Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(5): 10802–10821.
DOI: 10.3934/math.2023548
Received: 07 January 2023
Revised: 21 February 2023
Accepted: 22 February 2023
Published: 06 March 2023

Research article

A new discussion concerning to exact controllability for fractional mixed Volterra-Fredholm integrodifferential equations of order $r \in (1, 2)$ with impulses

Marimuthu Mohan Raja[1], **Velusamy Vijayakumar**[1], **Anurag Shukla**[2], **Kottakkaran Sooppy Nisar**[3,4,*], **Wedad Albalawi**[5] **and Abdel-Haleem Abdel-Aty**[6,7]

[1] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore, 632014, Tamil Nadu, India

[2] Department of Applied Science, Rajkiya Engineering College Kannauj, Kannauj-209732, India

[3] Department of Mathematics, College of Sciences and Humanities, Prince Sattam bin Abdulaziz University, Al Kharj, 16278, Saudi Arabia

[4] School of Technology, Woxsen University- Hyderabad-502345, Telangana State, India

[5] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia

[6] Department of Physics, College of Sciences, University of Bisha, P.O. Box 344, Bisha 61922, Saudi Arabia

[7] Physics Department, Faculty of Science, Al-Azhar University, Assiut, 71524, Egypt

* **Correspondence:** Email: n.sooppy@psau.edu.sa.

Abstract: In this article, we look into the important requirements for exact controllability of fractional impulsive differential systems of order $1 < r < 2$. Definitions of mild solutions are given for fractional integrodifferential equations with impulses. In addition, applying fixed point methods, fractional derivatives, essential conditions, mixed Volterra-Fredholm integrodifferential type, for exact controllability of the solutions are produced. Lastly, a case study is supplied to show the illustration of the primary theorems.

Keywords: fractional differential equations; controllability; impulsive system; measure of noncompactness; Volterra-Fredholm type; sectorial operators
Mathematics Subject Classification: 26A33, 34A08, 35R12, 47B12, 47H08, 46E36, 93B05

1. Introduction

Fractional derivatives have lately emerged as a very significant area of research as a result their steadily increasing numerous uses in several sectors of applied science and engineering. For more information, see the books [11, 16, 31, 47, 48]. Fractional differential systems produce a better and more accurate realistic scenario for understanding a wide variety of physical phenomena as compared to differential equations represented by regular integer order derivatives. Integer order definitions can be interpolated to non-integer order using a variety of techniques. Riemann-Liouville and Caputo derivatives are two of the most well-known. As a consequence, a large number of scholars have lately played an important role to fields like electromagnetic theory, rheology, image analysis, diffusion, data processing, porous materials, physiological engineering challenges, fluid mechanics, theology, and many others, see [4, 9, 10, 34, 35, 40, 41].

Recently, [5, 39, 45] the authors discussed some important methods for long-time anomalous heat flow study to the fractional derivatives, Laplace transformation, singular boundary approach, and dual reciprocity technique. Recently, the study of impulsive functional differential systems has offered a natural framework for the mathematical modelling of a variety of practical situations, particularly in the fields of control, biology, and medicine. The explanation for this applicability is that impulsive differential issues are a suitable model for explaining changes that occur their state rapidly at some points and can't be represented using traditional differential equations. For additional information on this theory as well as its implications, we suggest reading [7, 37, 38, 42, 43].

Mathematical control theory, a subfield of framework mathematics, focuses on the basic ideas that underlie the formulation and evaluation of control systems. While occasionally appearing to move in opposite directions, the two main study fields in control theory have typically been complementary. One of them assumes that a proper model of the object to be managed is offered and that the user wants to improve the behaviour of the object in some way. For instance, to design a spacecraft's trajectory that minimizes the overall trip time or fuel consumption, physical concepts and engineering standards are applied. The methods used here are strongly related to other branches of optimization theory as well as the classical calculus of variations; the result is typically a pre-programmed flight plan. The limitations imposed by uncertainty regarding the model or the environment in which the item operates provide the basis of the other key area of research. The use of feedback to correct for deviations from the anticipated behaviour in this case is the key strategy.

It is essential to carry out study on the consequences of such controllability of systems utilizing the resources at hand. As controllability is crucial to this theory, it makes sense to seek to generalize finite-dimensional control theory to infinite dimensions. Control system analysis and innovation have proven to benefit from the usage of controllability notation by employing fractional derivatives. It is used in a multitude of industries, as well as biochemistry, physicists, automation, electronics, transport, fields of study include economics, robotics, biology, physics, power systems, chemical outgrowth control, space technology and technology. As indicated by the researchers' papers [7, 9, 13, 29, 30, 33, 49], resolving these challenges has become a important undertaking for young researchers.

In addition, integrodifferential equations are used in many technological fields, including control theory, biology, medicine, and ecology, where a consequence or delay must be considered. In fact, it is always necessary to characterize a model with hereditary properties using integral-differential systems. Further, many researchers done the fruitful achievements in fractional Volterra-Fredholm integrodifferential systems with or without delay utilizing the mild solutions, semigroup theory, neutral systems, and fixed point theorems in [14, 15, 17–19, 21, 22, 33]. In [1, 25–28, 46], the authors discussed the solution of a functional derivatives utilizing to weak and strong convergence, Chebyshev quadrature collocation algorithm, mixed Volterra-Fredholm type, almost contraction mapping, the iterative method, weak w^2-stability, and faster iteration method.

In [35, 36], the researchers established the existence and uniqueness for fractional differential equations of $\alpha \in (1,2)$ by applying the upper and lower solution methods, sectorial operators, and nonlocal conditions. The authors [6, 49] established Caputo fractional derivative of $1 < \alpha < 2$ using nonlocal conditions, the Laplace transform, mild solutions, cosine families, measure of noncompactness(MNC), as well as other fixed point techniques. Additionally, using fractional derivatives, cosine functions, and Sobolev type, the authors discussed exact controllability outcomes for fractional differential systems of $(1,2)$ with finite delay in [13].

In [18], the authors explored the approximate controllability of Caputo fractional differential systems of $(1,2)$ by utilizing the impulsive system, sequence method, and cosine families. Furthermore, [40] proved fractional integrodifferential inclusions of $(1,2)$ using Laplace transforms, Fredholm integro-differential systems, and the fixed point approach. Moreover, in [29, 37], the researchers looked at Gronwall's inequality for the semilinear fractional derivatives of $(1,2]$, stochastic systems, asymptotic stability, optimal control concerns, Lipschitz continuity, and impulsive systems. The researchers are currently investigating the optimal controls for fractional derivative of $(1,2)$ with infinite delay in [19, 20].

In [35], the authors looked into the existence and uniqueness outcomes of fractional differential equations of $(1,2)$. In [12, 32], the fixed point theorem, Gronwall's inequality, impulsive systems, and sectorial operators are used to analyze optimal control for fractional derivatives of order $(1,2)$. To identify extremal solutions of fractional partial differential equations of order $(1,2)$, the authors of [36] used upper and lower solution approaches, sectorial operators, the Mittag-Leffler function, and mild solutions. The existence of positive mild solutions for Caputo fractional differential systems of order $r \in (1,2)$ was also addressed by the authors in [34].

Taking inspiration from the preceding information, let's investigate impulsive fractional integrodifferential systems of mixed type with order $r \in (1,2)$ with the following form:

$$\begin{cases} {}^{C}D^{r}_{\varrho} z(\varrho) = Az(\varrho) + \mathfrak{f}(\varrho, z(\varrho), (E_1 z)(\varrho), (E_2 z)(\varrho)) + \mathcal{B}x(\varrho), \ \varrho \text{ in } V, \ \varrho \neq \varrho_j, \\ \Delta z(\varrho_j) = m_j, \ \Delta z'(\varrho_j) = \widetilde{m}_j, \ j = 1, 2, \cdots, n, \\ z(0) = z_0, \ z'(0) = z_1. \end{cases} \tag{1.1}$$

In the above

$$(E_1 z)(\varrho) = \int_0^{\varrho} e_1(\varrho, s, z(s))ds, \ (E_2 z)(\varrho) = \int_0^{\sigma} e_2(\varrho, s, z(s))ds,$$

with ${}^{C}D^{r}_{\varrho}$ represents Caputo fractional derivative of order r in $(1,2)$; A maps from $D(A) \subset Q$ into Q denotes the sectorial operator of type (P, τ, r, ϕ) on the Banach space Q; the function $\mathfrak{f}$ maps from $[0,\sigma] \times Q \times Q \times Q$ into Q and $e_1, e_2 : \mathcal{S} \times Q \times Q \to Q$ are appropriate functions, in which $\mathcal{S} = \{(\varrho, s) : 0 \le s \le \varrho \le \sigma\}$. The bounded linear operator $\mathcal{B} : \mathbb{Y} \to Q$, the control function x in $L^2(V, \mathbb{Y})$, in which $\mathbb{Y}$ is also a Banach space. The continuous functions $m_j, \ \widetilde{m}_j : Q \to Q$ and $0 = \varrho_0 < \varrho_1 < \varrho_2 < \cdots <$

$\varrho_j < \cdots < \varrho_n = \sigma$; $\Delta z(\varrho_j) = z(\varrho_j^+) - z(\varrho_j^-)$, where $z(\varrho_j^+) = \lim_{\epsilon^+ \to 0} z(\varrho_\ell + \epsilon)$ and $z(\varrho_j^-) = \lim_{\epsilon^- \to 0} z(\varrho_\ell + \epsilon)$ denote the right and lift limits of $z(\varrho)$ at $\varrho = \varrho_j$. $\Delta z'(\varrho_j)$ has also a similar theories.

The following sections represent the remaining portions of this article: Section 2 starts with a description of some basic concepts and the results of the preparation. In Section 3, we look at the existence of mild solutions for the impulsive fractional Volterra-Fredholm type (1.1). Lastly, an application for establishing the theory of the primary results is shown.

2. Preliminaries

We will implement some definitions, sectorial operator assumptions, R-L and Caputo fractional derivative definitions, and preliminaries in this section, which will be used throughout the article.

The Banach space $C(V, Q)$ maps from V into Q is a continuous with $\|z\|_C = \sup_{\varrho \in V} \|z(\varrho)\|$.

$$PC(V, Q) = \{z : V \to \mathbb{R} : z \in C((\varrho_j, \varrho_{j+1}], \mathbb{R}),\ j = 0, \cdots n \text{ and } \exists\, z(\varrho_j^+) \text{ and } z(\varrho_j^-),$$
$$j = 1, \cdots, n \text{ with } z(\varrho_j^-) = z(\varrho_j)\},$$

with $\|z\|_{PC} = \sup_{\varrho \in V} \|z(\varrho)\|$. Consider $L(Q)$ represents the Banach space of every linear and bounded operators on Banach space Q.

Definition 2.1. *[31] The integral fractional order β with such a lower limit of zero for $\mathfrak{f}$ maps from $[0, \infty)$ into $\mathbb{R}^+$ is simply referred to as*

$$I^\beta \mathfrak{f}(\varrho) = \frac{1}{\Gamma(\beta)} \int_0^\varrho \frac{\mathfrak{f}(s)}{(\varrho - s)^{1-\beta}} ds, \quad \varrho > 0,\ \beta \in \mathbb{R}^+.$$

Definition 2.2. *[31] The fractional order β of R-L derivative with the lower limit of zero for $\mathfrak{f}$ is known as*

$$^L D^\beta \mathfrak{f}(\varrho) = \frac{1}{\Gamma(j - \beta)} \frac{d^j}{d\varrho^j} \int_0^\varrho \mathfrak{f}(s)(\varrho - s)^{j-\beta-1} ds, \quad \varrho > 0,\ \beta \in (j-1, j),\ \beta \in \mathbb{R}^+.$$

Definition 2.3. *[31] The fractional derivative of order β in Caputo's approach with the lower limit zero for $\mathfrak{f}$ is designated just for*

$$^C D^\beta \mathfrak{f}(\varrho) = {}^L D^\beta \left(\mathfrak{f}(\varrho) - \sum_{i=0}^{j-1} \frac{\mathfrak{f}^{(i)}(0)}{i!} \varrho^i \right), \quad \varrho > 0,\ \beta \in (j-1, j),\ \beta \in \mathbb{R}^+.$$

Definition 2.4. *[35] The closed and linear operator A is called the sectorial operator of type (P, τ, r, ϕ) provided that there exists ϕ in $\mathbb{R}$, τ in $(0, \frac{\pi}{2})$, and there exists a positive constant P such that the r-resolvent of A exists outside the sector*

$$\phi + \mathcal{S}_\tau = \{\tau + \rho^r : \rho \text{ in } C(V, Q),\ |Arg(-\rho^r)| < \tau\}, \tag{2.1}$$

and

$$\|(\rho^r I - A)^{-1}\| \le \frac{P}{|\rho^r - \phi|},\ \rho^r \notin \phi + \mathcal{S}_\tau.$$

It is also simple to prove that A represents the infinitesimal generator of an r-resolvent family $\{\mathbb{G}_r(\varrho)\}_{\varrho\geq 0}$ in Banach space if one assumes A stands for a sectorial operator of type (P, τ, r, ϕ), where

$$\mathbb{G}_r(\varrho) = \frac{1}{2\pi i}\int_c e^{\rho r} R(\rho^r, A)d\rho.$$

Definition 2.5. *A function z in $PC(V, Q)$ is called a mild solution of the system* (1.1) *provdied that*

$$z(\varrho) = \begin{cases} \mathbb{N}_r(\varrho)z_0 + \mathbb{M}_r(\varrho)z_1 + \int_0^{\varrho} \mathbb{G}_r(\varrho - s)\mathfrak{f}(s, z(s), (E_1 z)(s), (E_2 z)(s))ds \\ \quad + \int_0^{\varrho} \mathbb{G}_r(\varrho - s)\mathcal{B}x(s)ds, \ 0 \leq \varrho \leq \varrho_1, \\ \mathbb{N}_r(\varrho)z_0 + \mathbb{M}_r(\varrho)z_1 + \sum_{q=1}^{j} \mathbb{N}_r(\varrho - \varrho_j)m_j + \sum_{q=1}^{j} \mathbb{M}_r(\varrho - \varrho_j)\widetilde{m}_j \\ \quad + \int_0^{\varrho} \mathbb{G}_r(\varrho - s)\mathfrak{f}(s, z(s), (E_1 z)(s), (E_2 z)(s))ds \\ \quad + \int_0^{\varrho} \mathbb{G}_r(\varrho - s)\mathcal{B}x(s)ds, \ \varrho_j < \varrho \leq \varrho_{j+1}, \end{cases} \tag{2.2}$$

where

$$\mathbb{N}_r(\varrho) = \frac{1}{2\pi i}\int_c e^{\rho r}\rho^{r-1} R(\rho^r, A)d\rho, \ \mathbb{M}_r(\varrho) = \frac{1}{2\pi i}\int_c e^{\rho r}\rho^{r-2} R(\rho^r, A)d\rho,$$
$$\mathbb{G}_r(\varrho) = \frac{1}{2\pi i}\int_c e^{\rho r} R(\rho^r, A)d\rho,$$

with c being a suitable path such that $\rho^r \notin \phi + \mathcal{S}_\tau$ for ρ belongs to c.

We consider now definition of exact controllability.

Definition 2.6. *The system* (1.1) *is said to be controllable on V iff for all z_0, z_1, z_w in Q, there exists $x \in L^2(V, \mathbb{Y})$ such that a mild solution z of* (1.1) *fulfills $z(\sigma) = z_w$.*

Let us recall some notations about the measure of noncompactness (see [2, 3]).

Definition 2.7. *The Hausdorff MNC δ discovered on for every bounded subset θ of Q by*

$$\delta(\theta) = \inf\{\kappa > 0 : \ \text{A finite number of balls with radii smaller than } \kappa \text{ can cover } \theta\}.$$

Definition 2.8. *[7] Suppose that Q^+ is the positive cone of an ordered Banach space $(Q, \leq)$. The value of Q^+ is called MNC on Q of $\mathcal{N}$ defined on for any bounded subsets of the Banach space Q iff $\mathcal{N}(\overline{co}\,\theta) = \mathcal{N}(\theta)$ for every $\theta \subseteq Q$, in which $\overline{co}\,\theta$ denotes the closed convex hull of θ.*

Definition 2.9. *[2, 8] For every bounded subsets $\theta, \theta_1, \theta_2$ of Q.*

- **(i)** *monotone iff for every bounded subsets θ, θ_1, θ_2 of Q we obtain: $(\theta_1 \subseteq \theta_2) \Rightarrow (\mathcal{N}(\theta_1) \leq \mathcal{N}(\theta_2))$;*
- **(ii)** *non singular iff $\mathcal{N}(\{b\} \cup \theta) = \mathcal{N}(\theta)$ for every b in Q, $\theta \subset Q$;*
- **(iii)** *regular iff $\mathcal{N}(\theta) = 0$ iff θ in Q, which is relatively compact;*
- **(iv)** *$\delta(\theta_1 + \theta_2) \leq \delta(\theta_1) + \delta(\theta_2)$, where $\theta_1 + \theta_2 = \{u + v : u \text{ in } \theta_1, v \text{ in } \theta_2\}$;*
- **(v)** *$\delta(\theta_1 \cup \theta_2) \leq \max\{\delta(\theta_1), \delta(\theta_2)\}$;*
- **(vi)** *$\delta(\beta\theta) \leq |\beta|\delta(\theta)$, for every $\beta \in \mathbb{R}$;*
- **(vii)** *If the Lipschitz continuous function $\mathcal{T}$ maps from $G(\mathcal{T}) \subseteq Q$ into Banach space $\mathcal{X}$ along with $l > 0$, then $\delta_{\mathcal{X}}(\mathcal{T}\theta) \leq l\delta(\theta)$, for $\theta \subseteq G(\mathcal{T})$.*

Lemma 2.10. *[2] If $\mathcal{P}$ subset of $C([b,\sigma],Q)$ is bounded and equicontinuous, in addition, $\delta(\mathcal{P}(\varrho))$ is continuous for all $b \le \varrho \le \sigma$ and*

$$\delta(\mathcal{P}) = \sup\{\delta(\mathcal{P}(\varrho)), \sigma \in [b,\sigma]\}, \quad \textit{whereby} \quad \mathcal{P}(\varrho) = \{u(\varrho) : z \in \mathcal{P}\} \subseteq Q.$$

Lemma 2.11. *[24] Suppose that the functions $\{y_v\}_{v=1}^{\infty}$ is a sequence of Bochner integrable from $V \to Q$ including the evaluation $\|y_v(\varrho)\| \le \delta(\varrho)$, for every ϱ in V and for all $k \ge 1$, where $\delta \in L^1(V,\mathbb{R})$, then the function $\alpha(\varrho) = \delta(\{y_v(\varrho) : v \ge 1\})$ in $L^1(V,\mathbb{R})$ and fulfills*

$$\delta\Big(\Big\{\int_0^{\varrho} y_v(s)ds : v \ge 1\Big\}\Big) \le 2\int_0^{\varrho} \alpha(s)ds.$$

Now, we consider the some conditions of sectorial operator of type (P,τ,r,ϕ).

Theorem 2.12. *[35, 36] Assume that A is a sectorial operator of type (P,τ,r,ϕ). In addition, the subsequent on $\|\mathbb{N}_r(\varrho)\|$ hold:*

(i) *For $\zeta \in (0,\pi)$, and suppose that $\phi \ge 0$, we get*

$$\|\mathbb{N}_r(\varrho)\| \le \frac{M_1(\tau,\zeta)Pe^{[M_1(\tau,\zeta)(1+\phi\varrho^r)]^{\frac{1}{2}}\left[\left(1+\frac{\sin\zeta}{\sin(\zeta-\tau)}\right)^{\frac{1}{r}}-1\right]}}{\pi\sin^{1+\frac{1}{r}}\tau}(1+\tau\varrho^r) + \frac{\Gamma(r)P}{\pi(1+\phi\varrho^r)|\cos\frac{\pi-\zeta}{r}|^r\sin\tau\sin\zeta},$$

for $\varrho > 0$, and $M_1(\tau,\zeta) = \max\{1, \frac{\sin\zeta}{\sin(\zeta-\tau)}\}$.

(ii) *For $\zeta \in (0,\pi)$, and suppose that $\phi < 0$, we get*

$$\|\mathbb{N}_r(\varrho)\| \le \left(\frac{eP[(1+\sin\zeta)^{\frac{1}{r}}-1]}{\pi|\cos\zeta|^{1+\frac{1}{r}}} + \frac{\Gamma(r)P}{\pi|\cos\zeta||\cos\frac{\pi-\zeta}{r}|^r}\right)\frac{1}{1+|\phi|\varrho^r}, \ \varrho > 0.$$

Theorem 2.13. *[35, 36] Suppose that A is a sectorial operator of type (P,τ,r,ϕ). In addition, the subsequent on $\|\mathbb{G}_r(\varrho)\|$, and $\|\mathbb{M}_r(\varrho)\|$ hold:*

(i) *For $\zeta \in (0,\pi)$ and assume that $\phi \ge 0$. we get*

$$\|\mathbb{G}_r(\varrho)\| \le \frac{P\left[\left(1+\frac{\sin\zeta}{\sin(\zeta-\tau)}\right)^{\frac{1}{r}}-1\right]}{\pi\sin\tau}(1+\tau\varrho^r)^{\frac{1}{r}}\varrho^{r-1}e^{[M_1(\tau,\zeta)(1+\phi\varrho^r)]^{\frac{1}{r}}} + \frac{P\varrho^{r-1}}{\pi(1+\phi\varrho^r)|\cos\frac{\pi-\zeta}{r}|^r\sin\tau\sin\zeta},$$

$$\|\mathbb{M}_r(\varrho)\| \le \frac{P\left[\left(1+\frac{\sin\zeta}{\sin(\zeta-\tau)}\right)^{\frac{1}{r}}-1\right]M_1(\tau,\zeta)}{\pi\sin\tau^{\frac{r+2}{r}}}(1+\tau\varrho^r)^{\frac{r-1}{r}}\varrho^{r-1}e^{[M_1(\tau,\zeta)(1+\phi\varrho^r)]^{\frac{1}{r}}} + \frac{Pr\Gamma(r)}{\pi(1+\phi\varrho^r)|\cos\frac{\pi-\zeta}{r}|^r\sin\tau\sin\zeta},$$

for $\varrho > 0$, where $M_1(\tau,\zeta) = \max\{1, \frac{\sin\tau}{\sin(\zeta-\tau)}\}$.

(*ii*) *For* $0 < \zeta < \pi$, *assume that* $\phi < 0$, *we get*

$$\|\mathbb{G}_r(\varrho)\| \leq \Big(\frac{eP[(1+\sin\zeta)^{\frac{1}{r}} - 1]}{\pi|\cos\zeta|} + \frac{P}{\pi|\cos\zeta||\cos\frac{\pi-\zeta}{r}|}\Big)\frac{\varrho^{r-1}}{1+|\phi|\varrho^r},$$
$$\|\mathbb{M}_r(\varrho)\| \leq \Big(\frac{eP[(1+\sin\zeta)^{\frac{1}{r}} - 1]\varrho}{\pi|\cos\zeta|^{1+\frac{2}{r}}} + \frac{r\Gamma(r)P}{\pi|\cos\zeta||\cos\frac{\pi-\zeta}{r}|}\Big)\frac{1}{1+|\phi|\varrho^r},$$

for $\varrho > 0$.

Lemma 2.14. *[23] Suppose that* $\mathcal{T}$ *is closed convex subset of* Q *and zero in* $\mathcal{T}$, *let the continuous function* $\mathcal{E}$ *maps from* $\mathcal{T}$ *into* Q *and that fulfills Mönch's condition, which is,* ($N \subseteq \mathcal{T}$ *is countable,* $N \subseteq \overline{co}(\{0\} \cup \mathcal{E}(N)) \Rightarrow \overline{N}$ *is compact). Hence,* $\mathcal{E}$ *has a fixed point in* $\mathcal{T}$.

3. Exact controllability results

The existence of mild solutions for the Eq. (1.1) will be shown in this section. The following assumptions are required: It is straightforward to show that they are bounded because of the estimations on $\mathbb{N}_r(\varrho)$, $\mathbb{M}_r(\varrho)$ and $\mathbb{G}_r(\varrho)$ in Theorems 2.12 and 2.13.

(H₁) The operators $\mathbb{N}_r(\varrho)$, $\mathbb{M}_r(\varrho)$, and $\mathbb{G}_r(\varrho)$. for every $\varrho \in V$, there exists a $\widehat{P} > 0$ such that

$$\sup_{0\leq\varrho\leq\sigma} \|\mathbb{N}_r(\varrho)\| \leq \widehat{P}, \ \sup_{0\leq\varrho\leq\sigma} \|\mathbb{M}_r(\varrho)\| \leq \widehat{P}, \ \sup_{0\leq\varrho\leq\sigma} \|\mathbb{G}_r(\varrho)\| \leq \widehat{P}.$$

($\mathbf{H_2}$) $\mathfrak{f} : V \times Q \times Q \times Q \to Q$ fulfills:

- **(i)** $\mathfrak{f}(\cdot, \chi, u, z)$ is measurable for every (χ, u, z) in $Q \times Q$ and $\mathfrak{f}(\varrho, \cdot, \cdot, \cdot)$ is continuous for all $\varrho \in V$, $z \in Q$, $\mathfrak{f}(\cdot, \chi, u, z)$ is strongly measurable;
- **(ii)** there exists $p_1 \in (0, p)$ and $\varsigma_1 \in L^{\frac{1}{p_1}}([0, \sigma], \mathbb{R}^+)$, and $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is integrable function such that $\|\mathfrak{f}(\varrho, \chi, u, z)\| \leq \varsigma_1(\varrho)\omega(\|\chi\|_Q + \|u\| + \|z\|)$, for all (ϱ, χ, u, z) in $\mathcal{S} \times Q \times Q \times Q$, where ω satisfies $\lim_{v\to\infty} \inf \frac{\omega(v)}{v} = 0$;
- **(iii)** there exists $0 < p_2 < p$ and $\varsigma_2 \in L^{\frac{1}{p_2}}(V, \mathbb{R}^+)$ such that for every bounded subset $\mathcal{S}_1 \subset \mathcal{Y}$ and $\mathcal{W}_1 \subset Q$,

$$\delta(\mathfrak{f}(\varrho, \mathcal{W}_1, \mathcal{S}_1, \mathcal{S}_2)) \leq \varsigma_2(\varrho)\Big[\delta(\mathcal{W}_1) + \delta(\mathcal{S}_1) + \delta(\mathcal{S}_2)\Big], \text{ for a.e. } \varrho \in V,$$

$\mathcal{W}_1(\varphi) = \{e(\varphi) : e \in \mathcal{W}_1\}$ and δ is the Hausdorff MNC.

($\mathbf{H_3}$) $e_1 : \mathcal{S} \times Q \times Q \to Q$ satisfies:

- **(i)** $e_1(\cdot, s, z)$ is measurable for all $(s, z) \in Q \times Q$, $e_1(\varrho, \cdot, \cdot)$ is continuous for all $\varrho \in V$;
- **(ii)** there exists $F_0 > 0$ such that $\|e_1(\varrho, s, z)\| \leq F_0(1 + \|z\|_Q)$, for every ϱ in V, $z \in Q$;
- **(iii)** there exists $p_3 \in (0, p)$ and $\varsigma_3 \in L^{\frac{1}{p_3}}(V, \mathbb{R}^+)$ such that for every bounded subset $\mathcal{S}_3 \subset Q$,

$$\delta(e_1(\varrho, s, \mathcal{S}_3)) \leq \varsigma_3(\varrho, s)[\delta(\mathcal{S}_3)] \text{ for a.e. } \varrho \in V,$$

with $\varsigma_3^* = \sup_{s\in V} \int_0^s \varsigma_3(\varrho, s)ds < \infty$.

(H4) $e_2 : \mathcal{S} \times Q \times Q \to Q$ satisfies:

(i) $e_2(\cdot, s, z)$ is measurable for any (s, z) in $Q \times Q$, $e_2(\varrho, \cdot, \cdot)$ is continuous for all $\varrho \in V$;

(ii) there exists $F_1 > 0$ such that $\|e_2(\varrho, s, z)\| \leq F_1(1 + \|z\|_Q)$, for every $\varrho \in V$, z in Q;

(iii) there exists $p_4 \in (0, p)$ and $\varsigma_4 \in L^{\frac{1}{p_4}}(V, \mathbb{R}^+)$ such that for every bounded subset $\mathcal{S}_4 \subset \mathcal{S}$,

$$\delta(e_2(\varrho, s, \mathcal{S}_4)) \leq \varsigma_4(\varrho, s)[\delta(\mathcal{S}_4)] \text{ for a.e. } \sigma \in V,$$

with $\varsigma_4^* = \sup_{s \in V} \int_0^s \varsigma_4(\varrho, s)ds < \infty$;

(H5) the operator $\mathcal{B}$ maps from $L^2(V, \mathbb{Y})$ into $L^1(V, Q)$ is bounded and $\mathbb{W} : L^2(V, \mathbb{Y}) \to L^1(V, Q)$ is defined by

$$\mathbb{W}x = \int_0^\sigma \mathbb{G}(\sigma - s)\mathcal{B}x(s)ds,$$

fulfills:

(i) There exist a positive contants P_σ, P_x such that $\|B\| \leq P_\sigma$ and $\|\mathbb{W}^{-1}\| \leq P_x$ when $\mathbb{W}$ have an inverse $\mathbb{W}^{-1}$ acquires the value belongs to $L^2(V, \mathbb{Y})/Ker\mathbb{W}$.

(ii) For p_5 in $(0, p)$ and for all bounded subset $\mathcal{T} \in Q$, there exists $\varsigma_5 \in L^{\frac{1}{p_5}}(V, \mathbb{R}^+)$ such that $\delta(\mathbb{W}^{-1}(\mathcal{T})(\varrho)) \leq \varsigma_5(\varrho)\delta(\mathcal{T})$.

Consider the operator $\Pi : PC(V, Q) \to PC(V, Q)$ determined by

$$(\Pi z)(\varrho) = \begin{cases} \mathbb{N}_r(\varrho)z_0 + \mathbb{M}_r(\varrho)z_1 + \int_0^\varrho \mathbb{G}_r(\varrho - s)\mathfrak{f}(s, z(s), (E_1z)(s), (E_2z)(s))ds \\ \quad + \int_0^\varrho \mathbb{G}_r(\varrho - s)\mathcal{B}x(s)ds, \ 0 \leq \varrho \leq \varrho_1, \\ \mathbb{N}_r(\varrho)z_0 + \mathbb{M}_r(\varrho)z_1 + \sum_{q=1}^j \mathbb{N}_r(\varrho - \varrho_j)m_j + \sum_{q=1}^j \mathbb{M}_r(\varrho - \varrho_j)\widetilde{m}_j \\ \quad + \int_0^\varrho \mathbb{G}_r(\varrho - s)\mathfrak{f}(s, z(s), (E_1z)(s), (E_2z)(s))ds \\ \quad + \int_0^\varrho \mathbb{G}_r(\varrho - s)\mathcal{B}x(s)ds, \ \varrho_j < \varrho \leq \varrho_{j+1}. \end{cases} \tag{3.1}$$

Theorem 3.1. *If* **(H1)**–**(H5)** *are fuflilled. In addition, the system* (1.1) *is controllable if*

$$2\widehat{P}\|\varsigma_2\|[1 + 2\widehat{P}P_\sigma\|\varsigma_5\|](1 + (\varsigma_3^* + \varsigma_4^*)) < 1. \tag{3.2}$$

Proof. We introduce the control $x_z(\cdot)$ for arbitrary function $z \in PC(V, Q)$ and using **(H5)(i)**, presented by

$$x_z(\varrho) = \mathbb{W}^{-1} \begin{cases} z_w - \mathbb{N}_r(\varrho)z_0 - \mathbb{M}_r(\varrho)z_1 \\ \quad - \int_0^\varrho \mathbb{G}_r(\varrho - s)\mathfrak{f}(s, z(s), (E_1z)(s), (E_2z)(s))ds, \ 0 \leq \varrho \leq \varrho_1, \\ z_w - \mathbb{N}_r(\varrho)z_0 - \mathbb{M}_r(\varrho)z_1 - \sum_{q=1}^j \mathbb{N}_r(\varrho - \varrho_j)m_j - \sum_{q=1}^j \mathbb{M}_r(\varrho - \varrho_j)\widetilde{m}_j \\ \quad - \int_0^\varrho \mathbb{G}_r(\varrho - s)\mathfrak{f}(s, z(s), (E_1z)(s), (E_2z)(s))ds, \ \varrho_j < \varrho \leq \varrho_{j+1}. \end{cases}$$

We can see that the operator Π provided in (3.1) has a fixed point by using the control mentioned above. Moreover, if Π allows a fixed point, it is simple to deduce that $(\Pi z)(\sigma) = z_\sigma$, that suggests that $x_z(\varrho)$ drives the mild solution of (1.1) from the initial state z_0 and z_1 to the final state z_σ in time σ.

Step 1: There exists $\ell > 0$ such that $\Pi(\mathcal{G}_\ell) \subset \mathcal{G}_\ell$, where $\mathcal{G}_\ell = \{z \in PC(V, Q) : \ \|z\| \leq \ell\}$. Indeed, if the above assumption is fails, there is a function for every $z_\ell \in \mathcal{G}_\ell$ and $x_{z_\ell} \in L^2(V, \mathbb{Y})$ according to $\mathcal{G}_\ell$ such that $(\Pi z_\ell)(\varrho)$ not in $\mathcal{G}_\ell$, for every $\varrho \in [0, \varrho_1]$, we get

$$\begin{aligned}
\|x_z(\varrho)\| &= \|\mathbb{W}^{-1}\Big[z_w - \mathbb{N}_r(\varrho)z_0 - \mathbb{M}_r(\varrho)z_1 - \int_0^\varrho \mathbb{G}_r(\varrho - s)\mathfrak{f}(s, z(s), (E_1z)(s), (E_2z)(s))ds\Big]\| \\
&\leq P_x\Big[\|z_w\| + \widehat{P}\|z_0\| + \widehat{P}\|z_1\| + \widehat{P}\int_0^\varrho \|\mathfrak{f}(s, z(s), (E_1z)(s), (E_2z)(s))\|ds\Big] \\
&\leq P_x\Big[\|z_w\| + \widehat{P}\|z_0\| + \widehat{P}\|z_1\| + \widehat{P}\|\varsigma_1\|_{L^{\frac{1}{p_1}}(V, \mathbb{R}^+)}\omega(\ell + \sigma F_0(1 + \ell) + \sigma F_1(1 + \ell))\Big].
\end{aligned}$$

Likewise, for every $\varrho \in (\varrho_j, \varrho_{j+1}]$, $j = 1, 2, \cdots, n$, we can have

$$\begin{aligned}
\|x_z(\varrho)\| &= \|\mathbb{W}^{-1}\Big[z_w - \mathbb{N}_r(\varrho)z_0 - \mathbb{M}_r(\varrho)z_1 - \sum_{q=1}^{j}\mathbb{N}_r(\varrho - \varrho_j)m_j - \sum_{q=1}^{j}\mathbb{M}_r(\varrho - \varrho_j)\widetilde{m}_j \\
&\quad - \int_0^\varrho \mathbb{G}_r(\varrho - s)\mathfrak{f}(s, z(s), (E_1z)(s), (E_2z)(s))ds\Big]\| \\
&\leq P_x\Big[\|z_w\| + \widehat{P}\|z_0\| + \widehat{P}\|z_1\| + \widehat{P}\sum_{q=1}^{j}\|m_j\| + \widehat{P}\sum_{q=1}^{j}\|\widetilde{m}_j\| \\
&\quad + \widehat{P}\int_0^\varrho \|\mathfrak{f}(s, z(s), (E_1z)(s), (E_2z)(s))\|ds\Big] \\
&\leq P_x\Big[\|z_w\| + \widehat{P}\|z_0\| + \widehat{P}\|z_1\| + \widehat{P}\sum_{q=1}^{j}\|m_j\| + \widehat{P}\sum_{q=1}^{j}\|\widetilde{m}_j\| \\
&\quad + \widehat{P}\|\varsigma_1\|_{L^{\frac{1}{p_1}}(V, \mathbb{R}^+)}\omega(\ell + \sigma F_0(1 + \ell) + \sigma F_1(1 + \ell))\Big].
\end{aligned}$$

As a result, there exists $\jmath_1, \jmath_2 > 0$ such that

$$\|x_z(\varrho)\| = P_x\begin{cases}
\|z_w\| + \widehat{P}\|z_0\| + \widehat{P}\|z_1\| + \widehat{P}\|\varsigma_1\|_{L^{\frac{1}{p_1}}(V, \mathbb{R}^+)} \\
\quad \times\omega(\ell + \sigma F_0(1 + \ell) + \sigma F_1(1 + \ell)) = \jmath_1, \ 0 \leq \varrho \leq \varrho_1, \\
\|z_w\| + \widehat{P}\|z_0\| + \widehat{P}\|z_1\| + \widehat{P}\sum_{q=1}^{j}\|m_j\| + \widehat{P}\sum_{q=1}^{j}\|\widetilde{m}_j\| \\
\quad +\widehat{P}\|\varsigma_1\|_{L^{\frac{1}{p_1}}(V, \mathbb{R}^+)}\omega(\ell + \sigma F_0(1 + \ell) + \sigma F_1(1 + \ell)) = \jmath_2, \ \varrho_j < \varrho \leq \varrho_{j+1}.
\end{cases}$$

Using the assumptions $(\mathbf{H_1})$–$(\mathbf{H_5})$, and for every $\varrho \in [0, \varrho_1]$, we get

$$\begin{aligned}
\ell < \|(\Pi z)(\varrho)\| &\leq \|\mathbb{N}_r(\varrho)z_0\| + \|\mathbb{M}_r(\varrho)z_1\| + \int_0^\varrho \|\mathbb{G}_r(\varrho - s)\mathfrak{f}(s, z(s), (E_1z)(s), (E_2z)(s))\|ds \\
&\quad + \int_0^\varrho \|\mathbb{G}_r(\varrho - s)\mathcal{B}x_z(s)\|ds \\
&\leq \widehat{P}\|z_0\| + \widehat{P}\|z_1\| + \widehat{P}\int_0^\varrho \|\mathfrak{f}(s, z(s), (E_1z)(s), (E_2z)(s))\|ds \\
&\quad + \widehat{P}\int_0^\varrho \|\mathcal{B}x_z(s)\|ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \widehat{P}\|z_0\| + \widehat{P}\|z_1\| + \widehat{P}\|\varsigma_1\|_{L^{\frac{1}{p_1}}(V,\mathbb{R}^+)}\omega(\ell + \sigma F_0(1+\ell) + \sigma F_1(1+\ell)) \\
&\quad + \widehat{P}P_\sigma P_x\sigma[\|z_w\| + \widehat{P}\|z_0\| + \widehat{P}\|z_1\| + \widehat{P}\|\varsigma_1\|_{L^{\frac{1}{p_1}}(V,\mathbb{R}^+)} \\
&\quad \times \omega(\ell + \sigma F_0(1+\ell) + \sigma F_1(1+\ell))].
\end{aligned}
$$

Similarly, for every $\varrho \in (\varrho_j, \varrho_{j+1}]$, $j = 1, 2, \cdots, n$, we can have

$$
\begin{aligned}
\ell < (\Pi z)(\varrho) &\leq \|\mathbb{N}_r(\varrho)z_0\| + \|\mathbb{M}_r(\varrho)z_1\| + \|\sum_{q=1}^{j}\mathbb{N}_r(\varrho-\varrho_j)m_j\| + \|\sum_{q=1}^{j}\mathbb{M}_r(\varrho-\varrho_j)\widetilde{m}_j\| \\
&\quad + \int_0^\varrho \|\mathbb{G}_r(\varrho-s)\mathfrak{f}(s, z(s), (E_1z)(s), (E_2z)(s))\|ds \\
&\quad + \int_0^\varrho \|\mathbb{G}_r(\varrho-s)\mathcal{B}x_z(s)\|ds \\
&\leq \widehat{P}\|z_0\| + \widehat{P}\|z_1\| + \widehat{P}\sum_{q=1}^{j}\|m_j\| + \widehat{P}\sum_{q=1}^{j}\|\widetilde{m}_j\| \\
&\quad + \widehat{P}\int_0^\varrho \|\mathfrak{f}(s, z(s), (E_1z)(s), (E_2z)(s))\|ds + \widehat{P}\int_0^\varrho \|\mathcal{B}x_z(s)\|ds \\
&\leq \widehat{P}\|z_0\| + \widehat{P}\|z_1\| + \widehat{P}\sum_{q=1}^{j}\|m_j\| + \widehat{P}\sum_{q=1}^{j}\|\widetilde{m}_j\| \\
&\quad + \widehat{P}\|\varsigma_1\|_{L^{\frac{1}{p_1}}(V,\mathbb{R}^+)}\omega(\ell + \sigma F_0(1+\ell) + \sigma F_1(1+\ell)) \\
&\quad + \widehat{P}P_\sigma P_x\sigma[\|z_w\| + \widehat{P}\|z_0\| + \widehat{P}\|z_1\| + \widehat{P}\sum_{q=1}^{j}\|m_j\| + \widehat{P}\sum_{q=1}^{j}\|\widetilde{m}_j\| \\
&\quad + \widehat{P}\|\varsigma_1\|_{L^{\frac{1}{p_1}}(V,\mathbb{R}^+)}\omega(\ell + \sigma F_0(1+\ell) + \sigma F_1(1+\ell))].
\end{aligned}
$$

Therefore

$$
\ell < \begin{cases}
\widehat{P}\|z_0\| + \widehat{P}\|z_1\| + \widehat{P}\|\varsigma\|_{L^{\frac{1}{p_1}}(V,\mathbb{R}^+)}\omega(\ell + \sigma F_0(1+\ell) + \sigma F_1(1+\ell)) \\
\quad + \widehat{P}P_\sigma P_x\sigma[\|z_w\| + \widehat{P}\|z_0\| + \widehat{P}\|z_1\| + \widehat{P}\|\varsigma_1\|_{L^{\frac{1}{p_1}}(V,\mathbb{R}^+)} \\
\quad \times\omega(\ell + \sigma F_0(1+\ell) + \sigma F_1(1+\ell))],\ 0 \leq \varrho \leq \varrho_1, \\
\widehat{P}\|z_0\| + \widehat{P}\|z_1\| + \widehat{P}\sum_{q=1}^{j}\|m_j\| + \widehat{P}\sum_{q=1}^{j}\|\widetilde{m}_j\| \\
\quad + \widehat{P}\|\varsigma_1\|_{L^{\frac{1}{p_1}}(V,\mathbb{R}^+)}\omega(\ell + \sigma F_0(1+\ell) + \sigma F_1(1+\ell)) \\
\quad + \widehat{P}P_\sigma P_x\sigma[\|z_w\| + \widehat{P}\|z_0\| + \widehat{P}\|z_1\| + \widehat{P}\sum_{q=1}^{j}\|m_j\| + \widehat{P}\sum_{q=1}^{j}\|\widetilde{m}_j\| \\
\quad + \widehat{P}\|\varsigma_1\|_{L^{\frac{1}{p_1}}(V,\mathbb{R}^+)}\omega(\ell + \sigma F_0(1+\ell) + \sigma F_1(1+\ell))],\ \varrho_j < \varrho \leq \varrho_{j+1}.
\end{cases} \tag{3.3}
$$

If we divide (3.3) by ℓ, and assuming $\ell \to \infty$. In addition, by $(\mathbf{H_2})(\mathbf{ii})$, one can get $1 \leq 0$. This is a contradiction. Therefore, $\ell > 0$, $\Pi(\mathcal{G}_\ell) \subseteq \mathcal{G}_\ell$.

Step 2: We verify that Π is continuous on $PC(V, Q)$. For such a study, let $z^{(v)}$ tends to $z \in PC(V, Q)$. there exists $\ell > 0$ such that $\|z^{(v)}(\varrho)\| \leq \ell$ for any v and $\varrho \in V$, so $z^{(v)} \in PC(V, Q)$ and $z \in PC(V, Q)$. By the hypotheses ($\mathbf{H_2}$)–($\mathbf{H_5}$), we get

$$\mathcal{F}_v(s) = \mathfrak{f}(s, z(s), (Fz^{(v)})(s), (Hz^{(v)})(s)),$$
$$\mathcal{F}(s) = \mathfrak{f}(s, z(s), (E_1 z)(s), (E_2 z)(s)).$$

From Lebesgue's dominated convergence theorem, we get

$$\int_0^{\varrho} \|\mathcal{F}_v(s) - \mathcal{F}(s)\| ds \to 0 \text{ as } v \to \infty,\ \varrho_j < \varrho \leq \varrho_{j+1}.$$

Then,

$$\begin{aligned}
\|(\Pi z^{(v)})(\varrho) - (\Pi z)(\varrho)\| &= \|\mathbb{N}_r(\varrho) z_0 + \mathbb{M}_r(\varrho) z_1 + \sum_{q=1}^{j} \mathbb{N}_r(\varrho - \varrho_j) m_j + \sum_{q=1}^{j} \mathbb{M}_r(\varrho - \varrho_j) \widetilde{m}_j \\
&\quad + \int_0^{\varrho} \mathbb{G}_r(\varrho - s) \mathcal{F}_v(s) ds + \int_0^{\varrho} \mathbb{G}_r(\varrho - s) \mathcal{B} x_{z^{(v)}}(s) ds \\
&\quad - \mathbb{N}_r(\varrho) z_0 - \mathbb{M}_r(\varrho) z_1 - \sum_{q=1}^{j} \mathbb{N}_r(\varrho - \varrho_j) m_j - \sum_{q=1}^{j} \mathbb{M}_r(\varrho - \varrho_j) \widetilde{m}_j \\
&\quad - \int_0^{\varrho} \|\mathbb{G}_r(\varrho - s) \mathcal{F}(s) ds - \int_0^{\varrho} \mathbb{G}_r(\varrho - s) \mathcal{B} x_z(s) ds\| \\
&\leq \int_0^{\varrho} \|\mathbb{G}_r(\varrho - s)[\mathcal{F}_v(s) - \mathcal{F}(s)]\| ds \\
&\quad + \int_0^{\varrho} \|\mathbb{G}_r(\varrho - s) \mathcal{B}[x_{z^{(v)}}(s) - x_z(s)]\| ds \\
&\leq \widehat{P} \int_0^{\varrho} \|\mathcal{F}_v(s) - \mathcal{F}(s)\| ds + \widehat{P} P_{\sigma} \int_0^{\varrho} \|x_{z^{(v)}}(s) - x_z(s)\| ds,
\end{aligned} \tag{3.4}$$

where

$$\|x_{z^{(v)}}(s) - x_z(s)\| \leq P_x \widehat{P} \Big[\int_0^{\varrho} \|\mathcal{F}_v(s) - \mathcal{F}(s)\| ds \Big]. \tag{3.5}$$

Since the inequality (3.4) and (3.5), we get

$$\|(\Pi z^{(v)})(\varrho) - (\Pi z)(\varrho)\| \to 0 \text{ as } v \to \infty.$$

Then, Π is continuous on $PC(V, Q)$.
Step 3: Now, we show that $\{(\Pi z) : z \in \mathcal{G}_\ell\}$ is equicontinuous family.

$$(\Pi z)(\varrho) = \begin{cases} \mathbb{N}_r(\varrho) z_0 + \mathbb{M}_r(\varrho) z_1 + \int_0^{\varrho} \mathbb{G}_r(\varrho - s) \mathfrak{f}(s, z(s), (E_1 z)(s), (E_2 z)(s)) ds \\ \quad + \int_0^{\varrho} \mathbb{G}_r(\varrho - s) \mathcal{B} x(s) ds,\ 0 \leq \varrho \leq \varrho_1, \\ \mathbb{N}_r(\varrho) z_0 + \mathbb{M}_r(\varrho) z_1 + \sum_{q=1}^{j} \mathbb{N}_r(\varrho - \varrho_j) m_j + \sum_{q=1}^{j} \mathbb{M}_r(\varrho - \varrho_j) \widetilde{m}_j \\ \quad + \int_0^{\varrho} \mathbb{G}_r(\varrho - s) \mathfrak{f}(s, z(s), (E_1 z)(s), (E_2 z)(s)) ds \\ \quad + \int_0^{\varrho} \mathbb{G}_r(\varrho - s) \mathcal{B} x(s) ds,\ \varrho_j < \varrho \leq \varrho_{j+1}. \end{cases}$$

Suppose $0 \leq \hbar_1 < \hbar_2 \leq \varrho_1$. In addition, for every $\varrho \in [0, \varrho_1]$, we get

$$
\begin{aligned}
&\|(\Pi z)(\hbar_2) - (\Pi z)(\hbar_1)\| \\
&= \|\mathbb{N}_r(\hbar_2)z_0 + \mathbb{M}_r(\hbar_2)z_1 + \int_0^{\hbar_2} \mathbb{G}_r(\hbar_2 - s)\mathfrak{f}(s, z(s), (E_1 z)(s), (E_2 z)(s))ds \\
&\quad + \int_0^{\hbar_2} \mathbb{G}_r(\hbar_2 - s)\mathcal{B}x_z(s)ds - \mathbb{N}_r(\hbar_1)z_0 - \mathbb{M}_r(\hbar_1)z_1 \\
&\quad - \int_0^{\hbar_1} \mathbb{G}_r(\hbar_1 - s)\mathfrak{f}(s, z(s), (E_1 z)(s), (E_2 z)(s))ds - \int_0^{\hbar_1} \mathbb{G}_r(\hbar_1 - s)\mathcal{B}x_z(s)ds\| \\
&\leq \|[\mathbb{N}_r(\hbar_2) - \mathbb{N}_r(\hbar_1)]z_0\| + \|[\mathbb{M}_r(\hbar_2) - \mathbb{M}_r(\hbar_1)]z_1\| \\
&\quad + \int_{\hbar_1}^{\hbar_2} \|\mathbb{G}_r(\hbar_2 - s)\mathfrak{f}(s, z(s), (E_1 z)(s), (E_2 z)(s))\|ds \\
&\quad + \int_0^{\hbar_1} \|[\mathbb{G}_r(\hbar_2 - s) - \mathbb{G}_r(\hbar_1 - s)]\mathfrak{f}(s, z(s), (E_1 z)(s), (E_2 z)(s))\|ds \\
&\quad + \int_{\hbar_1}^{\hbar_2} \|\mathbb{G}_r(\hbar_2 - s)\mathcal{B}x_z(s)\|ds \\
&\quad + \int_0^{\hbar_1} \|[\mathbb{G}_r(\hbar_2 - s) - \mathbb{G}_r(\hbar_1 - s)]\mathcal{B}x_z(s)\|ds \\
&\leq \|\mathbb{N}_r(\hbar_2) - \mathbb{N}_r(\hbar_1)\|\|z_0\| + \|\mathbb{M}_r(\hbar_2) - \mathbb{M}_r(\hbar_1)\|\|z_1\| \\
&\quad + \widehat{P} \int_{\hbar_1}^{\hbar_2} \varsigma_1(s)\omega(\ell + \sigma F_0(1 + \ell) + \sigma F_1(1 + \ell))ds \\
&\quad + \int_0^{\hbar_1} \|\mathbb{G}_r(\hbar_2 - s) - \mathbb{G}_r(\hbar_1 - s)\|\varsigma_1(s)\omega(\ell + \sigma F_0(1 + \ell) + \sigma F_1(1 + \ell))ds \\
&\quad + \widehat{P}P_\sigma \|x_z\|_{L^\mu(V, \Upsilon)}(\hbar_2 - \hbar_1) + P_\sigma \int_0^{\hbar_1} \|\mathbb{G}_r(\hbar_2 - s) - \mathbb{G}_r(\hbar_1 - s)\|\|x_z(s)\|ds.
\end{aligned}
$$

Similarly, for every $\varrho \in (\varrho_j, \varrho_{j+1}]$, $j = 1, 2, \cdots, n$, we can have

$$
\begin{aligned}
&\|(\Pi z)(\hbar_2) - (\Pi z)(\hbar_1)\| \\
&= \|\mathbb{N}_r(\hbar_2)z_0 + \mathbb{M}_r(\hbar_2)z_1 + \sum_{0<\varrho_j<\hbar_2} \mathbb{N}_r(\hbar_2 - \varrho_j)m_j + \sum_{0<\varrho_j<\hbar_2} \mathbb{M}_r(\hbar_2 - \varrho_j)\widetilde{m}_j \\
&\quad + \int_0^{\hbar_2} \mathbb{G}_r(\hbar_2 - s)\mathfrak{f}(s, z(s), (E_1 z)(s), (E_2 z)(s))ds \\
&\quad + \int_0^{\hbar_2} \mathbb{G}_r(\hbar_2 - s)\mathcal{B}x(s)ds - \mathbb{N}_r(\hbar_1)z_0 - \mathbb{M}_r(\hbar_1)z_1 - \sum_{0<\varrho_j<\hbar_1} \mathbb{N}_r(\hbar_1 - \varrho_j)m_j \\
&\quad - \sum_{0<\varrho_j<\hbar_1} \mathbb{M}_r(\hbar_1 - \varrho_j)\widetilde{m}_j - \int_0^{\hbar_1} \mathbb{G}_r(\hbar_1 - s)\mathfrak{f}(s, z(s), (E_1 z)(s), (E_2 z)(s))ds \\
&\quad - \int_0^{\hbar_1} \mathbb{G}_r(\hbar_1 - s)\mathcal{B}x(s)ds\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|[\mathbb{N}_r(\hbar_2)-\mathbb{N}_r(\hbar_1)]z_0\| + \|[\mathbb{M}_r(\hbar_2)-\mathbb{M}_r(\hbar_1)]z_1\| + \sum_{\hbar_1<\varrho_j<\hbar_2}\|\mathbb{N}_r(\hbar_2-\varrho_j)m_j\| \\
&\quad + \sum_{0<\varrho_j<\hbar_2}\|[\mathbb{N}_r(\hbar_2-\varrho_j)-\mathbb{N}_r(\hbar_1-\varrho_j)]m_j\| + \sum_{\hbar_1<\varrho_j<\hbar_2}\|\mathbb{M}_r(\hbar_2-\varrho_j)\widetilde{m}_j\| \\
&\quad + \sum_{0<\varrho_j<\hbar_2}\|[\mathbb{M}_r(\hbar_2-\varrho_j)-\mathbb{M}_r(\hbar_1-\varrho_j)]\widetilde{m}_j\| \\
&\quad + \int_{\hbar_1}^{\hbar_2}\|\mathbb{G}_r(\hbar_2-s)\mathfrak{f}(s,z(s),(E_1z)(s),(E_2z)(s))\|ds \\
&\quad + \int_0^{\hbar_1}\|[\mathbb{G}_r(\hbar_2-s)-\mathbb{G}_r(\hbar_1-s)]\mathfrak{f}(s,z(s),(E_1z)(s),(E_2z)(s))\|ds \\
&\quad + \int_{\hbar_1}^{\hbar_2}\|\mathbb{G}_r(\hbar_2-s)\mathcal{B}x(s)\|ds + \int_0^{\hbar_1}\|[\mathbb{G}_r(\hbar_2-s)-\mathbb{G}_r(\hbar_1-s)]\mathcal{B}x(s)\|ds \\
&\leq \|\mathbb{N}_r(\hbar_2)-\mathbb{N}_r(\hbar_1)\|\|z_0\| + \|\mathbb{M}_r(\hbar_2)-\mathbb{M}_r(\hbar_1)\|\|z_1\| + \widehat{P}\sum_{\hbar_1<\varrho_j<\hbar_2}\|m_j\| \\
&\quad + \sum_{0<\varrho_j<\hbar_2}\|\mathbb{N}_r(\hbar_2-\varrho_j)-\mathbb{N}_r(\hbar_1-\varrho_j)\|\|m_j\| + \widehat{P}\sum_{\hbar_1<\varrho_j<\hbar_2}\|\widetilde{m}_j\| \\
&\quad + \sum_{0<\varrho_j<\hbar_2}\|\mathbb{M}_r(\hbar_2-\varrho_j)-\mathbb{M}_r(\hbar_1-\varrho_j)\|\|\widetilde{m}_j\| \\
&\quad + \widehat{P}\int_{\hbar_1}^{\hbar_2}\varsigma_1(s)\omega(\ell+\sigma F_0(1+\ell)+\sigma F_1(1+\ell))ds \\
&\quad + \int_0^{\hbar_1}\|\mathbb{G}_r(\hbar_2-s)-\mathbb{G}_r(\hbar_1-s)\|\varsigma_1(s)\omega(\ell+\sigma F_0(1+\ell)+\sigma F_1(1+\ell))ds \\
&\quad + \widehat{P}P_\sigma\|x\|_{L^\mu(V,\mathbb{Y})}(\hbar_2-\hbar_1) + P_\sigma\int_0^{\hbar_1}\|\mathbb{G}_r(\hbar_2-s)-\mathbb{G}_r(\hbar_1-s)\|\|x(s)\|ds.
\end{aligned}
$$

Thus, we get

$$
\|(\Pi z)(\hbar_2)-(\Pi z)(\hbar_1)\| = \begin{cases}
\|\mathbb{N}_r(\hbar_2)-\mathbb{N}_r(\hbar_1)\|\|z_0\| + \|\mathbb{M}_r(\hbar_2)-\mathbb{M}_r(\hbar_1)\|\|z_1\| \\
\quad +\widehat{P}\int_{\hbar_1}^{\hbar_2}\varsigma_1(s)\omega(\ell+\sigma F_0(1+\ell)+\sigma F_1(1+\ell))ds \\
\quad +\int_0^{\hbar_1}\|\mathbb{G}_r(\hbar_2-s)-\mathbb{G}_r(\hbar_1-s)\|\varsigma_1(s) \\
\quad \times\omega(\ell+\sigma F_0(1+\ell)+\sigma F_1(1+\ell))ds + \widehat{P}P_\sigma\|x_z\|_{L^\mu(V,\mathbb{Y})}(\hbar_2-\hbar_1) \\
\quad +P_\sigma\int_0^{\hbar_1}\|\mathbb{G}_r(\hbar_2-s)-\mathbb{G}_r(\hbar_1-s)\|\|x_z(s)\|ds,\ \varrho\in[0,\varrho_1], \\
\|\mathbb{N}_r(\hbar_2)-\mathbb{N}_r(\hbar_1)\|\|z_0\| + \|\mathbb{M}_r(\hbar_2)-\mathbb{M}_r(\hbar_1)\|\|z_1\| + \widehat{P}\sum_{\hbar_1<\varrho_j<\hbar_2}\|m_j\| \\
\quad +\sum_{0<\varrho_j<\hbar_2}\|\mathbb{N}_r(\hbar_2-\varrho_j)-\mathbb{N}_r(\hbar_1-\varrho_j)\|\|m_j\| + \widehat{P}\sum_{\hbar_1<\varrho_j<\hbar_2}\|\widetilde{m}_j\| \\
\quad +\sum_{0<\varrho_j<\hbar_2}\|\mathbb{M}_r(\hbar_2-\varrho_j)-\mathbb{M}_r(\hbar_1-\varrho_j)\|\|\widetilde{m}_j\| \\
\quad +\widehat{P}\int_{\hbar_1}^{\hbar_2}\varsigma_1(s)\omega(\ell+\sigma F_0(1+\ell)+\sigma F_1(1+\ell))ds \\
\quad +\int_0^{\hbar_1}\|\mathbb{G}_r(\hbar_2-s)-\mathbb{G}_r(\hbar_1-s)\|\varsigma_1(s) \\
\quad \times\omega(\ell+\sigma F_0(1+\ell)+\sigma F_1(1+\ell))ds + \widehat{P}P_\sigma\|x\|_{L^\mu(V,\mathbb{Y})}(\hbar_2-\hbar_1) \\
\quad +P_\sigma\int_0^{\hbar_1}\|\mathbb{G}_r(\hbar_2-s)-\mathbb{G}_r(\hbar_1-s)\|\|x(s)\|ds,\ \varrho\in(\varrho_j,\varrho_{j+1}].
\end{cases} \tag{3.6}
$$

The aforementioned inequality's RHS of the system (3.6) tends to zero independently of $z \in \mathcal{G}_\ell$ as $\hbar_2 \to \hbar_1$ by using the continuity of functions $\varrho \to \|\mathbb{N}_r(\varrho)\|$, $\varrho \to \|\mathbb{M}_r(\varrho)\|$, and $\varrho \to \|\mathbb{G}_r(\varrho)\|$. Therefore, $\Pi(\mathcal{G}_\ell)$ is equicontinuous.

Step 4: Next, we prove that Mönch's condition holds.

Consider $\mathcal{U} \subseteq \mathcal{G}_\ell$ is countable and $\mathcal{U} \subseteq conv(\{0\} \cup \Pi(\mathcal{U}))$, we show that $\lambda(\mathcal{U}) = 0$, where λ is the Hausdorff measure of noncompactness. Let $\mathcal{U} = \{z^v\}_{v=1}^{\infty}$. We check that $\Pi(\mathcal{U})(\sigma)$ is relatively compact in $PC(V, Q)$, for every $\varrho \in (\varrho_j, \varrho_{j+1}]$. From Theorem 2.11, and

$$
\begin{aligned}
&\delta(\{x_{z^{(v)}}(s)\}_{v=1}^{\infty})\\
&= \delta\Big\{\mathbb{W}^{-1}\Big(z_w - \mathbb{N}_r(\varrho)z_0 - \mathbb{M}_r(\varrho)z_1 - \sum_{0<\varrho_j<\hbar_2} \mathbb{N}_r(\varrho-\varrho_j)m_j - \sum_{0<\varrho_j<\hbar_2} \mathbb{M}_r(\varrho-\varrho_j)\widetilde{m}_j\\
&\quad - \int_0^{\varrho} \mathbb{G}_r(\varrho - s)\mathfrak{f}(s, z^{(v)}(s), (Fz^{(v)})(s), (Hz^{(v)})(s))ds\Big)\Big\}_{v=1}^{\infty}\\
&\le 2\varsigma_5(s)\widehat{P}\Big(\int_0^{\varrho} \varsigma_2(s)[\delta(\mathcal{U}(s)) + \delta(\{Fz^{(v)}(s)\}_{v=1}^{\infty}) + \delta(\{Hz^{(v)}(s)\}_{v=1}^{\infty})]ds\Big)\\
&\le 2\varsigma_5(s)\widehat{P}\Big(\int_0^{\varrho} \varsigma_2(s)\delta(\mathcal{U}(s))ds + 2\int_0^{\varrho} \varsigma_2(s)(\varsigma_3^* + \varsigma_4^*)\delta(\mathcal{U}(s))ds\Big).
\end{aligned}
$$

From Lemma 2.11, and assumptions $(\mathbf{H_1})$–$(\mathbf{H_5})$, we get

$$
\begin{aligned}
\delta(\{\Pi z^{(v)}(s)\}_{v=1}^{\infty}) &= \delta\Big(\Big\{\mathbb{N}_r(\varrho)z_0 + \mathbb{M}_r(\varrho)z_1 + \sum_{q=1}^{j} \mathbb{N}_r(\varrho-\varrho_j)m_j + \sum_{q=1}^{j} \mathbb{M}_r(\varrho-\varrho_j)\widetilde{m}_j\\
&\quad + \int_0^{\varrho} \mathbb{G}_r(\varrho - s)\mathfrak{f}(s, z(s), (E_1z)(s), (E_2z)(s))ds\\
&\quad + \int_0^{\varrho} \mathbb{G}_r(\varrho - s)\mathcal{B}x_z(s)ds\Big\}_{v=1}^{\infty}\Big)\\
&\le \delta\Big(\Big\{\int_0^{\varrho} \mathbb{G}_r(\varrho - s)\mathfrak{f}(s, z(s), (E_1z)(s), (E_2z)(s))ds\Big\}_{v=1}^{\infty}\Big)\\
&\quad + \delta\Big(\Big\{\int_0^{\varrho} \mathbb{G}_r(\varrho - s)\mathcal{B}x_z(s)ds\Big\}_{v=1}^{\infty}\Big)\\
&\le 2\widehat{P}\Big(\int_0^{\varrho} \varsigma_2(s)[\delta(\mathcal{U}(s)) + \delta(\{Fz^{(v)}(s)\}_{v=1}^{\infty}) + \delta(\{Hz^{(v)}(s)\}_{v=1}^{\infty})]ds\Big)\\
&\quad + 2\widehat{P}P_\sigma\Big(\int_0^{\varrho} \delta(\{x_{z^{(v)}}(s)\}_{v=1}^{\infty})ds\Big)\\
&\le 2\widehat{P}\Big(\int_0^{\varrho} \varsigma_2(s)\delta(\mathcal{U}(s))ds + \int_0^{\varrho} \varsigma_2(s)(\varsigma_3^* + \varsigma_4^*)\delta(\mathcal{U}(s))ds\Big) + 4\widehat{P}^2P_\sigma\\
&\quad \times\Big(\int_0^{\varrho} \varsigma_5(s)ds\Big)\Big(\int_0^{\varrho} \varsigma_2(s)\delta(\mathcal{U}(s))ds + \int_0^{\varrho} \varsigma_2(s)(\varsigma_3^* + \varsigma_4^*)\delta(\mathcal{U}(s))ds\Big)\\
&\le 2\widehat{P}\|\varsigma_2\|[1 + 2\widehat{P}P_\sigma\|\varsigma_5\|](1 + (\varsigma_3^* + \varsigma_4^*))\delta(\mathcal{U}(\varrho)).
\end{aligned}
$$

By Lemma 2.10, we get

$$
\delta(\{\Pi z^{(v)}(s)\}_{v=1}^{\infty}) \le M^*\delta(\mathcal{U}(\varrho)).
$$

Therefore by using Mönch's condition, one can obtain

$$\delta(\Pi) \leq \delta\Big(\mathrm{conv}(\{0\} \cup (\Pi(\mathcal{U}))\Big) = \delta(\Pi(\mathcal{U})) \leq M^* \delta(\mathcal{U}),$$

this implies $\delta(\mathcal{U}) = 0$. Hence, Π has a fixed point in $\mathcal{G}_\ell$. Thus, the fractional integrodifferential equations (1.1) has a fixed point fulfilling $z(\sigma) = z_\sigma$. Thus, the fractional integrodifferential equations (1.1) is exact controllable on $[0, \sigma]$. □

4. Application

Suppose the impulsive fractional mixed Volterra-Fredholm type integrodifferential systems of the form:

$$\begin{cases} \frac{\partial^r}{\partial \varrho^r} z(\varrho, \omega) = \frac{\partial^2}{\partial \varrho^2} z(\varrho, \omega) + \cos\Big[z(\varrho, \omega) + \int_0^\varrho (\varrho - \iota)^2 \sin z(\iota, \omega) d\iota\Big] + \frac{\partial}{\partial \omega} z(\varrho, \omega) \\ \quad + \int_0^\sigma \cos z(\varrho, \omega) d\iota + \xi \alpha(\varrho, \omega), \varrho \in V = [0, 1],\ \omega \in [0, \pi],\ \varrho \neq \varrho_j,\ j = 1, 2, \cdots, n, \\ z(\varrho, 0) = z(\varrho, 1) = 0,\ \varrho \in V, \\ z(\varrho_j^+, \omega) - z(\varrho_j^-, \omega) = m_j,\ z'(\varrho_j^+, \omega) - z'(\varrho_j^-, \omega) = \widetilde{m}_j,\ j = 1, 2, \cdots, n, \\ z(0, \omega) = z_0(\omega),\ z'(0, \omega) = z_1(\omega), \end{cases} \tag{4.1}$$

where $\frac{\partial^{\frac{3}{2}}}{\partial \varrho^{\frac{3}{2}}}$ denotes fractional partial derivative of $r = \frac{3}{2}$. $0 = \varrho_0 < \varrho_1 < \varrho_2 < \cdots < \varrho_j < \cdots < \varrho_n = \sigma$; $z(\varrho_j^+) = \lim_{(\epsilon^+, \omega) \to (0^+, \omega)} z(\varrho_j + \epsilon, \omega)$ and $z(\varrho_j^-) = \lim_{(\epsilon^-, \omega) \to (0^-, \omega)} z(\varrho_j + \epsilon, \omega)$.

Consider $Q = \mathbb{Y} = L^2([0, \pi])$, and let A maps from $D(A) \subset Q$ into Q be presented as $Az = z''$ along with domain $D(A)$, which is

$$D(A) = \{z \text{ in } Q : z,\ z' \text{ are absolutely continuous},\ z'' \text{ in } Q, z(0) = z(\pi) = 0\}.$$

Further, A stands for infinitesimal generator of an analytic semigroup $\{\mathbb{G}(\varrho),\ \varrho \geq 0\}$ determined by $\mathbb{G}(\varrho) z(s) = z(\varrho + s)$, for every z in Q. $\mathbb{G}(\varrho)$ is not compact semigroup on Q and $\delta(\mathbb{G}(\varrho)\mathcal{U}) \leq \delta(\mathcal{U})$, then δ stands for the Hausdorff MNC.

In addition, A has discrete spectrum with eigenvalues $-\mu^2$, $\mu \in \mathbb{N}$, and according normalized eigen functions given by $y_\mu(z) = \sqrt{(2/\pi)} \sin(\mu \pi z)$. Then, y_μ stands for an orthonormal basic of Q. For more details refer to [35].

$$\mathbb{G}(\varrho) = \sum_{\mu=1}^{\infty} e^{-\mu^2 \varrho} \langle z, y_\mu \rangle y_\mu,\ z \in Q,$$

$\mathbb{G}(\varrho)$ is compact for any $\varrho > 0$ and $\mathbb{G}(\varrho) \leq e^{-\varrho}$ for any $\varrho \geq 0$ [44].

$A = \frac{\partial^2}{\partial \varrho^2}$ represents sectorial operator of type (P, τ, r, ϕ) and generates r-resolvent families $\mathbb{N}_r(\varrho)$, $\mathbb{M}_r(\varrho)$, and $\mathbb{G}_r(\varrho)$ for $\varrho \geq 0$. Since $A = \frac{\partial^2}{\partial \varrho^2}$ is an m-accretive operator on Q with dense domain ($\mathbf{H_1}$) fulfilled.

$$Az = \sum_{\mu=1}^{\infty} \mu^2 \langle z, y_\mu \rangle y_\mu,\ z \in D(A).$$

Determine

$$\mathfrak{f}(\varrho, z(\varrho), (E_1z)(\varrho), (E_2z)(\varrho)(\omega)) = \cos\left[z(\varrho,\omega) + \int_0^{\varrho} (\varrho-\iota)^2 \sin z(\iota,\omega)d\iota\right] + \frac{\partial}{\partial\omega}z(\varrho,\omega) + \int_0^{\sigma} \cos z(\varrho,\omega)d\iota,$$

$$(E_1z)(\varrho) = \int_0^{\varrho} \sin z(\iota,\omega)d\iota,$$
$$(E_2z)(\varrho) = \int_0^{1} \cos z(\varrho,\omega)d\iota.$$

Assume that $\mathcal{B} : Q \to Q$ is determined by

$$(\mathcal{B}x)(\varrho)(\omega) = \xi\alpha(\varrho,\omega),\ \omega \in [0,\pi],$$

For $\omega \in [0,\pi]$, the linear operator $\mathbb{W}$ specified by

$$(\mathbb{W}x)(\omega) = \int_0^1 \mathbb{G}(1-s)\xi\alpha(s,\omega)ds,$$

fulfilling $(\mathbf{H_2})$–$(\mathbf{H_5})$. Thus, the system (4.1) can be rewritten as

$$\begin{cases} {}^{C}D^{r}_{\varrho}z(\varrho) = Az(\varrho) + \mathfrak{f}(\varrho, z(\varrho), (E_1z)(\varrho), (E_2z)(\varrho)) + \mathcal{B}x(\varrho),\ \varrho \in V,\ \ \varrho \neq \varrho_j, \\ \Delta z(\varrho_j) = m_j,\ \Delta z'(\varrho_j) = \widetilde{m}_j,\ j = 1, 2, \cdots, n, \\ z(0) = z_0,\ z'(0) = z_1. \end{cases} \tag{4.2}$$

As a result, Theorem 3.1's requirements are all fulfilled. The system (4.1) is therefore exact controllable on V according to Theorem 3.1.

5. Conclusions

In this study, we mainly concentrated on the exact controllability outcomes for fractional integrodifferential equations of mixed type via sectorial operators of type (P, τ, r, ϕ), employing fractional calculations, impulsive systems, sectorial operators, and fixed point technique. Lastly, an example for clarifying the theory of the important findings is constructed. The effectiveness of such research discoveries can be effectively increased to exact controllability using varied fractional differential structures (Hilfer system, A-B system, stochastic, etc.). Moreover, null controllability outcomes of impulsive fractional stochastic differential systems via sectorial operators will be the subject of future research.

Acknowledgments

This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1444). Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R157), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare no conflict of interest.

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