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Research article

Multiple periodic solutions of nonlinear second order differential equations

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Abstract: In this paper, we are interested in the existence of multiple nontrivial T-periodic solutions of the nonlinear second ordinary differential equation $\ddot{x} + V_x(t, x) = 0$ in $N(\geq 1)$ dimensions. Using homological linking and morse theory, we get at least two critical points of the functional corresponding to our problem. And, we also prove that two critical points are different by critical groups. Then, we obtain there are at least two nontrivial T-periodic solutions of the problem.

Keywords: multiple periodic solutions; morse theory; homological linking; critical groups; nontrivial periodic solutions
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1. Introduction and the main result

Let $T > 0$, $a(\cdot)$ be a T-periodic continous function defined in $\mathbf{R}$, and α $(\in \mathbf{R})$ be a positive constant. Define a potential function V as follows

$$V(t, x) = \frac{c(t)}{2}|x|^2 + \frac{a(t)}{P+1}|x|^{P+1}, \tag{1.1}$$

where $P > 1$ and $a(t) \geq \alpha > 0$. In this paper, we investigate the existence of multiple nontrivial T-periodic solutions of the following problem

$$\begin{cases} \ddot{x} + V_x(t, x) = 0, \\ x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T), \end{cases} \tag{1.2}$$

where $x \in \mathbf{R}^N$, $N \geq 1$, and $c(\cdot)$ satisfies the following condition, (H_c) $0 < c(t)$ is a T-periodic continous function and $c \in C(\mathbf{R}, \mathbf{R})$.

It's obvious that $x = 0$ is a trivial solution of (1.2). Indeed, we are interested in the multiplicity of nontrivial T-periodic solutions of (1.2), and will get that (1.2) has at least two nontrivial T-periodic solutions when $c(t)$ is near to any fixed eigenvalue of the linear periodic boundary value problem, for $t \in [0, T]$,

$$\begin{cases} \ddot{x} + \lambda x = 0, \\ x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T). \end{cases} \tag{1.3}$$

Actually, (1.3) has eigenvalues $\lambda_k = (\frac{2k\pi}{T})^2,\ k = 0, 1, 2, 3, ...$, and eigenfunctions

$$\cos \frac{2k\pi}{T} \bigotimes e, \quad \sin \frac{2k\pi}{T} \bigotimes f, \tag{1.4}$$

where, $e, f \in \mathbf{R}^N$, as $k > 1$; $e \in \mathbf{R}^N$, as $k = 0$.

Next, the variational framework for (1.2) will be given. Set

$$E := H^1_{\text{per}}((0, T), \mathbf{R}^N) = \{x \in H^1((0, T), \mathbf{R}^N) | x(0) = x(T)\}.$$

It's obvious that E is a Hilbert space with the inner product and norm listed below

$$\langle x, y \rangle = \int_0^T (\dot{x}\dot{y} + xy)dt, \quad \|x\|^2 = \langle x, x \rangle, \ x, y \in E.$$

By the compact embeddings

$$E \hookrightarrow C([0, T], \mathbf{R}^N), \ E \hookrightarrow L^q([0, T], \mathbf{R}^N), \ q \geq 1,$$

the T-periodic solutions of (1.2) correspond to the critical points of the functional

$$I(x) = \int_0^T \frac{1}{2}[|\dot{x}|^2 - c(t)|x|^2] - \frac{a(t)}{P+1}|x|^{P+1}dt.$$

It's obvious that as $c(t) \geq \lambda_1$, the trivial solution $x = 0$ is a critical point as a local saddle point of the functional I. By assumptions, we know $I \in C^2(E, \mathbf{R})$ and the derivatives

$$\langle I'(x), y \rangle = \int_0^T (\dot{x}\dot{y} - c(t)xy)dt - \int_0^T a(t)|x|^{P-1}xydt,$$

$$\langle I''(x)y, z \rangle = \int_0^T (\dot{y}\dot{z} - c(t)yz)dt - \int_0^T (a(t)|x|^{P-1}x)'_x yzdt,$$

where $x, y, z \in E$.

Our result is the following theorem.

Theorem 1.1. *Assume $k \geq 1$ and $V(t, x)$ satisfies the (1.1). Then, there exists a $\delta > 0$ such that as $c(t) \in (\lambda_{k+1} - \delta, \lambda_{k+1} + \delta)$, (1.2) has a nontrivial T-periodic solution x_1, and the critical group satisfies the following*

$$C_{i_{k+1}+1}(I, x_1) \ncong 0.$$

Furthermore, as $c(t) \in (\lambda_{k+1} - \delta, \lambda_{k+1})$, there exists another nontrivial T-periodic solutions x_2 of (1.2), and the critical group satisfies

$$C_{i_k+1}(I, x_2) \ncong 0.$$

This paper is directly motivated by [1, 2]. In [1], as $c(t)$ is a positive constant in $\mathbf{R}$, by linking theorem, the existence of one nontrivial T-periodic solutions of the problem (1.2) in one dimension is discussed. In [2], the authors studied the multiple periodic solutions of superlinear second order ODEs in one dimension by morse theory. Actually, a typical model in the applications of Morse theory and minimax methods is the second order Hamiltonian system. We can refer to [3–12] for some historical progress.

Here, applying morse theory and homological linking to look for multiple periodic solutions for ODEs in $N(\geq 1)$ dimensions. On the one hand, in our problem, $c(t)$ is a T-periodic vibrating function different from the previous case that $c(t)$ is a positive constant. As $c(t)$ is not the T-periodic function, the existence of unbounded solutions of higher order differential equations considered as perturbations of certain linear differential equations can be referred to [13]. On the other hand, the dimension in our problem is $N(\geq 1)$, which is more general than the dimension in [2]. To solve the problem, we need to construct new critical groups and direct sum decomposition.

Precisely, in Theorem 1.1, as $c(t)$ is near to λ_{k+1}, we can construct the homological linking with respect to $E_{k+1} \oplus E_{k+1}^{\perp}$. And we get at least two nontrivial T-periodic solutions, which is different from the result there is at least one nontrivial T-periodic solution in [1]. Furthermore, as $c(t) \in (\lambda_{k+1}-\delta, \lambda_{k+1})$, we can also construct the homological linking w.r.t. $E_k \oplus E_k^{\perp}$ and investigate the existence of the nontrivial T-periodic solutions. In the remarkable paper [14], the author obtained one nonconstant periodic solution as $c(t) = 0$ and the potential V was of class C^1, using a critical point theorem, which is now famous as the generalized mountain pass theorem. In [6], the author extended the existence result in [14] investigating (1.2) as $c(t)$ is a constant symmetric matrix by local linking argument, and got one nontrivial periodic solution as the potential V was of class C^1 and satisfied local sign condition [6] near the origin. The fundamental idea here is sources from [15], where the authors studied the superlinear elliptic problem with a saddle structure near zero by bifurcation methods, Morse theory, and topological linking.

The structure of the paper is arranged as follows. In Section 2, we recall the basic Morse theories and give a lemma on the (PS) condition. In Section 3, we prove the main result.

2. Preliminaries

In this section, we recall some preliminaries on Morse theory and homological linking in [7,8,16].

Assume the functional $I \in C^2(E, \mathbf{R})$, where E is a Hilbert space. Set $K = \{x \in E | I'(x) = 0\}, I^c = \{x \in E | I(x) \leq c\}$, and $K_c = \{x \in K | I(x) = c\}$. We recall that I satisfies $(PS)_c$ condition at the level $c \in \mathbf{R}$, if any sequence $\{x_n\} \subset E$ satisfying $I(x_n) \to c,\ I'(x_n) \to 0$ as $n \to \infty$, has a convergent subsequence. I satisfies (PS) if I satisfies $(PS)_c$ at any $c \in \mathbf{R}$.

Assume that the functional I satisfies (PS) and $\# K < \infty$ in this paper. Let $x_0 \in K$ with $I(x_0) = c \in \mathbf{R}$, and U be a neighborhood of x_0 such that $U \cap K = \{x_0\}$. Then, q-th critical group of I at x_0 is defined below

$$C_q(I, x_0) := H_q(I^c \cap U, I^c \cap U \setminus \{x_0\}),\ q \in \mathbf{Z},$$

where $H_*(A, B)$ denotes the singular relative homology group of the pair (A, B) with coefficient field $\mathbb{F}$ ([7,8,16]). We can distinguish critical points by critical groups. The multiplicity of critical points and critical groups can be referred to [16].

For the critical groups of I at an isolated critical point, the following basic conclusions hold.

Proposition 2.1. *Assume that x is an isolated critical point of $I \in C^2(E, \mathbf{R})$ with finite Morse index $i(x)$ and nullity $\nu(x)$. Then*
(1) $C_q(I, x) \cong \delta_{q,i(x)}\mathbb{F}$, if $\nu(x) = 0$;
(2) $C_q(I, x) \cong 0$ for $q \notin [i(x), i(x) + \nu(x)]$ (Gromoll and Meyer [17]).

From Theorems 1.1′ and 1.5 of Chapter II in [8], the following abstract linking theorem is easily obtained (See also [3, 18]).

Proposition 2.2. *([3, 8, 18]) Let E be a real Banach space with $E = X \oplus Y$ and suppose that $l = \dim X$ is finite. Suppose that $I \in C^1(E, \mathbf{R})$ satisfies (PS)condition and*
(H_1) there exist $\rho > 0, \alpha_0 > 0$ such that

$$I(x) \geq \alpha_0, \quad x \in S_\rho = Y \cap \partial B_\rho,$$

where $B_\rho = \{x \in E| \ \|x\| \leq \rho\}$,
(H_2) there exist $R > \rho > 0$, and $e \in Y$ with $\|e\| = 1$ such that

$$I(x) < \alpha_0, \quad x \in \partial Q,$$

where $Q = \{x = x_1 + se| \ \|x\| \leq R, x_1 \in X, \ 0 \leq s \leq R\}$.
Then I has a critical point x_* with $I(x_*) \geq \alpha_0$ and

$$C_{l+1}(I, x_*) \ncong 0.$$

Remark 2.1. *In proposition above, S_ρ and ∂Q are homotopically linked with respect to direct sum decomposition $E = X \oplus Y$.*

Lemma 2.1. *Assume that the T-periodic function $c(\cdot)$ satisfies (H_c), and the potential function V satisfies (1.1). Then, I satisfy the (PS) condition.*

Proof. Let $\{x_n\} \subset E = H^1_{per}((0, T), \mathbf{R}^N)$ satisfy the following,

$$\begin{cases} |I(x_n)| \leq C, & n \in \mathbf{N}, \\ I'(x_n) \to \theta, & \text{as } n \to \infty, \end{cases} \tag{2.1}$$

where θ is a zero vector, $C > 0$ is a constant. In fact, $\forall \varphi \in E = H^1_{\text{per}}((0, T), \mathbf{R}^N)$, we have

$$dI(x_n, \varphi) = \int_0^T [\dot{x}_n \dot{\varphi} - c(t) x_n \varphi - a(t)|x_n|^{P-1} x_n \varphi] dt \to 0. \tag{2.2}$$

Taking $\varphi = x_n$ in (2.2), we have

$$\int_0^T [|\dot{x}_n|^2 - c(t)|x_n|^2 - a(t)|x_n|^{P+1}] dt = o(\|x_n\|), \tag{2.3}$$

and by (2.1), the following holds

$$|\int_0^T [\frac{|\dot{x}_n|^2 - c(t)|x_n|^2}{2} - a(t)\frac{|x_n|^{P+1}}{P+1}] dt| \leq C, \tag{2.4}$$

i.e.,

$$-C \leq \int_0^T [\frac{|\dot{x}_n|^2 - c(t)|x_n|^2}{2} - a(t)\frac{|x_n|^{P+1}}{P+1}]dt \leq C. \tag{2.5}$$

Combining (2.3) and (2.4) (or (2.5)), we have

$$-C - o(\|x_n\|) \leq (\frac{1}{2} - \frac{1}{P+1}) \int_0^T |\dot{x}_n|^2 - c(t)|x_n|^2 dt \leq C + o(\|x_n\|). \tag{2.6}$$

Since $P > 1$, we have $\frac{1}{2} - \frac{1}{P+1} > 0$. So by (2.6), the following holds

$$-C_1 - o(\|x_n\|) \leq \int_0^T |\dot{x}_n|^2 - c(t)|x_n|^2 dt \leq C_1 + o(\|x_n\|), \tag{2.7}$$

where $C_1 > 0$ is a constant. Then, by (2.7) and (2.3), we have

$$\int_0^T a(t)|x_n|^{P+1} dt \leq C_2, \tag{2.8}$$

where $C_2 > 0$ is a constant. Furthermore, there exists a $C_M > 0$ such that $0 < c(t) \leq C_M, \forall\ t \in [0, T]$. By Hölder inequality, we get

$$\int_0^T |x_n(t)|^2 dt \leq (\int_0^T a(t)|x_n|^{P+1} dt)^{\frac{2}{P+1}} (\int_0^T a^{-\frac{2}{P-1}}(t) dt)^{\frac{P-1}{P+1}}, \tag{2.9}$$

and

$$\int_0^T c(t)|x_n(t)|^2 dt \leq C_M \int_0^T |x_n(t)|^2 dt \leq C_M (\int_0^T a(t)|x_n|^{P+1} dt)^{\frac{2}{P+1}} (\int_0^T a^{-\frac{2}{P-1}}(t) dt)^{\frac{P-1}{P+1}}. \tag{2.10}$$

Next, by (2.7), (2.8), and (2.10), we have

$$\int_0^T |\dot{x}_n(t)|^2 dt \leq C_3,$$

where $C_3 > 0$ is a constant. So, $\|x_n\| \leq C_4$, where $C_4 > 0$ is a constant, i.e., $\{x_n\}$ is bounded in E. Without loss of generality, up to a subsequence, assume that there exists a point $x_0 \in E$ such that as $n \to \infty,\ x_n \rightharpoonup x_0$ in E and $x_n \to x_0$ in $C([0, T], \mathbf{R}^N)$, and furthermore $x_n \to x_0$ in $L^2([0, T], \mathbf{R}^N)$. So

$$\begin{aligned} \|x_n - x_0\| &= \sup_{\|u\| \leq 1} < x_n - x_0, u >= \sup_{\|u\| \leq 1} [\int_0^T (\dot{x}_n - \dot{x}_0, \dot{u}) + (x_n - x_0, u) dt] \\ &= \sup_{\|u\| \leq 1} \{\int_0^T (a(t)[|x_n|^{P-1} x_n - |x_0|^{P-1} x_0], u) + (c(t)(x_n - x_0), u)) dt \\ &\quad + \langle I'(x_n) - I'(x_0), u \rangle + \int_0^T (x_n - x_0, u) dt\} \to 0, \ \text{ as } n \to \infty. \end{aligned}$$

Hence, $x_n \to x_0$ in E. The proof is completed. □

We need some notations to construct the linking. For $j \in \mathbf{N}$, set

$$E(\lambda_j) = ker(\frac{d^2}{dt^2} + \lambda_j), \quad E_j = \bigoplus_{i=1}^{j} E(\lambda_i), \quad E = E_j \bigoplus E_j^{\perp}.$$

For each $j \geq 1$, by (1.4), we have $\nu_j := \dim E(\lambda_j) = 2N$, $i_j := \dim E_j = \sum_{k=0}^{j} \nu_k = 2jN + N$. It's obvious that in $E_j^{\perp}$,

$$\int_0^T |\dot{x}(t)|^2 dt \geq \lambda_{j+1} \int_0^T |x(t)|^2 dt, \tag{2.11}$$

and in E_j,

$$\int_0^T |\dot{x}|^2 dt \leq \lambda_j \int_0^T |x|^2 dt. \tag{2.12}$$

3. Proof of the main result

At the beginning of this section, to verify the conditions (H_1) and (H_2) in Proposition 2.2, we give some preliminary lemmas below.

Lemma 3.1. *For any* $0 < c(t) < \lambda_{j+1}$, $t \in [0, T]$, *there exist* $r_j, \alpha_j > 0$ *such that*

$$\Phi(x) \geq \alpha_j, \ x \in \{x \in E_j^{\perp} |\ \|x\| = r_j\}.$$

Proof. By embedding theorem (Chapter 10 in [1], Chapter 1 in [16]), we have

$$\|x\|_{P+1} \leq C\|x\|, \tag{3.1}$$

where $C > 0$ is a constant, $\|x\|_{P+1} = (\int_0^T |x|^{P+1} dt)^{\frac{1}{P+1}}$, $\|x\| = (\int_0^T |\dot{x}|^2 + |x|^2 dt)^{\frac{1}{2}}$. Meanwhile, since $a(t)$ is a T-periodic continuous function. There exist m, M such that $0 < \alpha \leq m \leq a(t) \leq M$. So by (3.1), the following holds

$$\frac{1}{P+1} \int_0^T a(t)|x|^{P+1} dt \leq M' C^{P+1} \|x\|^{P+1}, \quad \text{where } M' = \frac{M}{P+1}.$$

In addition, since $c(t)$ is a T-periodic continuous function, for any $0 < c(t) < \lambda_{j+1}$, there exists a constant $c_M > 0$, which is the maximum of $c(t)$, $t \in [0, T]$, such that $0 < c(t) \leq c_M < \lambda_{j+1}$. So for $x \in E_j^{\perp}$, by (2.11), we have

$$\begin{aligned}
I(x) &= \frac{1}{2} \int_0^T [|\dot{x}|^2 - c(t)|x|^2] dt - \frac{1}{P+1} \int_0^T a(t)|x|^{P+1} dt \\
&= \frac{1}{2} \int_0^T [|\dot{x}|^2 + |x|^2 - (c(t)+1)|x|^2] dt - \frac{1}{P+1} \int_0^T a(t)|x|^{P+1} dt \\
&\geq \frac{1}{2} \int_0^T [|\dot{x}|^2 + |x|^2 - (c_M+1)|x|^2] dt - M' C^{P+1} \|x\|^{P+1} \qquad (3.2) \\
&\geq \frac{1}{2}(\|x\|^2 - \frac{c_M+1}{\lambda_{j+1}+1} \|x\|^2) - M' C^{P+1} \|x\|^{P+1}
\end{aligned}$$

$$\geq \quad \frac{1}{2}(\frac{\lambda_{j+1} - c_M}{\lambda_{j+1} + 1})\|x\|^2 - M'C^{P+1}\|x\|^{P+1}.$$

Next, taking $\alpha_* = \frac{\lambda_{j+1}-c_M}{\lambda_{j+1}+1} > 0$, $\beta = M'C^{P+1}$, we have

$$I(x) \geq \frac{\alpha_*}{2}\|x\|^2 - \beta\|x\|^{P+1}, \quad x \in E_j^{\perp}.$$

As a result of $P > 1$, the function $g(s) = \frac{\alpha_*}{2}s^2 - \beta s^{P+1}$ on $(0, \infty)$ has its maximum

$$\alpha_j := g_{\max} = g([\frac{\alpha_*}{(P+1)\beta}]^{\frac{1}{P-1}}) = \frac{P-1}{2(P+1)}\alpha_*^{\frac{P+1}{P-1}}(\beta(P+1)^{\frac{2}{1-P}}).$$

Set $r_j = [\frac{\alpha_*}{(P+1)\beta}]^{\frac{1}{P-1}}$. By the process above, we therefore have

$$I(x) \geq \alpha_j, \text{ for } x \in E_j^{\perp} \text{ with } \|x\| = r_j.$$

The proof is completed. □

Remark 3.1. *It's obvious that α_j and r_j decreasingly approach to zero as $c_M \to \lambda_{j+1}^-$.*

Now, for the eigenvalue λ_{j+1} of (1.3), we take the corresponding unit eigenfunction φ_{j+1}, i.e., $\|\varphi_{j+1}\| = 1$. For $j = k, k+1$, define

$$S_k = \{x \in E_k^{\perp} | \; \|x\| = r_k\}, \quad S_{k+1} = \{x \in E_{k+1}^{\perp} | \; \|x\| = r_{k+1}\},$$

$$V_k = E_k \oplus \text{span}\{\varphi_{k+1}\}, \quad V_{k+1} = E_{k+1} \oplus \text{span}\{\varphi_{k+2}\},$$

where r_j is defined in Lemma 3.1, $j = k, k+1$.

Lemma 3.2. *Assume $V(t,x)$ satisfies* (1.1) *and $c(t) \in (\lambda_k, \lambda_{k+2})$.There exist $\sigma \in \mathbf{R}$, $\delta > 0$, and $R > 0$ independent $c(t)$ and δ, such that as $c(t) \in (\lambda_{k+1} - \delta, \lambda_{k+1} + \delta)$, the following holds*

$$I(x) \leq \sigma < \alpha_{k+1}, \quad \textit{for } x \in \partial Q_{k+1},$$

where $Q_{k+1} = \{x \in V_{k+1} | \; \|x\| \leq R, x = x_1 + s\varphi_{k+2}, x_1 \in E_{k+1}, s \geq 0\}$. Furthermore, as $c(t) \in (\lambda_k, \lambda_{k+1})$, there exists a constant $R > 0$ independent of $c(t)$ such that

$$I(x) \leq 0, \; x \in \partial Q_k,$$

where $Q_k = \{x \in V_k | \; \|x\| \leq R, x = x_1 + s\varphi_{k+1}, x_1 \in E_k, s \geq 0\}$.

Proof. Since $a(\cdot) > 0$ is a T-periodic continuous function and $a(t) \geq \alpha, \; t \in [0, T]$, by (2.12) and $c(t) \in (\lambda_k, \lambda_{k+2})$, we easily get, for $x = x_1 + x_2 + x_3 \in V_{k+1}$, where $x_1 \in E_k, \; x_2 \in E(\lambda_{k+1}), \; x_3 \in \text{span}\{\varphi_{k+2}\}$,

$$\begin{aligned} I(x) &= \frac{1}{2}\int_0^T [|\dot{x}|^2] - c(t)|x|^2]dt - \int_0^T \frac{a(t)}{P+1}|x|^{P+1}dt \\ &\leq \frac{1}{2}\int_0^T (\lambda_k - c(t))|x_1|^2 + (\lambda_{k+1} - c(t))|x_3|^2 + (\lambda_{k+2} - c(t))|x_3|^2 dt - \frac{\alpha}{P+1}\int_0^T |x|^{P+1}dt \end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{2}\int_0^T (\lambda_{k+2} - c(t))|x|^2 dt - \frac{\alpha}{P+1}\int_0^T |x|^{P+1} dt \\
&\le \frac{1}{2}\int_0^T (\lambda_{k+2} - \lambda_k)|x|^2 dt - \frac{\alpha}{P+1}\int_0^T |x|^{P+1} dt. \qquad (3.3)
\end{aligned}$$

Since $P > 1, \dim V_{k+1} < \infty$, and all norms are equivalent in a space with a finite dimension, by (3.3), we have $I(x) \to -\infty,$ as $\|x\| \to \infty$. So, there exists a constant $R > 0$ independent of $c(t)$ such that

$$I(x) \le 0,\ x \in V_{k+1},\ \|x\| = R. \qquad (3.4)$$

Now, choosing $R > \max\{r_k, r_{k+1}\} > 0$. For $E_{k+1} \ni y = y_1 + y_2$ with $\|y\| \le R$, where $y_1 \in E_k, y_2 \in E(\lambda_{k+1})$.

Next, for $c(t) \in (\lambda_k, \lambda_{k+2})$, by (2.12), we have

$$\begin{aligned}
I(y) &\le \frac{1}{2}(\lambda_k - c(t))\|y_1\|_2^2 + \frac{1}{2}(\lambda_{k+1} - c(t))\|y_2\|_2^2 \\
&\le \frac{1}{2}|\lambda_{k+1} - c(t)|R^2.
\end{aligned}$$

Now, taking $\sigma = \frac{\alpha_{k+1}}{2}, \delta = \frac{2\sigma}{R^2}$, where α_{k+1} is defined in Lemma 3.1, we get as $|\lambda_{k+1} - c(t)| < \delta$,

$$I(y) \le \sigma < \alpha_{k+1},\ \text{for } y \in E_{k+1}, \|y\| \le R,\ \text{ as } |\lambda_{k+1} - c(t)| < \delta. \qquad (3.5)$$

Meanwhile, we have

$$\partial Q_{k+1} = \{x = x_1 + s\varphi_{k+2} |\ \|x\| = R, x_1 \in E_{k+1}, s \ge 0\} \cup \{x_1 \in E_{k+1} |\ \|x_1\| \le R\}.$$

Then, by (3.4) and (3.5), we obtain

$$I(x) \le \sigma < \alpha_{k+1}, \forall x \in \partial Q_{k+1},\ \text{as } c(t) \in (\lambda_{k+1} - \delta, \lambda_{k+1} + \delta).$$

Furthermore, as a matter of fact, we have

$$\partial Q_k = \{x = x_1 + t\varphi_{k+1} | \|x\| = R,\ x_1 \in E_k, t \ge 0\} \cup \{x_1 \in E_k | \|x\| \le R\}. \qquad (3.6)$$

On the one hand, since $V_k \subset V_{k+1}$, for $c(t) \in (\lambda_k, \lambda_{k+1}) \subset (\lambda_k, \lambda_{k+2}),\ R > 0$ mentioned above, by (3.4), it follows that

$$I(x) \le 0,\ x \in V_k, \|x\| = R. \qquad (3.7)$$

On the other hand, it's obvious that, for $\forall\ x_1 \in E_k$,

$$I(x_1) \le \frac{1}{2}(\lambda_k - c(t))\|x_1\|_2^2 - \int_0^T \frac{a(t)}{P+1}|x_1|^{P+1} dt \le 0. \qquad (3.8)$$

So, by (3.6), (3.7), and (3.8), we get

$$I(x) \le 0, \quad x \in \partial Q_k.$$

The proof is completed. □

Now, by two lemmas proved above, using the Proposition 2.2, we could prove the main result.

Proof. Proof of Theorem 1.1. To give a clear proof, we shall take several steps to finish it.

Firstly, by Lemma 2.1, I satisfies the (PS) condition.

Secondly, by Lemmas 3.1 and 3.2, I satisfies the (H_1) and (H_2) in Proposition 2.2. So the following holds

$$\inf_{x\in S_{k+1}} I(x) \geq \alpha_{k+1} > \frac{\alpha_{k+1}}{2} \geq \max_{x\in\partial Q_{k+1}} I(x).$$

Next, for the R mentioned in Lemma 3.2, it satisfies the following

$$R > \max\{r_k, r_{k+1}\} > 0. \tag{3.9}$$

So we have S_{k+1} and ∂Q_{k+1} homotopically link w.r.t. the decomposition $E = E_{k+1} \bigoplus E_{k+1}^{\perp}$. Furthermore, since $\dim V_{k+1} = i_{k+1} + 1$, by Proposition 2.2, we have a critical point x_1 of the functional I with $I(x_1) \geq \alpha_{k+1} > 0$ and

$$C_{i_{k+1}+1} \not\cong 0.$$

Thirdly, for the situation $c(t) \in (\lambda_{k+1} - \delta, \lambda_{k+1}) \subset (\lambda_k, \lambda_{k+1})$, I satisfies (H_1) and (H_2). What's more, the following holds

$$\inf_{x\in S_k} I(x) \geq \alpha_k > 0 \geq \max_{x\in\partial Q_k} I(x).$$

By (3.9), S_k and ∂Q_k homotopically link w.r.t. the decomposition $E = E_k \bigoplus E_k^{\perp}$. Furthermore, since $\dim V_k = i_k + 1$, by Proposition 2.2, we have a critical point x_2 of the functional I with $I(x_2) \geq \alpha_k > 0$ and

$$C_{i_k+1} \not\cong 0.$$

Finally, as a matter of fact, we have

$$(i_{k+1} + 1) - (i_k + 1) = i_{k+1} - i_k = [2(k+1)N + N] - (2kN + N) = 2N.$$

Combining the following fact,

$$\begin{aligned} \dim\ker I''(x_*) &= \dim\{x \in E | \ddot{x} + (c(t)I_N + (a(t)|x|^{P-1}x)'_x|_{x=x_*})x = 0\} \\ &= \dim\{x \in E | \ddot{x} + (c(t)I_N + B(t, x_*))x = 0\} \leq 2N, \end{aligned}$$

where x_* is a critical point of the functional I, and

$$B(t, x_c) = \begin{pmatrix} \frac{\partial a(t)|x|^{P-1}x_1}{\partial x_1} & \frac{\partial a(t)|x|^{P-1}x_1}{\partial x_2} & \cdots & \frac{\partial a(t)|x|^{P-1}x_1}{\partial x_N} \\ \frac{\partial a(t)|x|^{P-1}x_2}{\partial x_1} & \frac{\partial a(t)|x|^{P-1}x_2}{\partial x_2} & \cdots & \frac{\partial a(t)|x|^{P-1}x_2}{\partial x_N} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial |x|^{P-1}x_N}{\partial x_1} & \frac{\partial a(t)|x|^{P-1}x_N}{\partial x_1} & \cdots & \frac{\partial a(t)|x|^{P-1}x_N}{\partial x_N} \end{pmatrix}\Bigg|_{x=x_*},$$

by Proposition 2.1(2), we obtain $x_1 \neq x_2$. The proof is completed. □

Right now, we can directly have the following conclusion by Theorem 1.1.

Corollary 3.1. *Assume $V(t, x)$ satisfies (1.1) and $k(\in \mathbf{N}) \geq 1$. Then, there exists a $\delta > 0$ such that as $c(t) \in (\lambda_{k+1}, \lambda_{k+1} + \delta)$, (1.2) has at least one nontrivial T-periodic solution.*

4. Conclusions

In this paper, we prove the existence of multiple nontrivial T-periodic solutions of the equation $\ddot{x} + V_x(t, x) = 0$ in $\mathbf{R}^N$, $N(\geq 1)$, where $V_x(t, x)$ is the derivative of $V(t, x)$ satisfying (1.1) with respect to x. Since the nontrivial T-periodic solutions correspond to the critical points of the functional of the problem, to get the multiple T-periodic solutions, we prove the existence and multiplicity of critical points of the functional. Employing the homological linking and morse theory, we get at least two nontrivial T-periodic solutions and distinguish them by critical groups. By the way, as the range of the T-periodic function $c(t)$ satisfying (H_c) in (1.1), a right small neighborhood of the k-th eigenvalue of (1.3), there exists a nontrivial T-periodic solution. Here, our main result is different from the previous works (see [1, 2] and the references therein).

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Conflict of interest

The authors declare no conflicts of interest.

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