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Research article

Exponential sums involving the divisor function over arithmetic progressions

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Abstract: Let $\phi(x)$ be a smooth function supported on $[1, 2]$ with derivatives bounded by $\phi^{(j)}(x) \ll 1$ and $d_3(n)$ be the number of ways to write n as a product of three factors. We get the asymptotic formula for the nonlinear exponential sum $\sum\limits_{n \,\equiv\, l \bmod q} d_3(n)\phi\left(\frac{n}{X}\right)e\left(\frac{3\sqrt[3]{kn}}{q}\right)$.

Keywords: exponential sum; divisor function; arithmetic progressions
Mathematics Subject Classification: 11F30, 11L07, 11N37

1. Introduction

The divisor functions

$$d_k(n) = \sum_{n_1 n_2 \cdots n_k = n} 1$$

are the basic arithmetic functions in number theory, and it generate the Dirichlet series $\zeta^k(s)$ which are the simplest GL_k L-functions. Hence the behavior of the divisor functions are very important in the theory of automorphic L-funcions. In this article, we will study the sum of the type

$$\sum_{n \,\equiv\, l \bmod q} d_3(n)\phi\left(\frac{n}{X}\right)e\left(\frac{3\sqrt[3]{kn}}{q}\right), \tag{1.1}$$

where $\phi(x)$ is a C^∞-function supported on $[1, 2]$ with derivatives bounded by $\phi^{(j)}(x) \ll 1$ and $k \in \mathbb{Z}^+$.

Studying the asymptotic distribution of this type of sums that involving the Fourier coefficients and the nonlinear exponential functions are very classical in analytic number theory. The oscillation behavior of Fourier coefficients of GL_2 automorphic forms is studied by Ren and Ye [13] and they proved an asymptotic formula for the sum

$$\sum_{X<n\leq 2X} \lambda_f(n)e\left(\alpha\sqrt{n}\right),$$

where $\lambda_f(n)$ is the n-th Fourier coefficient of a holomorphic cusp form for GL_2. The analytic properties of $\lambda_f(n)$ were studied by many authors, see [7–10, 19–23]. For the Maass forms on GL_2, Sun and Wu [16] proved the similar asymptotic formula. Acharya and Singh [1] gave the upper bound of the sum

$$\sum_{N<n\leq 2N} \lambda_f(n)\nu(n)e\left(\alpha n^{\theta}\right),$$

where α, θ are real numbers with $0 < \theta < 1$, and $\nu(n)$ is either $\mu(n)$ or $\Lambda(n)$. If $f(x)$ is a Hecke-Maass cusp form for $SL(3,\mathbb{Z})$, Ren and Ye [14] proved an asymptotic formula for the sum

$$\sum_{n\geq 1} A_f(1;n)\phi\left(\frac{n}{X}\right)e\left(3\sqrt[3]{kn}\right),$$

where $A_f(1;n)$ is the $(1,n)$-th Fourier coefficients of f. Let f be a full-level cusp form for $GL_m(\mathbb{Z})$ with Fourier coefficients $A_f(n_1,\cdots,n_{m-1})$, Ren and Ye [15] considered the following exponential sums:

$$\sum_{X<|n|\leq 2X} A_f(n,1,\cdots,1)e\left(\pm\alpha|n|^{1/m}\right),$$

$$\sum_{n\neq 0} A_f(n,1,\cdots,1)e\left(\pm\alpha|n|^{1/m}\right)\phi\left(\frac{|n|}{X}\right).$$

They obtained the asymptotic formulas and upper bounds for these sums.

When the summation is restricted in arithmetic progressions, Yan [18] has proved an asymptotic formula for

$$\sum_{\substack{X<n\leq 2X \\ n\,\equiv\, l \bmod q}} \lambda_f(n)e\left(\pm\frac{2\sqrt{kn}}{q}\right),\quad k\in\mathbb{Z}^+,$$

where $\lambda_f(n)$ is the n-th Fourier coefficient of a holomorphic cusp form for $SL(2,\mathbb{Z})$. Ma and Yan [12] also focused on the oscillation behavior of the exponential sum twisted by $r(n)$ over the arithmetic progressions, where $r(n)$ denotes the number of representations of a positive integer n as a sum of two squares. He [3] studied the asymptotic formula for the corresponding GL_3 exponential sum

$$\sum_{\substack{X<n\leq 2X \\ n\,\equiv\, l \bmod q}} A_f(m;n)\phi\left(\frac{n}{X}\right)e\left(\frac{3\sqrt[3]{kn}}{q}\right),\quad k\in\mathbb{Z}^+.$$

The divisor functions are involving the theory of GL_k L-functions. Sun and Zhang [17] studied the average behavior of the divisor functions over values of quadratic forms and got its asymptotic formula. And the general divisor problems involving Hecke eigenvalues also have attracted many authors, see [4, 5]. In this paper, we consider the oscillation behavior of the divisor functions $d_k(n)$ in arithmetic progressions when $k = 3$. More precisely, the aim of this paper is to prove the following result.

Theorem 1. *Let $k, l, q \in \mathbb{N}$. Then for any $\epsilon > 0$ and $q \ll X^{\frac{1}{3}-\epsilon}$, we have*

$$\sum_{n\equiv l \bmod q} d_3(n)\phi\left(\frac{n}{X}\right)e\left(\frac{3\sqrt[3]{kn}}{q}\right) = \frac{1}{q^2}\sum_{\substack{a \bmod q\\(a,q)=1}} e\left(-\frac{al}{q}\right)\int_0^\infty P(\log x)\phi\left(\frac{x}{X}\right)e\left(\frac{3\sqrt[3]{kx}}{q}\right)dx$$
$$+\frac{\sqrt{3}X^{\frac{2}{3}}}{q^3k^{\frac{1}{3}}i}\sum_{\substack{a \bmod q\\(a,q)=1}} e\left(-\frac{al}{q}\right)\left(A\left(k,\frac{a}{q}\right)+B\left(k,\frac{a}{q}\right)\right)c(\phi)$$
$$+O\left(q^{\frac{5}{2}}X^{-\frac{2}{3}+\epsilon}+q^{\frac{1}{2}+\epsilon}k^{\frac{7}{12}+\epsilon}X^{\frac{1}{3}}\right),$$

where $A\left(k,\frac{a}{q}\right)$ and $B\left(k,\frac{a}{q}\right)$ are defined in Lemma 2.1 and $c(\phi)=\int_0^\infty u\phi(u^3)du$.

An interesting generalization of this problem is to replace the exponential function $e\left(\frac{3\sqrt[3]{kn}}{q}\right)$ in (1.1) with the q-exponential function $e_q(z)$ or the degenerate exponential function e_λ^t for their definitions and properties see Chung-Kim-Kwon [2] and Kim-Kim [11] respectively.

2. Some preliminary lemmas

To prove our theorem, we need the following lemmas.

Lemma 2.1. *Let $f(x)$ be a smooth function of compact support in $(0,\infty)$ and $d_3(n)$ be the number of ways to write n as a product of three factors. Then we have*

$$\sum_{n=1}^\infty f(n)d_3(n)e\left(\frac{hn}{k}\right) = \frac{1}{k}\int_0^\infty P(\log x)f(x)dx$$
$$+\frac{\pi^{\frac{3}{2}}}{k^3}\sum_{n=1}^\infty A\left(n,\frac{h}{k}\right)\int_0^\infty U\left(\frac{\pi^3nx}{k^3}\right)f(x)dx \tag{2.1}$$
$$+i\frac{\pi^{\frac{3}{2}}}{k^3}\sum_{n=1}^\infty B\left(n,\frac{h}{k}\right)\int_0^\infty V\left(\frac{\pi^3nx}{k^3}\right)f(x)dx,$$

where

$$A\left(n,\frac{h}{k}\right) = \frac{1}{2}\sum_{n_1n_2n_3=n}\sum_{x_1=1}^k\sum_{x_2=1}^k\sum_{x_3=1}^k\left\{e\left(\frac{n_1x_1+n_2x_2+n_3x_3+hx_1x_2x_3}{k}\right)\right.$$
$$\left.+e\left(\frac{n_1x_1+n_2x_2+n_3x_3-hx_1x_2x_3}{k}\right)\right\}, \tag{2.2}$$

$$B\left(n,\frac{h}{k}\right) = \frac{1}{2}\sum_{n_1n_2n_3=n}\sum_{x_1=1}^k\sum_{x_2=1}^k\sum_{x_3=1}^k\left\{e\left(\frac{n_1x_1+n_2x_2+n_3x_3+hx_1x_2x_3}{k}\right)\right.$$
$$\left.-e\left(\frac{n_1x_1+n_2x_2+n_3x_3-hx_1x_2x_3}{k}\right)\right\} \tag{2.3}$$

and

$$U(x) = \frac{1}{2\pi i}\int_{\frac{1}{3}-i\infty}^{\frac{1}{3}+i\infty} \frac{\Gamma^3\left(\frac{s}{2}\right)}{\Gamma^3\left(\frac{1-s}{2}\right)}\frac{ds}{x^s}, \qquad V(x) = \frac{1}{2\pi i}\int_{\frac{1}{3}-i\infty}^{\frac{1}{3}+i\infty} \frac{\Gamma^3\left(\frac{1+s}{2}\right)}{\Gamma^3\left(\frac{2-s}{2}\right)}\frac{ds}{x^s}. \tag{2.4}$$

Proof. See the Section 5 of Ivić [6]. □

Lemma 2.2. *If* $A\left(n, \frac{h}{k}\right)$ *is defined by* (2.2)*, then for* $(h, k) = 1$ *we have*

$$A\left(n, \frac{h}{k}\right) \ll_\epsilon k^{\frac{3}{2}+\epsilon} n^{\frac{1}{4}+\epsilon}.$$

Proof. See the Section 8 of Ivić [6]. □

Lemma 2.3. *If* $U(x)$ *and* $V(x)$ *are defined by* (2.4)*, then for any fixed integer* $K \geq 1$ *and* $x \geq x_0 > 0$

$$U(x) = \sum_{j=1}^{K} \frac{c_j \cos(6x^{1/3}) + d_j \sin(6x^{1/3})}{x^{j/3}} + O\left(\frac{1}{x^{(K+1)/3}}\right), \tag{2.5}$$

$$V(x) = \sum_{j=1}^{K} \frac{e_j \cos(6x^{1/3}) + f_j \sin(6x^{1/3})}{x^{j/3}} + O\left(\frac{1}{x^{(K+1)/3}}\right), \tag{2.6}$$

$$\int U(x)dx = \sum_{j=0}^{K} \frac{g_j \cos(6x^{1/3}) + h_j \sin(6x^{1/3})}{x^{(j-1)/3}} + O\left(\frac{1}{x^{K/3}}\right), \tag{2.7}$$

$$\int V(x)dx = \sum_{j=0}^{K} \frac{k_j \cos(6x^{1/3}) + l_j \sin(6x^{1/3})}{x^{(j-1)/3}} + O\left(\frac{1}{x^{K/3}}\right) \tag{2.8}$$

with suitable constants $c_j, \cdots, l_j$*, and in particular*

$$c_1 = 0, \quad d_1 = -\frac{2}{\sqrt{3}\pi}, \quad e_1 = -\frac{2}{\sqrt{3}\pi}, \quad f_1 = 0,$$
$$g_0 = \frac{1}{\sqrt{3}\pi}, \quad h_0 = 0, \quad k_0 = 0, \quad l_0 = -\frac{1}{\sqrt{3}\pi}.$$

Proof. See the Lemma 3 of Ivić [6]. □

3. The proof of Theorem 1

Denote $\alpha = \frac{3\sqrt[3]{k}}{q}$. We consider the sum

$$S := S(k, l, q, X) = \sum_{n \equiv l \ mod \ q} d_3(n)\phi\left(\frac{n}{X}\right)e\left(\alpha\sqrt[3]{n}\right). \tag{3.1}$$

Note that

$$\sum_{c|q}\sum_{\substack{a \ mod \ c \\ (a,c)=1}} e\left(\frac{an}{c}\right) = \begin{cases} q, & q|n; \\ 0, & otherwise. \end{cases}$$

Then

$$S = \frac{1}{q}\sum_{c|q}\sum_{\substack{a \bmod c\\(a,c)=1}} e\left(-\frac{al}{c}\right)\sum_{n\geq 1} d_3(n)e\left(\frac{an}{c}\right)\phi\left(\frac{n}{X}\right)e\left(\alpha\sqrt[3]{n}\right). \tag{3.2}$$

Applying Lemma 2.1 with $f(x) = \phi(x/X)e(\alpha\sqrt[3]{x})$, we have

$$\begin{aligned}\sum_{n\geq 1} d_3(n)e\left(\frac{an}{c}\right)f(n) =&\frac{1}{c}\int_0^\infty P(\log x)f(x)dx\\ &+ \frac{\pi^{\frac{3}{2}}}{c^3}\sum_{n\geq 1} A\left(n,\frac{a}{c}\right)\int_0^\infty U\left(\frac{\pi^3 nx}{c^3}\right)f(x)dx\\ &+ i\frac{\pi^{\frac{3}{2}}}{c^3}\sum_{n\geq 1} B\left(n,\frac{a}{c}\right)\int_0^\infty V\left(\frac{\pi^3 nx}{c^3}\right)f(x)dx\\ =&: \mathcal{S}_0(a,c) + \mathcal{S}_1(a,c) + \mathcal{S}_2(a,c),\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\mathcal{S}_0(a,c) =&\frac{1}{c}\int_0^\infty P(\log x)f(x)dx,\\ \mathcal{S}_1(a,c) =&\frac{\pi^{\frac{3}{2}}}{c^3}\sum_{n\geq 1} A\left(n,\frac{a}{c}\right)\psi_1\left(\frac{\pi\sqrt[3]{n}}{c}\right),\quad \psi_1\left(\frac{\pi\sqrt[3]{n}}{c}\right) = \int_0^\infty U\left(\frac{\pi^3 nx}{c^3}\right)f(x)dx,\\ \mathcal{S}_2(a,c) =&i\frac{\pi^{\frac{3}{2}}}{c^3}\sum_{n\geq 1} B\left(n,\frac{a}{c}\right)\psi_2\left(\frac{\pi\sqrt[3]{n}}{c}\right),\quad \psi_2\left(\frac{\pi\sqrt[3]{n}}{c}\right) = \int_0^\infty V\left(\frac{\pi^3 nx}{c^3}\right)f(x)dx.\end{aligned} \tag{3.4}$$

Applying Lemma 2.3 with $K = 3$, we obtain

$$\psi_1\left(\frac{\pi\sqrt[3]{n}}{c}\right) = \sum_{j=1}^{3}\int_0^\infty f(x)\frac{c_j\cos\left(\frac{6\pi\sqrt[3]{nx}}{c}\right) + d_j\sin\left(\frac{6\pi\sqrt[3]{nx}}{c}\right)}{\pi^j c^{-j}(nx)^{j/3}}dx + O\left(\pi^{-4}c^4 n^{-\frac{4}{3}}X^{-\frac{2}{3}+\epsilon}\right), \tag{3.5}$$

$$\psi_2\left(\frac{\pi\sqrt[3]{n}}{c}\right) = \sum_{j=1}^{3}\int_0^\infty f(x)\frac{e_j\cos\left(\frac{6\pi\sqrt[3]{nx}}{c}\right) + f_j\sin\left(\frac{6\pi\sqrt[3]{nx}}{c}\right)}{\pi^j c^{-j}(nx)^{j/3}}dx + O\left(\pi^{-4}c^4 n^{-\frac{4}{3}}X^{-\frac{2}{3}+\epsilon}\right). \tag{3.6}$$

By Lemma 2.2, the O-term in (3.5) contributes to $\mathcal{S}_1(a,c)$

$$\begin{aligned}&\ll c^{-3}\sum_{n\geq 1} A\left(n,\frac{a}{c}\right)\pi^{-4}c^4 n^{-\frac{4}{3}}X^{-\frac{2}{3}+\epsilon}\\ &\ll c^{\frac{5}{2}}X^{-\frac{2}{3}+\epsilon}.\end{aligned}$$

By Lemma 2.2, the O-term in (3.6) contributes to $\mathcal{S}_2(a,c)$

$$\begin{aligned}&\ll c^{-3}\sum_{n\geq 1} B\left(n,\frac{a}{c}\right)\pi^{-4}c^4 n^{-\frac{4}{3}}X^{-\frac{2}{3}+\epsilon}\\ &\ll c^{\frac{5}{2}}X^{-\frac{2}{3}+\epsilon}.\end{aligned}$$

For the integral in (3.5) and (3.6), we make a change of variable $x = Xu^3$ to get

$$\begin{aligned}&\int_0^\infty \phi\left(\frac{x}{X}\right) e\left(\alpha\sqrt[3]{x} \pm \frac{3\sqrt[3]{nx}}{c}\right) c^j \left(\pi^3 nx\right)^{-j/3} dx \\ &= 3X \int_0^\infty \phi\left(u^3\right) e\left(\left(\alpha \pm \frac{3\sqrt[3]{n}}{c}\right)\sqrt[3]{X}u\right) c^j \pi^{-j} (nX)^{-j/3} u^{-j} u^2 du \\ &= 3Xc^j \left(\pi^3 nX\right)^{-j/3} I_j^{\pm}\left(\frac{n}{c^3}\right),\end{aligned} \tag{3.7}$$

where

$$I_j^{\pm}\left(\frac{n}{c^3}\right) = \int_1^{\sqrt[3]{2}} u^{2-j}\phi\left(u^3\right) e\left(\left(\alpha \pm \frac{3\sqrt[3]{n}}{c}\right)\sqrt[3]{X}u\right) du. \tag{3.8}$$

By (3.4)–(3.7), we have

$$\begin{aligned}\mathcal{S}_1(a,c) =& \frac{3\pi^{\frac{3}{2}}}{c^3} \sum_{n\geq 1} A\left(n, \frac{a}{c}\right) \sum_{j=1}^{3} Xc^j \left(\pi^3 nX\right)^{-j/3} \left(\frac{c_j}{2} I_j^+\left(\frac{n}{c^3}\right) + \frac{c_j}{2} I_j^-\left(\frac{n}{c^3}\right)\right. \\ &\left. + \frac{d_j}{2i} I_j^+\left(\frac{n}{c^3}\right) - \frac{d_j}{2i} I_j^-\left(\frac{n}{c^3}\right)\right) + O\left(c^{\frac{5}{2}} X^{-\frac{2}{3}+\epsilon}\right),\end{aligned}$$

$$\begin{aligned}\mathcal{S}_2(a,c) =& i\frac{3\pi^{\frac{3}{2}}}{c^3} \sum_{n\geq 1} B\left(n, \frac{a}{c}\right) \sum_{j=1}^{3} Xc^j \left(\pi^3 nX\right)^{-j/3} \left(\frac{e_j}{2} I_j^+\left(\frac{n}{c^3}\right) + \frac{e_j}{2} I_j^-\left(\frac{n}{c^3}\right)\right. \\ &\left. + \frac{f_j}{2i} I_j^+\left(\frac{n}{c^3}\right) - \frac{f_j}{2i} I_j^-\left(\frac{n}{c^3}\right)\right) + O\left(c^{\frac{5}{2}} X^{-\frac{2}{3}+\epsilon}\right).\end{aligned}$$

Let

$$F_{\pm}(u) := F_{\pm}(u,n) = \left(\alpha \pm 3\frac{\sqrt[3]{n}}{c}\right)\sqrt[3]{X}u.$$

Note that

$$\alpha\sqrt[3]{X} = \frac{3\sqrt[3]{kX}}{q} > X^{\epsilon}$$

for $q < X^{\frac{1}{3}-\epsilon}$. Then

$$F_+'(u) = \left(\alpha + 3\frac{\sqrt[3]{n}}{c}\right)\sqrt[3]{X} \gg \alpha\sqrt[3]{X} \gg X^{\epsilon}$$

and by integration by parts many times, we show that $I_j^+\left(\frac{n}{c^3}\right)$ is negligible. For $n \geq \frac{2c^3\alpha^3}{27}$ or $n \leq \frac{c^3\alpha^3}{100}$, we also have

$$F_-'(u) = \left(\alpha - 3\frac{\sqrt[3]{n}}{c}\right)\sqrt[3]{X} \gg \alpha\sqrt[3]{X} \gg X^{\epsilon},$$

thus, $I_j^-\left(\frac{n}{c^3}\right)$ is negligible for $n \geq \frac{2c^3\alpha^3}{27}$ or $n \leq \frac{c^3\alpha^3}{100}$. Denote

$$I = \left(\frac{c^3\alpha^3}{100}, \frac{2c^3\alpha^3}{27}\right).$$

Then we can get

$$\begin{aligned}\mathcal{S}_1(a,c) =&\frac{3\pi^{\frac{3}{2}}}{2c^3}\sum_{n\in I}A\left(n,\frac{a}{c}\right)\sum_{j=1}^{3}c^j\pi^{-j}n^{-\frac{j}{3}}X^{1-\frac{j}{3}}\left(c_jI_j^-\left(\frac{n}{c^3}\right)+id_jI_j^-\left(\frac{n}{c^3}\right)\right)\\&+O\left(c^{\frac{5}{2}}X^{-\frac{2}{3}+\epsilon}\right),\end{aligned}\tag{3.9}$$

$$\begin{aligned}\mathcal{S}_2(a,c) =&i\frac{3\pi^{\frac{3}{2}}}{2c^3}\sum_{n\in I}B\left(n,\frac{a}{c}\right)\sum_{j=1}^{3}c^j\pi^{-j}n^{-\frac{j}{3}}X^{1-\frac{j}{3}}\left(e_jI_j^-\left(\frac{n}{c^3}\right)+if_jI_j^-\left(\frac{n}{c^3}\right)\right)\\&+O\left(c^{\frac{5}{2}}X^{-\frac{2}{3}+\epsilon}\right).\end{aligned}\tag{3.10}$$

For $j\geq 2$, we use the trivial bound $I_j^-\left(\frac{n}{c^3}\right)\ll 1$. Then the contribution from $I_j^-\left(\frac{n}{c^3}\right)$, $j=2,3$ to $\mathcal{S}_1(a,c)$ in (3.9) and $\mathcal{S}_2(a,c)$ in (3.10) is at most

$$\ll c^{\frac{9}{4}+\epsilon}\alpha^{\frac{7}{4}}X^{\frac{1}{3}}.$$

Then, by Lemma 2.3, (3.9) and (3.10), we have

$$\mathcal{S}_1(a,c)=\frac{\sqrt{3}}{c^2i}X^{\frac{2}{3}}\sum_{n\in I}A\left(n,\frac{a}{c}\right)n^{-\frac{1}{3}}I_1^-\left(\frac{n}{c^3}\right)+O\left(c^{\frac{5}{2}}X^{-\frac{2}{3}+\epsilon}+c^{\frac{9}{4}+\epsilon}\alpha^{\frac{7}{4}}X^{\frac{1}{3}}\right),\tag{3.11}$$

$$\mathcal{S}_2(a,c)=\frac{\sqrt{3}}{c^2i}X^{\frac{2}{3}}\sum_{n\in I}B\left(n,\frac{a}{c}\right)n^{-\frac{1}{3}}I_1^-\left(\frac{n}{c^3}\right)+O\left(c^{\frac{5}{2}}X^{-\frac{2}{3}+\epsilon}+c^{\frac{9}{4}+\epsilon}\alpha^{\frac{7}{4}}X^{\frac{1}{3}}\right).\tag{3.12}$$

Thus, by (3.11) and (3.12), we can get

$$\begin{aligned}\mathcal{S}_1(a,c)+\mathcal{S}_2(a,c) =&\frac{\sqrt{3}}{c^2i}X^{\frac{2}{3}}\sum_{n\in I}\left(A\left(n,\frac{a}{c}\right)+B\left(n,\frac{a}{c}\right)\right)n^{-\frac{1}{3}}I_1^-\left(\frac{n}{c^3}\right)\\&+O\left(c^{\frac{5}{2}}X^{-\frac{2}{3}+\epsilon}+c^{\frac{9}{4}+\epsilon}\alpha^{\frac{7}{4}}X^{\frac{1}{3}}\right).\end{aligned}\tag{3.13}$$

Note that the first term in (3.13) disappears when $\alpha\leq\frac{3}{\sqrt[3]{2}c}$. For $\alpha>\frac{3}{\sqrt[3]{2}c}$, let $n_\alpha\geq 1$ be the integer such that

$$\left(\frac{c\alpha}{3}\right)^3=n_\alpha+\lambda,\qquad -\frac{1}{2}<\lambda\leq\frac{1}{2}.\tag{3.14}$$

Then for any positive integer n, we have

$$\left|3\frac{\sqrt[3]{n}}{c}-\alpha\right|=\frac{3}{c}\left|\sqrt[3]{n}-\frac{c\alpha}{3}\right|=\frac{3|n-n_\alpha-\lambda|}{c\left(\sqrt[3]{n^2}+\sqrt[3]{n}\left(\frac{c\alpha}{3}\right)+\left(\frac{c\alpha}{3}\right)^2\right)}.$$

For $n\neq n_\alpha$, the right-hand side of above equation

$$\gg|n-n_\alpha|c^{-3}\alpha^{-2},$$

and then

$$F_-^{'}(u,n)\gg|n-n_\alpha|c^{-3}\alpha^{-2}\sqrt[3]{X}.$$

By integrating by parts, we can obtain

$$I_1^-\left(\frac{n}{c^3}\right) \ll \frac{c^3\alpha^2}{\sqrt[3]{X}|n-n_\alpha|}.$$

Thus, by Lemma 2.2, the contribution of the term $n \neq n_\alpha$ to (3.13) is at most

$$\begin{aligned} & c^{-2}X^{\frac{2}{3}}c^3\alpha^2X^{-\frac{1}{3}}\sum_{n\in I\setminus\{n_\alpha\}}\left(A\left(n,\frac{a}{c}\right)+B\left(n,\frac{a}{c}\right)\right)n^{-\frac{1}{3}}|n-n_\alpha|^{-1} \\ & \ll c^{1+\epsilon}\alpha^2X^{\frac{1}{3}}\sum_{n\in I\setminus\{n_\alpha\}}c^{\frac{3}{2}+\epsilon}n^{\frac{1}{4}-\frac{1}{3}+\epsilon}|n-n_\alpha|^{-1} \\ & \ll c^{\frac{9}{4}+\epsilon}\alpha^{\frac{7}{4}+\epsilon}X^{\frac{1}{3}}\sum_{n\in I\setminus\{n_\alpha\}}|n-n_\alpha|^{-1} \\ & \ll c^{\frac{9}{4}+\epsilon}\alpha^{\frac{7}{4}+\epsilon}X^{\frac{1}{3}}. \end{aligned}$$

When $n = n_\alpha$, we can get

$$\left|F_-'(u,n)\right| \asymp c^{-3}\alpha^{-2}X^{\frac{1}{3}}|\lambda|.$$

For $|\lambda| \geq \frac{1}{10}$, we have $I_1^-\left(\frac{n_\alpha}{c^3}\right) \ll c^3\alpha^2X^{-\frac{1}{3}}$ by integrating by parts. Hence, the first term in (3.13) is bounded by $c^{\frac{9}{4}+\epsilon}\alpha^{\frac{7}{4}+\epsilon}X^{\frac{1}{3}}$. If $c^{-3}\alpha^{-2}X^{\frac{1}{3}}|\lambda| \gg X^\epsilon$, then $I_1^-\left(\frac{n_\alpha}{c^3}\right)$ is negligible. This proves that

$$\begin{aligned} \mathcal{S}_1(a,c)+\mathcal{S}_2(a,c) = & \frac{\vartheta_\alpha\sqrt{3}}{ic^2\sqrt[3]{n_\alpha}}\sqrt[3]{X^2}\left(A\left(n_\alpha,\frac{a}{c}\right)+B\left(n_\alpha,\frac{a}{c}\right)\right)I_1^-\left(\frac{n_\alpha}{c^3}\right) \\ & + O\left(c^{\frac{5}{2}}X^{-\frac{2}{3}+\epsilon}+c^{\frac{9}{4}+\epsilon}\alpha^{\frac{7}{4}}X^{\frac{1}{3}}\right), \end{aligned} \tag{3.15}$$

where

$$\vartheta_\alpha = \begin{cases} 1, & \exists\, n_\alpha \in I,\ s.t. \left|n_\alpha - \frac{2c^3\alpha^3}{27}\right| < \min\left\{c^3\alpha^2X^{-\frac{1}{3}+\epsilon}, \frac{1}{10}\right\}; \\ 0, & otherwise. \end{cases}$$

By (3.2), (3.3) and (3.15), we have

$$\begin{aligned} S = & \frac{\vartheta_\alpha\sqrt{3}}{iq\sqrt[3]{n_\alpha}}\sqrt[3]{X^2}\sum_{c|q}\frac{1}{c^2}\sum_{\substack{a \bmod c \\ (a,c)=1}}e\left(-\frac{al}{c}\right)\left(A\left(n_\alpha,\frac{a}{c}\right)+B\left(n_\alpha,\frac{a}{c}\right)\right)I_1^-\left(\frac{n_\alpha}{c^3}\right) \\ & + \frac{1}{q}\sum_{c|q}\sum_{\substack{a \bmod c \\ (a,c)=1}}e\left(-\frac{al}{c}\right)\mathcal{S}_0(a,c) + O\left(q^{\frac{5}{2}}X^{-\frac{2}{3}+\epsilon}+q^{\frac{9}{4}+\epsilon}\alpha^{\frac{7}{4}}X^{\frac{1}{3}}\right). \end{aligned}$$

Since $\alpha = \frac{3\sqrt[3]{k}}{q}$ with $k \in \mathbb{Z}$, we have $c \neq q$. Then

$$\left|k - \frac{kc^3}{q^3}\right| = kc^3q^{-3}\left|\frac{c^3}{q^3}-1\right| \geq \frac{kc^3}{q^3} \geq c^3\alpha^2X^{-\frac{1}{3}+\epsilon}.$$

Thus, $\frac{kc^3}{q^3} = 0$ for $c \neq q$. Otherwise, we take

$$c = q, \quad n_\alpha = k.$$

Therefore,

$$\begin{aligned} S =& \frac{\sqrt{3}}{iq^3\sqrt[3]{k}}\sqrt[3]{X^2}\sum_{\substack{a \ mod \ q \\ (a,q)=1}} e\left(-\frac{al}{q}\right)\left(A\left(k,\frac{a}{q}\right)+B\left(k,\frac{a}{q}\right)\right)c(\phi) \\ &+\frac{1}{q}\sum_{\substack{a \ mod \ q \\ (a,q)=1}} e\left(-\frac{al}{q}\right)\mathcal{S}_0(a,q)+O\left(q^{\frac{5}{2}}X^{-\frac{2}{3}+\epsilon}+q^{\frac{1}{2}+\epsilon}k^{\frac{7}{12}+\epsilon}X^{\frac{1}{3}}\right), \end{aligned} \tag{3.16}$$

where

$$c(\phi)=\int_0^\infty u\phi(u^3)du$$

and

$$\mathcal{S}_0(a,q)=\frac{1}{q}\int_0^\infty P(\log x)\phi\left(\frac{x}{X}\right)e\left(\frac{3\sqrt[3]{kx}}{q}\right)dx.$$

This proves our theorem.

4. Conclusions

This paper focused on the study of the nonlinear exponential sum twisting the divisor functions $d_3(n)$ over the arithmetic progression and established its asymptotic formula. The main techniques used to prove the theorem were the estimation of exponential sum, Voronoi summation formula, and the weighted stationary phase. Moreover, it is also an interesting question to generalize the resonance theory to the divisor functions $d_k(n)$ if there is a better Voronoi summation formula for $k > 3$, and we can try to obtain a better bound for the error term respect to q which is a prime number.

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Conflict of interest

The authors declare no conflict of interest.

References

1. R. Acharya, S. Singh, An exponential sum involving Fourier coefficients of eigenforms for $SL(2,\mathbb{Z})$, *Ramanujan J.*, **54** (2021), 699–716. https://doi.org/10.1007/s11139-020-00255-0
2. W. Chung, T. Kim, H. Kwon, On the q-analog of the Laplace transform, *Russ. J. Math. Phys.*, **21** (2014), 156–168. https://doi.org/10.1134/S1061920814020034
3. X. He, Exponential sums involving automorphic forms for $GL(3)$ over arithmetic progressions, *Front. Math. China*, **13** (2018), 1355–1368. https://doi.org/10.1007/s11464-018-0732-x

4. J. Huang, T. Li, H. Liu, F. Xu, The general two-dimensional divisor problems involving Hecke eigenvalues, *AIMS Math.*, **7** (2022), 6396–6403. https://doi.org/10.3934/math.2022356

5. J. Huang, H. Liu, Divisor problems related to Hecke eigenvalues in three dimensions, *J. Math.*, **2021** (2021), 9928233. https://doi.org/10.1155/2021/9928233

6. A. Ivić, *On the ternary additive divisor problem and the sixth moment of the zeta-function*, Sieve Methods, Exponential Sums, and their Applications in Number Theory, London Math Soc Lecture Note Series, **237** (1997), 205–243. https://doi.org/10.1017/CBO9780511526091.015

7. H. Lao, The cancellation of Fourier coefficients of cusp forms over different sparse sequences, *Acta Math. Sin. (Engl. Ser.)*, **29** (2013), 1963–1972. https://doi.org/10.1007/s10114-013-2706-y

8. H. Lao, On comparing Hecke eigenvalues of cusp forms, *Acta Math. Hungar.*, **160** (2020), 58–71. https://doi.org/10.1007/s10474-019-00996-5

9. H. Lao, S. Luo, Sign changes and nonvanishing of Fourier coefficients of holomorphic cusp forms, *Rocky MT J. Math.*, **51** (2021), 1701–1714. https://doi.org/10.1216/rmj.2021.51.1701

10. H. Liu, Notes on general divisor problems related to Maass cusp forms, *Period. Math. Hung.*, In press.

11. T. Kim, D. Kim, Degenerate Laplace transform and degenerate gamma function, *Russ. J. Math. Phys.*, **24** (2017), 241–248. https://doi.org/10.1134/S1061920817020091

12. L. Ma, X. Yan, Resonance between the representation function and exponential functions over arithemetic progression, *J. Math.*, **2021** (2021), 6616348. https://doi.org/10.1155/2021/6616348

13. X. Ren, Y. Ye, Resonance between automorphic forms and exponential functions, *Sci. China Math.*, **53** (2010), 2463–2472. https://doi.org/10.1007/s11425-010-3150-4

14. X. Ren, Y. Ye, Resonance of automorphic forms for $GL(3)$, *Trans. Amer. Math. Soc.*, **367** (2015), 2137–2157. https://doi.org/10.1007/s11425-010-3150-4

15. X. Ren, Y. Ye, Resonance and rapid decay of exponential sums of Fourier coefficients of a Maass form for $GL(3)$, *Sci. China Math.*, **58** (2015), 2105–2124. https://doi.org/10.1007/s11425-014-4955-3

16. Q. Sun, Y. Wu, Exponential sums involving Maass forms, *Front. Math. China*, **9** (2014), 1349–1366. https://doi.org/10.1007/s11464-014-0360-z

17. Q. Sun, D. Zhang, Sums of the triple divisor function over values of a ternary quadratic form, *J. Number Theory*, **168** (2016), 215–246. https://doi.org/10.1016/j.jnt.2016.04.010

18. X. Yan, On some exponential sums involving Maass forms over arithmetic progressions, *J. Number Theory*, **160** (2016), 44–59. https://doi.org/10.1016/j.jnt.2015.08.010

19. D. Zhang, Y. Lau, Y. Wang, Remark on the paper "On products of Fourier coefficients of cusp forms", *Arch. Math. (Basel)*, **108** (2017), 263–269. https://doi.org/10.1007/s00013-016-0996-x

20. D. Zhang, Y. Wang, Ternary quadratic form with prime variables attached to Fourier coefficients of primitive holomorphic cusp form, *J. Number Theory*, **176** (2017), 211–225. https://doi.org/10.1016/j.jnt.2016.12.018

21. D. Zhang, Y. Wang, Higher-power moments of Fourier coefficients of holomorphic cusp forms for the congruence subgroup $\Gamma_0(N)$, *Ramanujan J.*, **47** (2018), 685–700. https://doi.org/10.1007/s11139-018-0051-6

22. D. Zhang, W. Zhai, On the distribution of Hecke eigenvalues over Piatetski-Shapiro prime twins, *Acta Math. Sin. (Engl. Ser.)*, **37** (2021), 1453–1464. https://doi.org/10.1007/s10114-021-0174-3

23. A. Zou, H. Lao, S. Luo, Some density results on sets of primes for Hecke eigenvalues, *J. Math.*, **2021** (2021), 2462693. https://doi.org/10.1155/2021/2462693