AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(5): 11238–11258.
DOI: 10.3934/math.2023569
Received: 30 August 2022
Revised: 04 December 2022
Accepted: 10 January 2023
Published: 09 March 2023

Research article

Set-valued minimax programming problems under σ-arcwisely connectivity

Koushik Das[1], **Savin Treanţă**[2,3,4] **and Thongchai Botmart**[5,*]

[1] Department of Mathematics, Taki Government College, Taki 743429, India
[2] Department of Applied Mathematics, University Politehnica of Bucharest, Bucharest 060042, Romania
[3] Academy of Romanian Scientists, 54 Splaiul Independentei, Bucharest 050094, Romania
[4] "Fundamental Sciences Applied in Engineering" Research Center (SFAI), University Politehnica of Bucharest, Bucharest 060042, Romania
[5] Department of Mathematics, Faculty of Science, Khon Kaen University, Khoon Kaen 40002, Thailand

* **Correspondence:** Email: thongbo@kku.ac.th.

Abstract: A set-valued minimax programming problem (in short, SVMP) is taken into consideration in this study. We present the idea of σ-arcwisely connectivity of set-valued maps (in short, SVM) in the broader sense of arcwisely connected SVMs. The sufficient criteria for Karush-Kuhn-Tucker (KKT) optimality are constituted for the problem (MP) under contingent epidifferentiation and σ-arcwisely connectivity suppositions. In addition, we develop the Mond-Weir (MWD), Wolfe (WD), and mixed (MD) kinds models of duality and verify the associated strong, weak, and converse theorems of duality among the primal (MP) and the associated figures of duals under σ-arcwisely connectivity supposition.

Keywords: convex cone; contingent epidifferentiation; arcwisely connectivity; duality
Mathematics Subject Classification: 26B25, 49N15

1. Introduction

An exclusive subset of optimization problems is the class of minimax programming problems. It is used in a variety of mathematical, economic, and operational research domains. In 1966, Bram [7] and Danskin [11, 12] constituted the necessary conditions of optimality of static minmax programming problems (in short, MPPs) with the help of Lagrange multiplier rule. Later, in 1977, Schmitendorf [25] derived the sufficient as well as necessary conditions of optimality for MPPs. Later, Tanimoto [26] studied theorems of duality of various kinds for the MPPs. After that, Demyanov and Malozehon [19], Datta and Bhatia [18], Zalmai [30], Bector and Bhatia [4], Bector et al. [5] and Chandra and Kumar [9] provided the sufficient as well as necessary conditions of optimality of MPPs and proved the theorem of duality. In 1999, Mehra and Bhatia [23] developed the necessary conditions of optimality for MPPs. They further formulated the Mond-Weir kind results of duality under arcwisely connectivity and generalized arcwisely connectivity suppositions.

Avriel [3] presented the idea of arcwisely connectivity in the theory of vector optimization. By substituting a continuous arc for the line segment connecting two points, the idea of convexity was generalized to arcwisely connectivity. Fu and Wang [20] and Lalitha et al. [22] proposed the idea of arcwisely connected SVMs. It is a development of the cone convex SVM class. Lalitha et al. [22] provided the sufficient condition of optimality for set-valued optimization problems (in short, SVOPs) via contingent epidifferentiation and cone arcwisely connectivity suppositions. Yu [28] provided the sufficient as well as necessary conditions of optimality for the identification of proper global points of efficiency of vector optimization problems (in short, VOPs) involving arcwisely connected SVMs. Yihong and Min [27] proposed the idea of nearly arcwisely connected SVMs of α-order. They also proved the sufficient as well as necessary conditions of optimality of SVOPs. Yu [29] provided the sufficient as well as necessary conditions of optimality for proper global efficiency in VOPs under cone arcwisely connectivity supposition. Peng and Xu [24] proposed the idea of cone subarcwisely connected SVMs. They also constituted the necessary conditions of optimality of second-order for the identification of local proper global efficient elements of SVOPs.

The notion of contingent derivative is basically an extension of the Frechet differentiability from single-valued to set-valued case. This concept has also been widely used in set-valued optimization theory. But the necessary and sufficient optimality conditions do not coincide under standard assumptions. So contingent derivatives are not completely the right tool for the formulation of optimality conditions in set-valued optimization. The notion of contingent epiderivative is one possible generalization of directional derivatives in the single-valued convex case. The main differences between the contingent derivative and the contingent epiderivative are that the graph is now replaced by the epigraph and the derivative is now single-valued. A special property of contingent epiderivatives in the cone-convex case is that they are sublinear, if they exist. So we are more interested in contingent epiderivative in the study of set-valued minimax programming problems.

The concept of arcwise connectedness is a generalization of convexity by replacing the line segment joining two points by a continuous arc. We introduce the notion of σ-cone arcwise connectedness of set-valued maps as a generalization of cone arcwise connected set-valued maps. For $\sigma = 0$, we get the usual notion of cone arcwise connected set-valued maps. Further, we construct an example of σ-cone arcwise connected set-valued map, which is not cone arcwise connected. We are mainly interested in finding the sufficient optimality conditions of set-valued minimax programming problems in more generalized case by using contingent epiderivative and σ-cone arcwise connectedness assumptions on the objective functions and constraints. We also study the weak, strong, and converse duality theorems of Mond-Weir, Wolfe, and mixed types under contingent epiderivative and σ-cone arcwise connectedness assumptions. For $\sigma = 0$, our results reduce the existing ones available in the literature.

The dual of dual need not be the original primal for the general nonlinear programming problems. Duality is not defined uniquely. This has led to the introduction of some other forms of dual such as the Lagrangian, Mangasarian, Wolfe, and Mond-Weir types. The beauty of Mond-Weir type dual is that the objective function is same as that of the primal problem. The mixed type dual is constructed as a combination of the Mond-Weir and Wolfe dual problems.

The Wolfe dual to a constrained scalar optimization problem was introduced by Wolfe, while the Mond-Weir dual followed twenty years later, due to Mond and Weir. In both cases the functions involved were considered differentiable and endowed with generalized convexity properties. Then these duality concepts evolved parallelly and one can distinguish two main directions for both of them. On one hand, considering the primal problem convex, the differentiability assumptions were dropped and the gradients that appear in the duals were replaced by subdifferentials. On the other hand, especially for Mond-Weir duality, the differentiability continued to play an important role and the convexity assumptions on the functions involved were weakened.

In this study, we determine the KKT conditions of sufficiency of the SVMPP (MP) under contingent epidifferentiation and σ-arcwisely connectivity suppositions. We construct the duals of Mond-Weir (MWD), Wolfe (WD), and mixed (MD) kinds and verify the associated theorems of duality under σ-arcwisely connectivity supposition.

The structure of this paper is as follows. SVM definitions and basic ideas are covered in Section 2. We present the notion of σ-arcwisely connectivity of SVMs in the broader sense of arcwisely connected SVMs in Section 3. We also give an illustration of σ-arcwisely connected SVM, which is not necessarily arcwisely connected. In Section 4, we give the primal problem SVMPP (MP) of set-valued minimax programming. In Section 5, the KKT requirements of sufficiency are constituted for the problem (MP). In Section 6, we establish the weak, strong, and converse duality results of Mond-Weir kind dual (MWD), where the SVMs $\psi(., q_m)$ and ω_n are contingent epidifferentiable using the assumptions of generalized cone arcwise connectivity. In Sections 7 and 8, we develop the duality results of Wolfe (WD), and mixed (MD) kinds models under σ-arcwisely connectivity supposition. Section 9 is the concluding remarks of our work.

2. Definitions and preliminaries

Let β be a real normed space (in short, RNS) and Q be a nonvoid subset of β. Then, Q is referred to as a cone if $\xi q \in Q$, for every $q \in Q$ and $\xi \in \mathbb{R}$, with $\xi \geq 0$. Moreover, Q is referred to as non-trivial if $Q \neq \{0_\beta\}$, proper if $Q \neq \beta$, pointed if $Q \cap (-Q) = \{0_\beta\}$, solid if $\text{int}(Q) \neq \emptyset$, closed if $\overline{Q} = Q$, and convex if $\xi Q + (1-\xi)Q \subseteq Q$, for every $\xi \in [0, 1]$, where $\text{int}(Q)$ and $\overline{Q}$ represent the interior and closure of Q, respectively and 0_β is the zero element of β.

Aubin [1, 2] proposed the idea of contingent cone in a RNS. Additionally, Aubin [1, 2] and Cambini et al. [8] proposed the idea of second-order contingent set in a RNS.

Definition 2.1. *[1, 2] Let β be a RNS, $\emptyset \neq N \subseteq \beta$, and $q' \in \overline{N}$. The contingent cone to N at q' is signified by $\eta(N, q')$ and is interpreted as follows: an element $q \in \eta(N, q')$ if there appear sequences $\{\xi_n\}$ in $\mathbb{R}$, with $\xi_n \to 0^+$ and $\{q_n\}$ in β, with $q_n \to q$, satisfying*

$$q' + \xi_n q_n \in N, \quad \forall n \in \mathbb{N},$$

or, there appear sequences $\{\nu_n\}$ in $\mathbb{R}$, with $\nu_n > 0$ and $\{q'_n\}$ in N, with $q'_n \to q'$, satisfying

$$\nu_n(q'_n - q') \to q.$$

Let α, β be RNSs, 2^β be the set of all subsets of β, and Q be a pointed solid convex cone in β. Let $\pi : \alpha \to 2^\beta$ be a SVM from α to β, i.e., $\pi(p) \subseteq \beta$, for every $p \in \alpha$. The domain, image, graph, and epigraph of π are interpreted by

$$\text{domain}(\pi) = \{p \in \alpha : \pi(p) \neq \emptyset\},$$

$$\pi(M) = \bigcup_{p \in M} \pi(p), \text{ for any } \emptyset \neq M \subseteq \alpha,$$

$$\text{grp}(\pi) = \{(p, q) \in \alpha \times \beta : q \in \pi(p)\},$$

and

$$\text{epigrp}(\pi) = \{(p, q) \in \alpha \times \beta : q \in \pi(p) + Q\}.$$

Jahn and Rauh [21] proposed the idea of contingent epidifferentiation of SVMs.

Definition 2.2. *[21] A function $\overrightarrow{\Delta}\pi(p', q') : \alpha \to \beta$ whose epigraph is identical with the contingent cone to the epigraph of π at (p', q'), i.e.,*

$$\text{epigrp}(\overrightarrow{\Delta}\pi(p', q')) = \eta(\text{epigrp}(\pi), (p', q')),$$

is presumed to be the contingent epidifferentiation of π at (p', q').

We now focus on the idea of the cone convexity of SVMs, developed by Borwein [6].

Definition 2.3. *[6] Let M be a nonvoid convex subset of a RNS α. A SVM $\pi : \alpha \to 2^{\beta}$, with $M \subseteq$ domain(π), is referred to as Q-convex on M if $\forall p_1, p_2 \in M$ and $\xi \in [0, 1]$,*

$$\xi\pi(p_1) + (1 - \xi)\pi(p_2) \subseteq \pi(\xi p_1 + (1 - \xi)p_2) + Q.$$

Avriel [3] presented the concept of arcwisely connectivity in the broader sense of convexity.

Definition 2.4. *A subset M of a RNS α is presumed to be an arcwisely connected set if for every $p_1, p_2 \in M$ there appears a continuous arc $\chi_{p_1,p_2}(\xi)$ defined on $[0, 1]$ with a value in M satisfying $\chi_{p_1,p_2}(0) = p_1$ and $\chi_{p_1,p_2}(1) = p_2$.*

Fu and Wang [20] and Lalitha et al. [22] proposed the idea of arcwisely connected SVMs as an elongation of the class of convex SVMs.

Definition 2.5. *[20, 22] Let M be an arcwisely connected subset of a RNS α and $\pi : \alpha \to 2^{\beta}$ be a SVM, with $M \subseteq$ domain(π). Then π is presumed to be Q-arcwisely connected on M if*

$$(1 - \xi)\pi(p_1) + \xi\pi(p_2) \subseteq \pi(\chi_{p_1,p_2}(\xi)) + Q, \quad \forall p_1, p_2 \in M \text{ and } \forall \xi \in [0, 1].$$

Peng and Xu [24] proposed the idea of cone subarcwisely connected SVMs.

Definition 2.6. *[24] Let M be an arcwisely connected subset of a RNS α, $e \in$ int(Q), and $\pi : \alpha \to 2^{\beta}$ be a SVM, with $M \subseteq$ domain(π). Then π is presumed to be Q-subarcwisely connected on M if*

$$(1 - \xi)\pi(p_1) + \xi\pi(p_2) + \epsilon e \subseteq \pi(\chi_{p_1,p_2}(\xi)) + Q, \ \forall p_1, p_2 \in M, \ \forall \epsilon > 0, \text{ and } \forall \xi \in [0, 1].$$

Remark 2.1. *Let α and β be real topological linear spaces (in short, RTLSs). A SVM $\pi : \alpha \to 2^{\beta}$ is referred to as upper semicontinuous if $\pi^{+}(N) = \{p \in \alpha : \pi(p) \subseteq N\}$ is an open set in α for an arbitrary open set N in β.*

Remark 2.2. *Let β be a RTLS, Q be a cone in β, and N be a nonvoid subset of β. Then, N is referred to as Q-semicompact if for every complements of all open cover having the form*

$$\{(q_m + \overline{Q})^c : q_m \in N, i \in I\},$$

a finite subcover exists.

Remark 2.3. *Let α and β be RTVSs and Q be a cone in β. A SVM $\pi : \alpha \to 2^\beta$ is referred to as Q-semicompact-valued if $\pi(p)$ is Q-semicompact for every $p \in \alpha$.*

Let α, β be RTVSs, M be a nonvoid subset of α, $\pi : \alpha \to 2^\beta$ be a SVM, and Q be a pointed convex cone in β. Consider a SVOP (P):

$$\max_{p \in M} \quad \pi(p). \tag{P}$$

The maximizer of the problem (P) is illustrated as follows:

Definition 2.7. *Let $p' \in M$ and $q' \in \pi(p')$. Then, (p', q') is referred to as to maximize the problem* (P) *if there appear no $p \in M$ and $q \in \pi(p)$ satisfying*

$$q' < q.$$

Corley [10] constituted necessary conditions of optimality for identification of maximal points of SVOPs in RTVSs.

Theorem 2.1. *[10] Let α, β be RTVSs, M be a nonvoid compact subset of α, and $\overline{Q}$ is pointed convex cone in β. Let $\pi : \alpha \to 2^\beta$ be Q-semicompact-valued and upper semicontinuous. Then there appears a maximal point for the problem* (P).

3. σ-arcwisely connectivity

Das and Nahak [13–17] proposed the idea of σ convexity of SVMs. They constituted the KKT conditions of sufficiency of optimality and verify the results of duality for various kinds of SVOPs under contingent epidifferentiation and σ convexity suppositions. For $\sigma = 0$, it diminishes to the common concept of cone convex SVMs, presented by Borwein [6].

We present the notion of σ-arcwisely connectivity of SVMs in the broader sense of arcwisely connected SVMs.

Definition 3.1. *Let M be an arcwisely connected subset of a RNS α, $p_1, p_2 \in M$, $e \in \operatorname{int}(Q)$, and $\pi : \alpha \to 2^\beta$ be a SVM, with $M \subseteq \operatorname{domain}(\pi)$. Then, π is presumed to be σ-Q-arcwisely connected w.r.t. e on M for p_1, p_2 if there appears $\sigma \in \mathbb{R}$, satisfying*

$$(1-\xi)\pi(p_1) + \xi\pi(p_2) \subseteq \pi(\chi_{p_1,p_2}(\xi)) + \sigma\xi(1-\xi)\|p_1 - p_2\|^2 e + Q, \ \forall \xi \in [0,1]. \tag{3.1}$$

Remark 3.1. *Let M be an arcwisely connected subset of a RNS α, $e \in \operatorname{int}(Q)$, and $\pi : \alpha \to 2^\beta$ be a SVM, with $M \subseteq \operatorname{domain}(\pi)$. Then, π is presumed to be σ-Q-arcwisely connected w.r.t. e on M if there appears $\sigma \in \mathbb{R}$, satisfying* (3.1) *for every $p_1, p_2 \in M$.*

Remark 3.2. *If $\sigma > 0$, then, π is presumed to be strongly σ-Q-arcwisely connected, if $\sigma = 0$, we have the common concept of Q-arcwisely connectivity , and if $\sigma < 0$, then π is presumed to be weakly σ-Q-arcwisely connected. Undoubtedly, strongly σ-Q-arcwisely connectivity $\Rightarrow$ Q-arcwisely connectivity $\Rightarrow$ weakly σ-Q-arcwisely connectivity.*

Additionally, we create an illustration of σ-arcwisely connected SVM, which is not necessarily arcwisely connected.

Example 3.1. Let $\alpha = \mathbb{R}^2,\ \beta = \mathbb{R},\ Q = \mathbb{R}_+$, and

$$M = \left\{p = (p_1, p_2) \mid p_1 + p_2 \geq \frac{1}{2},\ p_1 \geq 0,\ p_2 \geq 0\right\} \subseteq \alpha.$$

Define

$$\chi_{p,p'}(\xi) = (1 - \xi^2)p + \xi^2 p',$$

where $p = (p_1, p_2),\ p' = (p'_1, p'_2)$, and $\xi \in [0, 1]$. Evidently, M is an arcwisely connected set. Let us consider a SVM $\pi : \mathbb{R}^2 \to 2^{\mathbb{R}}$ as follows:

$$\pi(p) = \begin{cases} [0, 2], & \text{if } p_1 + p_2 \geq \frac{1}{2},\ p_1 \neq 3p_2,\ p = (p_1, p_2), \\ [3, 5], & \text{otherwise.} \end{cases}$$

We select $p = (1, 0),\ p' = (0, 1)$, and $\xi = \frac{1}{2}$. So,

$$\chi_{p,p'}\Big(\frac{1}{2}\Big) = \Big(\frac{3}{4}, \frac{1}{4}\Big)$$

and

$$\frac{1}{2}\pi(1, 0) + \frac{1}{2}\pi(0, 1) = \frac{1}{2}[0, 2] + \frac{1}{2}[0, 2] = [0, 2] \nsubseteq [3, 5] + \mathbb{R}_+ = \pi\Big(\frac{3}{4}, \frac{1}{4}\Big) + \mathbb{R}_+.$$

Hence, π is not $\mathbb{R}_+$-arcwisely connected. However, by taking into account $\sigma = -2$ and $e = 3$, we have that

$$(1 - \xi)\pi(1, 0) + \xi\pi(0, 1) = (1 - \xi)[0, 2] + \xi[0, 2] = [0, 2]$$

and

$$\pi(\chi_{p,p'}(\xi)) + \sigma\xi(1 - \xi)\|p - p'\|^2 e = \pi(1 - \xi^2, \xi^2) - 12\xi(1 - \xi).$$

For $\xi \neq \frac{1}{2}$,

$$\pi(1 - \xi^2, \xi^2) = [0, 2].$$

So,

$$(1 - \xi)\pi(1, 0) + \xi\pi(0, 1) + 12\xi(1 - \xi) = [0, 2] + 12\xi(1 - \xi) \subseteq [0, 2] + \mathbb{R}_+ = \mathbb{R}_+.$$

For $\xi = \frac{1}{2}$, we have

$$\pi(1 - \xi^2, \xi^2) = \pi\Big(\frac{3}{4}, \frac{1}{4}\Big) = [3, 5].$$

So,

$$(1 - \xi)\pi(1, 0) + \xi\pi(0, 1) + 12\xi(1 - \xi) = [0, 2] + 3 = [3, 5] \subseteq [3, 5] + \mathbb{R}_+.$$

Accordingly, π is a (-2)-$\mathbb{R}_+$-arcwisely connected SVM w.r.t. 3 on M for $(1, 0),\ (0, 1)$.

Theorem 3.1. *Let M be an arcwisely connected subset of a RNS α, $e \in \operatorname{int}(Q)$, and $\pi : \alpha \to 2^{\beta}$ be σ-Q-arcwisely connected w.r.t e on M. Let $p' \in M$ and $q' \in \pi(p')$. Then,*

$$\pi(p) - q' \subseteq \overrightarrow{\Delta}\pi(p', q')(\chi'_{p',p}(0+)) + \sigma\|p - p'\|^2 e + Q, \quad \forall p \in M,$$

where

$$\chi'_{p',p}(0+) = \lim_{\xi \to 0+} \frac{\chi_{p',p}(\xi) - \chi_{p',p}(0)}{\xi},$$

presuming that $\chi'_{p',p}(0+)$ exists for every $p, p' \in M$.

Proof. Let $p \in M$. As π is σ-Q-arcwisely connected w.r.t. e on M, we have

$$(1 - \xi)\pi(p') + \xi\pi(p) \subseteq \pi(\chi_{p',p}(\xi)) + \sigma\xi(1 - \xi)\|p - p'\|^2 e + Q, \ \forall \xi \in [0, 1].$$

Let $q \in \pi(p)$. Choose a sequence $\{\xi_n\}$, with $\xi_n \in (0, 1)$, $n \in \mathbb{N}$, satisfying $\xi_n \to 0+$ when $n \to \infty$. Let us consider

$$p_n = \chi_{p',p}(\xi_n)$$

and

$$q_n = (1 - \xi_n)q' + \xi_n q - \sigma\xi_n(1 - \xi_n)\|p - p'\|^2 e.$$

So,

$$q_n \in \pi(p_n) + Q.$$

It is undeniable that

$$p_n = \chi_{p',p}(\xi_n) \to \chi_{p',p}(0) = p', \ q_n \to q', \ \text{whenever } n \to \infty,$$

$$\frac{p_n - p'}{\xi_n} = \frac{\chi_{p',p}(\xi_n) - \chi_{p',p}(0)}{\xi_n} \to \chi'_{p',p}(0+), \ \text{whenever } n \to \infty,$$

and

$$\frac{q_n - q'}{\xi_n} = q - q' - \sigma(1 - \xi_n)\|p - p'\|^2 e \to q - q' - \sigma\|p - p'\|^2 e, \ \text{whenever } n \to \infty.$$

Therefore,

$$(\chi'_{p',p}(0+), q - q' - \sigma\|p - p'\|^2 e) \in \eta(\operatorname{epigrp}(\pi), (p', q')) = \operatorname{epigrp}(\overrightarrow{\Delta}\pi(p', q')).$$

Accordingly,

$$q - q' - \sigma\|p - p'\|^2 e \in \overrightarrow{\Delta}\pi(p', q')(\chi'_{p',p}(0+)) + Q,$$

that is accurate, for every $q \in \pi(p)$. So,

$$\pi(p) - q' \subseteq \overrightarrow{\Delta}\pi(p', q')(\chi'_{p',p}(0+)) + \sigma\|p - p'\|^2 e + Q, \quad \forall p \in M.$$

4. Construction of the primary problem

Let M be a nonvoid subset of $\mathbb{R}^i$ and N be a nonvoid compact subset of $\mathbb{R}^j$. Let $\psi : \mathbb{R}^i \times \mathbb{R}^j \to 2^{\mathbb{R}}$ and $\omega : \mathbb{R}^i \to 2^{\mathbb{R}^k}$ be two SVMs with

$$M \times N \subseteq \text{domain}(\psi) \text{ and } M \subseteq \text{domain}(\omega).$$

We assume the following SVMPP (MP):

$$\begin{aligned} &\underset{p\in M}{\text{minimize}} \quad \max \bigcup_{q\in N} \psi(p,q), \\ &\text{s. t.} \qquad\quad \omega(p) \cap (-\mathbb{R}^k_+) \neq \emptyset, \end{aligned} \tag{MP}$$

where the SVM $\psi(p,.) : \mathbb{R}^j \to 2^{\mathbb{R}}$ is $\mathbb{R}_+$-semicompact-valued and upper semicontinuous on N, for every $p \in M$.

The feasible set of (MP) can be constituted as

$$S = \{p \in M : \omega(p) \cap (-\mathbb{R}^k_+) \neq \emptyset\}.$$

For $p \in M$, suppose that

$$C(p) = \{n \in \mathbb{N} : 0 \in \omega_n(p), 1 \leq n \leq k\},$$
$$D(p) = \{1, ..., k\} \setminus C(p),$$

and

$$N(p) = \{n \in N : \max \bigcup_{q\in N} \psi(p,q) \subseteq \psi(p,n)\}.$$

Under these suppositions, $N(p) \neq \emptyset$, for every $p \in M$.

When ψ and ω becomes single-valued functions, the problem (MP) diminishes to the minimax programming problem.

The minimizer of the problem (MP) is illustrated as follows:

Definition 4.1. *Let $p' \in S$ and $r' \in \max \bigcup_{q\in N} \psi(p',q)$. Then, (p',r') is referred to as to minimize the problem (MP) if there appear no $p \in S$ and $r \in \max \bigcup_{q\in N} \psi(p,q)$ satisfying $r < r'$.*

5. Sufficient conditions of optimality

We determine the KKT conditions of sufficiency for the SVMPP (MP) under σ-arcwisely connectivity supposition.

Theorem 5.1. *(Sufficient conditions of optimality) Let M be an arcwisely connected subset of $\mathbb{R}^i$, $p' \in S$ and $r' \in \max \bigcup_{q\in N} \psi(p',q)$. Assume that there appear a natural number l, $r^*_m \geq 0$, $q_m \in N(p')$ $(1 \leq m \leq l)$ with $\sum_{m=1}^{l} r^*_m \neq 0$ and $s^*_n \geq 0$, $s'_n \in \omega_n(p') \cap (-\mathbb{R}_+)$ $(1 \leq n \leq k)$ satisfying*

$$\sum_{m=1}^{l} r^*_m \overrightarrow{\Delta}\psi(., q_m)(p',r')(\chi'_{p',p}(0+)) + \sum_{n=1}^{k} s^*_n \overrightarrow{\Delta}\omega_n(p',s'_n)(\chi'_{p',p}(0+)) \geq 0, \ \forall p \in M, \tag{5.1}$$

and

$$s_n^* s'_n = 0,\ \forall n = 1, ..., k. \tag{5.2}$$

If $\psi(., q_m)$ $(1 \leq m \leq l)$ and ω_n $(1 \leq n \leq k)$ are σ_m-$\mathbb{R}_+$-arcwisely connected and σ'_n-$\mathbb{R}_+$-arcwisely connected w.r.t. 1*, on M, respectively, together with*

$$\sum_{m=1}^{l} r_m^* \sigma_m + \sum_{n=1}^{k} s_n^* \sigma'_n \geq 0, \tag{5.3}$$

then, (p', r') minimizes the problem (MP).

Proof. Presume that (p', r') does not minimize the problem (MP). Then, there appear $p \in S$ and $r \in \max \bigcup_{q \in N} \psi(p, q)$ satisfying

$$r < r'.$$

As $q_m \in N(p')$ $(1 \leq m \leq l)$,

$$\max \bigcup_{q \in N} \psi(p', q) \subseteq \psi(p', q_m).$$

As $r' \in \max \bigcup_{q \in N} \psi(p', q)$,

$$r' \in \psi(p', q_m),\ \forall m = 1, ..., l.$$

Choose

$$r_m \in \psi(p, q_m),\ \forall m = 1, ..., l.$$

Again, as $r \in \max \bigcup_{q \in N} \psi(p, q)$ and $q_m \in N(p') \subseteq N$,

$$r_m \leq r.$$

Therefore,

$$r_m \leq r < r'.$$

So,

$$\sum_{m=1}^{l} r_m^* r_m < \sum_{m=1}^{l} r_m^* r'.$$

As $p \in S$, there appears

$$s_n \in \omega_n(p) \cap (-\mathbb{R}_+)\ (1 \leq n \leq k).$$

As $s_n^* \geq 0$ $(1 \leq n \leq k)$,

$$s_n^* s_n \leq 0,\ \forall n = 1, ..., k.$$

Hence,

$$\sum_{n=1}^{k} s_n^* s_n \leq 0.$$

As $s_n^* s'_n = 0,\ \forall n = 1, ..., k$,

$$\sum_{n=1}^{k} s_n^* s_n \leq \sum_{n=1}^{k} s_n^* s'_n.$$

So,

$$\sum_{m=1}^{l} r_m^* r_m + \sum_{n=1}^{k} s_n^* s_n < \sum_{m=1}^{l} r_m^* r' + \sum_{n=1}^{k} s_n^* s_n'. \tag{5.4}$$

Since $\psi(., q_m)$ $(1 \leq m \leq l)$ and ω_n $(1 \leq n \leq k)$ are σ_m-$\mathbb{R}_+$-arcwisely connected and σ_n'-$\mathbb{R}_+$-arcwisely connected w.r.t. 1, on M,

$$\psi(p, q_m) - r' \subseteq \overrightarrow{\Delta}\psi(., q_m)(p', r')(\chi'_{p',p}(0+)) + \sigma_m\|p - p'\|^2 + \mathbb{R}_+,$$

and

$$\omega_n(p) - s_n' \subseteq \overrightarrow{\Delta}\omega_n(p', s_n')(\chi'_{p',p}(0+)) + \sigma_n'\|p - p'\|^2 + \mathbb{R}_+.$$

Therefore,

$$r_m - r' \subseteq \overrightarrow{\Delta}\psi(., q_m)(p', r')(\chi'_{p',p}(0+)) + \sigma_m\|p - p'\|^2 + \mathbb{R}_+, \tag{5.5}$$

and

$$s_n - s_n' \subseteq \overrightarrow{\Delta}\omega_n(p', s_n')(\chi'_{p',p}(0+)) + \sigma_n'\|p - p'\|^2 + \mathbb{R}_+. \tag{5.6}$$

From (5.1), (5.3), (5.5) and (5.6), we have

$$\sum_{m=1}^{l} r_m^*(r_m - r') + \sum_{n=1}^{k} s_n^*(s_n - s_n') \geq 0,$$

which is in conflict with (5.4). Therefore, (p', r') minimizes the problem (MP).

6. Mond-Weir kind dual

We assume the Mond-Weir kind dual (MWD), where the SVMs $\psi(., q_m)$ and ω_n are contingent epidifferentiable.

$$\begin{aligned}
&\text{maximize } r', \\
&\text{s. t. } \sum_{m=1}^{l} r_m^* \overrightarrow{\Delta}\psi(., q_m)(p', r')(\chi'_{p',p}(0+)) + \sum_{n=1}^{k} s_n^* \overrightarrow{\Delta}\omega_n(p', s_n')(\chi'_{p',p}(0+)) \geq 0, \forall p \in M, \\
&\quad \text{for some } l \in \mathbb{N} \text{ and } q_m \in N(p')\ (1 \leq m \leq l), \\
&\quad \sum_{n=1}^{k} s_n^* s_n' \geq 0,\ p' \in M, r' \in \max \bigcup_{q \in N} \psi(p', q),\ s' = (s_1', ..., s_k'), s_n' \in \omega_n(p'), \\
&\quad r^* = (r_1^*, ..., r_l^*), s^* = (s_1^*, ..., s_k^*),\ r_m^* \geq 0, s_n^* \geq 0\ (1 \leq m \leq l, 1 \leq n \leq k), \text{ and } \sum_{m=1}^{l} r_m^* \neq 0.
\end{aligned} \tag{MWD}$$

A point (p', r', s', r^*, s^*) meeting all the requirements of (MWD) is referred to as feasible to the problem (MWD).

Definition 6.1. *A feasible point (p', r', s', r^*, s^*) of the problem (MWD) is said to maximize (MWD) if there appears no feasible point (p, r, s, r_1^*, s_1^*) of the problem (MWD) satisfying*

$$r' < r.$$

Theorem 6.1. *Let M be an arcwisely connected subset of $\mathbb{R}^i$, $p_0 \in S$ and (p', r', s', r^*, s^*) be feasible to the problem (MWD). If $\psi(., q_m)$ $(1 \le m \le l)$ and ω_n $(1 \le n \le k)$ are σ_m-$\mathbb{R}_+$-arcwisely connected and σ'_n-$\mathbb{R}_+$-arcwisely connected w.r.t. 1, on M, respectively and (5.3) is satisfied. Then,*

$$\max \bigcup_{q \in N} \psi(p_0, q) \nless r'.$$

Proof. Presume that for some $r_0 \in \max \bigcup_{q \in N} \psi(p_0, q)$,

$$r_0 < r'.$$

As $r' \in \max \bigcup_{q \in N} \psi(p', q)$,

$$r' \in \psi(p', q_m),\ \forall m = 1, ..., l.$$

Choose

$$r_m \in \psi(p_0, q_m),\ \forall m = 1, ..., l.$$

Since $r_0 \in \max \bigcup_{q \in N} \psi(p_0, q)$ and $q_m \in N(p') \subseteq N$,

$$r_m \le r_0.$$

Therefore,

$$r_m \le r_0 < r'.$$

Hence,

$$\sum_{m=1}^{l} r_m^* r_m < \sum_{m=1}^{l} r_m^* r'.$$

As $p_0 \in S$, there appears $s_n \in \omega_n(p_0) \cap (-\mathbb{R}_+)$ $(1 \le n \le k)$. Since $s_n^* \ge 0$ $(1 \le n \le k)$,

$$s_n^* s_n \le 0,\ \forall n = 1, ..., k.$$

Therefore,

$$\sum_{n=1}^{k} s_n^* s_n \le \sum_{n=1}^{k} s_n^* s'_n.$$

Hence,

$$\sum_{m=1}^{l} r_m^* r_m + \sum_{n=1}^{k} s_n^* s_n < \sum_{m=1}^{l} r_m^* r' + \sum_{n=1}^{k} s_n^* s'_n. \tag{6.1}$$

Since $\psi(., q_m)$ $(1 \le m \le l)$ and ω_n $(1 \le n \le k)$ are σ_m-$\mathbb{R}_+$-arcwisely connected and σ'_n-$\mathbb{R}_+$-arcwisely connected w.r.t. 1, on M,

$$\psi(p_0, q_m) - r' \subseteq \overrightarrow{\Delta}\psi(., q_m)(p', r')(\chi'_{p',p_0}(0+)) + \sigma_m \|p_0 - p'\|^2 + \mathbb{R}_+,$$

and

$$\omega_n(p_0) - s_n' \subseteq \overrightarrow{\Delta}\omega_n(p', s_n')(\chi'_{p',p_0}(0+)) + \sigma_n'\|p_0 - p'\|^2 + \mathbb{R}_+.$$

Therefore,

$$r_m - r' \subseteq \overrightarrow{\Delta}\psi(., q_m)(p', r')(\chi'_{p',p_0}(0+)) + \sigma_m\|p_0 - p'\|^2 + \mathbb{R}_+, \tag{6.2}$$

and

$$s_n - s_n' \subseteq \overrightarrow{\Delta}\omega_n(p', s_n')(\chi'_{p',p_0}(0+)) + \sigma_n'\|p_0 - p'\|^2 + \mathbb{R}_+. \tag{6.3}$$

From the requirements of (MWD) and Eqs (5.3), (6.2) and (6.3), we have

$$\sum_{m=1}^{l} r_m^*(r_m - r') + \sum_{n=1}^{k} s_n^*(s_n - s_n') \geq 0,$$

which is in conflict with (6.1). Hence,

$$r_0 \nless r'.$$

As $r_0 \in \max \bigcup_{q \in N} \psi(p_0, q)$ is arbitrary,

$$\max \bigcup_{q \in N} \psi(p_0, q) \nless r'.$$

Theorem 6.2. *Let (p', r') minimize the problem (MP) and $s_n' \in \omega_n(p') \cap (-\mathbb{R}_+)$ $(1 \leq n \leq k)$. Assume that for some natural number l, $r_m^* \geq 0$, $q_m \in N(p')$ $(1 \leq m \leq l)$ with $\sum_{m=1}^{l} r_m^* \neq 0$ and $s_n^* \geq 0$ $(1 \leq n \leq k)$, Eqs (5.1) and (5.2) are satisfied at (p', r', s', r^*, s^*). Then, (p', r', s', r^*, s^*) is a feasible to (MWD). If the Theorem 6.1 between (MP) and (MWD) holds, then, (p', r', s', r^*, s^*) maximizes (MWD).*

Proof. As the Eqs (5.1) and (5.2) are satisfied at (p', r', s', r^*, s^*),

$$\sum_{m=1}^{l} r_m^* \overrightarrow{\Delta}\psi(., q_m)(p', r')(\chi'_{p',p}(0+)) + \sum_{n=1}^{k} s_n^* \overrightarrow{\Delta}\omega_n(p', s_n')(\chi'_{p',p}(0+)) \geq 0, \ \forall p \in M,$$

and

$$s_n^* s_n' = 0, \ \forall n = 1, ..., k.$$

Hence, (p', r', s', r^*, s^*) is a feasible to (MWD). Presume that the Theorem 6.1 between (MP) and (MWD) holds and (p', r', s', r^*, s^*) does not maximize (MWD). Let (p, r, s, r_1^*, s_1^*) be feasible to (MWD) satisfying

$$r' < r.$$

It contradicts the Theorem 6.1 between (MP) and (MWD). Accordingly, (p', r', s', r^*, s^*) maximizes (MWD).

Theorem 6.3. *Let M be an arcwisely connected subset of $\mathbb{R}^i$ and (p', r', s', r^*, s^*) be a feasible point of (MWD). Assume that $\psi(., q_m)$ $(1 \leq m \leq l)$ and ω_n $(1 \leq n \leq k)$ are σ_m-$\mathbb{R}_+$-arcwisely connected and σ'_n-$\mathbb{R}_+$-arcwisely connected w.r.t.* 1*, on M, respectively and* (5.3) *is satisfied. If $p' \in S$, then (p', r') minimizes (MP).*

Proof. Presume that (p', r') does not minimize the problem (MP). Then, there appear $p \in S$ and $r \in \max \bigcup_{q \in N} \psi(p, q)$ satisfying $r < r'$.

As $q_m \in N(p')$ $(1 \leq m \leq l)$,

$$\max \bigcup_{q \in N} \psi(p', q) \subseteq \psi(p', q_m).$$

As $r' \in \max \bigcup_{q \in N} \psi(p', q)$,

$$r' \in \psi(p', q_m), \ \forall m = 1, ..., l.$$

Choose

$$r_m \in \psi(p, q_m), \ \forall m = 1, ..., l.$$

Since $r \in \max \bigcup_{q \in N} \psi(p, q)$ and $q_m \in N(p') \subseteq N$,

$$r_m \leq r.$$

Therefore,

$$r_m \leq r < r'.$$

So,

$$\sum_{m=1}^{l} r_m^* r_m < \sum_{m=1}^{l} r_m^* r'.$$

As $p \in S$, there appears

$$s_n \in \omega_n(p) \cap (-\mathbb{R}_+) \ (1 \leq n \leq k).$$

As $s_n^* \geq 0$ $(1 \leq n \leq k)$,

$$s_n^* s_n \leq 0, \ \forall n = 1, ..., k.$$

So,

$$\sum_{n=1}^{k} s_n^* s_n \leq 0.$$

From the requirements of (MWD),

$$\sum_{n=1}^{k} s_n^* s'_n \geq 0.$$

So,

$$\sum_{n=1}^{k} s_n^* s_n \leq \sum_{n=1}^{k} s_n^* s'_n.$$

Hence,

$$\sum_{m=1}^{l} r_m^* r_m + \sum_{n=1}^{k} s_n^* s_n < \sum_{m=1}^{l} r_m^* r' + \sum_{n=1}^{k} s_n^* s'_n. \tag{6.4}$$

Since $\psi(.,q_m)$ $(1 \leq m \leq l)$ and ω_n $(1 \leq n \leq k)$ are σ_m-$\mathbb{R}_+$-arcwisely connected and σ'_n-$\mathbb{R}_+$-arcwisely connected w.r.t. 1, on M,

$$\psi(p,q_m) - r' \subseteq \overrightarrow{\Delta}\psi(.,q_m)(p',r')(\chi'_{p',p}(0+)) + \sigma_m\|p-p'\|^2 + \mathbb{R}_+,$$

and

$$\omega_n(p) - s'_n \subseteq \overrightarrow{\Delta}\omega_n(p',s'_n)(\chi'_{p',p}(0+)) + \sigma'_n\|p-p'\|^2 + \mathbb{R}_+.$$

Therefore,

$$r_m - r' \subseteq \overrightarrow{\Delta}\psi(.,q_m)(p',r')(\chi'_{p',p}(0+)) + \sigma_m\|p-p'\|^2 + \mathbb{R}_+, \tag{6.5}$$

and

$$s_n - s'_n \subseteq \overrightarrow{\Delta}\omega_n(p',s'_n)(\chi'_{p',p}(0+)) + \sigma'_n\|p-p'\|^2 + \mathbb{R}_+. \tag{6.6}$$

From the requirements of (MWD) and Eqs (5.3), (6.5) and (6.6), we have

$$\sum_{m=1}^{l} r^*_m(r_m - r') + \sum_{n=1}^{k} s^*_n(s_n - s'_n) \geq 0,$$

which is in conflict with (6.4). Therefore, (p',r') minimizes the problem (MP).

7. Wolfe kind dual

We assume the Wolfe kind dual (WD) linked with the primal problem (MP), where the SVMs $\psi(.,q_m)$ and ω_n are contingent epidifferentiable.

$$\begin{aligned}
&\text{maximize } r' + \sum_{n=1}^{k} s^*_n s'_n,\\
&\qquad \text{s. t. } \sum_{m=1}^{l} r^*_m \overrightarrow{\Delta}\psi(.,q_m)(p',r')(\chi'_{p',p}(0+)) + \sum_{n=1}^{k} s^*_n \overrightarrow{\Delta}\omega_n(p',s'_n)(\chi'_{p',p}(0+)) \geq 0,\ \forall p \in M,\\
&\qquad\quad \text{for some } l \in \mathbb{N} \text{ and } q_m \in N(p')\ (1 \leq m \leq l),\\
&\qquad\quad p' \in M,\ r' \in \max \bigcup_{q \in N} \psi(p',q),\ s' = (s'_1,...,s'_k),\ s'_n \in \omega_n(p'),\\
&\qquad\quad r^* = (r^*_1,...,r^*_l),\ s^* = (s^*_1,...,s^*_k),\\
&\qquad\quad r^*_m \geq 0,\ s^*_n \geq 0 (1 \leq m \leq l, 1 \leq n \leq k), \text{ and } \sum_{m=1}^{l} r^*_m \neq 0.
\end{aligned} \tag{WD}$$

A point (p',r',s',r^*,s^*) meeting all the requirements of (WD) is referred to as feasible to the problem (WD).

Definition 7.1. *A feasible point (p',r',s',r^*,s^*) of the problem (WD) is said to maximize (WD) if there appears no feasible point $(p,r,s,\overline{r}^*,\overline{s}^*)$ of the problem (WD) satisfying*

$$r' + \sum_{n=1}^{k} s^*_n s'_n < r + \sum_{n=1}^{k} \overline{s}^*_n s_n,$$

where $s' = (s'_1,...,s'_k)$, $s^* = (s^*_1,...,s^*_k)$, $s = (s_1,...,s_k)$ and $\overline{s}^* = (\overline{s}^*_1,...,\overline{s}^*_k)$.

Theorem 7.1. *Let M be an arcwisely connected subset of $\mathbb{R}^i$ $p_0 \in S$ and (p', r', s', r^*, s^*) be feasible to the problem (WD). If $\psi(., q_m)$ $(1 \leq m \leq l)$ and ω_n, $(1 \leq n \leq k)$ are σ_m-$\mathbb{R}_+$-arcwisely connected and σ'_n-$\mathbb{R}_+$-arcwisely connected w.r.t. 1, on M, respectively and (5.3) is satisfied. Then,*

$$\max \bigcup_{q \in N} \psi(p_0, q) \nless r' + \sum_{n=1}^{k} s_n^* s_n'.$$

Proof. The proof is comparable to that of Theorem 6.1. It is therefore omitted.

Theorem 7.2. *Let (p', r') minimize the problem (MP) and $s_n' \in \omega_n(p') \cap (-\mathbb{R}_+)$ $(1 \leq n \leq k)$. Assume that for some natural number l, $r_m^* \geq 0$, $q_m \in N(p')$ $(1 \leq m \leq l)$ with $\sum_{m=1}^{l} r_m^* \neq 0$ and $s_n^* \geq 0$ $(1 \leq n \leq k)$, Eqs (5.1) and (5.2) are satisfied at (p', r', s', r^*, s^*). Then, (p', r', s', r^*, s^*) is a feasible to (WD). If the Theorem 7.1 between (MP) and (WD) holds, then, (p', r', s', r^*, s^*) maximizes (WD).*

Proof. The proof is comparable to that of Theorem 6.2. It is therefore omitted.

Theorem 7.3. *Let M be an arcwisely connected subset of $\mathbb{R}^i$ and (p', r', s', r^*, s^*) be a feasible point of (WD), with $\sum_{n=1}^{k} s_n^* s_n' \geq 0$. Assume that $\psi(., q_m)$ $(1 \leq m \leq l)$ and ω_n $(1 \leq n \leq k)$ are σ_m-$\mathbb{R}_+$-arcwisely connected and σ'_n-$\mathbb{R}_+$-arcwisely connected w.r.t. 1, on M, respectively and (5.3) is satisfied. If $p' \in S$, then (p', r') minimizes (MP).*

Proof. The proof is comparable to that of Theorem 6.3. It is therefore omitted.

8. Mixed kind dual

We assume the mixed kind dual (MD) linked with the primal problem (MP), where the SVMs $\psi(., q_m)$ and ω_n are contingent epidifferentiable.

$$\begin{aligned}
&\text{maximize } r' + \sum_{n=1}^{k} s_n^* s_n',\\
&\qquad \text{s. t. } \sum_{m=1}^{l} r_m^* \overrightarrow{\Delta} \psi(., q_m)(p', r')(\chi'_{p',p}(0+)) + \sum_{n=1}^{k} s_n^* \overrightarrow{\Delta} \omega_n(p', s_n')(\chi'_{p',p}(0+)) \geq 0, \forall p \in S,\\
&\qquad\quad \text{for some } l \in \mathbb{N} \text{ and } q_m \in N(p') \ (1 \leq m \leq l),\\
&\qquad\quad \sum_{n=1}^{k} s_n^* s_n' \geq 0,\ p' \in M,\ r' \in \max \bigcup_{q \in N} \psi(p', q),\\
&\qquad\quad s' = (s_1', ..., s_k'),\ s_n' \in \omega_n(p'), r^* = (r_1^*, ..., r_l^*),\ s^* = (s_1^*, ..., s_k^*),\\
&\qquad\quad r_m^* \geq 0,\ s_n^* \geq 0 \ (1 \leq m \leq l, 1 \leq n \leq k), \text{ and } \sum_{m=1}^{l} r_m^* \neq 0.
\end{aligned} \tag{MD}$$

A point (p', r', s', r^*, s^*) meeting all the requirements of (MD) is referred to as feasible to the problem (MD).

Definition 8.1. *A feasible point (p', r', s', r^*, s^*) of the problem (MD) is said to maximize (MD) if there appears no feasible point $(p, r, s, \overline{r}^*, \overline{s}^*)$ of the problem (MD) satisfying*

$$r' + \sum_{n=1}^{k} s_n^* s_n' < r + \sum_{n=1}^{k} \overline{s}_n^* s_n,$$

where $s' = (s_1', ..., s_k')$, $s^* = (s_1^*, ..., s_k^*)$, $s = (s_1, ..., s_k)$ and $\overline{s}^* = (\overline{s}_1^*, ..., \overline{s}_k^*)$.

Theorem 8.1. *Let M be an arcwisely connected subset of $\mathbb{R}^i$, $p_0 \in S$ and (p', r', s', r^*, s^*) be feasible to the problem (MD). If $\psi(., q_m)$ $(1 \leq m \leq l)$ and ω_n $(1 \leq n \leq k)$ are σ_m-$\mathbb{R}_+$-arcwisely connected and σ_n'-$\mathbb{R}_+$-arcwisely connected w.r.t. 1, on M, respectively and (5.3) is satisfied. Then,*

$$\max \bigcup_{q \in N} \psi(p_0, q) \nless r' + \sum_{n=1}^{k} s_n^* s_n'.$$

Proof. The proof is comparable to that of Theorem 6.1. It is therefore omitted.

Theorem 8.2. *Let (p', r') minimize the problem (MP) and $s_n' \in \omega_n(p') \cap (-\mathbb{R}_+)$ $(1 \leq n \leq k)$. Assume that for some natural number l, $r_m^* \geq 0$, $q_m \in N(p')$ $(1 \leq m \leq l)$ with $\sum_{m=1}^{l} r_m^* \neq 0$ and $s_n^* \geq 0$ $(1 \leq n \leq k)$, Eqs (5.1) and (5.2) are satisfied at (p', r', s', r^*, s^*). Then, (p', r', s', r^*, s^*) is a feasible to (MD). If the Theorem 8.1 between (MP) and (MD) holds, then, (p', r', s', r^*, s^*) maximizes (MD).*

Proof. The proof is comparable to that of Theorem 6.2. It is therefore omitted.

Theorem 8.3. *Let M be an arcwisely connected subset of $\mathbb{R}^i$ and (p', r', s', r^*, s^*) be a feasible point of (MD). Assume that $\psi(., q_m)$ $(1 \leq m \leq l)$ and ω_n $(1 \leq n \leq k)$ are σ_m-$\mathbb{R}_+$-arcwisely connected and σ_n'-$\mathbb{R}_+$-arcwisely connected w.r.t. 1, on M, respectively and (5.3) is satisfied. If $p' \in S$, then, (p', r') minimizes (MP).*

Proof. The proof is comparable to that of Theorem 6.3. It is therefore omitted.

Example 8.1. Let $M = [0, 1] \times [0, 1]$ and $N = [-1, 0] \times [-1, 0]$. Then, M is a nonvoid subset of $\mathbb{R}^2$ and N is a nonvoid compact subset of $\mathbb{R}^2$. We consider the SVMs $\psi : \mathbb{R}^2 \times \mathbb{R}^2 \to 2^{\mathbb{R}}$ and $\omega : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$, defined as

$$\forall (p, q) \in \mathbb{R}^2 \times \mathbb{R}^2, \ \psi(p, q) = \begin{cases} \{ \sqrt{t} + \|q\| : \sqrt{t} \geq \|p\|^2 \}, & \text{if } p \neq (0, 0), p \neq (\frac{1}{2}, \frac{1}{2}), \\ \{ \sqrt{-t} + \|q\| : t \in [-\frac{1}{4}, 0] \}, & \text{if } p = (\frac{1}{2}, \frac{1}{2}), \\ \{0\}, & \text{if } p = (0, 0), \\ \{0\}, & \text{if } q = (0, 0), \end{cases}$$

and

$$\forall p \in \mathbb{R}^2, \ \omega(p) = \begin{cases} \{(t^2 + \|p\|^2, t^2 + \|p\|^2) : t \in \mathbb{R}\}, & \text{if } p \neq (0, 0), p \neq (\frac{1}{2}, \frac{1}{2}), \\ \{(t + \frac{1}{4}, t + \frac{1}{4}) : t \geq 0\}, & \text{if } p = (\frac{1}{2}, \frac{1}{2}), \\ \{(0, 0)\}, & \text{if } p = (0, 0). \end{cases}$$

It is clear that for all $p \in M$, $\psi(p,.) : \mathbb{R}^2 \to 2^{\mathbb{R}}$ is upper semicontinuous and $\mathbb{R}_+$-semicompact-valued on N. The feasible set of the problem (MP) is

$$S = \{p \in M : \omega(p) \cap (-\mathbb{R}^2_+) \neq \emptyset\} = \{(0,0)\}.$$

Let $p = (p_1, p_2) \in \mathbb{R}^2$, and $p' = (p'_1, p'_2) \in \mathbb{R}^2$. Let us consider a continuous arc $\chi_{p',p} : [0,1] \to \mathbb{R}^2$ defined by

$$\chi_{p',p}(\xi) = (1 - \xi^3)p' + \xi^3 p, \ \forall \xi \in [0,1].$$

Clearly, M is an arcwisely connected subset of $\mathbb{R}^2$. Now,

$$\chi'_{p',p}(0+) = \lim_{\xi \to 0+} \frac{\chi_{p',p}(\xi) - \chi_{p',p}(0)}{\xi} = (0,0).$$

Let $p' = (0,0) \in S$, $r' = \max \bigcup_{q \in N} \psi(p',q) = 0$, and $s' = (s'_1, s'_2)$, with $s'_n = 0 \in \omega_n(p') \cap (-\mathbb{R}_+)$ $(n = 1,2)$. Obviously, $N(p') = \{n \in N : \{0\} = \max \bigcup_{q \in N} \psi(p',q) \subseteq \psi(p',n)\} = N$. We choose $l = 2$ and $q_m = -1 \in N(p')$, $m = 1,2$. We can show that $\psi(., q_m)$ is (-4)-$\mathbb{R}_+$-arcwisely connected and ω is 5-$\mathbb{R}^2_+$-arcwisely connected w.r.t. 1 on M. Hence, ω_n is 5-$\mathbb{R}_+$-arcwisely connected w.r.t. 1 on M, $(n = 1,2)$. So $\sigma_m = -4$ and $\sigma'_n = 5$, where $m = 1,2$ and $n = 1,2$. So, there exist $r^* = (r^*_1, r^*_2) = (\frac{1}{2}, \frac{1}{2}) \in \mathbb{R}^2_+$, with $\sum_{i=1}^{2} r^*_i = 1$ and $s^* = (s^*_1, s^*_2) = (1,1) \in \mathbb{R}^2_+$ such that Eqs (5.1)–(5.3) are satisfied. Then, the point $(p', r') = ((0,0), 0)$ minimizes the problem (MP).

Now, we have

$$\begin{aligned} \operatorname{epi}(\psi) &= \{((p,q),t) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} : (p,q) \in \mathbb{R}^2 \times \mathbb{R}^2, t \in \psi(p,q) + \mathbb{R}_+\} \\ &= \{((p,q),t) : p \neq (0,0), p \neq \left(\frac{1}{2}, \frac{1}{2}\right), t \geq \|p\|^2 + \|q\|\} \\ &\cup \{((p,q),t) : p = \left(\frac{1}{2}, \frac{1}{2}\right), t \geq \frac{1}{2} + \|q\|\} \\ &\cup \{((p,q),t) : p = (0,0), t \geq 0\} \\ &\cup \{((p,q),t) : q = (0,0), t \geq 0\}. \end{aligned}$$

Therefore,

$$T(\operatorname{epi}(\psi), ((p,q),t)) = \begin{cases} (\mathbb{R}^2_+ + p) \times (\mathbb{R}^2_+ + q) \times \mathbb{R}_+, & \text{if } p \neq (0,0), p \neq \left(\frac{1}{2}, \frac{1}{2}\right), \\ (\mathbb{R}^2_+ + (\frac{1}{2}, \frac{1}{2})) \times \mathbb{R}^2_+ \times (\mathbb{R}_+ + \frac{1}{2}), & \text{if } p = \left(\frac{1}{2}, \frac{1}{2}\right), \\ \mathbb{R}^2_+ \times (\mathbb{R}^2_+ + q) \times \mathbb{R}_+, & \text{if } p = (0,0), \\ (\mathbb{R}^2_+ + p) \times \mathbb{R}^2_+ \times \mathbb{R}_+, & \text{if } q = (0,0). \end{cases}$$

Let $(p', q') \in \mathbb{R}^2 \times \mathbb{R}^2$ and $t' \in \psi(p', q')$. Then, it is obvious that

$$\min\{t : ((p,q),t) \in T(\operatorname{epi}(\psi), ((p',q'),t'))\} = \begin{cases} \frac{1}{2}, & \text{if } p = \left(\frac{1}{2}, \frac{1}{2}\right), \\ 0, & \text{otherwise.} \end{cases}$$

Now, we have

$$\overrightarrow{\Delta}\psi((p',q'),t')(p,q)=\min\{t:((p,q),t)\in T(\mathrm{epi}(\psi),((p',q'),t'))\}=\begin{cases}\frac{1}{2}, & \text{if } p'=\left(\frac{1}{2},\frac{1}{2}\right),\\ 0, & \text{otherwise.}\end{cases}$$

Similarly, we can prove that

$$\overrightarrow{\Delta}\omega(p',t')(p) \quad =\begin{cases}\left(\frac{1}{4},\frac{1}{4}\right), & \text{if } p'=\left(\frac{1}{2},\frac{1}{2}\right),\\ (0,0), & \text{otherwise.}\end{cases}$$

Now, we have

$$\overrightarrow{\Delta}\psi((p',q'),t')(\chi'_{p',p}(0+))=\overrightarrow{\Delta}\psi((p',q'),t')(0,0)=0$$

and

$$\overrightarrow{\Delta}\omega(p',t')(\chi'_{p',p}(0+))=\overrightarrow{\Delta}\omega(p',t')(0,0)=(0,0).$$

So, we have

$$\overrightarrow{\Delta}\psi(.,q_m)(p',r')(\chi'_{p',p}(0+))=\overrightarrow{\Delta}\psi(.,q_m)(p',r')(0,0)=0$$

and

$$\overrightarrow{\Delta}\omega_n(p',s'_n)(\chi'_{p',p}(0+))=\overrightarrow{\Delta}\omega_n(p',s'_n)(0,0)=0.$$

It is clear that for $r^*=(r_1^*,r_2^*)=(\frac{1}{2},\frac{1}{2})\in\mathbb{R}^2_+$, with $\sum_{i=1}^{2}r_i^*=1$ and $s^*=(s_1^*,s_2^*)=(1,1)\in\mathbb{R}^2_+$, the sufficient optimality conditions (5.1)–(5.3) are satisfied and $\sum_{m=1}^{2}r_m^*\neq 0$. So (p',r',s',r^*,s^*) is a feasible solution of the (MWD). Again, $s_n^*s'_n=0,\forall n=1,2$. Hence, the weak duality Theorem 6.1 holds between (MP) and (MWD). We prove that (p',r',s',r^*,s^*) maximizes (MWD). Let any feasible point of the Mond-Weir type dual (MWD) be $(p,r,s,\overline{r}^*,\overline{s}^*)$, where $p\in M$, $r\in\max\bigcup_{q\in N}\psi(p,q)$, $s=(s_1,s_2)$, $s_n\in\omega_n(p)$, $\overline{r}^*=(\overline{r}_1^*,\overline{r}_2^*)$, $\overline{s}^*=(\overline{s}_1^*,\overline{s}_2^*)$, $\overline{r}_m^*\geq 0$, $\overline{s}_n^*\geq 0$ $(m=1,2;n=1,2)$, and $\sum_{m=1}^{2}\overline{r}_m^*\neq 0$. Now $\max\bigcup_{q\in N}\psi(p,q)$ exists only when $p=(0,0)$. So, $p=(0,0)$ and hence $r=0$. Hence, (p',r',s',r^*,s^*), $p'=(0,0)$, $r'=0$, $s'_n=0$, $(n=1,2)$, $r^*=(r_1^*,r_2^*)=(\frac{1}{2},\frac{1}{2})$, and $s^*=(s_1^*,s_2^*)=(1,1)$ maximizes the problem (MWD). Hence, Theorems 6.1–6.3 are satisfied. Similarly, the Wolfe and mixed kind duality results can be illustrated by this example.

9. Conclusions

In this paper, we have determined the KKT sufficient conditions of optimality of a SVMPP (MP) under contingent epidifferentiation and generalized cone arcwisely connectivity suppositions. We also formulated the results of duality of Mond-Weir (MWD), Wolfe (WD), and mixed (MD) kinds under the stated suppositions. We deal with σ-arcwisely connectivity of set-valued maps in finding the solutions of set-valued minimax programming problems. The idea of σ-arcwisely connectivity is more generalized than that of σ-cone convexity. It is shown by illustrating an example in our work. We basically work with σ-arcwisely connected set-valued maps as objective functions and constraints of set-valued minimax programming problems. This is the main advantage of our results. We assume the set-valued maps as contingent epi-derivable. We are not able to find out the solutions whenever set-valued maps fail to be contingent epi-derivable. From the analysis of work presented in this work, there are several scopes for extension. One can study optimality conditions for approximate efficient solutions and symmetric duals of set-valued minimax programming problems. One can study approximate duality results of set-valued minimax programming problems via approximate cone convexity assumptions. One can extend these results via various generalized notions of differentiability of set-valued maps, such as $\mathbb{T}$-epiderivative, generalized contingent epiderivative etc. The complementarity and control problems in the setting of set-valued maps can also be studied.

Author's contributions

All authors contributed equally to this article. They read and approved the final manuscript.

Conflict of interest

The authors declare that they have no competing interests.

References

1. J. P. Aubin, Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions, In: *Mathematical analysis and applications, Part A*, New York: Academic Press, 1981, 160–229.
2. J. P. Aubin, H. Frankowska, *Set-valued analysis*, Boston: Birhäuser, 1990.
3. M. Avriel, *Nonlinear programming: theory and method*, Englewood Cliffs, New Jersey: Prentice-Hall, 1976.
4. C. R. Bector, B. L. Bhatia, Sufficient optimality conditions and duality for a minmax Ž. problem, *Utilitas Math.*, **27** (1985), 229–247.
5. C. R. Bector, S. Chandra, I. Husain, Sufficient optimality conditions and duality for a continuous-time minmax programming problem, *Asia-Pac. J. Oper. Res.*, **9** (1992), 55–76.
6. J. Borwein, Multivalued convexity and optimization: a unified approach to inequality and equality constraints, *Math. Program.*, **13** (1977), 183–199. https://doi.org/10.1007/BF01584336
7. J. Bram, The Lagrange multiplier theorem for max-min with several constraints, *SIAM J. Appl. Math.*, **14** (1966), 665–667. https://doi.org/10.1137/0114054
8. A. Cambini, L. Martein, M. Vlach, Second order tangent sets and optimality conditions, *Math. Jpn.*, **49** (1999), 451–461.
9. S. Chandra, V. Kumar, Duality in fractional minimax programming, *J. Aust. Math. Soc.*, **58** (1995), 376–386. https://doi.org/10.1017/S1446788700038362
10. H. W. Corley, Existence and Lagrangian duality for maximizations of set-valued functions, *J. Optim. Theory Appl.*, **54** (1987), 489–501. https://doi.org/10.1007/BF00940198

11. J. M. Danskin, The theory of max-min with applications, *SIAM J. Appl. Math.*, **14** (1966), 641–644. https://doi.org/10.1137/0114053

12. J. M. Danskin, *The theory of max-min and its applications to weapons allocation problems*, Berlin, Heidelberg: Springer, 1967. https://doi.org/10.1007/978-3-642-46092-0

13. K. Das, C. Nahak, Sufficient optimality conditions and duality theorems for set-valued optimization problem under generalized cone convexity, *Rend. Circ. Mat. Palermo*, **63** (2014), 329–345. https://doi.org/10.1007/s12215-014-0163-9

14. K. Das, C. Nahak, Set-valued fractional programming problems under generalized cone convexity, *Opsearch*, **53** (2016), 157–177. https://doi.org/10.1007/s12597-015-0222-9

15. K. Das, C. Nahak, Set-valued minimax programming problems under generalized cone convexity, *Rend. Circ. Mat. Palermo*, **66** (2017), 361–374. https://doi.org/10.1007/s12215-016-0258-6

16. K. Das, C. Nahak, Optimality conditions for set-valued minimax fractional programming problems, *SeMA J.*, **77** (2020), 161–179. https://doi.org/10.1007/s40324-019-00209-7

17. K. Das, C. Nahak, Optimality conditions for set-valued minimax programming problems via second-order contingent epiderivative, *J. Sci. Res.*, **64** (2020), 313–321.

18. N. Datta, D. Bhatia, Duality for a class of nondifferentiable mathematical programming problems in complex space, *J. Math. Anal. Appl.*, **101** (1984), 1–11. https://doi.org/10.1016/0022-247X(84)90053-2

19. V. F. Demyanov, V. N. Malozehon, *Introduction to minmax*, New York: Wiley, 1974.

20. J. Y. Fu, Y. H. Wang, Arcwise connected cone-convex functions and mathematical programming, *J. Optim. Theory Appl.*, **118** (2003), 339–352. https://doi.org/10.1023/A:1025451422581

21. J. Jahn, R. Rauh, Contingent epiderivatives and set-valued optimization, *Math. Method Oper. Res.*, **46** (1997), 193–211. https://doi.org/10.1007/BF01217690

22. C. S. Lalitha, J. Dutta, M. G. Govil, Optimality criteria in set-valued optimization, *J. Aust. Math. Soc.*, **75** (2003), 221–232. https://doi.org/10.1017/S1446788700003736

23. A. Mehra, D. Bhatia, Optimality and duality for minmax problems involving arcwise connected and generalized arcwise connected functions, *J. Math. Anal. Appl.*, **231** (1999), 425–445. https://doi.org/10.1006/jmaa.1998.6231

24. Z. H. Peng, Y. H. Xu, Second-order optimality conditions for cone-subarcwise connected set-valued optimization problems, *Acta Math. Appl. Sin. Engl. Ser.*, **34** (2018), 183–196. https://doi.org/10.1007/s10255-018-0738-x

25. W. E. Schmitendorf, Necessary conditions and sufficient conditions for static minmax problems, *J. Math. Anal. Appl.*, **57** (1977), 683–693. https://doi.org/10.1016/0022-247X(77)90255-4

26. S. Tanimoto, Duality for a class of nondifferentiable mathematical programming problems, *J. Math. Anal. Appl.*, **79** (1981), 286–294. https://doi.org/10.1016/0022-247X(81)90025-1

27. Y. H. Xu, M. Li, Optimality conditions for weakly efficient elements of set-valued optimization with α-order near cone-arcwise connectedness, *J. Syst. Sci. Math. Sci.*, **36** (2016), 1721–1729. https://doi.org/10.12341/jssms12925

28. G. L. Yu, Optimality of global proper efficiency for cone-arcwise connected set-valued optimization using contingent epiderivative, *Asia-Pac. J. Oper. Res.*, **30** (2013), 1340004. https://doi.org/10.1142/S0217595913400046

29. G. L. Yu, Global proper efficiency and vector optimization with cone-arcwise connected set-valued maps, *Numer. Algebra Control Optim.*, **6** (2016), 35–44. https://doi.org/10.3934/naco.2016.6.35

30. G. J. Zalmai, *Optimality conditions for a class of continuous-time minmax programming problems*, New York: Washington State University, 1985.