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Research article

Decay estimate and blow-up for a fourth order parabolic equation modeling epitaxial thin film growth

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Abstract: In this paper, we study a fourth order parabolic equation modeling epitaxial thin film growth. By using the potential well method and some inequality techniques, we obtain the decay estimate of weak solutions. Meanwhile, the blow-up time is estimated from above and below. The blow-up rate is also derived.

Keywords: parabolic equation; epitaxial thin film growth; decay estimate; blow-up
Mathematics Subject Classification: 35B40, 35B44

1. Introduction

In this paper, we consider the following problem:

$$\begin{cases} u_t + \Delta^2 u - \Delta u_t = -div(|\nabla u|^{q-2}\nabla u \ln|\nabla u|), & x \in \Omega, t > 0, \\ u(x,t) = \Delta u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $q > 2$, $u_0(x) \in H^2(\Omega) \cap H_0^1(\Omega)$. The following equation is derived from the epitaxial growth of nanoscale thin films [1, 2]:

$$\frac{\partial u}{\partial t} + div[k\nabla\Delta u - |\nabla u|^{q-2}\nabla u] = 0. \tag{1.2}$$

The term $\Delta^2 u$ denotes the capillarity-driven surface diffusion, and the $div(|\nabla u|^{q-2}\nabla u)$ denotes the upward hopping of atoms. Liu et al. [3] studied the following equation modeling epitaxial thin film growth:

$$\begin{cases} u_t + \Delta^2 u = -div(|\nabla u|^{q-2}\nabla u \ln|\nabla u|), & x \in \Omega, t > 0, \\ u(x,t) = \Delta u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $2 < q < \frac{2(n+4)}{n+2}$, $u_0(x) \in (H_0^1(\Omega) \cap H^2(\Omega)) \setminus \{0\}$. The nonlinear term $div(|\nabla u|^{q-2}\nabla u)$ was replaced by $div(|\nabla u|^{q-2}\nabla u \ln|\nabla u|)$ when the influences of many factors, such as the molecular and ion effects, were considered by authors. They established a blow-up result for the initial and boundary value problem. Furthermore, the lower bound of the blow-up time and the blow-up rate are derived. In detail, on the condition of $2 < q < \frac{2(n+4)}{n+2}$, $u_0(x) \in (H^2(\Omega) \cap H_0^1(\Omega))$, $J(u_0) < d$ and $I(u_0) < 0$, they proved that the weak solution to problem (1.3) blows up at finite time. Moreover, by the Gagliardo-Nirenberg inequality, they obtained the lower bound of the blow-up time and blow-up rate.

It is well known that evolution equations with strong damping term Δu_t can be used to describe a lot of phenomena in some applied sciences, such as viscoelastic mechanics and quantum mechanics [4,5]. Therefore, many researchers have paid attention to such problems. We refer the interested reader to [6–10].

On the basis of (1.3), our equation considers the term Δu_t additionally. Local existence and uniqueness of weak solutions to problem (1.1) can be proved by using the Contraction Mapping Principle. We refer the interested reader to [7–9, 11, 12]. By using the potential well method and concavity argument, we derive the decay estimate and blow-up results. The upper bound and lower bound of blow-up time, and the blow-up rate are derived. In particular, we obtain the lower bound of blow-up time and blow-up rate similar to [3]. During the process of calculations, we find the condition $I(u_0) < 0$ can be removed by using Young's inequality with ε.

The rest of this paper is organized as follows. In Section 2, we state some definitions and lemmas. In Section 3, decay estimate of weak solution is derived. In Section 4, finite time blow-up of solutions and upper bound of blow-up time will be considered. In Section 5, the blow-up time and blow-up rate are estimated from below.

2. Preliminaries

First, we introduce the definitions of $L^q(\Omega)$, $H_1(\Omega)$, $H_2(\Omega)$:

$L^q(\Omega) := \{u : \Omega \to \mathbb{R} \mid u$ is Lebesgue measurable, $\|u\|_{L^q(\Omega)} < \infty\}$, where

$$\|u\|_{L^q(\Omega)} := \begin{cases} (\int_\Omega |u|^q\,dx)^{\frac{1}{q}} & (1 \le q < \infty), \\ \text{ess sup}_\Omega |u| & (q = \infty). \end{cases} \tag{2.1}$$

$H^1(\Omega) = W^{1,2}(\Omega) := \{u : \Omega \to \mathbb{R} \mid u$ is Lebesgue measurable, $\|u\|_{H^1(\Omega)} < \infty\}$, where

$$\|u\|_{H^1(\Omega)} = \|u\|_{W^{1,2}(\Omega)} := \begin{cases} (\sum_{|\alpha|\le 1} \int_\Omega |D^\alpha u|^2\,dx)^{\frac{1}{2}} & (1 \le q < \infty), \\ \sum_{|\alpha|\le 1} \text{ess sup}_\Omega |D^\alpha u| & (q = \infty). \end{cases} \tag{2.2}$$

$H^2(\Omega) = W^{2,2}(\Omega) := \{u : \Omega \to \mathbb{R} \mid u$ is Lebesgue measurable, $\|u\|_{H^2(\Omega)} < \infty\}$, where

$$\|u\|_{H^2(\Omega)} = \|u\|_{W^{2,2}(\Omega)} := \begin{cases} (\sum_{|\alpha|\le 2} \int_\Omega |D^\alpha u|^2\,dx)^{\frac{1}{2}} & (1 \le q < \infty), \\ \sum_{|\alpha|\le 2} \text{ess sup}_\Omega |D^\alpha u| & (q = \infty). \end{cases} \tag{2.3}$$

We denote by $H_0^1(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$. Throughout the whole paper, the following abbreviations are used for precise statement:

$$\begin{aligned}
&\|u\|_q = \|u\|_{L^q(\Omega)} = (\int_\Omega |u|^q\,dx)^{\frac{1}{q}},\\
&\|u\|_{H^1} = \|u\|_{H^1(\Omega)} = (\|u\|_2^2 + \|\nabla u\|_2^2)^{\frac{1}{2}},\\
&(u,v) = \int_\Omega uv\,dx, \quad \langle u,v\rangle = (u,v) + (\nabla u, \nabla v).
\end{aligned} \tag{2.4}$$

We denote by q^* the Sobolev conjugate of q, i.e., $q^* = +\infty$ for $n \le q$ and $q^* = \frac{nq}{n-q}$ for $n > q$.

Next, we define some functionals as follows:

$$I(u(t)) := \int_\Omega |\Delta u|^2\,dx - \int_\Omega |\nabla u|^q \ln|\nabla u|\,dx, \tag{2.5}$$

$$J(u(t)) := \frac{1}{2}\int_\Omega |\Delta u|^2\,dx - \frac{1}{q}\int_\Omega |\nabla u|^q \ln|\nabla u|\,dx + \frac{1}{q^2}\int_\Omega |\nabla u|^q\,dx. \tag{2.6}$$

By (2.5) and (2.6), we know

$$J(u) = \frac{1}{q}I(u) + \frac{q-2}{2q}\int_\Omega |\Delta u|^2\,dx + \frac{1}{q^2}\int_\Omega |\nabla u|^q\,dx, \tag{2.7}$$

and

$$J(u(t)) + \int_0^t \|u_\tau\|_{H^1}^2\,d\tau = J(u_0). \tag{2.8}$$

Now, we introduce the following definitions:

Definition 2.1. (Maximal existence time) For $u(x,t)$, we define the maximal existence time T_{max} of $u(x,t)$ as follows:

(i) If $u(x,t)$ exists for all $0 \le t < +\infty$, then $T_{max} = +\infty$.
(ii) If there exists $t_0 \in (0,+\infty)$ such that $u(x,t)$ exists for $0 \le t < t_0$, but does not exist at $t = t_0$, then $T_{max} = t_0$.

In what follows, the solution $u(x,t)$ to (1.1) in weak sense is considered.

Definition 2.2. (Weak solution) Function $u(x,t)$ is called a weak solution to (1.1) on $\Omega \times [0, T_{max}]$, if $u \in L^2(0,T_{max};(H^2(\Omega) \cap H_0^1(\Omega)))$, with $u_t \in L^2(0,T_{max};(H^2(\Omega) \cap H_0^1(\Omega)))$ such that $u(x,0) = u_0$ and

$$(u_t,\phi) + (\Delta u, \Delta\phi) + (\nabla u_t, \nabla\phi) = \int_\Omega \nabla\phi \cdot (|\nabla u|^{q-2}\nabla u \ln|\nabla u|)\,dx \tag{2.9}$$

for all $\phi \in (H^2(\Omega) \cap H_0^1(\Omega))$ a.e. $t \in [0,T]$.

Definition 2.3. (Blow-up) We say the weak solution $u(x,t)$ to (1.1) blows up at finite time if the maximal existence time T_{max} is finite, and

$$\lim_{t \to T_{max}^-} \|u(t)\|_{H^1} = +\infty. \tag{2.10}$$

Set

$$\mathcal{N} = \{u \in (H^2(\Omega) \cap H_0^1(\Omega)) | I(u) = 0\}, \tag{2.11}$$

$$d = \inf_{u \in \mathcal{N}} J(u), \tag{2.12}$$

where $\mathcal{N}$ is called the Nehari manifold, and $d > 0$ is the depth of the potential well. Next we give two lemmas. The first one gives some basic properties of the fibering maps $\lambda \mapsto J(\lambda u)$ for $\lambda > 0$, introduced by Drábek and Pohozaev [13]. The second one is about the functional $I(u)$ and potential well method.

Lemma 2.1. *Assume that* $u \in (H^2(\Omega) \cap H_0^1(\Omega))$, *and then*
(i) $\lim_{\lambda \to 0^+} J(\lambda u) = 0, \lim_{\lambda \to +\infty} J(\lambda u) = -\infty$;
(ii) *there exists a unique* $\lambda_* > 0$ *such that* $\frac{d}{d\lambda} J(\lambda u)|_{\lambda = \lambda_*} = 0$;
(iii) $J(\lambda u)$ *is increasing on* $(0, \lambda_*)$, *decresing on* $(\lambda_*, +\infty)$, *and attains the maximum at* $\lambda = \lambda_*$;
(iv) $I(\lambda u) > 0$ *on* $(0, \lambda_*)$, $I(\lambda u) < 0$ *on* $(\lambda_*, +\infty)$, *and* $I(\lambda_* u) = 0$.

Proof. (i) By the definition of $J(u)$, we get

$$J(\lambda u) = \frac{\lambda^2}{2} \|\Delta u\|_2^2 - \frac{\lambda^q}{q} \int_\Omega |\nabla u|^q \ln |\nabla u| \, dx - \frac{\lambda^q \ln \lambda}{q} \|\nabla u\|_q^q + \frac{\lambda^q}{q^2} \|\nabla u\|_q^q. \tag{2.13}$$

So, (i) holds.

(ii) Derivative of $J(\lambda u)$ with respect to λ,

$$\begin{aligned} \frac{d}{d\lambda} J(\lambda u) &= \lambda \|\Delta u\|_2^2 - \lambda^{q-1} \int_\Omega |\nabla u|^q \ln |\nabla u| \, dx - \lambda^{q-1} \ln \lambda \|\nabla u\|_q^q \\ &= \lambda (\|\Delta u\|_2^2 - \lambda^{q-2} \int_\Omega |\nabla u|^q \ln |\nabla u| \, dx - \lambda^{q-2} \ln \lambda \|\nabla u\|_q^q). \end{aligned} \tag{2.14}$$

Let $K(\lambda u) = \lambda^{-1} \frac{d}{d\lambda} J(\lambda u)$, and then we get

$$\begin{aligned} \frac{d}{d\lambda} K(\lambda u) &= -(q-2)\lambda^{q-3} \int_\Omega |\nabla u|^q \ln |\nabla u| \, dx - (q-2)\lambda^{q-3} \ln \lambda \|\nabla u\|_q^q - \lambda^{q-3} \|\nabla u\|_q^q \\ &= -\lambda^{q-3} [(q-2) \int_\Omega |\nabla u|^q \ln |\nabla u| \, dx + (q-2) \ln \lambda \|\nabla u\|_q^q + \|\nabla u\|_q^q]. \end{aligned} \tag{2.15}$$

Hence, by taking

$$\lambda_1 := exp\Big(\frac{(q-2) \int_\Omega |\nabla u|^q \ln |\nabla u| \, dx + \|\nabla u\|_q^q}{(2-q) \|\nabla u\|_q^q} \Big) \tag{2.16}$$

such that $\frac{d}{d\lambda} K(\lambda u) > 0$ on $(0, \lambda_1)$, $\frac{d}{d\lambda} K(\lambda u) < 0$ on $(\lambda_1, +\infty)$, and $\frac{d}{d\lambda} K(\lambda_1 u) = 0$. Combining $K(\lambda u)|_{\lambda=0} = \|\Delta u\|_2^2 \geq 0$ with $\lim_{\lambda \to +\infty} K(\lambda u) = -\infty$, there exists a unique $\lambda_* > 0$ such that $K(\lambda_* u) = 0$, as well as $\frac{d}{d\lambda} J(\lambda u)|_{\lambda = \lambda_*} = 0$.

(iii) It follows from the fact

$$\frac{d}{d\lambda} J(\lambda u) = \lambda K(\lambda u) \tag{2.17}$$

that $\frac{d}{d\lambda} J(\lambda u) > 0$ on $(0, \lambda_*)$, and $\frac{d}{d\lambda} J(\lambda u) < 0$ on $(\lambda_*, +\infty)$.

(iv) By the definition of $I(u)$, we get

$$\begin{aligned} I(\lambda u) &= \lambda^2 \|\Delta u\|_2^2 - \lambda^q \int_\Omega |\nabla u|^q \ln |\nabla u|\, dx - \lambda^q \ln \lambda \|\nabla u\|_q^q \\ &= \lambda \frac{d}{d\lambda} J(\lambda u). \end{aligned} \tag{2.18}$$

□

Lemma 2.2. *If $I(u_0) < 0, J(u_0) < d$, then $I(u) < 0$ for all $t \in [0, T_{max})$, and*

$$d \leq \frac{q-2}{2q} \|\Delta u\|_2^2 + \frac{1}{q^2} \|\nabla u\|_q^q. \tag{2.19}$$

Proof. It follows from (2.8) that

$$J(u(t)) \leq J(u_0) < d, \qquad t \in [0, T_{max}). \tag{2.20}$$

Now, we claim that $I(u) < 0$ for all $t \in [0, T_{max})$. Otherwise, there would exist a $t_0 \in (0, T_{max})$ such that $I(u) < 0$ for all $t \in [0, t_0)$, and $I(u(t_0)) = 0$. Then, from the definition of d,

$$d \leq J(u(t_0)), \tag{2.21}$$

which contradicts (2.20). Thus, $I(u) < 0$ for all $t \in [0, T_{max})$, and then we obtain

$$I(u) = I(\lambda u)|_{\lambda=1} < 0. \tag{2.22}$$

Combining with Lemma 2.1, we get $0 < \lambda_* < 1$, and

$$\begin{aligned} d \leq J(\lambda_* u) &= \frac{1}{q} I(\lambda_* u) + \frac{q-2}{2q} \lambda_*^2 \|\Delta u\|_2^2 + \frac{\lambda_*^q}{q^2} \|\nabla u\|_q^q \\ &= \frac{q-2}{2q} \lambda_*^2 \|\Delta u\|_2^2 + \frac{\lambda_*^q}{q^2} \|\nabla u\|_q^q \\ &\leq \frac{q-2}{2q} \|\Delta u\|_2^2 + \frac{1}{q^2} \|\nabla u\|_q^q. \end{aligned} \tag{2.23}$$

□

3. Decay estimate

Theorem 3.1. *Assume that $2 < q < 2^*$ (the Sobolev conjugate of 2), $I(u_0) > 0$, $J(u_0) \leq \frac{q-2}{2q} (\frac{e\delta_1}{C_1^{q+\delta_1}})^{\frac{2}{q+\delta_1-2}}$ and $0 < \delta_1 < 2^* - q$. Then, there exist two positive constants K_1 and K_2 such that $J(u)$ satisfies the following decay estimate:*

$$J(u) \leq K_1 e^{-K_2 t}, \quad \text{for all } t \in [0, \infty), \tag{3.1}$$

where the above constants will be given later.

Proof. We define

$$L(t) := J(u(t)) + \frac{1}{2}\|u\|_{H^1}^2. \tag{3.2}$$

Now, we claim that there exist two positive constants η_1, η_2 such that

$$\eta_1 J(u) \leq L(t) \leq \eta_2 J(u). \tag{3.3}$$

On the one hand, $L(t) \geq \eta_1 J(u)$ is obvious. On the other hand,

$$\begin{aligned} L(t) &= J(u) + \frac{1}{2}\|u\|_{H_1}^2 \\ &\leq J(u) + C\|\Delta u\|_2^2 \\ &\leq J(u) + C\frac{2q}{q-2}J(u) \\ &= \eta_2 J(u). \end{aligned} \tag{3.4}$$

Then,

$$\begin{aligned} L'(t) =& J'(u) + (u, u_t) + (\nabla u, \nabla u_t) \\ =& -\|u_t\|_2^2 - \|\nabla u_t\|_2^2 - \|\Delta u\|_2^2 + \int_\Omega |\nabla u|^q \ln|\nabla u|\,dx \\ =& -\alpha J(u) + \frac{\alpha}{2}\|\Delta u\|_2^2 - \frac{\alpha}{q}\int_\Omega |\nabla u|^q \ln|\nabla u|\,dx + \frac{\alpha}{q^2}\|\nabla u\|_q^q \\ & -\|u_t\|_2^2 - \|\nabla u_t\|_2^2 - \|\Delta u\|_2^2 + \int_\Omega |\nabla u|^q \ln|\nabla u|\,dx \\ =& -\alpha J(u) - \|u_t\|_2^2 - \|\nabla u_t\|_2^2 + (\frac{\alpha}{2} - 1)\|\Delta u\|_2^2 \\ & + (1 - \frac{\alpha}{q})\int_\Omega |\nabla u|^q \ln|\nabla u|\,dx + \frac{\alpha}{q^2}\|\nabla u\|_q^q, \end{aligned} \tag{3.5}$$

where α is a positive constant. We choose δ_1 small enough such that $q + \delta_1 \leq 2^*$. Using the basic inequality $e\delta_1 \ln x \leq x^{\delta_1}(x, \delta_1 > 0)$ and Sobolev inequality, we have

$$\int_\Omega |\nabla u|^q \ln|\nabla u|\,dx \leq \frac{1}{e\delta_1}\|\nabla u\|_{q+\delta_1}^{q+\delta_1} \leq \frac{C_1^{q+\delta_1}}{e\delta_1}\|\Delta u\|_2^{q+\delta_1} \leq \frac{C_1^{q+\delta_1}}{e\delta_1}[\frac{2q}{q-2}J(u_0)]^{\frac{q+\delta_1-2}{2}}\|\Delta u\|_2^2, \tag{3.6}$$

and

$$\|\nabla u\|_q^q \leq C_2^q\|\Delta u\|_2^q \leq C_2^q[\frac{2q}{q-2}J(u_0)]^{\frac{q-2}{2}}\|\Delta u\|_2^2, \tag{3.7}$$

where C_1 and C_2 are the optimal constants satisfying $\|\nabla u\|_{q+\delta_1} \leq C_1\|\Delta u\|_2$, $\|\nabla u\|_q \leq C_2\|\Delta u\|_2$. Inserting (3.6) and (3.7) into (3.5), we have

$$\begin{aligned} L'(t) \leq& -\alpha J(u) + \Big[\frac{\alpha}{2} - 1 + \frac{C_1^{q+\delta_1}}{e\delta_1}[\frac{2qJ(u_0)}{q-2}]^{\frac{q+\delta_1-2}{2}} \\ & - \frac{\alpha C_1^{q+\delta_1}}{qe\delta_1}[\frac{2qJ(u_0)}{q-2}]^{\frac{q+\delta_1-2}{2}} + \frac{\alpha C_2^q}{q^2}[\frac{2qJ(u_0)}{q-2}]^{\frac{q-2}{2}}\Big]\|\Delta u\|_2^2. \end{aligned} \tag{3.8}$$

It follows from the condition $J(u_0) \le \frac{q-2}{2q}(\frac{e\delta_1}{C_1^{q+\delta_1}})^{\frac{2}{q+\delta_1-2}}$ that

$$\frac{C_1^{q+\delta_1}}{e\delta_1}[\frac{2qJ(u_0)}{q-2}]^{\frac{q+\delta_1-2}{2}} - 1 \le 0. \tag{3.9}$$

We choose α small enough such that

$$\frac{\alpha}{2} + \frac{\alpha C_2^q}{q^2}[\frac{2qJ(u_0)}{q-2}]^{\frac{q-2}{2}} + \frac{C_1^{q+\delta_1}}{e\delta_1}[\frac{2qJ(u_0)}{q-2}]^{\frac{q+\delta_1-2}{2}} - 1 \le 0, \tag{3.10}$$

and then we obtain

$$L'(t) \le -\alpha J(u) \le -\frac{\alpha}{\eta_2}L(t), \tag{3.11}$$

which implies

$$J(u) \le K_1 e^{-K_2 t}, \tag{3.12}$$

where $K_1 = \frac{L(0)}{\eta_1}$ and $K_2 = \frac{\alpha}{\eta_2}$, $L(0) = J(u_0) + \frac{1}{2}\|u_0\|_{H^1}^2$. □

4. Blow-up and upper bound

Theorem 4.1. *Let $q > 2$ and $I(u_0) < 0$,*

(i) if $J(u_0) < d$. Then, the weak solution $u(x,t)$ to problem (1.1) *blows up at finite time. The blow-up time T_{max} can be estimated from above by*

$$T_{max} \le \frac{4\|u_0\|_{H^1}^2}{(q-2)^2(d - J(u_0))}; \tag{4.1}$$

(ii) if $\|u_0\|_{H^1}^2 > \frac{2qC_3^2}{q-2}J(u_0)$. Then, the weak solution $u(x,t)$ to problem (1.1) *blows up at finite time. The blow-up time T_{max} can be estimated from above by*

$$T_{max} \le \frac{64\|u_0\|_{H^1}^2}{(q-2)^2\omega_0}, \tag{4.2}$$

where the above constants will be given later.

Lemma 4.1. *[14, 15] Suppose that $0 < T \le +\infty$, and a nonnegative function $F(t) \in C^2[0,T)$ satisfies*

$$F(t)F''(t) - (1+\alpha)(F'(t))^2 \ge 0 \tag{4.3}$$

for constant $\alpha > 0$. If $F(0) > 0$ and $F'(0) > 0$, then $F(t) \to +\infty$ as $t \to T$, and

$$T \le \frac{F(0)}{\alpha F'(0)} < +\infty. \tag{4.4}$$

On the basis of Lemma 4.1, now we give the proof of Theorem 4.1 (i).

Case 1: $J(u_0) < d$.

Proof. Suppose that the weak solution $u(t)$ to (1.1) exists globally, and then $T_{max} = \infty$. For any $T > 0$, $\mu > 0$, $\nu > 0$, we define

$$F(t) := \int_0^t \|u(\tau)\|_{H^1}^2 \, d\tau + (T - t)\|u_0\|_{H^1}^2 + \mu(t + \nu)^2. \tag{4.5}$$

Taking the first derivative of $F(t)$, we obtain

$$F'(t) = \|u(t)\|_{H^1}^2 - \|u_0\|_{H^1}^2 + 2\mu(t + \nu) = 2\int_0^t \langle u, u_\tau\rangle \, d\tau + 2\mu(t + \nu). \tag{4.6}$$

Using Hölder's inequality and Young's inequality, we have

$$\begin{aligned}(F'(t))^2 &= 4[(\int_0^t \langle u, u_\tau\rangle \, d\tau)^2 + 2\mu(t+\nu)\int_0^t \langle u, u_\tau\rangle \, d\tau + \mu^2(t+\nu)^2] \\ &\le 4[\int_0^t \|u\|_{H^1}^2 \, d\tau \int_0^t \|u_\tau\|_{H^1}^2 \, d\tau + 2\mu(t+\nu)(\int_0^t \|u\|_{H^1}^2 \, d\tau)^{\frac{1}{2}}(\int_0^t \|u_\tau\|_{H^1}^2 \, d\tau)^{\frac{1}{2}} \\ &\quad + \mu^2(t+\nu)^2] \\ &\le 4[\int_0^t \|u\|_{H^1}^2 \, d\tau \int_0^t \|u_\tau\|_{H^1}^2 \, d\tau + \mu\int_0^t \|u\|_{H^1}^2 \, d\tau + \mu(t+\nu)^2\int_0^t \|u_\tau\|_{H^1}^2 \, d\tau \\ &\quad + \mu^2(t+\nu)^2] \\ &= 4[\int_0^t \|u\|_{H^1}^2 \, d\tau + \mu(t+\nu)^2][\int_0^t \|u_\tau\|_{H^1}^2 \, d\tau + \mu].\end{aligned} \tag{4.7}$$

Taking the second derivative of $F(t)$, and combining with (2.7) and (2.8), we have

$$\begin{aligned}F''(t) &= 2\langle u, u_t\rangle + 2\mu \\ &= -2I(u) + 2\mu \\ &= -2qJ(u_0) + 2q\int_0^t \|u_\tau\|_{H^1}^2 \, d\tau + (q-2)\int_\Omega |\Delta u|^2 \, dx \\ &\quad + \frac{2}{q}\int_\Omega |\nabla u|^q \, dx + 2\mu \\ &\ge -2qJ(u_0) + 2q\int_0^t \|u_\tau\|_{H^1}^2 \, d\tau + 2qd + 2\mu.\end{aligned} \tag{4.8}$$

Choosing $\mu = d - J(u_0)$, we get

$$F''(t) \ge 2q\int_0^t \|u_\tau\|_{H^1}^2 \, d\tau + 2q\mu = 2q(\int_0^t \|u_\tau\|_{H^1}^2 \, d\tau + \mu). \tag{4.9}$$

Combining (4.5) with (4.7) and (4.9), we have

$$\begin{aligned}&F(t)F''(t) - \frac{q}{2}(F'(t))^2 \\ &\ge 2q(T-t)\|u_0\|_{H^1}^2 \cdot (\int_0^t \|u_\tau\|_{H^1}^2 \, d\tau + \mu) \\ &\ge 0, t \in [0, T].\end{aligned} \tag{4.10}$$

Let

$$\nu > \frac{\|u_0\|_{H_1}^2}{(q-2)\mu}, \tag{4.11}$$

and we have

$$F(0) = T\|u_0\|_{H^1}^2 + \mu\nu^2 > 0, \tag{4.12}$$

$$F'(0) = 2\mu\nu > 0. \tag{4.13}$$

According to Lemma 4.1, we know $F(t)$ cannot exist globally. It should blow up at finite time. The blow-up time T_{max} satisfies

$$T_{max} \leq \frac{T\|u_0\|_{H^1}^2 + \mu\nu^2}{(q-2)\mu\nu}. \tag{4.14}$$

Similarly, we define

$$\tilde{F}(t) := \int_0^t \|u(\tau)\|_{H^1}^2 \, d\tau + (T_{max} - t)\|u_0\|_{H^1}^2 + \mu(t+\nu)^2. \tag{4.15}$$

As we discussed earlier, under the condition of

$$\begin{aligned} \mu &= d - J(u_0), \\ \nu &> \frac{\|u_0\|_{H_1}^2}{(q-2)\mu}, \end{aligned} \tag{4.16}$$

$\tilde{F}$ blows up at finite time. The blow-up time T_{max} satisfies

$$T_{max} \leq \frac{T_{max}\|u_0\|_{H^1}^2 + \mu\nu^2}{(q-2)\mu\nu}, \tag{4.17}$$

and equivalently

$$T_{max} \leq \frac{\mu\nu^2}{(q-2)\mu\nu - \|u_0\|_{H^1}^2} := f(\nu). \tag{4.18}$$

Some calculations show that

$$\min f(\nu) = f\left(\frac{2\|u_0\|_{H^1}^2}{(q-2)\mu}\right) = \frac{4\|u_0\|_{H^1}^2}{(q-2)^2(d - J(u_0))}, \tag{4.19}$$

which implies

$$T_{max} \leq \frac{4\|u_0\|_{H^1}^2}{(q-2)^2(d - J(u_0))}. \tag{4.20}$$

□

Case 2: $\|u_0\|_{H^1}^2 > \frac{2qC_3^2}{q-2} J(u_0)$.

Lemma 4.2. *If $I(u) < 0$ for all $t \in [0, T_{max})$, then $\|u(t)\|_{H^1}^2$ is strictly increasing on $[0, T_{max})$.*

Proof. By an easy calculation, we have

$$\begin{aligned}\frac{d}{dt}\|u(t)\|_{H^1}^2 &= 2[(u,u_t)+(\nabla u,\nabla u_t)]\\ &= 2(-\|\Delta u\|_2^2+\int_\Omega |\nabla u|^q \ln|\nabla u|\,dx)\\ &= -2I(u)\\ &> 0.\end{aligned} \tag{4.21}$$

We can deduce that $\|u(t)\|_{H^1}^2$ is strictly increasing on $[0,T_{max})$. □

Lemma 4.3. *Let $q>2$, suppose that the initial data satisfy*

$$\|u_0\|_{H^1}^2 > \frac{2qC_3^2}{q-2}J(u_0), \tag{4.22}$$

where C_3 is the optimal constant satisfying $\|u\|_{H^1}\le C_3\|\Delta u\|_2$. Then, $I(u_0)<0$ implies $I(u)<0$ for all $t\in[0,T_{max})$.

Proof. On the contrary, if it is false, there exists a $t_1\in[0,T_{max})$ such that $I(u)<0$ for all $t\in[0,t_1)$, and $I(u(t_1))=0$. Then, it follows from Lemma 4.2 that

$$\|u(t)\|_{H^1}^2 > \|u_0\|_{H^1}^2 > \frac{2qC_3^2}{q-2}J(u_0), \quad t\in(0,t_1). \tag{4.23}$$

By the monotonicity and continuity of $\|u(t)\|_{H^1}^2$, we obtain

$$\|u(t_1)\|_{H^1}^2 > \frac{2qC_3^2}{q-2}J(u_0). \tag{4.24}$$

On the other hand, a combination of $J(u)$, $I(u)$ and $\|u\|_{H^1}^2\le C_3^2\|\Delta u\|_2^2$ shows that

$$\begin{aligned}J(u_0) &\ge J(u(t_1))\\ &= \frac{1}{q}I(u(t_1))+\frac{q-2}{2q}\|\Delta u(t_1)\|_2^2+\frac{1}{q^2}\|\nabla u(t_1)\|_q^q\\ &\ge \frac{q-2}{2qC_3^2}\|u(t_1)\|_{H^1}^2,\end{aligned} \tag{4.25}$$

which contradicts (4.24). □

Next, we give the proof of Theorem 4.1 (ii).

Proof. Similarly, we suppose that $u(t)$ exists globally. For any $T>0$, $\omega>0$, $\rho>0$, we define

$$G(t) := \int_0^t \|u(\tau)\|_{H^1}^2\,d\tau + (T-t)\|u\|_{H^1}^2 + \omega(t+\rho)^2. \tag{4.26}$$

By a similar calculation, we have

$$
\begin{aligned}
&G(t)G''(t)-\frac{q+6}{8}(G'(t))^2\\
\geq &G(t)[G''(t)-\frac{q+6}{2}(\int_0^t \|u_\tau\|_{H^1}^2\,d\tau+\omega)]\\
=&G(t)[\frac{3q-6}{2}\int_0^t \|u_\tau\|_{H^1}^2\,d\tau+(q-2)\|\Delta u\|_2^2+\frac{2}{q}\|\nabla u\|_q^q-2qJ(u_0)-\frac{q+2}{2}\omega]\\
\geq &G(t)[(q-2)\|\Delta u\|_2^2-2qJ(u_0)-\frac{q+2}{2}\omega]\\
\geq &G(t)[\frac{q-2}{C_3^2}\|u\|_{H^1}^2-2qJ(u_0)-\frac{q+2}{2}\omega].
\end{aligned}
\tag{4.27}
$$

Considering the monotonicity of $\|u\|_{H^1}^2$ and the condition $\|u_0\|_{H^1}^2 > \frac{2qC_3^2}{q-2}J(u_0)$, choosing

$$
\omega \in \Big(0, \frac{2(q-2)}{(q+2)C_3^2}\|u_0\|_{H^1}^2 - \frac{4q}{q+2}J(u_0)\Big],
$$

we have

$$
G(t)G''(t)-\frac{q+6}{8}(G'(t))^2 \geq 0, \quad t\in[0,T]. \tag{4.28}
$$

Let

$$
\rho > \frac{4\|u_0\|_{H^1}^2}{(q-2)\omega}, \tag{4.29}
$$

we have

$$
G(0) = T\|u_0\|_{H^1}^2+\omega\rho^2 > 0, \tag{4.30}
$$

$$
G'(0) = 2\omega\rho > 0. \tag{4.31}
$$

According to Lemma 4.1, we know $G(t)$ cannot exists globally. It should blow up at finite time. The blow-up time T_{max} satisfies

$$
T_{max} \leq \frac{4(T\|u_0\|_{H^1}^2+\omega\rho^2)}{(q-2)\omega\rho}. \tag{4.32}
$$

Similarly, we define

$$
\tilde{G} := \int_0^t \|u(\tau)\|_{H^1}^2\,d\tau + (T_{max}-t)\|u\|_{H^1}^2 + \omega(t+\rho)^2. \tag{4.33}
$$

As we discussed earlier, under the condition of

$$
\begin{aligned}
&\omega \in \Big(0, \frac{2(q-2)}{(q+2)C_3^2}\|u_0\|_{H^1}^2 - \frac{4q}{q+2}J(u_0)\Big],\\
&\rho > \frac{4\|u_0\|_{H^1}^2}{(q-2)\omega},
\end{aligned}
\tag{4.34}
$$

$\tilde{G}$ blows up at finite time. The blow-up time T_{max} satisfies

$$
T_{max} \leq \frac{4(T_{max}\|u_0\|_{H^1}^2+\omega\rho^2)}{(q-2)\omega\rho}, \tag{4.35}
$$

and equivalently

$$T_{max} \leq \frac{4\omega\rho^2}{(q-2)\omega\rho - 4||u_0||_{H^1}^2} := g(\omega,\rho). \tag{4.36}$$

Some calculations show that $g(\omega,\rho)$ takes the minimum at

$$\begin{aligned} \omega_0 &= \frac{2(q-2)}{(q+2)C_3^2}||u_0||_{H^1}^2 - \frac{4q}{q+2}J(u_0), \\ \rho_0 &= \frac{8||u_0||_{H^1}^2}{(q-2)\omega_0}. \end{aligned} \tag{4.37}$$

Then,

$$\min g(\omega,\rho) = g(\omega_0,\rho_0) = \frac{64||u_0||_{H^1}^2}{(q-2)^2\omega_0}, \tag{4.38}$$

which implies

$$T_{max} \leq \frac{64||u_0||_{H^1}^2}{(q-2)^2\omega_0}. \tag{4.39}$$

□

5. Lower bound

Theorem 5.1. *Let $2 < q < \frac{2(n+4)}{n+2}$ and $J(u_0) < d$. Then, the weak solution $u(x,t)$ to problem* (1.1) *blows up at finite time. The blow-up time T_{max} can be estimated from below by*

$$T_{max} \geq \frac{||u_0||_{H^1}^{2(1-\beta)}}{(\beta-1)C_4}. \tag{5.1}$$

The blow-up rate can be estimated from below by

$$||u||_{H^1} \geq [C_4(\beta-1)]^{\frac{1}{2(1-\beta)}}(T_{max}-t)^{\frac{1}{2(1-\beta)}}, \tag{5.2}$$

where the above constants will be given later.

Lemma 5.1. *[16,17] (Gagliardo-Nirenberg inequality) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Suppose that l, k are any integers satisfying $0 \leq l < k$, $1 \leq q$, $\lambda \leq \infty$, and $p > 0$, $\frac{l}{k} \leq \theta^* \leq 1$ such that*

$$\frac{1}{p} - \frac{l}{n} = \theta^*(\frac{1}{q} - \frac{k}{n}) + \frac{1}{\lambda}(1-\theta^*). \tag{5.3}$$

Then, for any $\phi \in W^{k,p}(\Omega) \cap L^\lambda(\Omega)$, there exists a constant $C_{GN} > 0$ depending only on n, k, l, q, λ and Ω such that

$$||D^l\phi||_{p(\Omega)} \leq C_{GN}(||D^k\phi||_{q(\Omega)}^{\theta^*}||\phi||_{\lambda(\Omega)}^{1-\theta^*} + ||\phi||_{\lambda(\Omega)}). \tag{5.4}$$

Through the Lemma 5.1, choosing δ_2 small enough such that $q + \delta_2 < \frac{2(n+4)}{n+2}$, we obtain

$$||\nabla u||_{q+\delta_2} \leq C_{GN}||\Delta u||_2^{1-a}||u||_2^a, \tag{5.5}$$

where

$$a = \frac{1}{2} - \frac{n}{4} + \frac{n}{2(q+\delta_2)} \in (0, \frac{1}{2}). \tag{5.6}$$

Proof. We define

$$\Phi(t) := \|u\|_{H^1}^2. \tag{5.7}$$

Using the basic inequality $e\delta_2 \ln x \le x^{\delta_2} (x, \delta_2 > 0)$ and Young's inequality with ε

$$(\varepsilon a)(\frac{b}{\varepsilon}) \le \frac{\varepsilon^r a^r}{r} + \frac{\varepsilon^{-s} b^s}{s}, \tag{5.8}$$

and combining with Lemma 5.1, we have

$$\begin{aligned}
\frac{1}{2}\Phi'(t) &= \int_\Omega |\nabla u|^q \ln |\nabla u| \, dx - \int_\Omega |\Delta u|^2 \, dx \\
&\le \frac{1}{e\delta_2} \|\nabla u\|_{q+\delta_2}^{q+\delta_2} - \|\Delta u\|_2^2 \\
&\le \frac{C_{GN}^{q+\delta_2}}{e\delta_2} \|\Delta u\|_2^{(1-a)(q+\delta_2)} \|u\|_2^{a(q+\delta_2)} - \|\Delta u\|_2^2 \\
&\le \|\Delta u\|_2^2 + \frac{\varepsilon^{-s}}{s} (\frac{C_{GN}^{q+\delta_2}}{e\delta_2})^s (\|u\|_2^{a(q+\delta_2)})^s - \|\Delta u\|_2^2 \\
&= \frac{\varepsilon^{-s}}{s} (\frac{C_{GN}^{q+\delta_2}}{e\delta_2})^s (\|u\|_2^{a(q+\delta_2)})^s,
\end{aligned} \tag{5.9}$$

where a is given by (5.6) and

$$\begin{aligned}
r &= \frac{2}{(1-a)(q+\delta_2)}, \\
s &= \frac{r}{r-1} = \frac{2}{2-(1-a)(q+\delta_2)}, \\
\varepsilon &= r^{\frac{1}{r}} = (\frac{2}{(1-a)(q+\delta_2)})^{\frac{(1-a)(q+\delta_2)}{2}}, \\
0 &< \delta_2 < \frac{2(n+4)}{n+2} - q.
\end{aligned} \tag{5.10}$$

We have $r > 1$ because of

$$(1-a)(q+\delta_2) = (\frac{1}{2} + \frac{n}{4} - \frac{n}{2(q+\delta_2)})(q+\delta_2) = \frac{(n+2)(q+\delta_2)}{4} - \frac{n}{2} < 2. \tag{5.11}$$

Reviewing (5.9), we let

$$C_4 = 2\frac{\varepsilon^{-s}}{s} (\frac{C_{GN}^{q+\delta_2}}{e\delta_2})^s, \tag{5.12}$$

$$\beta = \frac{a(q+\delta_2)}{2-(1-a)(q+\delta_2)}. \tag{5.13}$$

It follows from $q + \delta_2 = a(q+\delta_2) + (1-a)(q+\delta_2) > 2$ and $(1-a)(q+\delta_2) < 2$ that $\beta > 1$. Therefore, we obtain

$$\Phi'(t) \le C_4(\|u\|_2^2)^\beta \le C_4 \Phi^\beta(t). \tag{5.14}$$

Integrating from 0 to t, we get

$$\Phi^{1-\beta}(0) - \Phi^{1-\beta}(t) \le (\beta - 1)C_4 t. \tag{5.15}$$

Letting $t \to T_{max}$ and recalling $\Phi(T_{max}) = +\infty$, we have

$$T_{max} \ge \frac{\|u_0\|_{H^1}^{2(1-\beta)}}{(\beta - 1)C_4}. \tag{5.16}$$

Similarly, integrating (5.14) from t to T_{max}, we have

$$\Phi(t) \ge [C_4(\beta - 1)]^{\frac{1}{1-\beta}} (T_{max} - t)^{\frac{1}{1-\beta}}, \tag{5.17}$$

which implies

$$\|u\|_{H^1} \ge [C_4(\beta - 1)]^{\frac{1}{2(1-\beta)}} (T_{max} - t)^{\frac{1}{2(1-\beta)}}. \tag{5.18}$$

□

6. Conclusions

This paper studies a fourth order parabolic equation modeling epitaxial thin film growth. By using some inequalities and methods, the decay estimate of energy functional is derived. In addition, the upper bound of blow-up time is obtained with lower initial energy and high initial energy respectively. Finally, the lower bound of blow-up time and blow-up rate are derived.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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