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Research article

On inextensible ruled surfaces generated via a curve derived from a curve with constant torsion

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Abstract: If both the arc length and the intrinsic curvature of a curve or surface are preserved, then the flow of the curve or surface is said to be inextensible. The absence of motion-induced strain energy is the physical characteristic of inextensible curve and surface flows. In this paper, we study inextensible tangential, normal and binormal ruled surfaces generated by a curve with constant torsion, which is also called a Salkowski curve. We investigate whether or not these surfaces are minimal or can be developed. In addition, we prove some theorems which are related to inextensible ruled surfaces within three-dimensional Euclidean space.

Keywords: ruled surface; Gaussian curvature; mean curvature; inextensible surface; constant curvature
Mathematics Subject Classification: 53A05

1. Introduction

Many fields, including computer vision [1], computer animation [2] and image processing [3] benefit greatly from the evolution of curves and surfaces. The movement of curves and surfaces in R^3 prompts nonlinear evolution equations, which are frequently integrable. There have been a lot of studies done on the connection between integrable systems and the differential geometry of curves.

The evolution of curves in the direction of their curvature vector field, also known as "curve shortening", "flow by curvature" and "heat flow", has been the subject of numerous studies in the literature. The approaches developed by Gage and Hamilton [4] and Grayson [5] to investigate the heat equation-based reduction of closed plane curves to a circle are particularly pertinent. In [6], Gage also investigates plane curve evolutions with area preservation.

For a curve whose length remains constant throughout time, which is called an inelastic plane curve the evolution equations are obtained. The partial differential equation involving curvature expresses the necessary and sufficient conditions for a flow of an inelastic curve in [7]. Physically, the absence of any strain energy caused by motion is what distinguishes inextensible curve and surface flows. Equivalent equations were derived for an inextensible flow of an inextensible surface, and it was shown that it suffices to describe the development of the surface in terms of two non-inextensible curve flows in [8].

The theories of curves and surfaces constitute an important field of study in differential geometry. In particularly, these theories are considered in the Euclidean, Minkowski and Galilean. See [9–14]. The evolution of space curves and ruled surfaces has been studied for many different frames and spaces. See also [15–21] for some related studies. In this article, we obtain and characterize the corresponding equations for inextensible flows of the tangential, the normal and the binormal ruled surfaces generated by the curve with constant torsion curve. We hope that this work will be useful for the specialists studying in this field.

2. Preliminaries

Let $\alpha : I \longrightarrow \mathbb{E}^3$ be a unit speed curve with an arc-length parameter s in three-dimensional Euclidean space such that I is an open interval in $\mathbb{R}$. The Frenet vectors of the curve α are $\{T_\alpha, N_\alpha, B_\alpha\}$, its curvature is κ_α and its torsion τ_α is constant. Let the curve $\overline{\alpha}$ which is generated by the curve α be defined as follows:

$$\overline{\alpha}(\overline{s}) = \frac{1}{\tau_\alpha} N_\alpha(s) - \int_0^s B_\alpha(u)du, \tag{2.1}$$

where $\overline{s}$ is the parameter of the curve $\overline{\alpha}$.

By the derivative of the curve $\overline{\alpha}$ with respect to the parameter s, we have

$$\overline{\alpha}'(\overline{s}) = \frac{d\overline{\alpha}}{d\overline{s}}\frac{d\overline{s}}{ds} = -\frac{\kappa_\alpha(s)}{\tau_\alpha} T_\alpha(s), \tag{2.2}$$

where κ_α and τ_α are the curvatures of the curve α. If we rearrange Eq (2.2), we get

$$\overline{\alpha}'(\overline{s}) = -\sigma \frac{\kappa_\alpha(s)}{\tau_\alpha} T_\alpha(s)$$

such that $\sigma = \dfrac{ds}{d\overline{s}}$. Additionally, the norm of the speed vector for the curve $\overline{\alpha}$ is $\vartheta = \sigma \dfrac{\kappa_\alpha}{\tau_\alpha}$. The Frenet vectors of the curve $\overline{\alpha}$ are calculated in Theorem 2.1; the curvatures of the curve $\overline{\alpha}$ are also calculated in Theorem 2.2

Theorem 2.1. *Let the curve $\overline{\alpha}$ be defined by Eq (2.1). There are the following relations between the Frenet vectors of the two curves $\overline{\alpha}$ and α in [22]:*

$$T_{\overline{\alpha}} = -T_\alpha, \qquad N_{\overline{\alpha}} = -N_\alpha, \qquad B_{\overline{\alpha}} = B_\alpha.$$

Theorem 2.2. *Let the curve $\overline{\alpha}$ be defined by Eq (2.1). There are the following relations between the curvatures of the two curves $\overline{\alpha}$ and α in [22]:*

$$\kappa_{\overline{\alpha}} = \tau_\alpha, \qquad \tau_{\overline{\alpha}} = -\frac{\tau_\alpha^2}{\kappa_\alpha}.$$

Since the curve $\overline{\alpha}$ is not a unit speed curve, the relationship between the derivation of the curve $\overline{\alpha}$ and its Frenet vectors is given as follows:

$$\begin{bmatrix} T'_{\overline{\alpha}} \\ N'_{\overline{\alpha}} \\ B'_{\overline{\alpha}} \end{bmatrix} = \begin{bmatrix} 0 & \vartheta\kappa_{\overline{\alpha}} & 0 \\ -\vartheta\kappa_{\overline{\alpha}} & 0 & \vartheta\tau_{\overline{\alpha}} \\ 0 & -\vartheta\tau_{\overline{\alpha}} & 0 \end{bmatrix} \begin{bmatrix} T_{\overline{\alpha}} \\ N_{\overline{\alpha}} \\ B_{\overline{\alpha}} \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_1 & 0 \\ -\varepsilon_1 & 0 & \varepsilon_2 \\ 0 & -\varepsilon_2 & 0 \end{bmatrix} \begin{bmatrix} T_{\overline{\alpha}} \\ N_{\overline{\alpha}} \\ B_{\overline{\alpha}} \end{bmatrix}, \tag{2.3}$$

where $\varepsilon_1 = \vartheta\kappa_{\overline{\alpha}}$ and $\varepsilon_2 = \vartheta\tau_{\overline{\alpha}}$.

Corollary 2.1. *The curve $\overline{\alpha}$ which is generated by the curve α is a Salkowski curve [22].*

Definition 2.3. For any differentiable two curves α and β, the surface

$$\phi(u, v) = \alpha(u) + v\beta(u) \tag{2.4}$$

defined by its parameterization in Eq (2.4) is called a ruled surface. The curve $\alpha(u)$ is called the base curve, and the curve $\beta(u)$ is called the directrix curve at the point $\alpha(u)$ of the surface $\phi(u, v)$ [23].

The unit normal vector field of a surface $\phi(u, v)$ is defined below such that $\phi_u = \frac{\partial\phi(u,v)}{\partial u}$ and $\phi_v = \frac{\partial\phi(u,v)}{\partial v}$:

$$U = \frac{\phi_u \times \phi_v}{||\phi_u \times \phi_v||}. \tag{2.5}$$

Also, the first and the second fundamental forms of the surface $\phi(u, v)$ are given respectively as

$$\begin{aligned} I &= Edu^2 + 2Fdudv + Gdv^2, \\ II &= edu^2 + 2fdudv + gdv^2 \end{aligned}$$

such that

$$E = \langle\phi_u, \phi_u\rangle, \quad F = \langle\phi_u, \phi_v\rangle, \quad G = \langle\phi_v, \phi_v\rangle, \quad e = \langle U, \phi_{uu}\rangle, \quad f = \langle U, \phi_{uv}\rangle, \quad g = \langle U, \phi_{vv}\rangle \tag{2.6}$$

are coefficients of the fundamental forms. The Gaussian curvature and the mean curvature of the surface $\phi(u, v)$ are calculated by the following equations, respectively [24]:

$$K = \frac{eg - f^2}{EG - F^2}, \qquad H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}. \tag{2.7}$$

Surfaces with zero Gaussian curvature at each point are called developable surfaces, and those with zero mean curvature at each point are called minimal surfaces [23].

Kwon and Park obtained fundamental results for inelastic flows of space curves. They clearly demonstrated the inelastic flows between the initial and final positions of the fixed-length plane and space curves [7, 8].

Definition 2.4. A curve evolution $F(u, t)$ and its flow $\frac{\partial F}{\partial t}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ are said to be inextensible if $\frac{\partial}{\partial t}|\frac{\partial F}{\partial u}| \equiv 0$ [8].

Definition 2.5. A surface evolution $\phi(s, v, t)$ and its flow $\dfrac{\partial \phi}{\partial t}$ are said to be inextensible [8] if its first fundamental coefficients $\{E, F, G\}$ satisfy

$$\frac{\partial E}{\partial t} = \frac{\partial F}{\partial t} = \frac{\partial G}{\partial t} = 0.$$

Definition 2.6. Let α be a curve and T_α, N_α and B_α be Frenet vectors of the curve α. Then, the tangential ruled surface, the normal ruled surface and the binormal ruled surface are respectively defined as follows [25]:

$$\phi_T = \alpha + vT_\alpha,$$

$$\phi_N = \alpha + vN_\alpha,$$

$$\phi_B = \alpha + vB_\alpha.$$

3. Characterizations of the ruled surface generated by the curve $\overline{\alpha}$ and its Frenet vectors

Let $\overline{\alpha}$ be a Salkowski curve given by Eq (2.1) in three-dimensional Euclidean space. In this part, we would like to examine some properties of the evolution of three different types of ruled surfaces generated by choosing the director curve as follows.

3.1. Tangential ruled surface

Evolution of a tangential ruled surface with the time parameter t of the curve $\overline{\alpha}$ defined by Eq (2.1) is given as follows:

$$\phi_T(\overline{s}, v, t) = \overline{\alpha}(\overline{s}, t) + vT_{\overline{\alpha}}(\overline{s}, t), \tag{3.1}$$

where $T_{\overline{\alpha}}$ is the tangent vector field of the curve $\overline{\alpha}$. By the derivative of the tangential ruled surface ϕ_T with respect to the parameter $\overline{s}$, we have

$$\phi_{T\overline{s}} = \frac{d\overline{\alpha}}{d\overline{s}} + vT'_{\overline{\alpha}}. \tag{3.2}$$

If we arrange Eq (3.2) and substitute it into Eq (2.3), we get

$$\phi_{T\overline{s}} = \vartheta T_{\overline{\alpha}} + v\varepsilon_1 N_{\overline{\alpha}}$$

such that $\dfrac{d\overline{\alpha}}{d\overline{s}} = \vartheta T_{\overline{\alpha}}$, where $\vartheta = \sigma\dfrac{\kappa_\alpha}{\tau_\alpha}$. Hence, we have

$$\phi_{T_{\overline{s}}} = -\sigma\frac{\varepsilon_1}{\varepsilon_2} T_{\overline{\alpha}} + v\varepsilon_1 N_{\overline{\alpha}} \tag{3.3}$$

such that $\vartheta = -\sigma\dfrac{\varepsilon_1}{\varepsilon_2}$, where $\varepsilon_1 = \vartheta\kappa_{\overline{\alpha}}$ and $\varepsilon_2 = \vartheta\tau_{\overline{\alpha}}$. If we take the derivative of the tangential ruled surface ϕ_T with respect to the parameter v, then we get

$$\phi_{T_v} = T_{\overline{\alpha}}. \tag{3.4}$$

The unit normal vector of the surface ϕ_T is calculated by Eq (2.5), so

$$U_{\phi_T} = -B_{\overline{\alpha}}$$

is obtained. Next, if we take the derivative of Eq (3.3) with respect to the parameters $\overline{s}$ and v, respectively, then we have

$$\phi_{T_{\overline{s}\overline{s}}} = \left(-\sigma\left(\frac{\varepsilon_1}{\varepsilon_2}\right)_{\overline{s}} - v\varepsilon_1^2\right)T_{\overline{\alpha}} + \left(-\sigma\frac{\varepsilon_1^2}{\varepsilon_2} + v\varepsilon_{1_{\overline{s}}}\right)N_{\overline{\alpha}} + v\varepsilon_1\varepsilon_2 B_{\overline{\alpha}},$$

$$\phi_{T_{\overline{s}v}} = \varepsilon_1 N_{\overline{\alpha}}.$$

The derivative of Eq (3.4) with respect to the parameter v gives rise to

$$\phi_{T_{vv}} = 0.$$

The coefficients of the first and second fundamental forms are calculated by using Eq (2.6), and they are given as follows:

$$E = \varepsilon_1^2 v^2 + \sigma^2\frac{\varepsilon_1^2}{\varepsilon_2^2}, \quad F = -\sigma\frac{\varepsilon_1}{\varepsilon_2}, \quad G = 1, \tag{3.5}$$

where $\sigma = \dfrac{ds}{d\overline{s}}$ and

$$e = -v\varepsilon_1\varepsilon_2, \quad f = 0, \quad g = 0.$$

The Gaussian curvature K and mean curvature H of the surface ϕ_T are calculated by using Eq (2.7), respectively, and

$$K = 0, \quad H = -\frac{\varepsilon_2}{2v\varepsilon_1}$$

are obtained.

Corollary 3.1. *The tangential ruled surface ϕ_T is developable.*

Corollary 3.2. *The tangential ruled surface ϕ_T is not a minimal surface.*

Proof. Since $\varepsilon_2 = \vartheta\tau_{\overline{\alpha}} = -\sigma\tau_\alpha$ is constant, $H \neq 0$. Therefore, the tangential ruled surface ϕ_T cannot be minimal. □

Theorem 3.1. *The tangent ruled surface ϕ_T is inextensible such that $\varepsilon_{1_t} = 0$.*

Proof. By partial derivation of Eq (3.5) with respect to the parameter t, we get the following respective equations:

$$\frac{\partial E}{\partial t} = 2\varepsilon_{1_t}\varepsilon_1\left(v^2 + \frac{\sigma^2}{\varepsilon_2^2}\right) - \frac{2\sigma^2\varepsilon_1^2\varepsilon_{2_t}}{\varepsilon_2^3},$$

$$\frac{\partial F}{\partial t} = -\sigma\frac{\varepsilon_{1_t}\varepsilon_2 - \varepsilon_1\varepsilon_{2_t}}{\varepsilon_2^2},$$

$$\frac{\partial G}{\partial t} = 0.$$

Since ε_2 is constant, $\varepsilon_{2_t} = 0$. By Definition 2.5, we have that $\varepsilon_{1_t} = 0$. □

3.2. Normal ruled surface

The evolution of a normal ruled surface with the time parameter t of the curve $\overline{\alpha}$ defined by Eq (2.1) is given as follows:

$$\phi_N(\overline{s}, v, t) = \overline{\alpha}(\overline{s}, t) + vN_{\overline{\alpha}}(\overline{s}, t), \tag{3.6}$$

where $N_{\overline{\alpha}}$ is the normal vector field of the curve $\overline{\alpha}$. Differentiating the normal ruled surface ϕ_N with respect to the parameter $\overline{s}$, we have

$$\phi_{N\overline{s}} = \frac{d\overline{\alpha}}{d\overline{s}} + vN'_{\overline{\alpha}}. \tag{3.7}$$

If we arrange Eq (3.7) and substitute it into Eq (2.3), we get

$$\phi_{N\overline{s}} = \vartheta T_{\overline{\alpha}} + v\left(-\varepsilon_1 T_{\overline{\alpha}} + \varepsilon_2 B_{\overline{\alpha}}\right)$$

such that $\frac{d\overline{\alpha}}{d\overline{s}} = \vartheta T_{\overline{\alpha}}$, where $\vartheta = \sigma\frac{\kappa_\alpha}{\tau_\alpha}$. Hence, we have

$$\phi_{N_{\overline{s}}} = -\left(\sigma\frac{\varepsilon_1}{\varepsilon_2} + v\varepsilon_1\right)T_{\overline{\alpha}} + v\varepsilon_2 B_{\overline{\alpha}} \tag{3.8}$$

such that $\vartheta = -\sigma\frac{\varepsilon_1}{\varepsilon_2}$, where $\varepsilon_1 = \vartheta\kappa_{\overline{\alpha}}$ and $\varepsilon_2 = \vartheta\tau_{\overline{\alpha}}$. By taking the derivative of the normal ruled surface ϕ_N with respect to the parameter v, we get

$$\phi_{N_v} = N_{\overline{\alpha}}. \tag{3.9}$$

The unit normal vector of the surface ϕ_N is calculated by using Eq (2.5), so

$$U_{\phi_N} = \frac{1}{\sqrt{\left(v\varepsilon_1 + \sigma\frac{\varepsilon_1}{\varepsilon_2}\right)^2 + v^2\varepsilon_2^2}}\left(-v\varepsilon_2 T_{\overline{\alpha}} - \left(v\varepsilon_1 + \sigma\frac{\varepsilon_1}{\varepsilon_2}\right)B_{\overline{\alpha}}\right)$$

is obtained. Next, differentiating Eq (3.8) again with respect to the parameters $\overline{s}$ and v, respectively, we have

$$\phi_{N_{\overline{s}\overline{s}}} = \left(-\varepsilon_{1_{\overline{s}}}\left(\frac{\sigma}{\varepsilon_2} + v\right) + \sigma\frac{\varepsilon_1\varepsilon_{2_{\overline{s}}}}{\varepsilon_2^2}\right)T_{\overline{\alpha}} + \left(-\sigma\frac{\varepsilon_1^2}{\varepsilon_2} - v(\varepsilon_1^2 + \varepsilon_2^2)\right)N_{\overline{\alpha}} + v\varepsilon_{2_{\overline{s}}}B_{\overline{\alpha}},$$

$$\phi_{N_{\overline{s}v}} = -\varepsilon_1 T_{\overline{\alpha}} + \varepsilon_2 B_{\overline{\alpha}}.$$

By differentiating Eq (3.9) with respect to the parameter v, the equation

$$\phi_{N_{vv}} = 0$$

is obtained. By using Eq (2.6), the following coefficients of the first and the second fundamental forms are calculated, respectively,

$$E = \left(v\varepsilon_1 + \sigma\frac{\varepsilon_1}{\varepsilon_2}\right)^2 + v^2\varepsilon_2^2, \quad F = 0, \quad G = 1, \tag{3.10}$$

$$e = \frac{v\varepsilon_{1_{\bar{s}}}\varepsilon_2\left(v + \dfrac{\sigma}{\varepsilon_2}\right) - v\varepsilon_1\varepsilon_{2_{\bar{s}}}\left(\dfrac{2\sigma}{\varepsilon_2} + v\right)}{\sqrt{\left(v\varepsilon_1 + \sigma\dfrac{\varepsilon_1}{\varepsilon_2}\right)^2 + v^2\varepsilon_2^2}}, \quad f = \frac{-\sigma\varepsilon_1}{\sqrt{\left(v\varepsilon_1 + \sigma\dfrac{\varepsilon_1}{\varepsilon_2}\right)^2 + v^2\varepsilon_2^2}}, \quad g = 0.$$

From Eq (2.7), the Gaussian curvature K and mean curvature H of the normal ruled surface ϕ_N are found, respectively, and the equations

$$K = -\frac{\sigma^2\varepsilon_1^2}{\left(\left(v\varepsilon_1 + \sigma\dfrac{\varepsilon_1}{\varepsilon_2}\right)^2 + v^2\varepsilon_2^2\right)^2}, \quad H = \frac{v\varepsilon_{1_{\bar{s}}}\varepsilon_2\left(v + \dfrac{\sigma}{\varepsilon_2}\right) - v\varepsilon_1\varepsilon_{2_{\bar{s}}}\left(\dfrac{2\sigma}{\varepsilon_2} + v\right)}{2\left(\left(v\varepsilon_1 + \sigma\dfrac{\varepsilon_1}{\varepsilon_2}\right)^2 + v^2\varepsilon_2^2\right)^{3/2}} \tag{3.11}$$

are obtained.

Corollary 3.3. *The normal ruled surface ϕ_N is not developable.*

Proof. Assume that the normal ruled surface ϕ_N is developable. Then, the Gaussian curvature K of the surface ϕ_N vanishes such that we have $\sigma^2\varepsilon_1^2 = 0$ by Eq (3.11). Since $\varepsilon_1 \neq 0$ and $\sigma \neq 0$, it is a contradiction. Hence, the normal ruled surface ϕ_N is not developable. □

Theorem 3.2. *The normal ruled surface ϕ_N is a minimal surface, where $\sigma = \dfrac{ds}{d\bar{s}}$ and v is the parameter of that surface if and only if the curve α is a circular helix.*

Proof. Assume that the normal ruled surface ϕ_N is a minimal surface. Then, $H = 0$ from the definition of a minimal surface. In this case, by using Eq (3.11), we get

$$v\varepsilon_{1_{\bar{s}}}\varepsilon_2\left(v + \frac{\sigma}{\varepsilon_2}\right) - v\varepsilon_1\varepsilon_{2_{\bar{s}}}\left(\frac{2\sigma}{\varepsilon_2} + v\right) = 0.$$

Since ε_2 is constant, the equation $\varepsilon_{1_{\bar{s}}} = 0$ is obtained. Then, ε_1 is constant such that $\kappa_\alpha = c$. Therefore, the curve α, which has constant torsion τ_α, is a circular helix.

Suppose that the curve α is a helix. Then, we have that $H = 0$ by calculating the mean curvature of the surface ϕ_N. Therefore, the surface ϕ_N is minimal. □

Theorem 3.3. *If the curve $\overline{\alpha}$ is a circular helix, then the normal ruled surface ϕ_N is inextensible.*

Proof. Suppose that the curve $\overline{\alpha}$ is a circular helix with curvatures $\kappa_{\overline{\alpha}}$ and $\tau_{\overline{\alpha}}$ that are constants, given that the surface ϕ_N is defined by Eq (3.6). Then, we have Eq (3.10), which gives the coefficients of the first fundamental form of the surface ϕ_N. By taking the derivative of Eq (3.10) with respect to the t parameter, we have that

$$\frac{\partial E}{\partial t} = 2\left(\sigma\frac{\varepsilon_1}{\varepsilon_2} + v\varepsilon_1\right)\left(\frac{\sigma(\varepsilon_{1_t}\varepsilon_2 - \varepsilon_1\varepsilon_{2_t})}{\varepsilon_2^2} + v\varepsilon_{1_t}\right) + 2v^2\varepsilon_2\varepsilon_{2_t}$$

is obtained. Since ε_1 and ε_2 are constant, the partial differential equation $\dfrac{\partial E}{\partial t} = 0$. Additionally, $\dfrac{\partial F}{\partial t} = 0$ and $\dfrac{\partial G}{\partial t} = 0$. Therefore, it is an inextensible surface by Definition 2.5. □

3.3. Binormal ruled surface

The evolution of a binormal ruled surface with the time parameter t of the curve $\overline{\alpha}$ defined by Eq (2.1) is given as follows:

$$\phi_B(\overline{s}, v, t) = \overline{\alpha}(\overline{s}, t) + vB_{\overline{\alpha}}(\overline{s}, t), \tag{3.12}$$

where $B_{\overline{\alpha}}$ is the binormal vector field of the curve $\overline{\alpha}$. By taking the derivative of the binormal ruled surface ϕ_T with respect to the parameter $\overline{s}$, we have

$$\phi_{B\overline{s}} = \frac{d\overline{\alpha}}{d\overline{s}} + vB'_{\overline{\alpha}}. \tag{3.13}$$

If we arrange Eq (3.13) and substitute it into Eq (2.3), we get

$$\phi_{B\overline{s}} = \vartheta T_{\overline{\alpha}} - v\varepsilon_2 N_{\overline{\alpha}}$$

such that $\frac{d\overline{\alpha}}{d\overline{s}} = \vartheta T_{\overline{\alpha}}$, where $\vartheta = \sigma\frac{\kappa_\alpha}{\tau_\alpha}$. Hence, we have

$$\phi_{B_{\overline{s}}} = -\sigma\frac{\varepsilon_1}{\varepsilon_2}T_{\overline{\alpha}} - v\varepsilon_2 N_{\overline{\alpha}} \tag{3.14}$$

such that $\vartheta = -\sigma\frac{\varepsilon_1}{\varepsilon_2}$, where $\varepsilon_1 = \vartheta\kappa_{\overline{\alpha}}$ and $\varepsilon_2 = \vartheta\tau_{\overline{\alpha}}$. Differentiating the binormal ruled surface ϕ_B with respect to the parameter v, we get

$$\phi_{B_v} = B_{\overline{\alpha}}. \tag{3.15}$$

The unit normal vector of the surface ϕ_B is calculated by using Eq (2.5), so

$$U_{\phi_B} = \frac{1}{\sqrt{\sigma^2\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2 + v^2\varepsilon_2^2}}\left(-v\varepsilon_2 T_{\overline{\alpha}} + \sigma\frac{\varepsilon_1}{\varepsilon_2}N_{\overline{\alpha}}\right)$$

is obtained. Next, by differentiating Eq (3.14) again with respect to the parameters $\overline{s}$ and v, respectively, then we have

$$\phi_{B_{\overline{s}\overline{s}}} = \left(-\sigma\left(\frac{\varepsilon_{1_{\overline{s}}}\varepsilon_2 - \varepsilon_1\varepsilon_{2_{\overline{s}}}}{\varepsilon_2^2}\right) + v\varepsilon_1\varepsilon_2\right)T_{\overline{\alpha}} + \left(-\sigma\frac{\varepsilon_1^2}{\varepsilon_2} - v\varepsilon_{2_{\overline{s}}}\right)N_{\overline{\alpha}} - v\varepsilon_2^2 B_{\overline{\alpha}},$$

$$\phi_{B_{\overline{s}v}} = -\varepsilon_2 N_{\overline{\alpha}}.$$

By taking the derivative of Eq (3.15) with respect to the parameter v gives rise to

$$\phi_{B_{vv}} = 0.$$

By using Eq (2.6), the following coefficients of the first and the second fundamental forms of the binormal ruled surface ϕ_B are obtained:

$$E = \sigma^2\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2 + v^2\varepsilon_2^2, \quad F = 0, \quad G = 1, \tag{3.16}$$

$$e=\frac{-\nu\varepsilon_2\left(\nu\varepsilon_1\varepsilon_2-\frac{\sigma}{\varepsilon_2}\varepsilon_{1_{\bar{s}}}\right)-\sigma^2\frac{\varepsilon_1^3}{\varepsilon_2^2}-2\sigma\nu\varepsilon_{2_{\bar{s}}}\frac{\varepsilon_1}{\varepsilon_2}}{\sqrt{\sigma^2\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2+\nu^2\varepsilon_2^2}},\quad f=\frac{-\sigma\varepsilon_1}{\sqrt{\sigma^2\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2+\nu^2\varepsilon_2^2}},\quad g=0.$$

From Eq (2.7), the Gaussian curvature K and the mean curvature H of the binormal surface ϕ_B are found, respectively, and

$$K=-\frac{\sigma^2\varepsilon_1^2}{\left(\sigma^2\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2+\nu^2\varepsilon_2^2\right)^2},\quad H=\frac{\nu\sigma\varepsilon_2^2\varepsilon_{1_{\bar{s}}}-\nu^2\varepsilon_1\varepsilon_2^4-\sigma^2\varepsilon_1^3-2\sigma\nu\varepsilon_1\varepsilon_2\varepsilon_{2_{\bar{s}}}}{2\varepsilon_2^2\left(\sigma^2\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2+\nu^2\varepsilon_2^2\right)^{3/2}} \tag{3.17}$$

are obtained.

Corollary 3.4. *The binormal ruled surface ϕ_B is not developable.*

Proof. Assume that the binormal ruled surface ϕ_B is developable. Then, the Gaussian curvature K of the surface ϕ_B vanishes such that we have $\sigma^2\varepsilon_1^2=0$ by Eq (3.17). Since $\sigma^2\varepsilon_1\neq 0$, it is a contradiction. Hence, the binormal ruled surface ϕ_B is not developable. □

Theorem 3.4. *If the curve $\overline{\alpha}$ is a circular helix, then the binormal ruled surface ϕ_B is inextensible.*

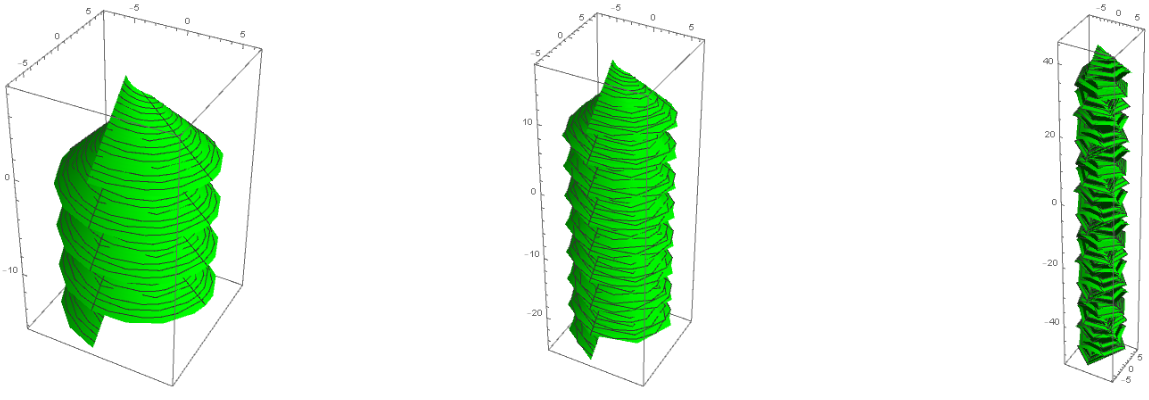
Proof. Suppose that the curve $\overline{\alpha}$ is a circular helix with curvatures $\kappa_{\overline{\alpha}}$ and $\tau_{\overline{\alpha}}$ that are constants so that ε_1 and ε_2, and given that the binormal ruled surface ϕ_B is defined by Eq (3.12). Then, we have Eq (3.16), which gives the coefficients of the first fundamental form of the binormal ruled surface ϕ_B. Differentiating Eq (3.16) with respect to the parameter t,

$$\frac{\partial E}{\partial t}=2\sigma^2\left(\frac{\varepsilon_{1_t}\varepsilon_2-\varepsilon_1\varepsilon_{2_t}}{\varepsilon_2^2}\right)\frac{\varepsilon_1}{\varepsilon_2}+2\nu^2\varepsilon_{2_t}\varepsilon_2$$

is obtained. Since ε_1 and ε_2 are constant, the partial differential equation $\frac{\partial E}{\partial t}=0$. In addition, $\frac{\partial F}{\partial t}=0$ and $\frac{\partial G}{\partial t}=0$. Therefore, it is an inextensible surface by Definition 2.5.

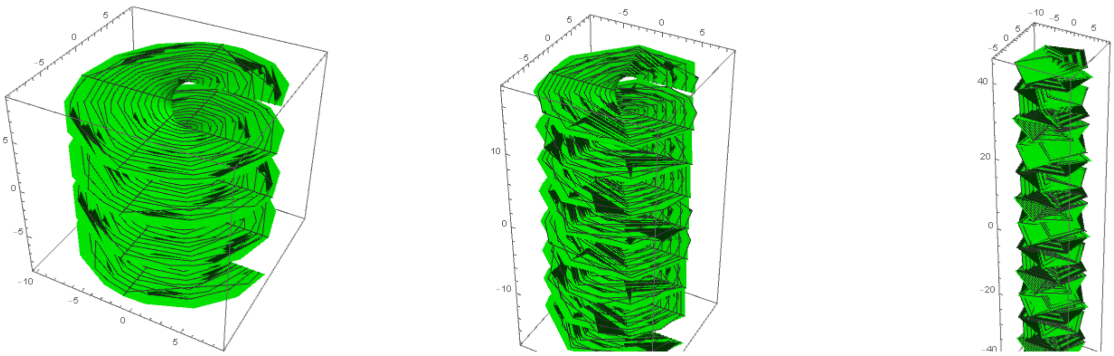
□

Example. Let the curve α be given by the parametric equation $\alpha(s)=(\frac{1}{\sqrt{2}}\cos(s),\frac{1}{\sqrt{2}}\sin(s),\frac{s}{\sqrt{2}})$. Let $\overline{\alpha}$ be the curve which is generated by the curve α, as defined in Eq (2.1). Additionally, we obtain some ruled surfaces generated by the curve $\overline{\alpha}$ and its Frenet vectors. Now, we give some figures with time t that belong to the tangential ruled surface, the normal ruled surface and the binormal ruled surface, respectively, in Figures 1–3.



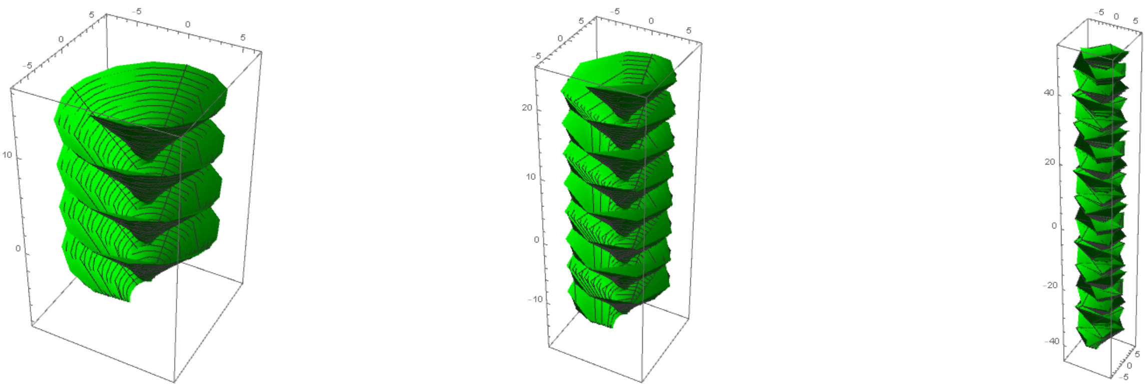
(a) $t = 0$ **(b)** $t = 3$ **(c)** $t = 9$

Figure 1. Tangential ruled surface as defined in Eq (3.1) with the parameters $s \in (-\pi, \pi)$, $v \in (1, 10)$ and $t = 0, 3, 9$.



(a) $t = 0$ **(b)** $t = 3$ **(c)** $t = 9$

Figure 2. Normal ruled surface as defined in Eq (3.6) with the parameters $s \in (-\pi, \pi)$, $v \in (1, 10)$ and $t = 0, 3, 9$.



(a) $t = 0$ **(b)** $t = 3$ **(c)** $t = 9$

Figure 3. Binormal ruled surface as defined in Eq (3.12) with the parameters $s \in (-\pi, \pi)$, $v \in (1, 10)$ and $t = 0, 3, 9$.

4. Conclusions

The curve $\overline{\alpha}$ generated by a curve α with constant torsion is called a Salkowski curve. In this study, we have obtained some associated ruled surfaces whose base curves are $\overline{\alpha}$ and directrix curves are its Frenet vectors. We have examined the first and the second fundamental forms of those surfaces. First, it is shown that the tangential ruled surface is a developable but not minimal surface. In addition, a necessary condition is given for inextensible tangential ruled surfaces. Then, it is shown that the normal ruled surface is not developable, and we obtained that the curve α should be a circular helix for the normal ruled surface to be minimal. Next, if the curve $\overline{\alpha}$ is a circular helix, then it is seen that the normal ruled surface is inextensible. Finally, it is found that the binormal ruled surface is not developable, and this surface is obtained to be an inextensible ruled surface if the curve $\overline{\alpha}$ is a circular helix.

We need to point out that the relationship between our work and singular theory and soliton theory would be interesting. In literature, there are some references about the latter theories. Therefore, the references [26–30] might be useful for developing further works.

Conflict of interest

The authors declare that there is no conflict of interest.

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