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Research article

Exponential stability analysis for nonlinear time-varying perturbed systems on time scales

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Abstract: This paper is concerned with the stability of nonlinear time-varying perturbed system on time scales under the assumption that the corresponding linear time-varying nominal system is uniformly exponentially stable. Some less conservative sufficient conditions for uniform exponential stability and uniform practical exponential stability are proposed by imposing different assumptions on the perturbation term. Compared with the traditional exponential stability results of perturbed systems, the time derivatives of related Lyapunov functions in this paper are not required to be negative definite for all time. The main tools employed are two Gronwall's inequalities on time scales. Some examples are also given to illustrate the effectiveness of the theoretical results.

Keywords: time scale; nonlinear time-varying perturbed system; exponential stability; Lyapunov function
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1. Introduction

Stability analysis of dynamical systems has always been a hot and important topic in theoretical research and practical application. As is well-known, Lyapunov's second method is not only used to investigate the stability in the sense of Lyapunov [1–3], but also used to discuss the practical stability [4–8]. When utilizing this method to analyse the stability of dynamical systems, one needs to find a suitable positive definite Lyapunov function, whose time derivative along the solutions of the considered system is negative definite. Lyapunov's second method is powerful, yet in some cases, the construction of Lyapunov function is not a simple task since there are no general rules, except for some special cases. Moreover, from a practical point of view, the condition that the time derivative of Lyapunov function is negative definite may be conservative and difficult to be satisfied. Thus, it is necessary and interesting to come up with some less conservative stability conditions. In recent years, some research work has been done to weaken the requirement on negative definiteness of the time derivatives of related Lyapunov functions. For example, some new stability conditions for continuous-time and discrete-time linear time-varying systems were derived in [9–11], respectively. In [12, 13], the traditional Lyapunov stability theorems for continuous-time nonlinear time-varying systems were generalized in the sense that the time derivatives of Lyapunov functions were allowed to be indefinite. With the help of the new notion of practical stable functions, some differential Lyapunov inequalities based necessary and sufficient conditions were derived in [14] for testing global uniform practical asymptotic stability and practical exponential stability of general continuous-time nonlinear systems. For some other relevant works, one can refer to [15–18].

In 1988, Stefan Hilger introduced the theory of time scales in his Ph.D. thesis to unify continuous and discrete analysis. Later, Bohner and Peterson summarized and organized a great deal of time scale calculus in their books [19, 20]. With the development of the theory of time scales, the stability of dynamical systems on time scales, such as asymptotic stability, exponential stability, practical stability and h-stability, has received much attention from numerous authors and a large number of results have been obtained in [21–33]. In particular, Pötzsche et al. [21] studied exponential stability of linear time-invariant systems using the standard exponential function, while DaCunha [22] discussed uniform exponential stability of linear time-varying systems employing the generalized exponential function on time scales. In addition, Ben Nasser et al. [28] introduced the concepts for local uniform exponential stability, uniform exponential stability and uniform practical exponential stability of nonlinear time-varying systems utilizing a more general exponential function, and studied exponential stability of the closed-loop systems on arbitrary time scales. Recently, the authors in [26, 27] investigated uniform exponential stability of linear time-varying system with nonlinear perturbation using the generalized exponential function on time scales. However, the uniform exponential stability results established in [26,27] were based on the negative definiteness restriction on the time derivative of related Lyapunov function. In 2020, by means of a less conservative Lyapunov inequality on time scales, Lu et al. [33] proposed an improved uniform exponential stability criteria for linear time-varying system on time scales, which weakened the negative definiteness restriction on the time derivative of Lyapunov function. Furthermore, to the best of our knowledge, the research on practical stability of nonlinear time-varying perturbed systems on time scales is inadequate. But as emphasized in [14], practical stability is a significant performance specification from an engineering point of view. Therefore, it is necessary and helpful to propose some other improved stability theorems for nonlinear time-varying perturbed systems on time scales.

Motivated greatly by the above-mentioned works, in this paper, we consider the stability of nonlinear time-varying perturbed system on time scales under the assumption that the corresponding linear time-varying nominal system is uniformly exponentially stable. The main contributions are summarized as follows:

(i) Compared with [33], the cases of linear time-varying nominal system under vanishing and non-vanishing perturbations are investigated in this note, respectively.

(ii) Compared with [26,27], a less conservative sufficient condition for uniform exponential stability of perturbed system is derived with the help of the uniformly exponentially stable function on time scales. Moreover, based on an improved Gronwall-type integral inequality on time scales, a new sufficient condition for uniform practical exponential stability is explored.

(iii) The advantage of the theorems proposed in this paper is that the time derivatives of related Lyapunov functions are not required to be negative definite for all time. Owing to the complexity and generality of time scales, the proofs need to be more technical.

The rest of this paper is organized as follows. In Section 2, some foundational definitions and results about time scales are briefly outlined. In Section 3, the studied problem is stated. Our main results are obtained in Section 4. Three illustrative examples are presented in Section 5. The paper is concluded in Section 6.

In the sequel, the following notations will be used. $\mathbb{N}_0$ and $\mathbb{Z}$ denote the sets of nonnegative integers and integers, respectively. $\mathbb{R}_+$ and $\mathbb{R}$ denote the sets of nonnegative real numbers and real numbers, respectively. $\mathbb{R}^n$ denotes the space of n-dimensional real vectors. The norm of an $n \times 1$ vector x is defined to be $\|x\| = \sqrt{x^{\mathrm{T}} x}$, where x^{T} stands for the transpose of x. $\mathbb{R}^{m\times n}$ denotes the space of $m \times n$ real matrices and I_n is an $n \times n$ identity matrix. The norm of an $n \times n$ matrix A is defined to be $\|A\| = [\lambda_{\max}(A^{\mathrm{T}}A)]^{\frac{1}{2}}$, where A^{T} stands for the transpose of A and $\lambda_{\max}(A^{\mathrm{T}}A)$ is the maximum eigenvalue of $A^{\mathrm{T}}A$. Moreover, for symmetric matrices $P,\ Q \in \mathbb{R}^{n\times n}$, the notation $P \geq Q$ ($P \leq Q$) means that $P - Q$ is a positive semi-definite (negative semi-definite) matrix. $\exp(t)$ represents the standard exponential function e^t.

2. Preliminaries

In this section, we shall provide some foundational definitions and results on time scales, which will be useful for the following sections. For more details, one can refer to [19, 29]. A time scale is an arbitrary nonempty closed subset of the real numbers and is often denoted by the symbol $\mathbb{T}$, which has the topology that it inherits from the real numbers with the standard topology. Also, $\mathbb{T}$ takes the sets $\mathbb{R}$ and $\mathbb{Z}$ as its special cases. Throughout this paper, for $a \in \mathbb{T}$, we define the set $\mathbb{T}_a^+ = [a, +\infty) \cap \mathbb{T}$.

Definition 2.1. *Let $t \in \mathbb{T}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by*

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition, it is assumed that $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. If $\sigma(t) > t$, then t is called right-scattered, while if $\rho(t) < t$, then t is called left-scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$. Finally, the graininess function $\mu : \mathbb{T} \to \mathbb{R}_+$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Definition 2.2. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then $f^{\Delta}(t)$ is defined to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$\left|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)\,[\sigma(t) - s]\right| \leq \varepsilon|\sigma(t) - s| \text{ for all } s \in U.$$

In this case, $f^{\Delta}(t)$ is called the delta derivative of f at t.

Moreover, f is called delta differentiable (or in short: differentiable) on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is called the (delta) derivative of f on $\mathbb{T}^k$.

Remark 2.1. *If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f'(t)$ is the usual derivative. If $\mathbb{T} = \mathbb{Z}$, then $f^{\Delta}(t) = \Delta f(t)$ is the usual forward difference operator.*

Lemma 2.1. *[19] Assume $f : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}^k$. Then $f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$.*

Lemma 2.2. *[19] Assume $f,\ g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then the product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

Definition 2.3. *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.*

Definition 2.4. *A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided*

$$F^{\Delta}(t) = f(t) \text{ holds for all } t \in \mathbb{T}^k.$$

If $F : \mathbb{T} \to \mathbb{R}$ is an antiderivative of $f : \mathbb{T} \to \mathbb{R}$, then the Cauchy integral is defined by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \text{ for all } s,\ r \in \mathbb{T}.$$

Remark 2.2. *Let $r, s \in \mathbb{T}$ and $f \in C_{rd}(\mathbb{T}, \mathbb{R})$. Then*
(i) *if $\mathbb{T} = \mathbb{R}$, then $\int_r^s f(t)\Delta t = \int_r^s f(t)dt$;*
(ii) *if $\mathbb{T} = \mathbb{Z}$, then*

$$\int_r^s f(t)\Delta t = \begin{cases} \sum_{t=r}^{s-1} f(t),\ r < s, \\ 0,\ r = s, \\ -\sum_{t=s}^{r-1} f(t),\ r > s. \end{cases}$$

Definition 2.5. *A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided*

$$1 + \mu(t)p(t) \neq 0 \text{ for all } t \in \mathbb{T}^k$$

holds. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R})$. The set of positively regressive functions $\mathcal{R}^+(\mathbb{T}, \mathbb{R})$ is defined as the set consisting of those $p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$ satisfying

$$1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}.$$

Definition 2.6. *If $p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$, then the exponential function is defined by*

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \text{ for } t, s \in \mathbb{T}$$

with the cylinder transformation $\xi_h(z)$ defined by

$$\xi_h(z) = \begin{cases} \dfrac{1}{h} Log(1 + zh),\ h > 0, \\ z,\ h = 0, \end{cases}$$

where Log is the principal logarithm function.

Remark 2.3. *Let $p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$. Then*
(i) *if $\mathbb{T} = \mathbb{R}$, then $e_p(t, s) = e^{\int_s^t p(\tau)d\tau}$;*
(ii) *if $\mathbb{T} = \mathbb{Z}$, then $e_p(t, s) = \prod_{\tau=s}^{t-1}(1 + p(\tau))$.*

Definition 2.7. *If $p, q \in \mathcal{R}(\mathbb{T}, \mathbb{R})$, then the functions $p \oplus q$ and $\ominus p$ are defined by*

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t) \text{ for all } t \in \mathbb{T}^k,$$

$$(\ominus p)(t) := -\frac{p(t)}{1 + \mu(t)p(t)} \text{ for all } t \in \mathbb{T}^k.$$

Lemma 2.3. *[19] Let $p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$ and $s, r \in \mathbb{T}$. Then*
(i) *$e_p(t, t) \equiv 1$ for all $t \in \mathbb{T}$;*
(ii) *$e_p^{\Delta}(t, s) = p(t)e_p(t, s)$ for all $t \in \mathbb{T}^k$;*
(iii) *$e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$ for all $t \in \mathbb{T}^k$;*
(iv) *$e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$ for all $t \in \mathbb{T}$;*
(v) *$e_p(t, s)e_p(s, r) = e_p(t, r)$ for all $t \in \mathbb{T}$;*
(vi) *$e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$ for all $t \in \mathbb{T}^k$.*
Moreover, if $p \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$ and $t_0 \in \mathbb{T}$, then $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$.

Lemma 2.4. *[19] If $p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$ and $a, b, c \in \mathbb{T}$, then*

$$\int_a^b p(t)e_p(c, \sigma(t))\Delta t = e_p(c, a) - e_p(c, b).$$

Lemma 2.5. *[19] Let $t_0 \in \mathbb{T}$, y, $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ and $p \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$. Then*

$$y^{\Delta}(t) \leq p(t)y(t) + f(t) \text{ for all } t \in \mathbb{T}$$

implies

$$y(t) \leq y(t_0)e_p(t, t_0) + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau)\Delta\tau \text{ for all } t \in \mathbb{T}.$$

Lemma 2.6. *[19]* (Gronwall's inequality) *Let $t_0 \in \mathbb{T}$, y, $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ and $p \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$, $p \geq 0$. Then*

$$y(t) \leq f(t) + \int_{t_0}^t y(\tau)p(\tau)\Delta\tau \text{ for all } t \in \mathbb{T}$$

implies

$$y(t) \leq f(t) + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau)p(\tau)\Delta\tau \text{ for all } t \in \mathbb{T}.$$

Lemma 2.7. *[27] Let $t_0 \in \mathbb{T}$. If p is rd-continuous and nonnegative, then*

$$1 + \int_{t_0}^t p(\tau)\Delta\tau \leq e_p(t, t_0) \leq \exp\Big(\int_{t_0}^t p(\tau)\Delta\tau\Big) \text{ for all } t \in \mathbb{T}_{t_0}^+.$$

Lemma 2.8. *[34] Let $t_0 \in \mathbb{T}$. If $p \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$ and $p(t) \leq q(t)$ for all $t \in \mathbb{T}$, then*

$$e_p(t, t_0) \leq e_q(t, t_0) \text{ for all } t \in \mathbb{T}_{t_0}^+.$$

The following is an improved Gronwall-type integral inequality on time scales.

Lemma 2.9. *[27] Let ϕ, φ, ψ, α, $\chi \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$ and $t_0 \in \mathbb{T}$. Then*

$$\phi(t) \leq \varphi(t) + \psi(t) \int_{t_0}^{t} [\alpha(\tau)\phi(\tau) + \chi(\tau)]\Delta\tau$$

implies

$$\phi(t) \leq \varphi(t) + \psi(t) \int_{t_0}^{t} [\alpha(\tau)\varphi(\tau) + \chi(\tau)] \exp\Big(\int_{\sigma(\tau)}^{t} \alpha(s)\psi(s)\Delta s\Big)\Delta\tau$$

for all $t \in \mathbb{T}_{t_0}^+$.

Definition 2.8. *Let A be an $m \times n$-matrix-valued function on $\mathbb{T}$. A is called rd-continuous on $\mathbb{T}$ if each entry of A is rd-continuous on $\mathbb{T}$, and the class of all such rd-continuous $m \times n$-matrix-valued function on $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R}^{m\times n})$. Moreover, A is called differentiable on $\mathbb{T}$ provided each entry of A is differentiable on $\mathbb{T}$, and in this case $A^{\Delta} = (a_{ij}^{\Delta})_{1\leq i\leq m, 1\leq j\leq n}$, where $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$. The set of $m \times n$-matrix-valued functions that are differentiable and whose derivative is rd-continuous is denoted by $C_{rd}^1(\mathbb{T}, \mathbb{R}^{m\times n})$.*

Lemma 2.10. *[19] Let A be an $m\times n$-matrix-valued function on $\mathbb{T}$. If A is differentiable at $t \in \mathbb{T}^k$, then $A(\sigma(t)) = A(t) + \mu(t)A^{\Delta}(t)$.*

Definition 2.9. *An $n\times n$-matrix-valued function A on $\mathbb{T}$ is called regressive (with respect to $\mathbb{T}$) provided*

$$I_n + \mu(t)A(t) \text{ is invertible for all } t \in \mathbb{T}^k.$$

Definition 2.10. *A function $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is called rd-continuous, if g defined by $g(t) = f(t, x(t))$ is rd-continuous for any continuous function $x : \mathbb{T} \to \mathbb{R}^n$.*

3. Problem statement

Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$ and $\sup \mathbb{T} = +\infty$. Assume that the graininess function μ is bounded on $\mathbb{T}$. First, we consider the following nonlinear time-varying system:

$$\begin{cases} x^{\Delta}(t) = f(t, x(t)), \\ x(t_0) = x_0, \end{cases} \tag{3.1}$$

where $t_0 \in \mathbb{T}_a^+$, $x_0 \in \mathbb{R}^n$, $x : \mathbb{T}_a^+ \to \mathbb{R}^n$ is the state vector and $f : \mathbb{T}_a^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is an rd-continuous vector-valued function. It is assumed that the conditions for the existence of a unique solution of system (3.1) on $\mathbb{T}_{t_0}^+$ ($\mathbb{T}_{t_0}^+ = [t_0, +\infty) \cap \mathbb{T}_a^+$) are satisfied. For the existence, uniqueness and extensibility of the solutions of system (3.1), one can refer to [19]. Hereafter, $x(t) := x(t, t_0, x_0)$ denotes the solution of system (3.1) starting from an arbitrary initial state x_0 at initial time t_0.

The following definition is given in [24] with an additional concept of uniform practical exponential stability.

Definition 3.1. *System* (3.1) *is said to be*
(i) *exponentially stable if there exists a positive constant α with $-\alpha \in \mathcal{R}^+(\mathbb{T}_a^+, (-\infty, 0))$ such that for every $t_0 \in \mathbb{T}_a^+$, there exists $N = N(t_0) \geq 1$ such that the solution of* (3.1) *satisfies*

$$\|x(t)\| \leq N \|x(t_0)\| e_{-\alpha}(t, t_0) \text{ for all } t \in \mathbb{T}_{t_0}^+;$$

(ii) *uniformly exponentially stable if it is exponentially stable and constant N can be chosen independently of $t_0 \in \mathbb{T}_a^+$;*
(iii) *uniformly practically exponentially stable if there exist constants $N \geq 1$, $\alpha > 0$ with $-\alpha \in \mathcal{R}^+(\mathbb{T}_a^+, (-\infty, 0))$ and $r > 0$ such that for every $t_0 \in \mathbb{T}_a^+$, the solution of* (3.1) *satisfies*

$$\|x(t)\| \leq N \|x(t_0)\| e_{-\alpha}(t, t_0) + r \text{ for all } t \in \mathbb{T}_{t_0}^+.$$

In particular, if $f(t, x(t)) = A(t)x(t) + g(t, x(t))$ for $t \in \mathbb{T}_a^+$, where $A : \mathbb{T}_a^+ \to \mathbb{R}^{n\times n}$ is an rd-continuous and regressive matrix-valued function, and $g : \mathbb{T}_a^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is an rd-continuous vector-valued function, then system (3.1) is transformed into the following nonlinear time-varying perturbed system:

$$\begin{cases} x^\Delta(t) = A(t)x(t) + g(t, x(t)), \\ x(t_0) = x_0. \end{cases} \tag{3.2}$$

Here, the perturbation term g could result from modeling errors, aging, or uncertainties and disturbances, which exist in any realistic problem. If $g(t, x(t)) \equiv 0$ for all $t \in \mathbb{T}_a^+$, then system (3.2) is transformed into the following linear time-varying system:

$$\begin{cases} x^\Delta(t) = A(t)x(t), \\ x(t_0) = x_0, \end{cases} \tag{3.3}$$

which is regarded as the corresponding nominal system of perturbed system (3.2).

Next, we consider the following scalar linear time-varying system:

$$\begin{cases} y^\Delta(t) = \beta(t)y(t), \\ y(t_0) = y_0, \end{cases} \tag{3.4}$$

where $t_0 \in \mathbb{T}_a^+$, $y_0 \in \mathbb{R}$, $y : \mathbb{T}_a^+ \to \mathbb{R}$ is the state variable and $\beta \in \mathcal{R}^+(\mathbb{T}_a^+, \mathbb{R})$. As stated in [19], the unique solution of system (3.4) is given by $y(t) = e_\beta(t, t_0)y_0$.

The exponential stability and uniform exponential stability criteria of system (3.4) in terms of the exponential function are characterized in the following lemma.

Lemma 3.1. *[33] System* (3.4) *is*
(i) *exponentially stable if and only if for any $t_0 \in \mathbb{T}_a^+$, there exist constants $\eta(t_0) \geq 1$ and $\alpha > 0$ with $-\alpha \in \mathcal{R}^+(\mathbb{T}_a^+, (-\infty, 0))$ such that*

$$e_\beta(t, t_0) \leq \eta(t_0) e_{-\alpha}(t, t_0), \ t \in \mathbb{T}_{t_0}^+;$$

(ii) *uniformly exponentially stable if and only if η in Item* (i) *is independent of t_0.*

Definition 3.2. *[33] The function β is said to be*
(i) *exponentially stable if system* (3.4) *is exponentially stable*;
(ii) *uniformly exponentially stable if system* (3.4) *is uniformly exponentially stable.*

At the end of this section, we notice that a very interesting question comes into focus. When linear time-varying nominal system (3.3) is uniformly exponentially stable, what about the stability behavior of nonlinear time-varying perturbed system (3.2)? The answer to this question depends crucially on whether the perturbation term vanishes at the origin. Moreover, a natural approach to address this question is to use a Lyapunov function for nominal system (3.3) as a Lyapunov function candidate for perturbed system (3.2). In the next section, by imposing different assumptions on the perturbation term, we shall investigate the cases of nominal system (3.3) under vanishing and non-vanishing perturbations, respectively.

4. Main results

Inspired by the improved uniform exponential stability criteria of nominal system (3.3) in [33], we make the following assumption:
(A_1) There exist two positive constants λ_1 and λ_2, a symmetric matrix function $Q \in C^1_{rd}(\mathbb{T}^+_a, \mathbb{R}^{n\times n})$, and a uniformly exponentially stable function $\beta \in \mathcal{R}^+(\mathbb{T}^+_a, \mathbb{R})$ such that for all $t \in \mathbb{T}^+_a$,

$$\lambda_1 I_n \leq Q(t) \leq \lambda_2 I_n, \tag{4.1}$$

$$A^{\mathrm{T}}(t)Q(t) + (I_n + \mu(t)A^{\mathrm{T}}(t))(Q^{\Delta}(t) + Q(t)A(t) + \mu(t)Q^{\Delta}(t)A(t)) \leq \tilde{\beta}(t)Q(t), \tag{4.2}$$

where $\tilde{\beta}(t) = (2 + \mu(t)\beta(t))\beta(t)$, here, $\beta(t)$ is given in system (3.4) and $\mu(t)$ is the graininess function denoted in Definition 2.1.

When assumption (A_1) holds, according to Theorem 4.2 in [33], we know that nominal system (3.3) is uniformly exponentially stable. Moreover, it follows from $\beta \in \mathcal{R}^+(\mathbb{T}^+_a, \mathbb{R})$ and $\tilde{\beta}(t) = (2+\mu(t)\beta(t))\beta(t)$ that $\tilde{\beta} \in \mathcal{R}^+(\mathbb{T}^+_a, \mathbb{R})$. At the same time, if $t_0 \in \mathbb{T}^+_a$, then by Lemma 2.3, we have

$$e_{\beta}(t, t_0) > 0 \text{ for all } t \in \mathbb{T}^+_a \tag{4.3}$$

and

$$e_{\tilde{\beta}}(t, t_0) = e_{\beta\oplus\beta}(t, t_0) = (e_{\beta}(t, t_0))^2 \text{ for all } t \in \mathbb{T}^+_a. \tag{4.4}$$

For convenience, in the remainder of this paper, we denote

$$w_1(t) := \mu^2(t)\|Q^{\Delta}(t)\| + \lambda_2\mu(t), \ t \in \mathbb{T}^+_a,$$

$$w_2(t) := \mu^2(t)\|A(t)\| \cdot \|Q^{\Delta}(t)\| + \lambda_2\mu(t)\|A(t)\| + \mu(t)\|Q^{\Delta}(t)\| + \lambda_2, \ t \in \mathbb{T}^+_a.$$

Now, let us start with the case of nominal system (3.3) under vanishing perturbation. In this case, the origin is an equilibrium point of perturbed system (3.2). It is assumed that the perturbation term is bounded as follows:
(A_2) There exists an rd-continuous function $k : \mathbb{T}^+_a \to \mathbb{R}_+$ such that for all $t \in \mathbb{T}^+_a$,

$$\|g(t, x(t))\| \leq k(t)\|x(t)\|,$$

where $x(t)$ is an arbitrary solution of system (3.2).

Theorem 4.1. *Suppose that* (A_1) *and* (A_2) *hold. If there exists a positive constant* M *such that for all* $t \in \mathbb{T}_a^+$,

$$\int_a^t \frac{w_1(\tau)k^2(\tau) + 2w_2(\tau)k(\tau)}{1 + \mu(\tau)\tilde{\beta}(\tau)} \Delta\tau \leq M, \tag{4.5}$$

then system (3.2) *is uniformly exponentially stable.*

Proof. Construct the following Lyapunov function:

$$V(t, x) = x^{\mathrm{T}} Q(t) x \text{ for } (t, x) \in \mathbb{T}_a^+ \times \mathbb{R}^n.$$

Let $t_0 \in \mathbb{T}_a^+$ be arbitrarily given. Then by Lemmas 2.2 and 2.10, we know that the delta derivative of the function $V(t, x)$ along the solutions of system (3.2) is as follows:

$$\begin{aligned}
V^{\Delta}(t, x(t)) \mid_{(3.2)} =& (x^{\mathrm{T}}(t)Q(t))^{\Delta} x(t) + x^{\mathrm{T}}(\sigma(t))Q(\sigma(t))x^{\Delta}(t) \\
=& [(x^{\mathrm{T}}(t))^{\Delta} Q(t) + x^{\mathrm{T}}(\sigma(t))Q^{\Delta}(t)]x(t) + x^{\mathrm{T}}(\sigma(t))Q(\sigma(t))x^{\Delta}(t) \\
=& (x^{\Delta}(t))^{\mathrm{T}} Q(t)x(t) + [x^{\mathrm{T}}(t) + \mu(t)(x^{\Delta}(t))^{\mathrm{T}}][Q^{\Delta}(t)x(t) + (Q(t) + \mu(t)Q^{\Delta}(t))x^{\Delta}(t)] \\
=& [x^{\mathrm{T}}(t)A^{\mathrm{T}}(t) + g^{\mathrm{T}}(t, x(t))]Q(t)x(t) + \{x^{\mathrm{T}}(t) + \mu(t)[x^{\mathrm{T}}(t)A^{\mathrm{T}}(t) + g^{\mathrm{T}}(t, x(t))]\} \\
& \cdot \{Q^{\Delta}(t)x(t) + Q(t)[A(t)x(t) + g(t, x(t))] + \mu(t)Q^{\Delta}(t)[A(t)x(t) + g(t, x(t))]\} \\
=& x^{\mathrm{T}}(t)[Q^{\Delta}(t) + A^{\mathrm{T}}(t)Q(t) + Q(t)A(t) + \mu(t)Q^{\Delta}(t)A(t) \\
& + \mu(t)A^{\mathrm{T}}(t)Q^{\Delta}(t) + \mu(t)A^{\mathrm{T}}(t)Q(t)A(t) + \mu^2(t)A^{\mathrm{T}}(t)Q^{\Delta}(t)A(t)]x(t) \\
& + \mu^2(t)g^{\mathrm{T}}(t, x(t))Q^{\Delta}(t)g(t, x(t)) + \mu(t)g^{\mathrm{T}}(t, x(t))Q(t)g(t, x(t)) \\
& + \mu^2(t)g^{\mathrm{T}}(t, x(t))Q^{\Delta}(t)A(t)x(t) + \mu^2(t)x^{\mathrm{T}}(t)A^{\mathrm{T}}(t)Q^{\Delta}(t)g(t, x(t)) \\
& + \mu(t)g^{\mathrm{T}}(t, x(t))Q(t)A(t)x(t) + \mu(t)x^{\mathrm{T}}(t)A^{\mathrm{T}}(t)Q(t)g(t, x(t)) \\
& + \mu(t)g^{\mathrm{T}}(t, x(t))Q^{\Delta}(t)x(t) + \mu(t)x^{\mathrm{T}}(t)Q^{\Delta}(t)g(t, x(t)) \\
& + g^{\mathrm{T}}(t, x(t))Q(t)x(t) + x^{\mathrm{T}}(t)Q(t)g(t, x(t)) \text{ for } t \in \mathbb{T}_{t_0}^+.
\end{aligned} \tag{4.6}$$

In view of (4.6), (A_1) and (A_2), we can obtain

$$\begin{aligned}
V^{\Delta}(t, x(t)) \mid_{(3.2)} \leq & x^{\mathrm{T}}(t)\tilde{\beta}(t)Q(t)x(t) + [\mu^2(t)\|Q^{\Delta}(t)\| + \lambda_2\mu(t)]k^2(t)\|x(t)\|^2 \\
& + 2[\mu^2(t)\|A(t)\| \cdot \|Q^{\Delta}(t)\| + \lambda_2\mu(t)\|A(t)\| + \mu(t)\|Q^{\Delta}(t)\| + \lambda_2]k(t)\|x(t)\|^2 \\
=& \tilde{\beta}(t)V(t, x(t)) + [w_1(t)k^2(t) + 2w_2(t)k(t)]\|x(t)\|^2 \text{ for all } t \in \mathbb{T}_{t_0}^+.
\end{aligned} \tag{4.7}$$

Now, by means of (4.7) and Lemma 2.5, we get

$$V(t, x(t)) \leq V(t_0, x(t_0))e_{\tilde{\beta}}(t, t_0) + \int_{t_0}^t e_{\tilde{\beta}}(t, \sigma(\tau))[w_1(\tau)k^2(\tau) + 2w_2(\tau)k(\tau)]\|x(\tau)\|^2 \Delta\tau \text{ for all } t \in \mathbb{T}_{t_0}^+,$$

which together with (4.1) and Lemma 2.3 shows that

$$\begin{aligned}
\lambda_1 \|x(t)\|^2 \leq & V(t, x(t)) \\
\leq & V(t_0, x(t_0))e_{\tilde{\beta}}(t, t_0) + \int_{t_0}^t e_{\tilde{\beta}}(t, \sigma(\tau))[w_1(\tau)k^2(\tau) + 2w_2(\tau)k(\tau)] \|x(\tau)\|^2 \Delta\tau \\
\leq & \lambda_2 \|x(t_0)\|^2 e_{\tilde{\beta}}(t, t_0)
\end{aligned}$$

$$+ e_{\tilde{\beta}}(t, t_0) \int_{t_0}^{t} e_{\tilde{\beta}}(t_0, \tau) \|x(\tau)\|^2 e_{\tilde{\beta}}(\tau, \sigma(\tau))[w_1(\tau)k^2(\tau) + 2w_2(\tau)k(\tau)]\Delta\tau \text{ for all } t \in \mathbb{T}_{t_0}^+.$$

In view of (4.3) and (4.4), we know that $e_{\tilde{\beta}}(t, t_0) > 0$ for all $t \in \mathbb{T}_{t_0}^+$. So,

$$\frac{\|x(t)\|^2}{e_{\tilde{\beta}}(t, t_0)} \le \frac{\lambda_2}{\lambda_1} \|x(t_0)\|^2 + \int_{t_0}^{t} \frac{\|x(\tau)\|^2}{e_{\tilde{\beta}}(\tau, t_0)} \cdot \frac{w_1(\tau)k^2(\tau) + 2w_2(\tau)k(\tau)}{\lambda_1(1 + \mu(\tau)\tilde{\beta}(\tau))}\Delta\tau \text{ for all } t \in \mathbb{T}_{t_0}^+. \tag{4.8}$$

If we define

$$p(t) = \frac{w_1(t)k^2(t) + 2w_2(t)k(t)}{\lambda_1(1 + \mu(t)\tilde{\beta}(t))} \text{ for } t \in \mathbb{T}_a^+,$$

then it is not difficult to know that $p \in \mathcal{R}^+(\mathbb{T}_a^+, \mathbb{R}_+)$. On the one hand, by (4.8) and Lemma 2.6, we have

$$\frac{\|x(t)\|^2}{e_{\tilde{\beta}}(t, t_0)} \le \frac{\lambda_2}{\lambda_1} \|x(t_0)\|^2 + \int_{t_0}^{t} \frac{\lambda_2}{\lambda_1} e_p(t, \sigma(\tau)) \|x(t_0)\|^2 p(\tau)\Delta\tau \text{ for all } t \in \mathbb{T}_{t_0}^+,$$

which together with Lemma 2.4 and $e_p(t, t) \equiv 1$ for all $t \in \mathbb{T}_a^+$ implies that

$$\frac{\|x(t)\|^2}{e_{\tilde{\beta}}(t, t_0)} \le \frac{\lambda_2}{\lambda_1} \|x(t_0)\|^2 [1 + e_p(t, t_0) - e_p(t, t)] = \frac{\lambda_2}{\lambda_1} \|x(t_0)\|^2 e_p(t, t_0) \text{ for all } t \in \mathbb{T}_{t_0}^+,$$

so,

$$\|x(t)\|^2 \le \frac{\lambda_2}{\lambda_1} e_p(t, t_0) \|x(t_0)\|^2 e_{\tilde{\beta}}(t, t_0) \text{ for all } t \in \mathbb{T}_{t_0}^+. \tag{4.9}$$

On the other hand, by Lemma 2.7 and (4.5), we know

$$e_p(t, t_0) \le \exp\left(\int_{t_0}^{t} p(\tau)\Delta\tau\right) \le e^{\frac{M}{\lambda_1}} \text{ for all } t \in \mathbb{T}_{t_0}^+. \tag{4.10}$$

Moreover, since β is uniformly exponentially stable, there exist constants $\eta \ge 1$ and $\alpha > 0$ with $-\alpha \in \mathcal{R}^+(\mathbb{T}_a^+, (0, -\infty))$ such that

$$e_\beta(t, t_0) \le \eta e_{-\alpha}(t, t_0) \text{ for all } t \in \mathbb{T}_{t_0}^+,$$

which together with (4.3) and (4.4) shows that

$$e_{\tilde{\beta}}(t, t_0) \le \eta^2 (e_{-\alpha}(t, t_0))^2 \text{ for all } t \in \mathbb{T}_{t_0}^+. \tag{4.11}$$

So, in view of (4.9)–(4.11), we obtain

$$\|x(t)\|^2 \le \frac{\eta^2 \lambda_2 e^{\frac{M}{\lambda_1}}}{\lambda_1} \|x(t_0)\|^2 (e_{-\alpha}(t, t_0))^2 \text{ for all } t \in \mathbb{T}_{t_0}^+,$$

and so,

$$\|x(t)\| \le \frac{\eta(\lambda_1\lambda_2)^{\frac{1}{2}} e^{\frac{M}{2\lambda_1}}}{\lambda_1} \|x(t_0)\| e_{-\alpha}(t, t_0) \text{ for all } t \in \mathbb{T}_{t_0}^+,$$

which indicates that system (3.2) is uniformly exponentially stable. □

Next, let us turn to the case of nominal system (3.3) under non-vanishing perturbation. In this situation, the origin is no longer the equilibrium point of perturbed system (3.2). We assume that the perturbation term is bounded as follows:
(A_3) There exist two rd-continuous functions $l_1, l_2 : \mathbb{T}_a^+ \to \mathbb{R}_+$ such that for all $t \in \mathbb{T}_a^+$,

$$\|g(t, x(t))\| \leq l_1(t)\,\|x(t)\| + l_2(t),$$

where $x(t)$ is an arbitrary solution of system (3.2).

Theorem 4.2. *Suppose that* (A_1) *and* (A_3) *hold. If there exist two positive constants* N_1 *and* N_2 *such that for all* $t \in \mathbb{T}_a^+$,

$$\int_a^t \frac{u_1(\tau)}{1 + \mu(\tau)\tilde{\beta}(\tau)} \Delta\tau \leq N_1, \tag{4.12}$$

$$\int_a^t e_{\tilde{\beta}}(t, \sigma(\tau))u_2(\tau)\Delta\tau \leq N_2, \tag{4.13}$$

where

$$u_1(t) = w_1(t)l_1^2(t) + 2w_2(t)l_1(t) + \frac{\lambda_2}{\lambda_1}[w_1(t)l_1(t) + w_2(t)]l_2(t),\ t \in \mathbb{T}_a^+,$$

$$u_2(t) = [w_1(t)l_1(t) + w_2(t)]l_2(t) + w_1(t)l_2^2(t),\ t \in \mathbb{T}_a^+,$$

then system (3.2) *is uniformly practically exponentially stable.*

Proof. Choose the following Lyapunov function:

$$V(t, x) = x^{\mathrm{T}}Q(t)x \text{ for } (t, x) \in \mathbb{T}_a^+ \times \mathbb{R}^n.$$

Then it follows from (4.1) that

$$\|x\| \leq \left(\frac{1}{\lambda_1}V(t, x)\right)^{\frac{1}{2}} \leq \frac{1}{2\lambda_1}V(t, x) + \frac{1}{2} \leq \frac{\lambda_2}{2\lambda_1}\|x\|^2 + \frac{1}{2} \text{ for all } (t, x) \in \mathbb{T}_a^+ \times \mathbb{R}^n. \tag{4.14}$$

Let $t_0 \in \mathbb{T}_a^+$ be arbitrarily given. Then as calculated in Theorem 4.1, the delta derivative of the function $V(t, x)$ along the solutions of system (3.2) is as follows:

$$\begin{aligned}
V^{\Delta}(t, x(t))\,|_{(3.2)} = &x^{\mathrm{T}}(t)[Q^{\Delta}(t) + A^{\mathrm{T}}(t)Q(t) + Q(t)A(t) + \mu(t)Q^{\Delta}(t)A(t) \\
&+ \mu(t)A^{\mathrm{T}}(t)Q^{\Delta}(t) + \mu(t)A^{\mathrm{T}}(t)Q(t)A(t) + \mu^2(t)A^{\mathrm{T}}(t)Q^{\Delta}(t)A(t)]x(t) \\
&+ \mu^2(t)g^{\mathrm{T}}(t, x(t))Q^{\Delta}(t)g(t, x(t)) + \mu(t)g^{\mathrm{T}}(t, x(t))Q(t)g(t, x(t)) \\
&+ \mu^2(t)g^{\mathrm{T}}(t, x(t))Q^{\Delta}(t)A(t)x(t) + \mu^2(t)x^{\mathrm{T}}(t)A^{\mathrm{T}}(t)Q^{\Delta}(t)g(t, x(t)) \\
&+ \mu(t)g^{\mathrm{T}}(t, x(t))Q(t)A(t)x(t) + \mu(t)x^{\mathrm{T}}(t)A^{\mathrm{T}}(t)Q(t)g(t, x(t)) \\
&+ \mu(t)g^{\mathrm{T}}(t, x(t))Q^{\Delta}(t)x(t) + \mu(t)x^{\mathrm{T}}(t)Q^{\Delta}(t)g(t, x(t)) \\
&+ g^{\mathrm{T}}(t, x(t))Q(t)x(t) + x^{\mathrm{T}}(t)Q(t)g(t, x(t)) \text{ for } t \in \mathbb{T}_{t_0}^+,
\end{aligned}$$

which together with (A_1) and (A_3) implies that

$$
\begin{aligned}
V^{\Delta}(t, x(t)) \mid_{(3.2)} \leq & x^{\mathrm{T}}(t)\tilde{\beta}(t)Q(t)x(t) + [\mu^2(t)\|Q^{\Delta}(t)\| + \lambda_2\mu(t)][l_1(t)\|x(t)\| + l_2(t)]^2 \\
& + 2[\mu^2(t)\|A(t)\| \cdot \|Q^{\Delta}(t)\| + \lambda_2\mu(t)\|A(t)\| + \mu(t)\|Q^{\Delta}(t)\| + \lambda_2] \\
& \cdot [l_1(t)\|x(t)\| + l_2(t)]\|x(t)\| \\
= & \tilde{\beta}(t)V(t, x(t)) + [w_1(t)l_1^2(t) + 2w_2(t)l_1(t)] \, \|x(t)\|^2 \\
& + 2[w_1(t)l_1(t) + w_2(t)]l_2(t) \, \|x(t)\| + w_1(t)l_2^2(t) \text{ for all } t \in \mathbb{T}_{t_0}^+.
\end{aligned} \tag{4.15}
$$

In view of (4.14) and (4.15), we have

$$
\begin{aligned}
V^{\Delta}(t, x(t)) \mid_{(3.2)} \leq & \tilde{\beta}(t)V((t, x(t)) + \{w_1(t)l_1^2(t) + 2w_2(t)l_1(t) + \frac{\lambda_2}{\lambda_1}[w_1(t)l_1(t) + w_2(t)]l_2(t)\} \, \|x(t)\|^2 \\
& + [w_1(t)l_1(t) + w_2(t)]l_2(t) + w_1(t)l_2^2(t) \\
= & \tilde{\beta}(t)V(t, x(t)) + u_1(t) \, \|x(t)\|^2 + u_2(t) \text{ for all } t \in \mathbb{T}_{t_0}^+.
\end{aligned} \tag{4.16}
$$

Now, by means of (4.16) and Lemma 2.5, we get

$$
V(t, x(t)) \leq V(t_0, x(t_0))e_{\tilde{\beta}}(t, t_0) + \int_{t_0}^{t} e_{\tilde{\beta}}(t, \sigma(\tau))[u_1(\tau) \, \|x(\tau)\|^2 + u_2(\tau)]\Delta\tau \text{ for all } t \in \mathbb{T}_{t_0}^+,
$$

which together with (4.1) and Lemma 2.3 shows that

$$
\begin{aligned}
\lambda_1\|x(t)\|^2 \leq & V(t, x(t)) \\
\leq & V(t_0, x(t_0))e_{\tilde{\beta}}(t, t_0) + \int_{t_0}^{t} e_{\tilde{\beta}}(t, \sigma(\tau))[u_1(\tau) \, \|x(\tau)\|^2 + u_2(\tau)]\Delta\tau \\
\leq & \lambda_2 \, \|x(t_0)\|^2 \, e_{\tilde{\beta}}(t, t_0) \\
& + e_{\tilde{\beta}}(t, t_0) \int_{t_0}^{t} [\frac{1}{\lambda_1} e_{\tilde{\beta}}(t_0, \sigma(\tau))u_1(\tau)\lambda_1 \, \|x(\tau)\|^2 + e_{\tilde{\beta}}(t_0, \sigma(\tau))u_2(\tau)]\Delta\tau \text{ for all } t \in \mathbb{T}_{t_0}^+.
\end{aligned} \tag{4.17}
$$

By (4.17) and Lemma 2.9, we get

$$
\begin{aligned}
\lambda_1 \, \|x(t)\|^2 \leq & \lambda_2 \, \|x(t_0)\|^2 \, e_{\tilde{\beta}}(t, t_0) \\
& + e_{\tilde{\beta}}(t, t_0) \int_{t_0}^{t} [\frac{\lambda_2}{\lambda_1} e_{\tilde{\beta}}(t_0, \sigma(\tau))u_1(\tau) \, \|x(t_0)\|^2 \, e_{\tilde{\beta}}(\tau, t_0) + e_{\tilde{\beta}}(t_0, \sigma(\tau))u_2(\tau)] \\
& \cdot \exp\Big(\int_{\sigma(\tau)}^{t} \frac{1}{\lambda_1} e_{\tilde{\beta}}(t_0, \sigma(s))u_1(s)e_{\tilde{\beta}}(s, t_0)\Delta s \Big)\Delta\tau \text{ for all } t \in \mathbb{T}_{t_0}^+,
\end{aligned}
$$

so,

$$
\begin{aligned}
\|x(t)\|^2 \leq & \frac{\lambda_2}{\lambda_1} \, \|x(t_0)\|^2 \, e_{\tilde{\beta}}(t, t_0) \\
& + \Big[\frac{\lambda_2}{\lambda_1^2} \, \|x(t_0)\|^2 \, e_{\tilde{\beta}}(t, t_0) \int_{t_0}^{t} \frac{u_1(\tau)}{1 + \mu(\tau)\tilde{\beta}(\tau)}\Delta\tau + \frac{1}{\lambda_1} \int_{t_0}^{t} e_{\tilde{\beta}}(t, \sigma(\tau))u_2(\tau)\Delta\tau \Big] \\
& \cdot \exp\Big(\frac{1}{\lambda_1} \int_{t_0}^{t} \frac{u_1(s)}{1 + \mu(s)\tilde{\beta}(s)}\Delta s \Big) \text{ for all } t \in \mathbb{T}_{t_0}^+,
\end{aligned}
$$

which together with (4.12) and (4.13) indicates that

$$\|x(t)\|^2 \leq \frac{\lambda_2}{\lambda_1} \|x(t_0)\|^2 e_{\tilde{\beta}}(t, t_0) + \frac{\lambda_2 N_1 e^{\frac{N_1}{\lambda_1}}}{\lambda_1^2} \|x(t_0)\|^2 e_{\tilde{\beta}}(t, t_0) + \frac{N_2 e^{\frac{N_1}{\lambda_1}}}{\lambda_1} \text{ for all } t \in \mathbb{T}_{t_0}^+. \tag{4.18}$$

Furthermore, it follows from the fact β is uniformly exponentially stable that there exist constants $\eta \geq 1$ and $\alpha > 0$ with $-\alpha \in \mathcal{R}^+(\mathbb{T}_a^+, (0, -\infty))$ such that

$$e_\beta(t, t_0) \leq \eta e_{-\alpha}(t, t_0) \text{ for all } t \in \mathbb{T}_{t_0}^+. \tag{4.19}$$

By (4.3), (4.4) and (4.19), we have

$$e_{\tilde{\beta}}(t, t_0) \leq \eta^2 (e_{-\alpha}(t, t_0))^2 \text{ for all } t \in \mathbb{T}_{t_0}^+. \tag{4.20}$$

So, in view of (4.18) and (4.20), we can obtain

$$\|x(t)\|^2 \leq \frac{\eta^2 \lambda_1 \lambda_2 + \eta^2 \lambda_2 N_1 e^{\frac{N_1}{\lambda_1}}}{\lambda_1^2} \|x(t_0)\|^2 (e_{-\alpha}(t, t_0))^2 + \frac{N_2 e^{\frac{N_1}{\lambda_1}}}{\lambda_1} \text{ for all } t \in \mathbb{T}_{t_0}^+,$$

and so,

$$\|x(t)\| \leq \frac{\eta \left(\lambda_1 \lambda_2 + \lambda_2 N_1 e^{\frac{N_1}{\lambda_1}} \right)^{\frac{1}{2}}}{\lambda_1} \|x(t_0)\| \, e_{-\alpha}(t, t_0) + \frac{(\lambda_1 N_2)^{\frac{1}{2}} e^{\frac{N_1}{2\lambda_1}}}{\lambda_1} \text{ for all } t \in \mathbb{T}_{t_0}^+,$$

which implies that system (3.2) is uniformly practically exponentially stable. □

Remark 4.1. *Note that inequality* (4.2) *is a generalised form and covers many special cases. For example, when* $\mathbb{T} = \mathbb{R}$, *inequality* (4.2) *is transformed as* $\dot{Q}(t) + A^{\mathrm{T}}(t)Q(t) + Q(t)A(t) \leq 2\beta(t)Q(t)$. *When* $\mathbb{T} = \mathbb{Z}$, *inequality* (4.2) *is transformed as* $[I_n + A^{\mathrm{T}}(t)]Q(t+1)[I_n + A(t)] \leq (1 + \beta(t))^2 Q(t)$.

Remark 4.2. *The sufficient conditions for uniform exponential stability of perturbed system* (3.2) *proposed in [26, 27] require the time derivatives of the quadratic Lyapunov functions to be negative definite, which are conservative. By* (4.7), *we know that the time derivative of the quadratic Lyapunov function along the trajectories of system* (3.2) *is allowed to be indefinite. Thus, the sufficient conditions for uniform exponential stability of system* (3.2) *derived in Theorem* 4.1 *are less conservative than those in [26, 27].*

Remark 4.3. *If* $l_1(t) \equiv 0$ *for all* $t \in \mathbb{T}_a^+$ *in assumption* (A_3), *then integral inequalities (4.12) and (4.13) in Theorem 4.2 can be simplified.*

Remark 4.4. *We say system (3.2) is uniformly exponentially stable if the origin is uniformly exponentially stable. However, we say system (3.2) is uniformly practically exponentially stable if a neighborhood of the origin is uniformly exponentially stable.*

Remark 4.5. *The main results obtained in this paper are of great generality, since they can be effective for not only trivial continuous-time and discrete-time nonlinear time-varying perturbed systems, but also some other nontrivial cases, such as systems on hybrid time domains.*

5. Some illustrative examples

In this section, we provide three examples to demonstrate the effectiveness of the results obtained.

Example 5.1. Let $\mathbb{T} = \bigcup_{k\in\mathbb{N}_0} [k, k + 0.6]$ and $a = 0$. We consider the following nonlinear time-varying perturbed system:

$$\begin{cases} x^{\Delta}(t) = A(t)x(t) + g(t, x(t)), \\ x(t_0) = x_0, \end{cases} \tag{5.1}$$

where $t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}^2$, $x : \mathbb{T} \to \mathbb{R}^2$,

$$A(t) = \begin{pmatrix} -1 - 0.9\sin t & 0 \\ 0 & -1 - 0.9\sin t \end{pmatrix}$$

and

$$g(t, x(t)) = \begin{pmatrix} \sqrt{2}e_{-1}(t, 0)\|x(t)\| \\ \sqrt{2}e_{-1}(t, 0)\|x(t)\| \end{pmatrix}.$$

First, if we choose $\lambda_1 = \lambda_2 = 1$, $Q(t) = I_2$ and $\beta(t) = -1 - 0.9\sin t$ for $t \in \mathbb{T}$, then we may assert that all conditions of assumption (A_1) are satisfied.

In fact, it is obvious that $Q \in C^1_{rd}(\mathbb{T}, \mathbb{R}^{2\times 2})$ is symmetric and (4.1) is satisfied. At the same time, we know that $-1.9 \le \beta(t) \le -0.1$ for all $t \in \mathbb{T}$. Since $\mathbb{T} = \bigcup_{k\in\mathbb{N}_0} [k, k + 0.6]$, we have

$$\mu(t) = \begin{cases} 0, \ t \in \bigcup_{k\in\mathbb{N}_0} [k, k + 0.6), \\ 0.4, \ t \in \bigcup_{k\in\mathbb{N}_0} \{k + 0.6\}. \end{cases}$$

So, it is not difficult to check that $\beta \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$, which together with Lemma 2.8 indicates that for any $t_0 \in \mathbb{T}$,

$$e_{\beta}(t, t_0) \le e_{-0.1}(t, t_0) \text{ for all } t \in \mathbb{T}^+_{t_0}. \tag{5.2}$$

By (5.2) and Lemma 3.1, we know that β is uniformly exponentially stable. Moreover, for all $t \in \mathbb{T}$, we have

$$\begin{aligned} & A^{\mathrm{T}}(t)Q(t) + (I_2 + \mu(t)A^{\mathrm{T}}(t))(Q^{\Delta}(t) + Q(t)A(t) + \mu(t)Q^{\Delta}(t)A(t)) \\ =& A^{\mathrm{T}}(t) + A(t) + \mu(t)A^{\mathrm{T}}(t)A(t) \\ =& \left[-2(1 + 0.9\sin t) + \mu(t)(1 + 0.9\sin t)^2\right] Q(t) \\ =& \tilde{\beta}(t)Q(t), \end{aligned}$$

which implies that (4.2) is fulfilled.

Next, we let $k(t) = 2e_{-1}(t, 0)$ for $t \in \mathbb{T}$. Then $k : \mathbb{T} \to \mathbb{R}_+$ is rd-continuous and $\|g(t, x(t))\| = k(t)\,\|x(t)\|$ for $t \in \mathbb{T}$. That is, (A_2) holds.

Finally, we choose $M = 150$. Then in view of $w_1(t) = \mu(t)$ and $w_2(t) = \mu(t)(1 + 0.9\sin t) + 1$ for $t \in \mathbb{T}$, we can obtain

$$\begin{aligned}
&\int_0^t \frac{w_1(\tau)k^2(\tau)+2w_2(\tau)k(\tau)}{1+\mu(\tau)\tilde{\beta}(\tau)}\Delta\tau\\
\leq &\int_0^t \frac{1.6(e_{-1}(\tau,0))^2+4[0.4(1+0.9\sin t)+1]e_{-1}(\tau,0)}{(1+0.4\beta(\tau))^2}\Delta\tau\\
\leq &150\int_0^t e_{-1}(\tau,0)\Delta\tau\\
= &150(1-e_{-1}(t,0))\\
< &150 \text{ for all } t\in\mathbb{T},
\end{aligned}$$

which indicates that (4.5) is fulfilled.

Therefore, it follows from Theorem 4.1 that system (5.1) is uniformly exponentially stable.

Example 5.2. Let $\mathbb{T}=\mathbb{R}$ and $a=0$. We consider the following continuous-time nonlinear time-varying perturbed system:

$$\begin{cases} x'(t)=A(t)x(t)+g(t,x(t)),\\ x(t_0)=x_0, \end{cases} \tag{5.3}$$

where $t_0\in\mathbb{R}_+$, $x_0\in\mathbb{R}$, $x:\mathbb{R}_+\to\mathbb{R}$, $A(t)=\frac{t\cos t^2-2}{2}$ and $g(t,x(t))=e^{-2t}+\frac{1}{6(t^2+1)}\tanh(x(t))x(t)$.

Choose $\lambda_1=2$, $\lambda_2=3$, $Q(t)=e^{-t}+2$, $\beta(t)=\frac{t\cos t^2-2}{2}$, $l_1(t)=\frac{1}{6(t^2+1)}$ and $l_2(t)=e^{-2t}$ for $t\in\mathbb{R}_+$, $N_1=\frac{2\pi+9}{4}$ and $N_2=\frac{3}{2}$. Then $Q\in C^1(\mathbb{R}_+,\mathbb{R})$, $\beta\in C(\mathbb{R}_+,\mathbb{R})$ and $l_1,l_2\in C(\mathbb{R}_+,(0,+\infty))$. In view of $\mu(t)\equiv 0$, we know that $\tilde{\beta}(t)=t\cos t^2-2$, $w_1(t)=0$ and $w_2(t)=3$ for $t\in\mathbb{R}_+$. In what follows, we will verify that all conditions of Theorem 4.2 are satisfied.

First, (4.1) is obviously satisfied. Moreover, for any $t_0\in\mathbb{R}_+$ and $t\in[t_0,+\infty)$, we have

$$e_\beta(t,t_0)=\exp\left(\int_{t_0}^t \frac{\tau\cos\tau^2-2}{2}d\tau\right)\leq \sqrt{e}e^{-(t-t_0)}=\sqrt{e}e_{-1}(t,t_0),$$

which together with Lemma 3.1 implies that β is uniformly exponentially stable. At the same time, for all $t\in\mathbb{R}_+$, we get

$$A^{\mathrm{T}}(t)Q(t)+Q'(t)+Q(t)A(t)=(t\cos t^2-2)Q(t)-e^{-t}<2\beta(t)Q(t),$$

which shows that (4.2) is fulfilled. Thus, (A_1) holds.

Next, for all $t\in\mathbb{R}_+$, we have

$$|g(t,x(t))|\leq l_1(t)|x(t)|+l_2(t),$$

which shows that (A_3) holds.

Finally, in view of $w_1(t)=0$ and $w_2(t)=3$ for $t\in\mathbb{R}_+$, we know that $u_1(t)=\frac{1}{t^2+1}+\frac{9}{2}e^{-2t}$ and $u_2(t)=3e^{-2t}$ for $t\in\mathbb{R}_+$. So, by some calculations, for all $t\in\mathbb{R}_+$,

$$\int_0^t u_1(\tau)d\tau=\arctan t+\frac{9}{4}(1-e^{-2t})<N_1$$

and

$$\int_0^t e_{\tilde{\beta}}(t,\tau)u_2(\tau)d\tau\leq 3e\int_0^t e^{-2(t-\tau)}e^{-2\tau}d\tau=3ete^{-2t}\leq N_2,$$

which implies that (4.12) and (4.13) are satisfied.

Therefore, it follows from Theorem 4.2 that system (5.3) is uniformly practically exponentially stable.

Example 5.3. Let $\mathbb{T}$ be a time scale with a non-uniform step size and $a = 0$. The graininess function is bounded as follows:

$$0 \le \mu(t) \le 0.25 \text{ for all } t \in \mathbb{T}.$$

We consider the following nonlinear time-varying perturbed system:

$$\begin{cases} x^{\Delta}(t) = A(t)x(t) + g(t, x(t)), \\ x(t_0) = x_0, \end{cases} \tag{5.4}$$

where $t_0 \in \mathbb{T}_0^+$, $x_0 \in \mathbb{R}^2$, $x : \mathbb{T}_0^+ \to \mathbb{R}^2$,

$$A(t) = \begin{pmatrix} -a(t) & -1 \\ 1 & -a(t) \end{pmatrix}$$

and

$$g(t, x(t)) = \begin{pmatrix} \dfrac{-0.125(a(t) - 1)\cos t}{1 + \|x(t)\|} \\ \dfrac{0.125(a(t) - 1)\sin t}{1 + \|x(t)\|} \end{pmatrix},$$

here, $a(t) = e_{\ominus 8}(t, 0) + 1$, $t \in \mathbb{T}_0^+$.

First, if we choose $\lambda_1 = \lambda_2 = 1$, $Q(t) = I_2$ and $\beta(t) = -e_{\ominus 8}(t, 0) - \frac{1}{2}$ for $t \in \mathbb{T}_0^+$, then we may assert that all conditions of assumption (A_1) are satisfied. In fact, (4.1) is obviously satisfied and for all $t \in \mathbb{T}_0^+$,

$$\begin{aligned} & A^{\mathrm{T}}(t)Q(t) + (I_2 + \mu(t)A^{\mathrm{T}}(t))(Q^{\Delta}(t) + Q(t)A(t) + \mu(t)Q^{\Delta}(t)A(t)) \\ = & \left[-2a(t) + \mu(t)(a^2(t) + 1)\right] Q(t) \\ < & \tilde{\beta}(t)Q(t), \end{aligned}$$

where $\tilde{\beta}(t) = (2 + \mu(t)\beta(t))\beta(t)$. Moreover, in view of $\ominus 8 = -\frac{8}{1+8\mu(t)}$, we get $\ominus 8 \in \mathcal{R}^+(\mathbb{T}_0^+, \mathbb{R})$ and $-8 \le \ominus 8 \le -\frac{8}{3}$ for all $t \in \mathbb{T}_0^+$. So, $0 < e_{\ominus 8}(t, 0) \le 1$ for all $t \in \mathbb{T}_0^+$, and so, $-\frac{3}{2} \le \beta(t) < -\frac{1}{2}$ for all $t \in \mathbb{T}_0^+$. At the same time, we have $-\frac{39}{16} \le \tilde{\beta}(t) < -\frac{15}{16}$ for all $t \in \mathbb{T}_0^+$. Then it follows from Lemmas 2.8 and 3.1 that $\beta \in \mathcal{R}^+(\mathbb{T}_0^+, \mathbb{R})$ is uniformly exponentially stable.

Next, since

$$\|g(t, x(t))\| = \frac{e_{\ominus 8}(t, 0)}{8(1 + \|x(t)\|)} \le \frac{1}{8} e_{\ominus 8}(t, 0) \text{ for all } t \in \mathbb{T}_0^+,$$

we can choose $l_1(t) \equiv 0$ and $l_2(t) = \frac{1}{8} e_{\ominus 8}(t, 0)$ for $t \in \mathbb{T}_0^+$. Obviously, $l_2 : \mathbb{T}_0^+ \to \mathbb{R}_+$ is rd-continuous. This implies that assumption (A_3) holds.

Finally, in view of $w_1(t) = \mu(t)$ and $w_2(t) = \mu(t)\sqrt{(e_{\ominus 8}(t, 0) + 1)^2 + 1} + 1$ for $t \in \mathbb{T}_0^+$, we can obtain

$$u_1(t) = \frac{1}{8}[\mu(t)\sqrt{(e_{\ominus 8}(t, 0) + 1)^2 + 1} + 1]e_{\ominus 8}(t, 0), \ t \in \mathbb{T}_0^+,$$

and

$$u_2(t) = \frac{1}{8}[\mu(t)\sqrt{(e_{\ominus 8}(t,0)+1)^2+1}+1]e_{\ominus 8}(t,0) + \frac{1}{64}\mu(t)(e_{\ominus 8}(t,0))^2,\ t \in \mathbb{T}_0^+.$$

So, if we choose $N_1 = \frac{12+3\sqrt{5}}{100}$ and $N_2 = \frac{33+8\sqrt{5}}{240}$, then for all $t \in \mathbb{T}_0^+$, we can obtain

$$\int_0^t \frac{u_1(\tau)}{1+\mu(\tau)\tilde{\beta}(\tau)}\Delta\tau \le \frac{4+\sqrt{5}}{32}\int_0^t \frac{e_{\ominus 8}(\tau,0)}{(1+\mu(\tau)\beta(\tau))^2}\Delta\tau \le \frac{8+2\sqrt{5}}{25}\int_0^t e_{-\frac{8}{3}}(\tau,0)\Delta\tau \le N_1,$$

$$\int_0^t e_{\tilde{\beta}}(t,\sigma(\tau))u_2(\tau)\Delta\tau < -\frac{33+8\sqrt{5}}{240}\int_0^t \tilde{\beta}(\tau)e_{\tilde{\beta}}(t,\sigma(\tau))\Delta\tau = -\frac{33+8\sqrt{5}}{240}(e_{\tilde{\beta}}(t,0)-e_{\tilde{\beta}}(t,t)) \le N_2,$$

which shows that (4.12) and (4.13) are satisfied.

Therefore, it follows from Theorem 4.2 that system (5.4) is uniformly practically exponentially stable.

Remark 5.1. *When the origin is an equilibrium point of perturbed system, the uniform exponential stability has been studied . When the origin is no longer the equilibrium point of perturbed system, the uniform practical exponential stability has been discussed, which enriches the existing results on time scales.*

Remark 5.2. *By the above examples, we know that the time derivatives of related Lyapunov functions are allowed to be nonnegative definite on some time intervals.*

Remark 5.3. *Note that the time scale and the corresponding linear time-varying nominal system considered in Example 5.3 are the same as those in [26]. The authors in [26] have only discussed the case of nominal system under vanishing perturbation, while we have investigated the case of nominal system under non-vanishing perturbation.*

6. Conclusions

This paper is concerned with nonlinear time-varying perturbed systems on time scales, which can cover not only trivial continuous-time and discrete-time dynamical perturbed systems, but also some other nontrivial cases, such as systems on hybrid time domains, where time is partly continuous and partly discrete.

Inspired by the uniform exponential stability criteria for linear time-varying nominal system in [33], we use a Lyapunov function for nominal system as a Lyapunov function candidate for perturbed system. Although the work of this paper is motivated by linear time-varying system in [33], the system considered in this article is quite different from it. By imposing different assumptions on the perturbation term, we investigate the cases of nominal system under vanishing and non-vanishing perturbations, respectively. It should be pointed out that if applying the existing methods in [26, 27] to analyse the exponential stability of nonlinear time-varying perturbed system, we must guarantee that the time derivatives of related Lyapunov functions are negative definite for all time. However, by inequalities (4.7) and (4.16), we know that the time derivatives of Lyapunov functions in this paper are allowed to be nonnegative definite on some time intervals. Compared with [26, 27], a less conservative sufficient condition for uniform exponential stability of perturbed system is derived with the help of the uniformly exponentially stable function on time scales. In addition, it should be emphasized that the research about the practical stability problem of nonlinear time-varying perturbed systems on time scales is inadequate. Based on an improved Gronwall-type integral inequality and the uniformly exponentially stable function on time scales, a new sufficient condition for uniform practical exponential stability of perturbed system is explored, which enriches the existing results on time scales. Finally, some examples are included to illustrate the effectiveness of the results obtained.

Our possible future work is to investigate the stability for nonlinear systems with time-delay on time scales. Besides, since the conclusions obtained in this paper are all theoretical results, we will pay more attention to the combination of theoretical research and practical application in the future.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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