfrak{I}_1} \\ \|r(\tau)-\widetilde{r}(\tau)\|_{\mathfrak{I}_2} \end{bmatrix} \leq \begin{bmatrix} \frac{1}{\beth} & \frac{\eth}{1-\wp}\frac{1}{\beth} \\ \frac{\eth^*}{1-\wp^*}\frac{1}{\beth} & \frac{1}{\beth} \end{bmatrix} \begin{bmatrix} \frac{S\varepsilon}{1-\wp} \\ \frac{S^*\varepsilon^*}{1-\wp^*} \end{bmatrix}, \tag{4.4}$$

where $\beth = 1 - \frac{\eth\eth^*}{(1-\wp)(1-\wp^*)} > 0$. Based on System (4.4), one can write

$$\|z(\tau)-\widetilde{z}(\tau)\|_{\mathfrak{I}_1} \leq \frac{1}{\beth}\frac{S\varepsilon}{1-\wp} + \frac{\eth S^*\varepsilon^*}{(1-\wp^*)(1-\wp)}\frac{1}{\beth},$$

and

$$\|r(\tau)-\widetilde{r}(\tau)\|_{\mathfrak{I}_2} \leq \frac{\eth^* S\varepsilon}{(1-\wp^*)(1-\wp)}\frac{1}{\beth} + \frac{1}{\beth}\frac{S^*\varepsilon^*}{1-\wp^*},$$

which implies that

$$\begin{aligned} \|z(\tau)-\widetilde{z}(\tau)\|_{\mathfrak{I}_1} + \|r(\tau)-\widetilde{r}(\tau)\|_{\mathfrak{I}_2} \quad \leq \quad & \frac{1}{\beth}\frac{S\varepsilon}{1-\wp} + \frac{1}{\beth}\frac{S^*\varepsilon^*}{1-\wp^*} \\ & + \frac{\eth S^*\varepsilon^*}{(1-\wp^*)(1-\wp)}\frac{1}{\beth} + \frac{\eth^* S\varepsilon}{(1-\wp^*)(1-\wp)}\frac{1}{\beth}. \end{aligned}$$

Let us consider $\widehat{\varepsilon} = \max\{\varepsilon, \varepsilon^*\}$ and

$$\widehat{\Delta} = \frac{1}{\beth}\frac{S}{1-\wp} + \frac{1}{\beth}\frac{S^*}{1-\wp^*} + \frac{\eth S^*}{(1-\wp^*)(1-\wp)}\frac{1}{\beth} + \frac{\eth^* S}{(1-\wp^*)(1-\wp)}\frac{1}{\beth} > 0.$$

Then, we have

$$\|(z,r)-(\widetilde{z},\widetilde{r})\|_{\mathfrak{I}} \leq \widehat{\Delta\epsilon}, \text{ for all } \upsilon \in U,$$

which yields that the CBVP (4.1) is HU stable. This completes the required proof.

5. Supportive example

Example 5.1. Consider the CBVP

$$\begin{cases} \frac{57}{64}D^{2.6}(z(\tau)) + \frac{7}{64}D^{2.1}(z(\tau)) = \tau^2\left[\sin z(\tau) + \cos r(\tau)\right], \ \tau \in [0, \frac{1}{5}], \\ \frac{47}{54}D^{2.7}(z(\tau)) + \frac{7}{54}D^{2.2}(z(\tau)) = \tau^2\left[\sin r(\tau) + \cos z(\tau)\right], \ \tau \in [0, \frac{1}{5}], \\ z(0) = 0, \ \ \varrho_1 D^{\frac{1}{4}} z(\frac{1}{5}) + (1-\varrho_1)\, D^{\frac{1}{8}} z(\frac{1}{5}) = \frac{1}{18}, \\ r(0) = 0, \ \ \varrho_1^* D^{\frac{1}{3}} r(\frac{1}{5}) + (1-\varrho_{*1}) D^{\frac{1}{6}} r(\frac{1}{5}) = \frac{1}{16}, \\ \varrho_2 I^{\frac{4}{5}} z(\frac{1}{5}) + (1-\varrho_2) I^{\frac{5}{3}} z(\frac{1}{5}) = \frac{5}{13}, \ \ \varrho_2^* I^{\frac{3}{4}} r(\frac{1}{5}) + (1-\varrho_2^*) I^{\frac{7}{3}} r(\frac{1}{5}) = \frac{5}{17}. \end{cases} \tag{5.1}$$

where $\rho = 2.6$, $\theta = 2.1$, $\rho^* = 2.7$, $\theta^* = 2.2$, $\upsilon = \frac{57}{64}$, $\upsilon^* = \frac{47}{54}$, $\eta_1 = \frac{1}{4}$, $\eta_1^* = \frac{1}{3}$, $\eta_2 = \frac{1}{8}$, $\eta_2^* = \frac{1}{6}$, $s_1 = \frac{4}{5}$, $s_1^* = \frac{3}{4}$, $s_2 = \frac{5}{3}$, $s_2^* = \frac{7}{3}$, $\xi_1 = \frac{1}{18}$, $\xi_1^* = \frac{1}{16}$, $\xi_2 = \frac{5}{13}$, $\xi_1^* = \frac{5}{17}$ and $G = \frac{1}{5}$. Clearly $2 < \theta < \rho$, $2 < \theta^* < \rho^*$, $\upsilon, \upsilon^* \in (0, 1]$, $0 \leq \eta_1, \eta_2 < \rho - \theta$, $0 \leq \eta_1^*, \eta_2^* < \rho^* - \theta^*$, and $s_1, s_2, s_1^*, s_2^* \in \mathbb{R}^+$. Also, we have

$$\begin{aligned}
|\Xi(\tau, z(\tau), r(\tau)) - \Xi(\tau, \widetilde{z}(\tau), \widetilde{r}(\tau))| &\leq \left(\frac{1}{5}\right)^2 (|\sin z(\tau) - \sin \widetilde{z}(\tau)| + |\cos r(\tau) - \cos \widetilde{r}(\tau)|), \\
|\Xi^*(\tau, r(\tau), z(\tau)) - \Xi(\tau, \widetilde{r}(\tau), \widetilde{z}(\tau))| &\leq \left(\frac{1}{5}\right)^2 (|\sin r(\tau) - \sin \widetilde{r}(\tau)| + |\cos z(\tau) - \cos \widetilde{z}(\tau)|).
\end{aligned}$$

It follows that $T = T^* = \widehat{T} = \frac{1}{25}$ and

$$\begin{aligned}
|\Xi(\tau, z(\tau), r(\tau))| &= \left|\tau^2 [\sin z(\tau) + \cos r(\tau)]\right| \leq \tau^2 (|\sin z(\tau)| + |\cos r(\tau)|) \leq \tau^2 = V(\tau), \\
|\Xi^*(\tau, r(\tau), z(\tau))| &= \left|\tau^2 [\sin r(\tau) + \cos z(\tau)]\right| \leq \tau^2 (|\sin r(\tau)| + |\cos z(\tau)|) \leq \tau^2 = V^*(\tau).
\end{aligned}$$

If we take $\varrho_1 = \varrho_1^* = \frac{1}{4}$ and $\varrho_2 = \varrho_2^* = \frac{3}{4}$, we have $\varrho_1, \varrho_2, \varrho_1^*, \varrho_2^* \in (0, 1]$. We can easily calculate

$$\begin{aligned}
&\nabla_1 \approx 0.110255, \quad \nabla_2 \approx 0.494979, \quad \nabla_3 \approx 0.007777, \quad \nabla_4 \approx 0.058922, \\
&\nabla_1^* \approx 0.107356, \quad \nabla_2^* \approx 0.734601, \quad \nabla_3^* \approx 0.007162, \quad \nabla_4^* \approx 0.044779, \\
&\Phi \approx 0.002646, \quad \Phi^* \approx 0.000454, \quad \Lambda_1 = 0.332710, \quad \Lambda_2 = 0.300271, \\
&\Lambda_1^* = 0.512841, \quad \Lambda_2^* = 0.530105, \quad \Lambda_3 = 0.845551, \quad \Lambda_4 = 0.830376.
\end{aligned}$$

Hence, $\widehat{T}\Lambda_4 + \Lambda_3 \approx 0.878766 < 1$. From Theorem 3.1, the CBVP (5.1) has a US.
If we take $\varrho_1 = \varrho_1^* = 1$ and $\varrho_2 = \varrho_2^* = 1$, we get

$$\eth \approx 0.007583, \; \eth^* \approx 0.058179, \; \wp^* \approx 0.036841 \text{ and } \wp \approx 0.0782149.$$

Since $\daleth = 1 - \frac{0.0004412}{0.8878256} \approx 0.999503 > 0$, then by Theorem 4.1, the CBVP (5.1) is HU stable with

$$\widehat{\Delta} = \frac{1}{\daleth}\left(\frac{S}{1-\wp} + \frac{S^*}{1-\wp^*} + \frac{\eth S^*}{(1-\wp^*)(1-\wp)} + \frac{\eth^* S}{(1-\wp^*)(1-\wp)}\right) = 0.0258741 > 0.$$

6. Conclusions

Fractional calculus has found numerous miscellaneous applications connected with real-world problems as they appear in many fields of science and engineering, including fluid flow, signal and image processing, fractal theory, control theory, electromagnetic theory, fitting of experimental data, optics, potential theory, biology, chemistry, diffusion, and viscoelasticity. Due to the many applications that have been mentioned, this branch has become of interest to many writers. Therefore, in this paper, the existence of solutions to a system of two-term FDEs with a fractional bi-order involving the Riemann-Liouville derivative has been established. Also, the considered boundaries are mixed Riemann-Liouville integro-derivative conditions with four different orders. Further, HU stability is studied, and an illustrative example has been introduced. Ultimately, we conclude that our results are new and are considered a further development of the qualitative analysis of fractional differential equations.

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Conflict of interest

The authors declare that they have no conflict of interests.

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