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Research article

Intuitionistic fuzzy k-ideals of right k-weakly regular hemirings

Bander Almutairi[1], **Rukhshanda Anjum**[2,*], **Qin Xin**[3] **and Muhammad Hassan**[2]

[1] Department of Mathematics, College of Sciences, King Saud University, P. O. Box 2455 Riyadh 11451, Saudi Arabia

[2] Department of Mathematics and Statistics, University of Lahore, Lahore, Pakistan

[3] Faculty of Science and Technology, University of the Faroe Islands, Faroe Islands, Denmark

* **Correspondence:** Email: rukhshanda.anjum@math.uol.edu.pk.

Abstract: In this work, we will examine the concept of intuitionistic fuzzy k-ideals in the context of right k-weakly regular hemirings. We will investigate the properties of these ideals and how they relate to other concepts such as fuzzy prime k-ideals, intuitionistic fuzzy prime k-ideals, intuitionistic fuzzy right pure k-ideals, and purely prime intuitionistic fuzzy k-ideals in hemirings. We will also explore how the regularity of a k-weakly regular hemiring can be characterized through its intuitionistic fuzzy k-ideals.

Keywords: fuzzy ideal; hemiring; right k-weakly regular hemirings; intuitionistic fuzzy k-ideal; intuitionistic fuzzy prime k-ideal; intuitionistic fuzzy right pure k-ideal; purely prime intuitionistic fuzzy k-ideal
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1. Introduction

In 1934, H. S. Vandiver presented the concept of semiring [1]. In abstract algebra, a semiring is an algebraic structure similar to a ring, but without the requirement that each element must have an additive inverse. Various branches of applied mathematics, formal languages, theory of automata and optimization use distributive lattice in some applications. Hemiring is a special semiring with commutative addition and zero. In applications, hemirings are helpful in formal languages and automata theory [2].

Atanassov gave the idea of intuitionistic fuzzy sets by generalizing the concept of fuzzy sets. In fuzzy sets, the degree of membership of an element in a given set is considered, while in intuitionistic fuzzy sets, both the degree of membership and the degree of non-membership of an element in a given set are taken into account. Decision Making (DM) is regarded as the cognitive process used to solve problems that we face in our daily life. Due to the complexity of the current socio-economic environment, DM is one of the most prominent endeavours, whose aim is to get an optimal or at least satisfactory solution by identifying and choosing alternatives.

Decision making becomes more complicated when uncertainty is involved. Although to handle uncertainty, Zadeh's fuzzy set is a good tool, but as time passes it is observed that it is insufficient. To overcome the problems thus occurred, intuitionistic fuzzy sets (IFS) have proved their effectiveness. In order to handle the information provided without loss and to end up with the optimal alternative, that is, the desirable result in DM process, aggregation operators such as fuzzy weighted arithmetic average operators based on the concepts of t-norm, Einstein operations, and copula are used. In recent years there have appeared many generalizations of IFSs, like Pythagorean fuzzy sets, q-rung orthopair fuzzy sets, picture fuzzy sets, spherical fuzzy sets, T-spherical fuzzy sets, neutrosophic sets, and dual hesitant fuzzy sets. Recent developments in DM can be found in social sciences, computer sciences, biosciences, information sciences, and related fields based on intuitionistic fuzzy sets and their generalizations (IFSGs). Many applications of intuitionistic fuzzy sets are carried out using distance measures approach. Distance measure between intuitionistic fuzzy sets is an important concept in fuzzy mathematics because of its wide applications in real world, such as pattern recognition, machine learning, decision making and market prediction [3–6].

The concept of semiring was studied in the context of intuitionistic fuzzy sets by many researchers. In [7], authors explained the methods to characterize the intuitionistic fuzzy left k-ideals and also described some techniques to construct the intuitionistic fuzzy sets. They also discussed the division of intuitionistic fuzzy left k-ideals in their natural classification. Pyung Ki Lim and others explained the concept of intuitionistic fuzzy k-ideals in the context of semirings and gave the concept about the intuitionistic fuzzy prime k-ideals of semiring and also described its properties [8]. In [9–11], R. Anjum, W. A. Dudek and J. Zhan et al. discussed their properties and connections with chains of left h-ideals of the corresponding hemirings.

In [11], J. Zhan et al. presented the notion of h-hemiring to generalize the ring in its regular form. They also explained hemirings with respect to prime fuzzy h-ideals and also made characterization of prime fuzzy h-ideals of h-hemiregular hemirings by using fuzzy h-ideals. They described the characteristics of hemirings with respect to the normal and maximal fuzzy left h-ideals. In [12], M. Shabir et al. worked on the characterization of hemirings that each k-ideal and each fuzzy k-ideal is idempotent. They provided some basic idea of fuzzy sets and hemiring. They also explained the hemirings in terms of k-sum and k-product of fuzzy sets in a hemiring.

In [13], S. Ghosh explained the concept of fuzzy and prime fuzzy k-ideals and showed that semirings are semiregular. They used fuzzy k-ideals for the characterization of prime fuzzy k-ideals, noetherian semirings, semiregular semiring and artinian semirings. Semirings have received a lot of attention by the researchers [14–16]. Ideals of hemi-rings are an important part of the theory of algebraic structures, and many properties of hemi-rings are related to their ideals. In contrast to rings, the ideals in hemi-rings do not generally correspond to ideals in rings. L. A. Zadeh [17] introduced in 1965 fuzzy set theory as an alternative to probability theory. His fundamental idea consists in understanding lattice valued maps as generalized characteristic functions of some new kind of subsets of so-called fuzzy sets of a given universe. Fuzzy sets are very useful in a range of fields, applied mathematics, control engineering, information sciences, expert systems, and automata theory. While there have been many variations of fuzzy sets, none of them address the issue of contradictory characteristics among members with a membership degree of 0. Henriksen [18], defined k-ideals in the context of hemi-rings. Iizuka [19], introduced a type of hemi-ring ideal called h-ideals, which are a more restricted version of k-ideals. According to the new concept of ideals, La Torre [20], thoroughly examined h-ideals and k-ideals in hemi-rings using the new concept of ideals. In 2014, M. Zhou and others [21], explored the use of bipolar fuzzy theory in the context of hemi-rings. The authors introduced basic definitions, theorems, and examples related to bipolar-valued fuzzy h-ideals, and used these ideals to characterize the properties of h-hemi-regular and h-hemi-simple hemi-rings. They also used bipolar-valued h-bi-ideals and bipolar-valued h-quasi-ideals in their analysis. There are many other applications and references available on this topic [22–26].

In this article, after abstract and brief introduction we will move to the section of preliminaries where some definitions will be discussed. The main research work will be presented in the next four sections related to the primeness and pureness of intuitionistic fuzzy k-ideals. Also purely prime intuitionistic fuzzy k-ideal will be discussed. Then we will conclude our research work with the future directions.

2. Preliminaries

In this section, we will review some important definitions that will be used later in the paper.

Definition 2.1. [7] Let $(R, +, \times)$ is a semiring if the following conditions are met:

1) R is a semigroup under $+$
2) R is a semigroup under $\times$
3) $\times$ is distributive with respect to $+$, i.e., $\forall\ j, k, l \in R$

$$j \times (k + l) = j \times k + j \times l \qquad \text{(Left distributive law)}$$
$$(j + k) \times l = j \times l + k \times l \qquad \text{(Right distributive law)}$$

Definition 2.2. [6] A non-empty set $\check{R}$ with two binary operations $+$ and $\times$ defined on $\check{R}$ is called a hemiring if:

1) $(\check{R}, +, \times)$ is a commutative semiring
2) 0 element exist in $\check{R}$ such that $p + 0 = 0 + p = p$ and $p \times 0 = 0 \times p = 0\ \forall\ p \in\ \check{R}$.

A hemiring contains an identity element then it is called a hemiring with identity such that $1 \in \check{R}$, and $1 \times p = p \times 1 = p\ \forall\ p \in \check{R}$.

A hemiring is called a commutative hemiring if $\times$ is commutative in $\check{R}$. A non-empty subset A of a hemiring $\check{R}$ is called a subhemiring of $\check{R}$ if A itself is a hemiring with respect to the induced operations of $\check{R}$.

Definition 2.3. [3] A non-empty subset M of a hemiring $\check{R}$ is called a left (right) *ideal* of $\check{R}$ if

1) $p + q \in M\ \forall\ p, q \in M$ and
2) $rp \in M\ (pr \in M)\ \forall\ a \in M\ , r \in \check{R}$. Obviously $0 \in M$ for any left (right) *ideal* M of $\check{R}$.

Definition 2.4. [3] A left (right) *ideal* M of a hemiring $\check{R}$ is called a left (right) $k-ideal$ of $\check{R}$ if for any $p, q \in M$ and $s \in \check{R}$ from $s+p=q$ it follows $s \in M$.

Definition 2.5. [3] An element b of a hemiring $\check{R}$ is called regular if there exist $u \in \check{R}$ such that $b = bub$. A hemiring $\check{R}$ is called regular if each element of $\check{R}$ is regular.

Definition 2.6. [3] A hemiring $\check{R}$ is said to be $k-regular$ hemiring if for each $b \in \check{R}$ $\exists$ $u, v \in \check{R}$ such that $b + bub = bvb$.

If H is not empty set and $\phi^{''}$ and $\psi^{'}$ are fuzzy subsets of H then

$$\phi^{''} \leqq \psi^{'} \Leftrightarrow \phi^{''}(k^{\urcorner}) \leqq \psi^{'}(k^{\urcorner})$$
$$(\phi^{''} \wedge \psi^{'})(k^{\urcorner}) = \phi^{''}(k^{\urcorner}) \wedge \psi^{'}(k^{\urcorner}) = \min\left\{\phi^{''}(k^{\urcorner}), \psi^{'}(k^{\urcorner})\right\}$$
$$(\phi^{''} \vee \psi^{'})(k^{\urcorner}) = \phi^{''}(k^{\urcorner}) \vee \psi^{'}(k^{\urcorner}) = \max\left\{\phi^{''}(k^{\urcorner}), \psi^{'}(k^{\urcorner})\right\}$$

for any $k^{\urcorner} \in H$.

More often, if collection $\left\{\phi^{''} : i^{''} \in j\right\} \subset H$ then union and intersection of fuzzy sets are defined as

$$(\bigwedge_{i^{''} \in j} \phi^{''}_{i^{''}})(k^{\urcorner}) = \bigwedge_{i^{''} \in j} \phi^{''}_{i^{''}}(k^{\urcorner}) = \inf_{i^{''} \in j}\left\{\phi^{''}_{i^{''}}(k^{\urcorner})\right\}$$
$$(\bigvee_{i^{''} \in j} \phi^{''}_{i^{''}})(k^{\urcorner}) = \bigvee_{i^{''} \in j} \phi^{''}_{i^{''}}(k^{\urcorner}) = \sup_{i^{''} \in j}\left\{\phi^{''}_{i^{''}}(k^{\urcorner})\right\}$$

respectively.

Definition 2.7. [3] In a hemiring $\check{R}$, $\phi^{''}$is called a fuzzy right (left) ideal of $\check{R}$ if $\forall$ $c^{'}, d^{'} \in \check{R}$ we have

1) $\phi^{''}(c^{'} + d^{'}) \geqq \phi^{''}(c^{'}) \wedge \phi^{''}(d^{'})$
2) $\phi^{''}(c^{'}d^{'}) \geqq \phi^{''}(c^{'}), (\phi^{''}(c^{'}d^{'}) \geqq \phi^{''}(d^{'}))$.

Note that $\phi^{''}(0) \geqq \phi^{''}(x)$ $\forall x \in \check{R}$.

Definition 2.8. [3] A fuzzy right (left) ideal $\phi^{'}$ of a hemiring $\check{R}$ is called a fuzzy right (left) $k-$ideal if

$$u + v = w \Longrightarrow \phi^{'}(u) \geqq \phi^{'}(v) \wedge \phi^{'}(w) \ holds \ \forall \ u, v, w \in \check{R}.$$

Definition 2.9. [3] Suppose fuzzy subset $\phi^{'}$of a universe X and $s \in [0, 1]$. Then

$$V(\phi^{'}; s) = \left\{x^{'} \in X : \phi(x^{'}) \geqq s\right\}$$

is said to be level subset of $\phi^{'}$.

As an impotant generalization of the notion of fuzzy sets, Atanassov introduced in [26] the concept of an intuitionistic fuzzy set (IFS) defined as:

$$J = (\mu_J, \rho_J) = \{(x, \mu_J(x), \rho_J(x))|x \in R\}$$

where the fuzzy sets μ_J and ρ_J denote the degree of membership and the degree of non-membership of each element $x \in R$ to the set J respectively, $0 \leq \mu_J(x) + \rho_J(x) \leq 1 \forall x \in R$.

Definition 2.10. Assume that $J = (\mu_J, \rho_J)$ is a non-empty intuitionistic fuzzy set in a hemiring $\check{R}$. Then J is said to be an intuitionistic fuzzy right $k-$ideal if,

1) $\mu_J(x'' + y'') \geqq \min(\mu_J(x''), \mu_J(y''))$

2) $\rho_J(x'' + y'') \leqq \max(\rho_J(x''), \rho_J(y'')$

3) $\mu_J(x''y'') \geqq \mu_J(x'')$

4) $\rho_J(x''y'') \leqq \rho_J(x'')$

5) $x'' + y'' = z'' \Rightarrow \mu_J(x'') \geqq \min(\mu_J(y''), \mu_J(z''))$

6) $x'' + y'' = z'' \Rightarrow \rho_J(x'') \leqq \max(\rho_J(y''), \rho_J(z''))$

$\forall x'', y'', z'' \in \check{R}$.

An IFS $J = (\mu_J, \rho_J)$ satisfies the first four axioms then we said $J = (\mu_J, \rho_J)$ is an intuitionistic fuzzy right ideal.

In hemiring $\check{R}$, $IFI(\check{R})$ denote the famlly of all intuitionistic fuzzy right $k-$ideals.

Now $\mu_J(x'') \leqq \mu_J(0)$ and $\rho_J(x'') \geqq \rho_J(0)$ for each $J \in IFI(\check{R})$ and $x'' \in \check{R}$.

Assume that H is not an empty set. A complex mapping $E = (\mu_E, \rho_E) : H \to I \times I$ is said to be an intuitionistic fuzzy set (IFS) in H if $\mu_E(x'') + \rho_E \leqq 1$ for each $x' \in X$, where the mapping $\mu_E(x'') : H \to I$ (degree of membership (μ_E)) and $\rho_E : H \to I$ (degree of non-membership (ρ_E)) for each $x'' \in H$ to E. In particular, $\underset{\sim}{0}$ and $\underset{\sim}{1}$ are empty set and whole set of the intuitionistic fuzzy in H, respectively, defined by $\underset{\sim}{0}(x'') = (0, 1)$ and $\underset{\sim}{1}(x'') = (1, 0)$ for any $x'' \in H$. All IFS in X will be denoted by $IFS(X)$ (IFS stands for intuitionistic fuzzy set).

Lemma 2.11. *[11]*

Let X be a non-empty set and let $U' = (\mu_{U'}, \rho_{U'})$ and $V' = (\mu_{V'}, \rho_{V'})$ be $IFS\,s$ on X. Then

1) $U' \subset V' \Longleftrightarrow \mu_{U'} \leqq \mu_{U'}$ and $\rho_{V'} \geqq \rho_{V'}$.

2) $U' = V' \Longleftrightarrow U' \subset V'$ and $V' \subset U'$.

3) $V'^{c} = (\rho_{V'}, \mu_{V'})$.

4) $U' \cap V' = (\mu_{U'} \wedge \mu_{V'}, \rho_{U'} \vee \rho_{V'})$.

5) $U' \cup V' = (\mu_{U'} \vee \rho_{V'}, \rho_{U'} \wedge \rho_{V'})$.

Theorem 2.12. *[10] A hemiring $\underset{\leftrightarrow}{H}$ is k-regular hemiring if and only if for any fuzzy right $k-$ideal L and any fuzzy left $k-$ideal M, we have $\overline{LM} = L \cap M$.*

Definition 2.13. Suppose that $L = (\mu_L, \rho_L)$ and $M = (\mu_M, \rho_M)$ be the intuitionistic fuzzy sets on $\check{R}$ then $L \odot_k M$ is called $k-$ *product* of intuitionistic fuzzy sets if

$$(\mu_{L'\backslash} \odot_k \mu_{M'\backslash})(x') = \bigvee_{x' + \sum_{i'=1}^{p'} a_{i'} b_{i'} = \sum_{j'=1}^{q'} c_{j'} d_{j'}} \left\{ \bigwedge_{i'=1}^{p'} (\mu_L(a_{i'}) \wedge \mu_M(b_{i'})) \wedge \bigwedge_{j'=1}^{q'} \left(\mu_L(c_{j'}) \wedge \mu_M(d_{j'})\right) \right\},$$

$$(\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(x') = \bigwedge_{x' + \sum_{i'=1}^{p'} a_{i'} b_{i'} = \sum_{j'=1}^{q'} c_{j'} d_{j'}} \left\{ \bigvee_{i'=1}^{p'} (\rho_L(a_{i'}) \vee \rho_M(b_{i'})) \vee \bigvee_{j'=1}^{q'} \left(\rho_L(c_{j'}) \vee \rho_M(d_{j'})\right) \right\}$$

Theorem 2.14. *If $L'^{\backslash} = (\mu_{L'\backslash}, \rho_{L'\backslash})$ and $M'^{\backslash} = (\mu_{M'\backslash}, \rho_{M'\backslash})$ are intuitionistic fuzzy $k-$ideals of $\underset{\leftrightarrow}{H}$ then $\mu_{L'\backslash} \odot_k \mu_{M'\backslash}, \rho_{L'\backslash} \odot_k \rho_{M'\backslash}$ are intuitionistic fuzzy $k-$ideal of $\underset{\leftrightarrow}{H}$ and $\mu_{L'\backslash} \odot_k \mu \leqq \mu_{L'\backslash} \wedge \mu_{M'\backslash}$, and $\rho_{L'\backslash} \odot_k \rho_{M'\backslash} \geqq \rho_{L'\backslash} \vee \rho_{M'\backslash}$.*

Proof. Suppose $L'^{\backslash} = (\mu_{L'\backslash}, \rho_{L'\backslash})$, $M'^{\backslash} = (\mu_{M'\backslash}, \rho_{M'\backslash})$ be the intuitionistic fuzzy $k-$ideals of $\underset{\leftrightarrow}{H}$. Suppose $u, v \in \underset{\leftrightarrow}{H}$ then we have

$$(\mu_{L'\backslash} \odot_k \mu_{M'\backslash})(u) = \bigvee_{u + \sum_{\sigma=1}^{\Omega} c_\sigma d_\sigma = \sum_{\varrho=1}^{\Delta} c'_\varrho d'_\varrho} \left\{ \bigwedge_{\sigma=1}^{\Omega} (\mu_{L'\backslash}(c_\sigma) \wedge \mu_{M'\backslash}(d_\sigma)) \wedge \bigwedge_{\varrho=1}^{\Delta} (\mu_{L'\backslash}(c'_\varrho) \wedge \mu_{M'\backslash}(d'_\varrho)) \right\},$$

$$(\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(u) = \bigwedge_{u + \sum_{\sigma=1}^{\Omega} c_\sigma d_\sigma = \sum_{\varrho=1}^{\Delta} c'_\varrho d'_\varrho} \left\{ \bigvee_{\sigma=1}^{\Omega} (\rho_{L'\backslash}(c_\sigma) \vee \rho_{M'\backslash}(d_\sigma)) \vee \bigvee_{\varrho=1}^{\Delta} (\rho_{L'\backslash}(c'_\varrho) \vee \rho_{M'\backslash}(d'_\varrho)) \right\}$$

$$(\mu_{L'\backslash} \odot_k \mu_{M'\backslash})(v) = \bigvee_{v + \sum_{\varepsilon=1}^{p''} a_\varepsilon b_\varepsilon = \sum_{l''=1}^{q''} a'_{l''} b'_{l''}} \left\{ \bigwedge_{\varepsilon=1}^{p''} (\mu_{L'\backslash}(a_\varepsilon) \wedge \mu_{M'\backslash}(b_\varepsilon)) \wedge \bigwedge_{l''=1}^{q''} (\mu_{L'\backslash}(a'_{l''}) \wedge \mu_{M'\backslash}(b'_{l''})) \right\},$$

$$(\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(v) = \bigwedge_{v + \sum_{\varepsilon=1}^{p''} a_\varepsilon b_\varepsilon = \sum_{\tau=1}^{q''} a'_\tau b'_\tau} \left\{ \bigvee_{\varepsilon=1}^{p''} (\rho_{L'\backslash}(a_\varepsilon) \vee \rho_{M'\backslash}(b_\varepsilon)) \vee \bigvee_{\tau=1}^{q''} (\rho_{L'\backslash}(a'_\tau) \vee \rho_{M'\backslash}(b'_\tau)) \right\}$$

Now

$$(\mu_{L'\backslash} \odot_k \mu_{M'\backslash})(u) \wedge (\mu_{L'\backslash} \odot_k \mu_{M'\backslash})(v) = \bigvee_{u + \sum_{\sigma=1}^{\Omega} c_\sigma d_\sigma = \sum_{\varrho=1}^{\Delta} c'_\varrho d'_\varrho} \left\{ \begin{array}{c} \bigwedge_{\sigma=1}^{\Omega} (\mu_{L'\backslash}(c_\sigma) \wedge \mu_{M'\backslash}(d_\sigma)) \wedge \\ \bigwedge_{\varrho=1}^{\Delta} (\mu_{L'\backslash}(c'_\varrho) \wedge \mu_{M'\backslash}(d'_\varrho)) \end{array} \right\} \wedge$$

$$\bigvee_{v + \sum_{\varepsilon=1}^{p''} a_\varepsilon b_\varepsilon = \sum_{\tau=1}^{q''} a'_\tau b'_\tau} \left\{ \begin{array}{c} \bigwedge_{\varepsilon=1}^{p''} (\mu_{L'\backslash}(a_\varepsilon) \wedge \mu_{M'\backslash}(b_\varepsilon)) \\ \wedge \bigwedge_{\tau=1}^{q''} (\mu_{L'\backslash}(a'_\tau) \wedge \mu_{M'\backslash}(b'_\tau)) \end{array} \right\}$$

$$(\rho_{L'\backslash}\odot_k \rho_{M'\backslash})(u)\vee(\rho_{L'\backslash}\odot_k \rho_{M'\backslash})(v)=\bigwedge_{u+\sum_{\sigma=1}^{\Omega}c_\sigma d_\sigma=\sum_{\varrho=1}^{\Delta}c'_\varrho d'_\varrho}\left\{\begin{array}{c}\bigvee_{\sigma=1}^{\Omega}(\rho_{L'\backslash}(c_\sigma)\vee\rho_{M'\backslash}(d_\sigma))\vee\\ \bigvee_{\varrho=1}^{\Delta}(\rho_{L'\backslash}(c'_\varrho)\vee\rho_{M'\backslash}(d'_\varrho))\end{array}\right\}\vee$$
$$\bigwedge_{v+\sum_{\varepsilon=1}^{p''}a_\varepsilon b_\varepsilon=\sum_{\tau=1}^{q''}a'_\tau b'_\tau}\left\{\begin{array}{c}\bigvee_{\varepsilon=1}^{p''}(\rho_{L'\backslash}(a_\varepsilon)\vee\rho_{M'\backslash}(b_\varepsilon))\\ \vee\bigvee_{\tau=1}^{q''}(\rho_{L'\backslash}(a'_\tau)\vee\rho_{M'\backslash}(b'_\tau))\end{array}\right\}$$

$$(\mu_{L'\backslash}\odot_k \mu_{M'\backslash})(u)\wedge(\mu_{L'\backslash}\odot_k \mu_{M'\backslash})(v)=\bigvee_{u+\sum_{\sigma=1}^{\Omega}c_\sigma d_\sigma=\sum_{\varrho=1}^{\Delta}c'_\varrho d'_\varrho}[\bigvee_{v+\sum_{\varepsilon=1}^{p''}a_\varepsilon b_\varepsilon=\sum_{\tau=1}^{q''}a'_\tau b'_\tau}\{\bigwedge_{\sigma=1}^{\Omega}(\mu_{L'\backslash}(c_\sigma)\wedge\mu_{M'\backslash}(d_\sigma))\wedge$$
$$\bigwedge_{\varrho=1}^{\Delta}(\mu_{L'\backslash}(c'_\varrho)\wedge\mu_{M'\backslash}(d'_\varrho))\wedge\bigwedge_{\varepsilon=1}^{p''}(\mu_{L'\backslash}(a_\varepsilon)\wedge\mu_{M'\backslash}(b_\varepsilon))\wedge$$
$$\bigwedge_{\tau=1}^{q''}(\mu(a'_\tau)\wedge\mu_{M'\backslash}(b'_\tau))\}]$$

and

$$(\rho_{L'\backslash}\odot_k \rho_{M'\backslash})(u)\vee(\rho_{L'\backslash}\odot_k \rho_{M'\backslash})(v)=\bigwedge_{u+\sum_{\sigma=1}^{\Omega}c_\sigma d_\sigma=\sum_{\varrho=1}^{\Delta}c'_\varrho d'_\varrho}[\bigwedge_{v+\sum_{\varepsilon=1}^{p''}a_\varepsilon b_\varepsilon=\sum_{\tau=1}^{q''}a'_\tau b'_\tau}\{\bigvee_{\sigma=1}^{\Omega}(\rho_{L'\backslash}(c_\sigma)\vee\rho_{M'\backslash}(d_\sigma))\vee$$
$$\bigvee_{\varrho=1}^{\Delta}(\rho_{L'\backslash}(c'_\varrho)\vee\rho_{M'\backslash}(d'_\varrho))\vee\bigvee_{\varepsilon=1}^{p''}(\rho_{L'\backslash}(a_\varepsilon)\vee\rho_{M'\backslash}(b_\varepsilon))\vee$$
$$\bigvee_{\tau=1}^{q}(\rho_{L'\backslash}(a'_\tau)\vee\rho_{M'\backslash}(b'_\tau))\}]$$

Since for each expression $u+\sum_{\sigma=1}^{\Omega}c_\sigma d_\sigma=\sum_{\varrho=1}^{\Delta}c'_\varrho d'_\varrho$ and $v+\sum_{\varepsilon=1}^{p''}a_\varepsilon b_\varepsilon=\sum_{\tau=1}^{q''}a'_\tau b'_\tau$ we have

$$u+v+\sum_{\sigma=1}^{\Omega}c_\sigma d_\sigma+\sum_{\varepsilon=1}^{p''}a_\varepsilon b_\varepsilon=\sum_{\varrho=1}^{\Delta}c'_\varrho d'_\varrho+\sum_{\tau=1}^{q''}a'_\tau b'_\tau$$

So

$$(\mu_{L'\backslash}\odot_k \mu_{M'\backslash})(u)\wedge(\mu_{L'\backslash}\odot_k \mu_{M'\backslash})(v)\leqq\bigvee_{u+v+\sum_{s=1}^{q}g_s h_s=\sum_{t=1}^{r}g'_t h'_t}\left\{\begin{array}{c}\bigwedge_{s=1}^{q}(\mu_{L'\backslash}(g_s)\wedge\mu_{M'\backslash}(h_s))\\ \wedge\bigwedge_{t=1}^{r}(\mu_{L'\backslash}(g'_t)\wedge\mu_{M'\backslash}(h'_t))\end{array}\right\},$$

$$(\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(u) \vee (\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(v) \geqq \bigwedge_{u+v+\sum_{s=1}^{q} g_s h_s = \sum_{t=1}^{r} g_t' h_t'} \left\{ \begin{array}{c} \bigvee_{s=1}^{q} (\rho_{L'\backslash}(g_s) \vee \rho_{M'\backslash}(h_s)) \\ \vee \bigvee_{t=1}^{r} (\rho_{L'\backslash}(g_t') \vee \rho_{M'\backslash}(h_t')) \end{array} \right\}$$

Again

$$(\mu_{L'\backslash} \odot_k \mu_{M'\backslash})(u) = \bigvee_{u+\sum_{\sigma=1}^{\Omega} c_\sigma d_\sigma = \sum_{\varrho=1}^{\Delta} c_\varrho' d_\varrho'} \left\{ \bigwedge_{\sigma=1}^{\Omega} (\mu_{L'\backslash}(c_\sigma) \wedge \mu_{M'\backslash}(d_\sigma)) \wedge \bigwedge_{\varrho=1}^{\Delta} (\mu_{L'\backslash}(c_\varrho') \wedge \mu_{M'\backslash}(d_\varrho')) \right\},$$

$$(\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(u) = \bigwedge_{u+\sum_{\sigma=1}^{\Omega} c_\sigma d_\sigma = \sum_{\varrho=1}^{\Delta} c_\varrho' d_\varrho'} \left\{ \bigvee_{\sigma=1}^{\Omega} (\rho_{L'\backslash}(c_\sigma) \vee \rho_{M'\backslash}(d_\sigma)) \vee \bigvee_{\varrho=1}^{\Delta} (\rho_{L'\backslash}(c_\varrho') \vee \rho_{M'\backslash}(d_\varrho')) \right\}$$

implies that

$$(\mu_{L'\backslash} \odot_k \mu_{M'\backslash})(u) \leqq \bigvee_{u+\sum_{\sigma=1}^{\Omega} c_\sigma d_\sigma = \sum_{\varrho=1}^{\Delta} c_\varrho' d_\varrho'} \left\{ \bigwedge_{\sigma=1}^{\Omega} (\mu_{L'\backslash}(c_\sigma) \wedge \mu_{M'\backslash}(d_\sigma r)) \wedge \bigwedge_{\varrho=1}^{\Delta} (\mu_{L'\backslash}(c_\varrho') \wedge \mu_{M'\backslash}(d_\varrho' r)) \right\},$$

$$(\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(u) \geqq \bigwedge_{u+\sum_{\sigma=1}^{\Omega} c_i d_i = \sum_{\varrho=1}^{\Delta} c_j' d_j'} \left\{ \bigvee_{\sigma=1}^{\Omega} (\rho_{L'\backslash}(c_\sigma) \vee \rho_{M'\backslash}(d_\sigma r)) \vee \bigvee_{\varrho=1}^{\Delta} (\rho_{L'\backslash}(c_\varrho') \vee \rho_{M'\backslash}(d_\varrho' r)) \right\}$$

So we get

$$(\mu_{L'\backslash} \odot_k \mu_{M'\backslash})(u) \leqq \bigvee_{ur+\sum_{s'=1}^{k''} g_{s'} h_{s'} = \sum_{t'=1}^{l''} g_{t'}' h_{t'}'} \left\{ \bigwedge_{s'=1}^{k''} (\mu_{L'\backslash}(g_{s'}) \wedge \mu_{M'\backslash}(h_{s'})) \wedge \bigwedge_{t'=1}^{l''} (\mu_{L'\backslash}(g_{t'}') \wedge \mu_{M'\backslash}(h_{t'}')) \right\}$$

Which implies that

$$(\mu_{L'\backslash} \odot_k \mu_{M'\backslash})(u) \leqq (\mu_{L'\backslash} \odot_k \mu_{M'\backslash})(ur)$$

and

$$(\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(u) \geqq \bigwedge_{ur+\sum_{s'=1}^{k''} g_{s'} h_{s'} = \sum_{t'=1}^{l''} g_{t'}' h_{t'}'} \left\{ \bigvee_{s'=1}^{k''} (\rho_{L'\backslash}(g_{s'}) \vee \rho_{M'\backslash}(h_{s'})) \vee \bigvee_{t'=1}^{l''} (\rho_{L'\backslash}(g_{t'}') \vee \rho_{M'\backslash}(h_{t'}')) \right\}$$

implies that

$$(\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(u) \geqq (\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(ur).$$

This means that $\mu_{L'\backslash} \odot_k \mu_{M'\backslash}, \rho_{L'\backslash} \odot_k \rho_{M'\backslash}$ are intuitionistic fuzzy ideal of $\underset{\leftrightarrow}{H}$. Now will prove that

$$u + b'' = a'' \Longrightarrow (\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(u) \leqq (\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(b'') \vee (\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(a'')$$

observe that

$$b^{''}+\sum_{\sigma=1}^{\Omega}c_{\sigma}d_{\sigma}=\sum_{\varrho=1}^{\Delta}c^{'}_{\varrho}d^{'}_{\varrho}\text{ and }a^{''}+\sum_{\varphi=1}^{l^{''}}a_{\varphi}b_{\varphi}=\sum_{\pi=1}^{p^{''}}a^{'}_{\pi}b^{'}_{\pi}$$

together with $u+b^{''}=a^{''}$ gives $u+b^{''}+\sum_{\sigma=1}^{\Omega}c_{\sigma}d_{\sigma}=a^{''}+\sum_{\sigma=1}^{\Omega}c_{\sigma}d_{\sigma}$ thus

$$u+\sum_{\varrho=1}^{\Delta}c^{'}_{\varrho}d^{'}_{\varrho}=a^{''}+\sum_{\sigma=1}^{\Omega}c_{\sigma}d_{\sigma}$$

and accordingly

$$u+\sum_{\varrho=1}^{\Delta}c^{'}_{\varrho}d^{'}_{\varrho}+\sum_{\varphi=1}^{l^{''}}a_{\varphi}b_{\varphi}=a^{''}+\sum_{\varphi=1}^{l^{''}}a_{\varphi}b_{\varphi}+\sum_{\sigma=1}^{\Omega}c_{\sigma}d_{\sigma}=\sum_{\pi=1}^{p^{''}}a^{'}_{\pi}b^{'}_{\pi}+\sum_{\sigma=1}^{\Omega}c_{\sigma}d_{\sigma}$$

Therefore

$$u+\sum_{\varrho=1}^{\Delta}c^{'}_{\varrho}d^{'}_{\varrho}+\sum_{\pi=1}^{l^{''}}a_{\varphi}b_{\varphi}=\sum_{\sigma=1}^{\Omega}c_{\sigma}d_{\sigma}+\sum_{\pi=1}^{p^{''}}a^{'}_{\pi}b^{'}_{\pi}$$

$$(\mu_{L'\backslash}\odot_k\mu_{M'\backslash})(b^{''})\wedge(\mu_{L'\backslash}\odot_k\mu_{M'\backslash})(a^{''})=\bigvee_{b''+\sum_{\sigma=1}^{e'}c_{\sigma}d_{\sigma}=\sum_{\varrho=1}^{f'}c'_{\varrho}d'_{\varrho}}\left\{\begin{array}{l}\bigwedge_{\sigma=1}^{e^{'}}(\mu_{L'\backslash}(c_{\sigma})\wedge\mu_{M'\backslash}(d_{\sigma}))\\ \wedge\bigwedge_{\varrho=1}^{\Delta}(\mu_{L'\backslash}(c^{'}_{\varrho})\wedge\mu_{M'\backslash}(d^{'}_{\varrho}))\end{array}\right\}\wedge$$

$$\bigvee_{a''+\sum_{\varphi=1}^{l^{''}}a_{\varphi}b_{\varphi}=\sum_{\pi=1}^{p^{''}}a'_{\pi}b'_{\pi}}\left\{\begin{array}{l}\bigwedge_{\varphi=1}^{l^{''}}(\mu_{L'\backslash}(a_{\varphi})\wedge\mu_{M'\backslash}(b_{\varphi}))\\ \wedge\bigwedge_{\pi=1}^{p^{''}}(\mu_{L'\backslash}(a^{'}_{\pi})\wedge\mu_{M'\backslash}(b^{'}_{\pi}))\end{array}\right\}$$

$$(\rho_{L'\backslash}\odot_k\rho_{M'\backslash})(b^{''})\vee(\rho_{L'\backslash}\odot_k\rho_{M'\backslash})(a^{''})=\bigwedge_{b''+\sum_{\sigma=1}^{e'}c_{\sigma}d_{\sigma}=\sum_{\varrho=1}^{f'}c'_{\varrho}d'_{\varrho}}\left\{\begin{array}{l}\bigvee_{\sigma=1}^{\Omega}(\rho_{L'\backslash}(c_{\sigma})\vee\rho_{M'\backslash}(d_{\sigma}))\\ \vee\bigvee_{\varrho=1}^{\Delta}(\rho_{L'\backslash}(c^{'}_{\varrho})\vee\rho_{M'\backslash}(d^{'}_{\varrho}))\end{array}\right\}\vee$$

$$\bigwedge_{a''+\sum_{\varphi=1}^{l^{''}}a_{\varphi}b_{\varphi}=\sum_{\pi=1}^{p^{''}}a'_{\pi}b'_{\pi}}\left\{\begin{array}{l}\bigvee_{\varphi=1}^{l^{''}}\left(\rho_{L'\backslash}(a_{\varphi})\vee\rho_{M'\backslash}(b_{\varphi})\right)\\ \vee\bigvee_{\pi=1}^{p^{''}}(\rho_{L'\backslash}(a^{'}_{\pi})\vee\rho_{M'\backslash}(b^{'}_{\pi}))\end{array}\right\}$$

$$(\mu_{L'\backslash}\odot_k\mu_{M'\backslash})(b^{''})\wedge(\mu_{L'\backslash}\odot_k\mu_{M'\backslash})(a^{''})=\bigwedge_{b''+\sum_{\sigma=1}^{e'}c_{\sigma}d_{\sigma}=\sum_{\varrho=1}^{f'}c'_{\varrho}d'_{\varrho}}[\bigwedge_{a''+\sum_{\varphi=1}^{l^{''}}a_{\varphi}b_{\varphi}=\sum_{\pi=1}^{p^{''}}a'_{\pi}b'_{\pi}}\{\bigwedge_{\sigma=1}^{\Omega}(\mu_{L'\backslash}(c_{\sigma})\wedge\mu_{M'\backslash}(d_{\sigma}))$$

$$\wedge \bigwedge_{\varrho=1}^{\Delta}(\mu_{L'\backslash}(c_{\varrho}^{'}) \wedge \mu_{M'\backslash}(d_{\varrho}^{'})) \wedge \bigwedge_{\varphi=1}^{l''}(\mu_{L'\backslash}(a_{\varphi}) \wedge \mu_{M'\backslash}(b_{\varphi}))$$
$$\wedge \bigwedge_{\pi=1}^{p''}(\mu_{L'\backslash}(a_{\pi}^{'}) \wedge \mu_{M'\backslash}(b_{\pi}^{'}))\}]$$

and

$$(\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(b^{''}) \vee (\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(a^{''}) = \bigwedge_{b''+\sum_{\sigma=1}^{e'}c_{\sigma}d_{\sigma}=\sum_{\varrho=1}^{f'}c'_{\varrho}d'_{\varrho}} [\bigwedge_{a''+\sum_{\varphi=1}^{l''}a_{\varphi}b_{\varphi}=\sum_{\pi=1}^{p''}a'_{\pi}b'_{\pi}} \{\bigvee_{\sigma=1}^{\Omega}(\rho_{L'\backslash}(c_{\sigma}) \vee \rho_{M'\backslash}(d_{\sigma}))$$
$$\vee \bigvee_{\varrho=1}^{\Delta}(\rho_{L'\backslash}(c_{\varrho}^{'}) \vee \rho_{M'\backslash}(d_{\varrho}^{'})) \vee \bigvee_{k''=1}^{l''}(\rho_{L'\backslash}(a_{k''}) \vee \rho_{M'\backslash}(b_{k''}))$$
$$\vee \bigvee_{q''=1}^{p''}(\rho_{L'\backslash}(a_{q''}^{'}) \vee \rho_{M'\backslash}(b_{q''}^{'}))\}]$$

$$(\mu_{L'\backslash} \odot_k \mu_{M'\backslash})(b^{''}) \wedge (\mu_{L'\backslash} \odot_k \mu_{M'\backslash})(a^{''}) \leqq \bigvee_{u+\sum_{t=1}^{q}h_tg_t=\sum_{s^*=1}^{r}h'_{s^*}g'_{s^*}} \left\{ \begin{array}{l} \bigwedge_{t=1}^{q}(\mu_{L'\backslash}(h_t) \wedge \mu_{M'\backslash}(g_t))\wedge \\ \bigwedge_{s^*=1}^{r}(\mu_{L'\backslash}(h_{s^*}^{'}) \wedge \mu_{M'\backslash}(g_{s^*}^{'})) \end{array} \right\}$$
$$\leqq (\mu_{L'\backslash} \odot_k \mu_{M'\backslash})(u)$$

$$(\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(b^{''}) \vee (\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(a^{''}) \geqq \bigwedge_{u+\sum_{t=1}^{q}h_tg_t=\sum_{s^*=1}^{r}h'_{s^*}g'_{s^*}} \left\{ \begin{array}{l} \bigvee_{t=1}^{q}((\rho_{L'\backslash}(h_t) \vee \rho_{M'\backslash}(g_t))\vee \\ \bigvee_{s^*=1}^{r}(\rho_{L'\backslash}(h_{s^*}^{'}) \vee \rho_{M'\backslash}(g_{s^*}^{'})) \end{array} \right\}$$
$$\geqq (\rho_{L'\backslash} \odot_k \rho_{M'\backslash})(u).$$

Thus $(\mu_{L'\backslash} \odot_k \mu_{M'\backslash}), (\rho_{L'\backslash} \odot_k \rho_{M'\backslash})$ are intuitionistic fuzzy k−ideal of $\underset{\leftrightarrow}{H}$. □

3. Intuitionistic fuzzy $k - ideals$ of right $k - weakly$ regular hemirings

Definition 3.1. [3] For every $\acute{z} \in \underset{\leftrightarrow}{H}$, $x^{'''} \in \overline{(\acute{z}r)^2}$ $\left(x^{'''} \in \overline{(r\acute{z})^2}\right)$ the hemiring $\underset{\leftrightarrow}{H}$ is called right (left) $k - weakly$ regular if for $\acute{z} \in \underset{\leftrightarrow}{H}$ we have $p_i, q_i, u_j, v_j \in \underset{\leftrightarrow}{H}$ such that

$$\acute{z} + \sum_{p''=1}^{m} x^{'''}p_ix^{'''}q_i = \sum_{q''=1}^{n} x^{'''}u_jx^{'''}v_j \ (\acute{z} + \sum_{p''=1}^{m} p_ix^{'''}q_ix^{'''} = \sum_{q''=1}^{n} u_jx^{'''}v_jx^{'''}).$$

Thus every $k - regular$ hemiring with identity is right $k - weakly$ regular but not conversely true. However both the concepts coincide for a commutative hemiring.

Definition 3.2. Suppose that intuitionistic fuzzy sets $U^{''} = (\mu_{U''}, \rho_{U''})$ and $V^{''} = (\mu_{V''}, \rho_{V''})$ of $\underset{\leftrightarrow}{H}$ then $U^{''} +_k V^{''}$ is called $k-sum$ of intuitionistic fuzzy sets if

$$\mu_{U''} +_k \mu_{V''} = \bigvee_{x^{'}+(a^{'}+b^{'})=(c^{'}+d^{'})} [\mu_{U''}(a^{''}) \wedge \mu_{U''}(c^{''}) \wedge \mu_{V''}(b^{''}) \wedge \mu_{V''}(d^{''})],$$

$$\rho_{U''} +_k \rho_{V''} = \bigwedge_{x^{''}+(a^{''}+b^{''})=(c^{''}+d^{''})} [\rho_{U''}(a^{''}) \vee \rho_{U''}(c^{''}) \vee \rho_{V''}(b^{''}) \vee \rho_{V''}(d^{''})]$$

where $x^{''}, a^{''}, b^{''}, c^{''}, d^{''} \in \underset{\leftrightarrow}{H}$.

Theorem 3.3. *The $k-$ sum of intuitionistic fuzzy $k-$ideals of hemiring $\underset{\leftrightarrow}{H}$ is also a intuitionistic fuzzy $k-$ideal of $\underset{\leftrightarrow}{H}$.*

Proof. Suppose $G^{'\backslash} = (\mu_{G'\backslash}, \rho_{G'\backslash})$, $H^{'\backslash} = (\mu_{H'\backslash}, \rho_{H'\backslash})$ are intuitionistic fuzzy $k-$ideals of $\underset{\leftrightarrow}{H}$. Therefore $y^{''}, z^{''}, r^{''} \in \underset{\leftrightarrow}{H}$ we have

$$\begin{aligned}(\mu_{G'\backslash} +_k \mu_{H'\backslash})(y^{''}) \wedge (\mu_{G'\backslash} +_k \mu_{H'\backslash})(z^{''}) = {} & [\bigvee_{y^{''}+(c_{11}+d_{11})=(c_{22}+d_{22})} \{\mu_{G'\backslash}(c_{11}) \wedge \mu_{G'\backslash}(c_{22}) \\ & \wedge \mu_{H'\backslash}(d_{11}) \wedge \mu_{H'\backslash}(d_{22})\}] \wedge \\ & [\bigvee_{z^{''}+(c_1^{'}+d_1^{'})=(c_2^{'}+d_2^{'})} \{\mu_{G'\backslash}(c_1^{'}) \wedge \mu_{G'\backslash}(c_2^{'}) \\ & \wedge \mu_{H'\backslash}(d_1^{'}) \wedge \mu_{H'\backslash}(d_2^{'})\}],\end{aligned}$$

and

$$\begin{aligned}(\rho_{G'\backslash} +_k \rho_{H'\backslash})(y^{''}) \vee (\rho_{G'\backslash} +_k \rho_{H'\backslash})(z^{''}) = {} & [\bigwedge_{y^{''}+(c_{11}+d_{11})=(c_{22}+d_{22})} \{\rho_{G'\backslash}(c_{11}) \vee \rho_{G'\backslash}(c_{22}) \\ & \vee \rho_{H'\backslash}(d_{11}) \vee \rho_{H'\backslash}(d_{22})\}] \vee \\ & [\bigwedge_{z^{'}+(c_1^{'}+d_1^{'})=(c_2^{'}+d_2^{'})} \{\rho_{G'\backslash}(c_1^{'}) \vee \rho_{G'\backslash}(c_2^{'}) \vee \\ & \rho_{H'\backslash}(d_1^{'}) \vee \rho_{H'\backslash}(d_2^{'})\}]\end{aligned}$$

Now

$$\begin{aligned}(\mu_{G'\backslash} +_k \mu_{H'\backslash})(y^{''}) \wedge (\mu_{G'\backslash} +_k \mu_{H'\backslash})(z^{''}) = {} & \bigvee_{\substack{y^{''}+(c_{11}+d_{11})=(c_{22}+d_{22}) \\ z^{''}+(c_1^{'}+d_2^{'})=(c_2^{'}+d_2^{'})}} \{\mu_{G'\backslash}(c_{11}) \wedge \mu_{G'\backslash}(c_{22}) \wedge \mu_{H'\backslash}(d_{11}) \wedge \\ & \mu_{H'\backslash}(d_{22}) \wedge \mu_{G'\backslash}(c_1^{'}) \wedge \mu_{G'\backslash}(c_2^{'}) \wedge \mu_{H'\backslash}(d_1^{'}) \wedge \mu_{H'\backslash}(d_2^{'})\},\end{aligned}$$

and

$$(\rho_{G'\backslash} +_k \rho_{H'\backslash})(y^{''}) \vee (\rho_{G'\backslash} +_k \rho_{H'\backslash})(z^{''}) = \bigwedge_{\substack{y^{''}+(c_{11}+d_{11})=(c_{22}+d_{22}) \\ z^{''}+(c_1^{'}+d_1^{'})=(c_2^{'}+d_2^{'})}} \{\rho_{G'\backslash}(c_{11}) \vee \rho_{G'\backslash}(c_{22}) \vee \rho_{H'\backslash}(d_{11}) \vee$$

$$\rho_{H'\backslash}(d_{22}) \vee \rho_{G'\backslash}(c_1^{'}) \vee \rho_{G'\backslash}(c_2^{'}) \vee \rho_{H'\backslash}(d_1^{'}) \vee \rho_{H'\backslash}(d_2^{'})\}$$

This implies that

$$(\mu_{G'\backslash} +_k \mu_{H'\backslash})(y^{''}) \wedge (\mu_{G'\backslash} +_k \mu_{H'\backslash})(z^{''}) = \bigvee_{\substack{y^{''}+(c_{11}+d_{11})=(c_{21}+d_{22}) \\ z^{''}+(c_1^{'}+d_1^{'})=(c_2^{'}+d_2^{'})}} \{\mu_{G'\backslash}(c_{11}+c_1^{'}) \wedge \mu_{G'\backslash}(c_{22}+c_2^{'}) \wedge \mu_{H'\backslash}(d_{11}+d_1^{'}) \wedge \mu_{H'\backslash}(d_{22}+d_2^{'})\},$$

and

$$(\rho_{G'\backslash} + \rho_{H'\backslash})(y^{''}) \vee (\rho_{G'\backslash} +_k \rho_{H'\backslash})(z^{''}) = \bigwedge_{\substack{y^{''}+(c_{11}+d_{11})=(c_{22}+d_{22}) \\ z^{''}+(c_1^{'}+d_1^{'})=(c_2^{'}+d_2^{'})}} \{\rho_{G'\backslash}(c_{11}+c_1^{'}) \vee \rho_{G'\backslash}(c_{22}+c_2^{'}) \vee \rho_{H'\backslash}(d_{11}+d_1^{'}) \vee \rho_{H'\backslash}(d_{22}+d_2^{'})\}$$

This implies that

$$(\mu_{G'\backslash} +_k \mu_{H'\backslash})(y^{''}) \wedge (\mu_{G'\backslash} +_k \mu_{H'\backslash})(z^{''}) \leqq \bigvee_{y^{''}+z^{''}+(a_1^{'}+b_1^{'})=(a_2^{'}+b_2^{'})} \{\mu_{G'\backslash}(a_1^{'}) \wedge \mu_{G'\backslash}(a_2^{'}) \wedge \mu_{H'\backslash}(b_1^{'}) \wedge \mu_{H'\backslash}(b_2^{'})\},$$

and

$$(\rho_{G'\backslash} + \rho_{H'\backslash})(y^{''}) \vee (\rho_{G'\backslash} +_k \rho_{H'\backslash})(z^{''}) \geqq \bigwedge_{y^{''}+z^{''}+(a_1^{'}+b_1^{'})=(a_2^{'}+b_2^{'})} \{\rho_{G'\backslash}(a_1^{'}) \vee \rho_{G'\backslash}(a_2^{'}) \vee \rho_{H'\backslash}(b_1^{'}) \vee \rho_{H'\backslash}(b_2^{'})\}$$

Hence

$$(\mu_{G'\backslash} +_k \mu_{H'\backslash})(y^{''}) \wedge (\mu_{G'\backslash} +_k \mu_{H'\backslash})(z^{''}) \leqq (\mu_{G'\backslash} +_k \mu_{H'\backslash})(y^{''}+z^{''}),$$
$$(\rho_{G'\backslash} +_k \rho_{H'\backslash})(y^{''}) \vee (\rho_{G'\backslash} +_k \rho_{H'\backslash})(z^{''}) \geqq (\rho_{G'\backslash} +_k \rho_{H'\backslash})(y^{''}+z^{''}).$$

Similarly

$$\begin{aligned}(\mu_{G'\backslash} +_k \mu_{H'\backslash})(y^{''}) &= \bigvee_{y^{''}+(c_{11}+d_{11})=(c_{22}+d_{22})} \{\mu_{G'\backslash}(c_{11}) \wedge \mu_{G'\backslash}(c_{22}) \wedge \mu_{H'\backslash}(d_{11}) \wedge \mu_{H'\backslash}(d_{22})\} \\ &\leqq \bigvee_{y^{''}+(c_{11}+d_{11})=(c_{22}+d_{22})} \{\mu_{G'\backslash}(r^{'}c_{11}) \wedge \mu_{G'\backslash}(r^{'}c_{22}) \wedge \mu_{H'\backslash}(r^{'}d_{11}) \wedge \mu_{H'\backslash}(r^{'}d_{22})\} \\ &\leqq \bigvee_{r^{'}y^{''}+(c_1^{''}+d_1^{''})=(c_2^{''}+d_2^{''})} \{\mu_{G'\backslash}(c_1^{''}) \wedge \mu_{G'\backslash}(c_2^{''}) \wedge \mu_{H'\backslash}(d_1^{''}) \wedge \mu_{H'\backslash}(d_2^{''})\},\end{aligned}$$

and

$$\begin{aligned}(\rho_{G'\backslash} +_k \rho_{H'\backslash})(y^{''}) &= \bigwedge_{y^{''}+(c_{11}+d_{11})=(c_{22}+d_{22})} \{\rho_{G'\backslash}(c_{11}) \vee \rho_{G'\backslash}(c_{22}) \vee \rho_{H'\backslash}(d_{11}) \vee \rho_{H'\backslash}(d_{22})\} \\ &\geqq \bigwedge_{y^{''}+(c_{11}+d_{11})=(c_{22}+d_{22})} \{\rho_{G'\backslash}(r^{'}c_{11}) \vee \rho_{G'\backslash}(r^{'}c_{22}) \vee \rho_{H'\backslash}(r^{'}d_{11}) \vee \rho_{H'\backslash}(r^{'}d_{22})\}\end{aligned}$$

$$\geqq \bigwedge_{r' y'' + (c_1'' + d_1'') = (c_2'' + d_2'')} \{\rho_{G'\backslash}(c_1'') \vee \rho_{G'\backslash}(c_2'') \vee \rho_{H'\backslash}(d_1'') \vee \rho_{H'\backslash}(d_2'')\}$$

So we get

$$(\mu_{G'\backslash} +_k \mu_{H'\backslash})(y'') \leqq (\mu_{G'\backslash} +_k \mu_{H'\backslash})(r' y''),$$
$$(\rho_{G'\backslash} +_k \rho_{H'\backslash})(y'') \geqq (\rho_{G'\backslash} +_k \rho_{H'\backslash})(r' y'').$$

Analogously

$$(\mu_{G'\backslash} +_k \mu_{H'\backslash})(y'') \leqq (\mu_{G'\backslash} +_k \mu_{H'\backslash})(y'' r'),$$
$$(\rho_{G'\backslash} +_k \rho_{H'\backslash})(y'') \geqq (\rho_{G'\backslash} +_k \rho_{H'\backslash})(y'' r').$$

Hence we have proved that $(\mu_{G'\backslash} +_k \mu_{H'\backslash}), (\rho_{G'\backslash} + \rho_{H'\backslash})$ is an intuitionistic fuzzy ideal of $\underset{\leftrightarrow}{H}$. Next we show that

$$y'' + p'' = q'' \Longrightarrow (\mu_{G'\backslash} +_k \mu_{H'\backslash})(y'') \geqq (\mu_{G'\backslash} +_k \mu_{H'\backslash})(p'') \wedge (\mu_{G'\backslash} +_k \mu_{H'\backslash})(q''),$$
$$(\rho_{G'\backslash} + \rho_{H'\backslash})(y'') \leqq (\rho_{G'\backslash} + \rho_{H'\backslash})(p'') \vee (\rho_{G'\backslash} + \rho_{H'\backslash})(q'').$$

Suppose that

$$p'' + (c_1' + d_1') = (c_2' + d_2') \text{ and } q'' + (a_1' + b_1') = (a_2' + b_2')$$

then

$$y'' + p'' + (a_1' + b_1') = (a_2' + b_2')$$

Now

$$y'' + p'' + (a_1' + b_1') + (c_1' + d_1') = (a_2' + b_2') + (c_1' + d_1')$$
$$y'' + (p'' + c_1' + d_1') + (a_1' + b_1') = (c_1' + d_1') + (a_2' + b_2')$$
$$y'' + (c_2' + d_2') + (a_1' + b_1') = (c_1' + d_1') + (a_2' + b_2')$$

Thus

$$y'' + (c_2' + a_1') + (d_2' + b_1') = (c_1' + a_2') + (d_1' + b_2').$$

Therefore

$$\begin{aligned}(\mu_{G'\backslash} +_k \mu_{H'\backslash})(p'') \wedge (\mu_{G'\backslash} +_k \mu_{H'\backslash})(q'') = [&\bigvee_{p'' + (c_1' + d_1') = (c_2' + d_2')} \{\mu_{G'\backslash}(c_1') \wedge \mu_{G'\backslash}(c_2') \\ &\wedge \mu_{H'\backslash}(d_1') \wedge \mu_{H'\backslash}(d_2')\}] \wedge \\ &[\bigvee_{q'' + (a_1' + b_1') = (a_2' + b_2')} \{\mu_{G'\backslash}(a_1') \wedge \mu_{G'\backslash}(a_2') \\ &\wedge \mu_{H'\backslash}(b_1') \wedge \mu_{H'\backslash}(b_2')\}],\end{aligned}$$

$$\begin{aligned}(\rho_{G'\backslash} + \rho_{H'\backslash})(p'') \vee (\rho_{G'\backslash} + \rho_{H'\backslash})(q'') = [&\bigwedge_{p'' + (c_1' + d_1') = (c_2' + d_2')} \{\rho_{G'\backslash}(c_1') \vee \rho_{G'\backslash}(c_2') \\ &\vee \rho_{H'\backslash}(d_1') \vee \rho_{H'\backslash}(d_2')\}] \vee\end{aligned}$$

$$[\bigwedge_{q''+(a_1'+b_1')=(a_2'+b_2')} \{\rho_{G'\setminus}(a_1') \vee \rho_{G'\setminus}(a_2') \vee \rho_{H'\setminus}(b_1') \vee \rho_{H'\setminus}(b_2')\}]$$

$$(\mu_{G'\setminus} +_k \mu_{H'\setminus})(p'') \wedge (\mu_{G'\setminus} +_k \mu_{H'\setminus})(q'') = \bigwedge_{\substack{p''+(c_1'+d_1')=(c_2'+d_2') \\ q''+(a_1'+b_1')=(a_2'+b_2')}} \{\mu_{G'\setminus}(c_1') \wedge \mu_{G'\setminus}(c_2') \wedge \mu_{H'\setminus}(d_1') \wedge \mu_{H'\setminus}(d_2') \wedge \mu_{G'\setminus}(a_1') \wedge \mu_{G'\setminus}(a_2') \wedge \mu_{H'\setminus}(b_1') \wedge \mu_{H'\setminus}(b_2')\},$$

$$(\rho_{G'\setminus} + \rho_{H'\setminus})(p'') \vee (\rho_{G'\setminus} + \rho_{H'\setminus})(q'') = \bigwedge_{\substack{p''+(c_1'+d_1')=(c_2'+d_2') \\ q''+(a_1'+b_1')=(a_2'+b_2')}} \{\rho_{G'\setminus}(c_1') \vee \rho_{G'\setminus}(c_2') \vee \rho_{H'\setminus}(d_1') \vee \rho_{H'\setminus}(d_2') \vee \rho_{G'\setminus}(a_1') \vee \rho_{G'\setminus}(a_2') \vee \rho_{H'\setminus}(b_1') \vee \rho_{H'\setminus}(b_2')\}$$

$$(\mu_{G'\setminus} +_k \mu_{H'\setminus})(p'') \wedge (\mu_{G'\setminus} +_k \mu_{H'\setminus})(q'') = \bigvee_{\substack{p''+(c_1'+d_1')=(c_2'+d_2') \\ q''+(a_1'+b_1')=(a_2'+b_2')}} \{\mu_{G'\setminus}(c_2' + a_1') \wedge \mu_{G'\setminus}(c_1' + a_2') \wedge \mu_{H'\setminus}(d_2' + b_1') \wedge \mu_{H'\setminus}(d_1' + b_2')\}$$

$$(\rho_{G'\setminus} + \rho_{H'\setminus})(p'') \vee (\rho_{G'\setminus} + \rho_{H'\setminus})(q'') = \bigwedge_{\substack{p''+(c_1'+d_1')=(c_2'+d_2') \\ q''+(a_1'+b_1')=(a_2'+b_2')}} \{\rho_{G'\setminus}(c_2' + a_1') \vee \rho_{G'\setminus}(c_1' + a_2') \vee \rho_{H'\setminus}(d_2' + b_1') \vee \rho_{H'\setminus}(d_1' + b_2')\}$$

$$(\mu_{G'\setminus} +_k \mu_{H'\setminus})(p'') \wedge (\mu_{G'\setminus} +_k \mu_{H'\setminus})(q'') \leqq [\bigvee_{y''+(c''+d'')=(c'''+d''')} \{\mu_{G'\setminus}(c'') \wedge \mu_{G'\setminus}(c''') \wedge \mu_{H'\setminus}(d'') \wedge \mu_{H'\setminus}(d''')\}]$$

$$(\rho_{G'\setminus} + \rho_{H'\setminus})(p'') \vee (\rho_{G'\setminus} + \rho_{H'\setminus})(q'') \geqq [\bigwedge_{y''+(c''+d'')=(c'''+d''')} \{\rho_{G'\setminus}(c'') \vee \rho_{G'\setminus}(c''') \vee \rho_{H'\setminus}(d'') \vee \rho_{H'\setminus}(d''')\}]$$

So

$$(\mu_{G'\setminus} +_k \mu_{H'\setminus})(p'') \wedge (\mu_{G'\setminus} +_k \mu_{H'\setminus})(q'') \leqq (\mu_{G'\setminus} +_k \mu_{H'\setminus})(y''),$$

$$(\rho_{G'\setminus} + \rho_{H'\setminus})(p'') \vee (\rho_{G'\setminus} + \rho_{H'\setminus})(q'') \geqq (\rho_{G'\setminus} + \rho_{H'\setminus})(y'').$$

Thus $(\mu_{G'\setminus} +_k \mu_{H'\setminus})$, $(\rho_{G'\setminus} + \rho_{H'\setminus})$ are intuitionistic fuzzy $k-$ideals of $\underset{\leftrightarrow}{H}$. □

Lemma 3.4. *[4] Suppose $\underset{\leftrightarrow}{H}$ be a hemiring and $S'', T'' \subseteq \underset{\leftrightarrow}{H}$. Then the following is true,*

N_1 $S'' \subseteq T'' \Longleftrightarrow \chi_{S''} \subseteq \chi_{T''}$

N_2 $\chi_{S''} \cap \chi_{T''} = \chi_{S'' \cap T''}$

N$_3$ $\chi_{S''} \odot_k \chi_{T''} = \chi_{\overline{S''T''}}$.

Proposition 3.5. *[3] For a hemiring $\underset{\leftrightarrow}{H}$ including indentity given statements are equivalent:*

N$_1$ $\underset{\leftrightarrow}{H}$ is a right $k-$ *weakly* regular hemiring.

N$_2$ every right $k-$ideal of $\underset{\leftrightarrow}{H}$ is $k-$ *idempotent*.

N$_3$ $\overline{C^{'\backslash}D^{'\backslash}} = C^{'\backslash} \cap D^{'\backslash}$ for every right $k-$ideal $C^{'\backslash}$ and two-sided k-ideal $D^{'\backslash}$.

Theorem 3.6. *For a hemiring $\underset{\leftrightarrow}{H}$ including indentity given statements are equivalent:*

N$_1$ $\underset{\leftrightarrow}{H}$ is right $k-$ *weakly* regular hemiring.

N$_2$ every intuitionistic fuzzy right $k-$ideals of $\underset{\leftrightarrow}{H}$ is idempotent.

N$_3$ $\mu_{G'\backslash} \odot_k \mu_{H'\backslash} = \mu_{G'\backslash} \wedge \mu_{H'\backslash}$, $\rho_{G'\backslash} \odot_k \rho_{H'\backslash} = \rho_{G'\backslash} \vee \rho_{H'\backslash}$ for every intuitionistic fuzzy right $k-$ideal $G^{'\backslash} = (\mu_{G'\backslash}, \rho_{G'\backslash})$ and every two sided intuitionistic fuzzy $k-$ideal $H^{'\backslash} = (\mu_{H'\backslash}, \rho_{H'\backslash})$ of $\underset{\leftrightarrow}{H}$.

Proof. $N_1 \Longrightarrow N_2$

Suppose $G^{'\backslash} = (\mu_{G'\backslash}, \rho_{G'\backslash})$ and $H^{'\backslash} = (\mu_{H'\backslash}, \rho_{H'\backslash})$ are two intuitionistic fuzzy sets of $\underset{\leftrightarrow}{H}$. Then

$$\mu_{G'\backslash} \odot_k \mu_{G'\backslash} \leqq \mu_{G'\backslash} and \rho_{G'\backslash} \odot_k \rho_{G'\backslash} \geqq \rho_{G'\backslash}.$$

Assume that $y^{''} \in \underset{\leftrightarrow}{H}$ and $\underset{\leftrightarrow}{H}$ is a right $k-$ *weakly* regular then $\exists\, a_i, b_i, c_j, d_j \in \underset{\leftrightarrow}{H}$ such that

$$y^{''} + \sum_{\sigma=1}^{p^{''}} y^{''} a_\sigma y^{''} b_\sigma = \sum_{\varrho=1}^{q^{''}} y^{''} c_\varrho y^{''} d_\varrho.$$

Hence

$$\mu_{G'\backslash}(y^{''}) = \mu_{G'\backslash}(y^{''}) \wedge \mu_{G'\backslash}(y^{''}) \leqq \bigwedge_{\sigma=1}^{p^{''}} (\mu_{G'\backslash}(y^{''} a_\sigma) \wedge \mu_{G'\backslash}(y^{''} b_\sigma)),$$

$$\rho_{G'\backslash}(y^{''}) = \rho_{G'\backslash}(y^{''}) \vee \rho_{G'\backslash}(y^{''}) \geqq \bigvee_{\sigma=1}^{p^{''}} (\mu_{G'\backslash}(y^{''} a_\sigma) \vee \mu_{G'\backslash}(y^{''} b_\sigma))$$

Also

$$\mu_{G'\backslash}(y^{''}) = \mu_{G'\backslash}(y^{''}) \wedge \mu_{G'\backslash}(y^{''}) \leqq \bigwedge_{\varrho=1}^{q^{''}} (\mu_{G'\backslash}(y^{''} c_\varrho) \wedge \mu_{G'\backslash}(y^{''} d_\varrho)),$$

$$\rho_{G'\backslash}(y^{''}) = \rho_{G'\backslash}(y^{''}) \vee \rho_{G'\backslash}(y^{''}) \geqq \bigvee_{\varrho=1}^{q^{''}} (\mu_{G'\backslash}(y^{''} c_\varrho) \vee \mu_{G'\backslash}(y^{''} d_\varrho))$$

Therefore

$$\mu_{G'\backslash}(y^{''}) \leqq \bigwedge_{\sigma=1}^{p^{''}} (\mu_{G'\backslash}(y^{''} a_\sigma) \wedge \mu(y^{''} b_\sigma)) \wedge \bigwedge_{\varrho=1}^{q^{''}} (\mu_{G'\backslash}(y^{''} c_\varrho) \wedge \mu_{G'\backslash}(y^{''} d_\varrho)),$$

$$\rho_{G'\backslash}(y^{''}) \geqq \bigvee_{\sigma=1}^{p^{''}} (\mu_{G'\backslash}(y^{''} a_\sigma) \vee \mu_{G'\backslash}(y^{''} b_\sigma)) \vee \bigvee_{\varrho=1}^{q^{''}} (\mu_{G'\backslash}(y^{''} c_\varrho) \vee \mu_{G'\backslash}(y^{''} d_\varrho))$$

$$\mu_{G'\backslash}(y'') \leqq \bigvee_{y''+\sum_{\sigma=1}^{p''} y'' a_\sigma y'' b_\sigma = \sum_{\varrho=1}^{q''} y'' c_{j'} y'' d_\varrho} \left[\bigwedge_{\sigma=1}^{p''} (\mu_{G'\backslash}(y'' a_\sigma) \wedge \mu_{G'\backslash}(y'' b_\sigma)) \wedge \bigwedge_{\varrho=1}^{q''} (\mu_{G'\backslash}(y'' c_\varrho) \wedge \mu_{G'\backslash}(y'' d_\varrho)) \right],$$

$$\rho_{G'\backslash}(y'') \geqq \bigwedge_{y''+\sum_{\sigma=1}^{p''} y'' a_\sigma y'' b_\sigma = \sum_{\varrho=1}^{q''} y'' c_{j'} y'' d_\varrho} \left[\bigvee_{\sigma=1}^{p''} (\mu_{G'\backslash}(y'' a_\sigma) \vee \mu_{G'\backslash}(y'' b_\sigma)) \vee \bigvee_{\varrho=1}^{q''} (\mu_{G'\backslash}(y'' c_\varrho) \vee \mu_{G'\backslash}(y'' d_\varrho)) \right]$$

$$\mu_{G'\backslash}(y'') \leqq (\mu_{G'\backslash} \odot_k \mu_{G'\backslash})(y''),$$
$$\rho_{G'\backslash}(y'') \geqq (\rho_{G'\backslash} \odot_k \rho_{G'\backslash})(y'').$$

Hence

$$\mu_{G'\backslash} = \mu_{G'\backslash} \odot_k \mu_{G'\backslash},$$
$$\rho_{G'\backslash} = \rho_{G'\backslash} \odot_k \rho_{G'\backslash}.$$

$N_2 \Longrightarrow N_3$ Assume that $P^{'\backslash} = (\mu_{P'\backslash}, \rho_{P'\backslash})$ and $Q^{'\backslash} = (\mu_{Q'\backslash}, \rho_{Q'\backslash})$ be the intuitionistic fuzzy right and two sided $k-$ideal of $\underset{\leftrightarrow}{H}$. Since for $\mu_{P'\backslash} \wedge \mu_{Q'\backslash}, \rho_{P'\backslash} \vee \rho_{Q'\backslash}$

$$\mu_{P'\backslash} \wedge \mu_{Q'\backslash} \geqq \mu_{P'\backslash} \odot_k \mu_{Q'\backslash}, \rho_{P'\backslash} \vee \rho_{Q'\backslash} \leqq \rho_{P'\backslash} \odot_k \rho_{Q'\backslash}.$$

By assumption,

$$\mu_{P'\backslash} \wedge \mu_{Q'\backslash} = (\mu_{P'\backslash} \wedge \mu_{Q'\backslash}) \odot_k (\mu_{P'\backslash} \wedge \mu_{Q'\backslash}) \leqq \mu_{P'\backslash} \odot_k \mu_{Q'\backslash}$$
$$\rho_{P'\backslash} \vee \rho_{Q'\backslash} = (\rho_{P'\backslash} \vee \rho_{Q'\backslash}) \odot_k (\rho_{P'\backslash} \vee \rho_{Q'\backslash}) \geqq \rho_{P'\backslash} \odot_k \rho_{Q'\backslash}.$$

Hence

$$\mu_{G'\backslash} \wedge \mu_{H'\backslash} = \mu_{G'\backslash} \odot_k \mu_{H'\backslash}, \rho_{G'\backslash} \vee \rho_{H'\backslash} = \rho_{G'\backslash} \odot_k \rho_{H'\backslash}.$$

$N_3 \Longrightarrow N_1$ Assume L and M are right $k-$ideal and two sided $k-$ideal of $\underset{\leftrightarrow}{H}$ respectively. Then the characteristic functions χ_L and χ_M of L and M are intuitionistic fuzzy right and intuitionistic fuzzy two sided $k-$ideal of $\underset{\leftrightarrow}{H}$ (resp.). Accordingly the assumption and Lemma 3.4 we get $\chi_L \odot_k \chi_M = \chi_L \wedge \chi_M$ i.e., $\chi_{\overline{LM}} = \chi_{L \cap M}$,which implies that $\overline{LM} = L \cap M$. □

Theorem 3.7. *For a hemiring $\underset{\leftrightarrow}{H}$ including indentity given statements are equivalent:*

N_1 $\underset{\leftrightarrow}{H}$ is right $k-$ *weakly* regular hemiring.

N_2 every right $k-$ideal of $\underset{\leftrightarrow}{H}$ is $k-$ *idempotent*.

N_3 $\overline{C'\backslash D'\backslash} = C^{'\backslash} \cap D^{'\backslash}$ for every right $k-$ideals $C^{'\backslash}$ and $D^{'\backslash}$.

N_4 every intuitionistic fuzzy right $k-$ideals of $\underset{\leftrightarrow}{H}$ is idempotent.

N_5 $\mu_{A'\backslash} \odot_k \mu_{B'\backslash} = \mu_{A'\backslash} \wedge \mu_{B'\backslash}$, $\rho_{A'\backslash} \odot_k \rho_{B'\backslash} = \rho_{A'\backslash} \vee \rho_{B'\backslash}$ for every intuitionistic fuzzy right $k-$ideal $A^{'\backslash} = (\mu_{A'\backslash}, \rho_{A'\backslash})$ and every two sided intuitionistic fuzzy $k-$ideal $B^{'\backslash} = (\mu_{B'\backslash}, \rho_{B'\backslash})$ of $\underset{\leftrightarrow}{H}$.

Equivalent to the above statements.

N_6 If commutative property hold in $\underset{\leftrightarrow}{H}$ then $\underset{\leftrightarrow}{H}$ is $k-$ *regular*.

Proof. By proposition 3.5, $(N_1),(N_2),(N_3)$ are equivalent and by theorem 2.14, $(N_1),(N_4),(N_5)$ are equivalent. If commutative property holds in $\underset{\leftrightarrow}{H}$ then (N_1) and (N_6) are equivalent. □

Theorem 3.8. *The set $_r$ of all intuitionistic fuzzy $k-$ideals of $\underset{\leftrightarrow}{H}$ (odered by $\leqq$) is a distributive lattice if $\underset{\leftrightarrow}{H}$ is a right $k-$ weakly regular hemiring.*

Proof. It is clear the set $_r$ of all intuitionistic fuzzy $k-$ideals of $\underset{\leftrightarrow}{H}$ (oderded by $\leqq$) is lattice with operation $+_k$ and $\wedge$ of intuitionistic fuzzy $k-$ideals. Now we will prove $_r$ is distributive lattice for any intuitionistic fuzzy $k-$ideals $S^{'\setminus} = (\mu_{S'\setminus},\rho_{S'\setminus}), T^{'\setminus} = (\mu_{T'\setminus},\rho_{T'\setminus})$ and $U^{'\setminus} = (\mu_{U'\setminus},\rho_{U'\setminus})$ of $\underset{\leftrightarrow}{H}$, then

$$(\mu_{S'\setminus} \wedge \mu_{T'\setminus}) + \mu_{U'\setminus} = (\mu_{S'\setminus} + \mu_{U'\setminus}) \wedge (\mu_{T'\setminus} + \mu_{U'\setminus}),$$

$$(\rho_{S'\setminus} \vee \rho_{T'\setminus}) + \rho_{U'\setminus} = (\rho_{S'\setminus} + \rho_{U'\setminus}) \vee (\rho_{T'\setminus} + \rho_{U'\setminus})$$

for any $y \in \underset{\leftrightarrow}{H}$

$$[(\mu_{S'\setminus} \wedge \mu_{T'\setminus}) + \mu_{U'\setminus}](y) = \bigvee_{y+(a''+b'')=(c''+d'')} \left[(\mu_{S'\setminus} \wedge \mu_{T'\setminus})(a'') \wedge (\mu_{S'\setminus} \wedge \mu_{T'\setminus})(c'') \wedge \mu_{U'\setminus}(b'') \wedge \mu_{U'\setminus}(d'')\right],$$

$$[(\rho_{S'\setminus} \vee \rho_{T'\setminus}) + \rho_{U'\setminus}](y) = \bigwedge_{y+(a''+b'')=(c''+d'')} \left[(\rho_{S'\setminus} \vee \rho_{T'\setminus})(a'') \vee (\rho_{S'\setminus} \vee \rho_{T'\setminus})(c'') \vee \rho_{U'\setminus}(b'') \wedge \rho_{U'\setminus}(d'')\right]$$

$$[(\mu_{S'\setminus} \wedge \mu_{T'\setminus}) + \mu_{U'\setminus}](y) = \bigvee_{y+(a''+b'')=(c''+d'')} \begin{bmatrix} \mu_{S'\setminus}(a'') \wedge \mu_{S'\setminus}(c'') \wedge \mu_{U'\setminus}(b') \wedge \mu_{U'\setminus}(d'') \wedge \\ \mu_{T'\setminus}(a'') \wedge \mu_{T'\setminus}(c'') \wedge \mu_{U'\setminus}(b'') \wedge \mu_{U'\setminus}(d'') \end{bmatrix},$$

$$[(\rho_{S'\setminus} \vee \rho_{T'\setminus}) + \rho_{U'\setminus}](y) = \bigwedge_{y+(a''+b'')=(c''+d'')} \begin{bmatrix} \rho_{S'\setminus}(a'') \vee \rho_{S'\setminus}(c'') \vee \rho_{U'\setminus}(b'') \vee \rho_{U'\setminus}(d'') \vee \\ \rho_{T'\setminus}(a'') \vee \rho_{T'\setminus}(c'') \vee \rho_{U'\setminus}(b'') \vee \rho_{U'\setminus}(d'') \end{bmatrix}$$

$$[(\mu_{S'\setminus} \wedge \mu_{T'\setminus}) + \mu_{U'\setminus}](y) = (\mu_{S'\setminus} + \mu_{U'\setminus})(y) \wedge (\mu_{T'\setminus} + \mu_{U'\setminus})(y) = \left[(\mu_{S'\setminus} + \mu_{U'\setminus}) \wedge (\mu_{T'\setminus} + \mu_{U'\setminus})\right](y),$$

$$[(\rho_{S'\setminus} \vee \rho_{T'\setminus}) + \rho_{U'\setminus}](y) = (\rho_{S'\setminus} + \rho_{U'\setminus})(y) \vee (\rho_{T'\setminus} + \rho_{U'\setminus})(y) = \left[(\rho_{S'\setminus} + \rho_{U'\setminus}) \vee (\rho_{T'\setminus} + \rho_{U'\setminus})\right](y).$$

Hence proved. □

4. Prime and intuitionistic fuzzy prime right $k-$ideals

Definition 4.1. In a hemiring $\underset{\leftrightarrow}{H}$ an intuitionistic fuzzy $k-$ideal $S^{'\setminus} = (\mu_{S'\setminus},\rho_{S'\setminus})$ of $\underset{\leftrightarrow}{H}$ is called an intuitionistic fuzzy k-prime (k-semiprime) if we take two intuitionistic fuzzy right $k-$ideals $T^{'\setminus} = (\mu_{T'\setminus},\rho_{T'\setminus})$, $U^{'\setminus} = (\mu_{U'\setminus},\rho_{U'\setminus})$ of $\underset{\leftrightarrow}{H}$ then

$$\mu_{T'\setminus} \odot_k \mu_{U'\setminus} \leqq \mu_{S'\setminus} \implies \mu_{T'\setminus} \leqq \mu_{S'\setminus} \; or \; \mu_{U'\setminus} \leqq \mu_{S'\setminus} \; (\mu_{T'\setminus} \odot_k \mu_{T'\setminus} \leqq \mu_{S'\setminus} \implies \mu_{T'\setminus} \leqq \mu_{S'\setminus}),$$

$$\rho_{T'\setminus} \odot_k \rho_{U'\setminus} \geqq \rho_{S'\setminus} \implies \rho_{T'\setminus} \geqq \rho_{S'\setminus} \; or \; \rho_{U'\setminus} \geqq \rho_{S'\setminus} \; (\rho_{T'\setminus} \odot_k \rho_{T'\setminus} \geqq \rho_{S'\setminus} \implies \rho_{T'\setminus} \geqq \rho_{S'\setminus}).$$

Proposition 4.2. *Suppose $\underset{\leftrightarrow}{H}$ is a right $k-$ weakly regular hemiring. If $J = (\mu_J, \rho_J)$ is a intuitionistic fuzzy right $k-$ ideal of $\underset{\leftrightarrow}{H}$ with $\mu_J(b^{'}) = \beta, \rho_J(b^{'}) = \beta$, where $b^{'}$ is any element of $\underset{\leftrightarrow}{H}$ and $\beta \in (0, 1]$, then $\exists$ an intuitionistic fuzzy k-irreducible right $k-$ideal $K = (\mu_K, \rho_K)$ of $\underset{\leftrightarrow}{H}$ such that $\mu_J \leqq \mu_K, \rho_J \geqq \rho_K$ and $\mu_K(a^{'}) = \alpha, \rho_K(a^{'}) = \alpha$.*

Proof. Suppose

$$Y = \{H = (\mu_H, \rho_H) : \mu_H(b^{'}) = \beta,\ \rho_H(b^{'}) = \beta\ \ \&\ \ \mu_J \leqq \mu_H,\ \rho_J \geqq \rho_H\}$$

where H is an intuitionistic fuzzy $k-$ideal of $\underset{\leftrightarrow}{H}$. Then $Y \neq \{\}$, since $J \in Y$. Suppose $F \subseteq Y$, F be a totally odered set such that $F = \{J_{t^{''}} = (\mu_{J_{t^{''}}}, \rho_{J_{t^{''}}}) : t^{''} \in T^{''}\}$. We prove that $\bigvee_{t' \in T''} \mu_{J_{t''}}, \bigwedge_{t'' \in T''} \rho_{J_{t''}}$ is an intuitionistic fuzzy right $k-$ideal of $\underset{\leftrightarrow}{H}$. For any $y^{\upsilon}, s^{\upsilon} \in \underset{\leftrightarrow}{H}$, we have

$$(\bigvee_{t'' \in T''} \mu_{Jt''})(y^{\upsilon}) = \bigvee_{t'' \in T''} (\mu_{Jt''}(y^{\upsilon})) \leqq \bigvee_{t'' \in T''} (\mu_{Jt''}(y^{\upsilon} s^{\upsilon})) = (\bigvee_{t'' \in T''} \mu_{Jt''})(y^{\upsilon} s^{\upsilon}),$$

$$(\bigwedge_{t'' \in T''} \rho_{Jt''})(y^{\upsilon}) = \bigwedge_{t'' \in T''} (\rho_{Jt''}(y^{\upsilon})) \geqq \bigwedge_{t'' \in T''} (\rho_{J't}(y^{\upsilon} s^{\upsilon})) = (\bigwedge_{t'' \in T''} \rho_{Jt''})(y^{\upsilon} s^{\upsilon}).$$

Suppose $y^{\upsilon}, z^{\upsilon} \in \underset{\leftrightarrow}{H}$,

$$(\bigvee_{t'' \in T''} \mu_{Jt''})(y^{\upsilon}) \wedge (\bigvee_{k'' \in T''} \mu_{J_{k''}})(z^{\upsilon}) = (\bigvee_{t'' \in T''} \mu_{Jt''}(y^{\upsilon}) \wedge (\bigvee_{k'' \in T''} \mu_{J_{k''}}(z^{\upsilon}))$$

$$(\bigvee_{t'' \in T''} \mu_{Jt''})(y^{\upsilon}) \wedge (\bigvee_{k'' \in T''} \mu_{J_{k''}})(z^{\upsilon}) = \bigvee_{k'' \in T''} (\bigvee_{t'' \in T''} (\mu_{Jt''}(y^{\upsilon}) \wedge \mu_{J_{k''}}(z^{\upsilon})))$$

$$(\bigvee_{t'' \in T''} \mu_{Jt''})(y^{\upsilon}) \wedge (\bigvee_{k'' \in T''} \mu_{J_{k''}})(z^{\upsilon}) \leqq \bigvee_{k'' \in T''} (\bigvee_{t'' \in T''} (\max\{\mu_{Jt''}(y^{\upsilon}), \mu_{J_{k''}}(y^{\upsilon})\} \wedge \max\{\mu_{Jt''}(z^{\upsilon}), \mu_{J_{k''}}(z^{\upsilon})\})$$

$$(\bigvee_{t'' \in T''} \mu_{Jt''})(y^{\upsilon}) \wedge (\bigvee_{k'' \in T''} \mu_{J_{k''}})(z^{\upsilon}) \leqq \bigvee_{k'' \in T''} (\bigvee_{t'' \in T''} (\max\{\mu_{Jt''}(y^{\upsilon} + z^{\upsilon}), \mu_{J_{k''}}(y^{\upsilon} + z^{\upsilon})\})$$

$$(\bigvee_{t'' \in T''} \mu_{Jt''})(y^{\upsilon}) \wedge (\bigvee_{k'' \in T''} \mu_{J_{k''}})(z^{\upsilon}) \leqq \bigvee_{t'', k'' \in T''} \max\{\mu_{Jt''}(y^{\upsilon} + z^{\upsilon}), \mu_{J_{k''}}(y^{\upsilon} + z^{\upsilon})\}$$

$$(\bigvee_{t'' \in T''} \mu_{Jt''})(y^{\upsilon}) \wedge (\bigvee_{k'' \in T''} \mu_{J_{k''}})(z^{\upsilon}) \leqq (\bigvee_{t'' \in T''} \mu_{Jt''})(y^{\upsilon} + z^{\upsilon})$$

and

$$(\bigwedge_{t'' \in T''} \rho_{J_{t''}})(y^{\upsilon}) \vee (\bigwedge_{k'' \in T''} \rho_{J_{k''}})(z^{\upsilon}) = (\bigwedge_{t'' \in T''} \rho_{J_{t''}}(y^{\upsilon})) \vee (\bigwedge_{k'' \in T''} \rho_{J_{k''}}(z^{\upsilon}))$$

$$(\bigwedge_{t'' \in T''} \rho_{J_{t''}})(y^{\upsilon}) \vee (\bigwedge_{k'' \in T''} \rho_{J_{k''}})(z^{\upsilon}) = \bigwedge_{k'' \in T''} (\bigwedge_{t'' \in T''} (\rho_{J_{t''}}(y^{\upsilon}) \vee \rho_{J_{k''}}(z^{\upsilon})))$$

$$(\bigwedge_{t'' \in T''} \rho_{J_{t''}})(y^{\upsilon}) \vee (\bigwedge_{k'' \in T''} \rho_{J_{k''}})(z^{\upsilon}) \geqq \bigwedge_{k'' \in T''} (\bigwedge_{t'' \in T''} (\min\{\rho_{J_{t''}}(y^{\upsilon}), \rho_{J_{k''}}(y^{\upsilon})\} \vee \min\{\rho_{J_{t''}}(z^{\upsilon}), \rho_{J_{k''}}(z^{\upsilon})\})$$

$$(\bigwedge_{t'' \in T''} \rho_{J_{t''}})(y^{\upsilon}) \vee (\bigwedge_{k'' \in T''} \rho_{J_{k''}})(z^{\upsilon}) \geqq \bigwedge_{k'' \in T''} (\bigwedge_{t'' \in T''} (\min\{\rho_{J_{t''}}(y^{\upsilon} + z^{\upsilon}), \rho_{J_{k''}}(y^{\upsilon} + z^{\upsilon})\})$$

$$(\bigwedge_{t''\in T''}\rho_{J_{t''}})(y^{\upsilon})\vee(\bigwedge_{k''\in T''}\rho_{J_{k''}})(z^{\upsilon})\geqq\bigwedge_{t'',k''\in T''}\min\{\rho_{J_{t''}}(y^{\upsilon}+z^{\upsilon}),\rho_{J_{k''}}(y^{\upsilon}+z^{\upsilon})\}$$

$$(\bigwedge_{t''\in T''}\rho_{J_{t''}})(y^{\upsilon})\vee(\bigwedge_{k''\in T''}\rho_{J_{k''}})(z^{\upsilon})\geqq(\bigwedge_{t''\in T''}\rho_{J_{t''}})(y^{\upsilon}+z^{\upsilon}).$$

Now Let $y^{\upsilon}+c=d$ for $c,d\in\underset{\leftrightarrow}{H}$ then

$$(\bigvee_{t''\in T''}\mu_{J_{t''}})(c)\wedge(\bigvee_{k''\in T''}\mu_{J_{k''}})(d)=(\bigvee_{t''\in T''}\mu_{J_{t''}}(c))\wedge(\bigvee_{k''\in T''}\mu_{J_{k''}}(d))$$

$$(\bigvee_{t''\in T''}\mu_{J_{t''}})(c)\wedge(\bigvee_{k''\in T''}\mu_{J_{k''}})(d)=\bigvee_{k''\in T''}(\bigvee_{t''\in T''}(\mu_{J_{t''}}(c)\wedge\mu_{J_{k''}}(d)))$$

$$(\bigvee_{t''\in T''}\mu_{J_{t''}})(c)\wedge(\bigvee_{k''\in T''}\mu_{J_{k''}})(d)\leqq\bigvee_{k''\in T''}(\bigvee_{t''\in T''}(\max\{\mu_{J_{t''}}(c),\mu_{J_{k''}}(c)\}\wedge\max\{\mu_{J_{t''}}(d),\mu_{J_{k''}}(d)\})$$

$$(\bigvee_{t''\in T''}\mu_{J_{t''}})(c)\wedge(\bigvee_{k''\in T''}\mu_{J_{k''}})(d)\leqq\bigvee_{t'',k''\in T''}\max\{\mu_{J_{t''}}(y^{\upsilon}),\mu_{J_{k''}}(y^{\upsilon})\}$$

$$(\bigvee_{t''\in T''}\mu_{J_{t''}})(c)\wedge(\bigvee_{k''\in T''}\mu_{J_{k''}})(d)\leqq\bigvee_{t''\in T''}\mu_{J_{t''}}(y^{\upsilon})$$

and

$$(\bigwedge_{t''\in T''}\rho_{J_{t''}})(c)\vee(\bigwedge_{k\in T''}\rho_{J_{k}})(d)=(\bigwedge_{t''\in T''}\rho_{J_{t''}}(c))\vee(\bigwedge_{k\in T''}\rho_{J_{k}}(d))$$

$$(\bigwedge_{t''\in T''}\rho_{J_{t''}})(c)\vee(\bigwedge_{k\in T''}\rho_{J_{k''}})(d)=\bigwedge_{k''\in T''}(\bigwedge_{t''\in T''}(\rho_{J_{t''}}(c)\vee\rho_{J_{k''}}(d)))$$

$$(\bigwedge_{t''\in T''}\rho_{J_{t''}})(c)\vee(\bigwedge_{k''\in T''}\rho_{J_{k''}})(d)\geqq\bigwedge_{k''\in T''}(\bigwedge_{t''\in T''}(\min\{\rho_{J_{t''}}(c),\rho_{J_{k''}}(c)\}\vee\min\{\rho_{J_{t''}}(d),\rho_{J_{k''}}(d)\})$$

$$(\bigwedge_{t''\in T''}\rho_{J_{t''}})(c)\vee(\bigwedge_{k''\in T''}\rho_{J_{k''}})(d)\geqq\bigwedge_{t'',k''\in T''}\min\{\rho_{J_{t''}}(y^{\upsilon}),\rho_{J_{k''}}(y^{\upsilon})\}$$

$$(\bigwedge_{t''\in T''}\rho_{J_{t''}})(c)\vee(\bigwedge_{k''\in T''}\rho_{J_{k''}})(d)\geqq\bigwedge_{t''\in T''}\rho_{J_{t''}})(y^{\upsilon}).$$

Thus $\bigvee_{t''\in T''}\mu_{J_{t''}},\bigwedge_{t''\in T''}\rho_{J_{t''}}$ is an intuitionistic fuzzy right $k-$ideal of $\underset{\leftrightarrow}{H}$. Clearly

$$\mu_J\leqslant\mu_{J_{t''}},\rho_J\geqq\bigwedge_{t''\in T''}\rho_{J_{t''}}\text{ and }\bigvee_{t''\in T''}\mu_{J_{t''}}(a)=\alpha,\ \bigwedge_{t''\in T''}\rho_{J_{t''}}(a)=\alpha.$$

Thus $\bigvee_{t''\in T''}\mu_{J_{t''}}$ is the least upper bound, $\bigwedge_{t''\in T''}\rho_{J_{t''}}$ is the greatest lower bound. Hence by Zorn' s lemma there exists an intuitionistic fuzzy right $k-$ideal $K^{'\backslash}=(\mu_{K'\backslash},\rho_{K'\backslash})$ of $\underset{\leftrightarrow}{H}$. By the property that $\mu_J\leqq\mu_{K'\backslash},\rho_J\geqq\rho_{K'\backslash}$ and $\mu_{K'\backslash}(b')=\beta,\rho_{K'\backslash}(b')=\beta$, $K^{'\backslash}=(\mu_{K'\backslash},\rho_{K'\backslash})$ is maximal.

Now we prove that $K^{'\backslash}=(\mu_{K'\backslash},\rho_{K'\backslash})$ is an intuitionistic fuzzy $k-$ *irreducible* right $k-$ideal of $\underset{\leftrightarrow}{H}$. Suppose $E^{'\backslash}=(\mu_{E'\backslash},\rho_{E'\backslash})$ and $F^{'\backslash}=(\mu_{F'\backslash},\rho_{F'\backslash})$ are intuitionistic fuzzy $k-$ *ideals* of $\underset{\leftrightarrow}{H}$. Let $\mu_{K'\backslash}=\mu_{E'\backslash}\wedge\mu_{F'\backslash}$ and $\rho_{K'\backslash}=\rho_{E'\backslash}\vee\rho_{F'\backslash}$. Thus $\mu_{K'\backslash}\leqq\mu_{E'},\mu_{K'\backslash}=\mu_{F'\backslash}$ and $\rho_{K'\backslash}\geqq\rho_{E'\backslash},\rho_{K'\backslash}\geqq\rho_{F'\backslash}$. We claim that either $\mu_{K'\backslash}=\mu_{E'\backslash}$ or $\mu_{K'\backslash}=\mu_{F'\backslash}$ and $\rho_{K'\backslash}=\rho_{E'\backslash}$ or $\rho_{K'\backslash}=\rho_{F'\backslash}$. Suppose $\mu_{K'\backslash}\neq\mu_{E'\backslash}$ &

$\mu_{K'\setminus} \neq \mu_{F'\setminus}$ and $\rho_{K'\setminus} \neq \rho_{E'\setminus}$ & $\rho_{K'\setminus} \neq \rho_{F'\setminus}$. By the property that $\mu_J(b') = \beta, \rho_J(b') = \beta$, $K'^{\setminus}$ is maximal and since $\mu_{K'\setminus} \lneqq \mu_{E'\setminus}, \mu_{K'\setminus} \lneqq \mu_{F'\setminus}$ and $\rho_{K'\setminus} \gneqq \rho_{E'\setminus}, \rho_{K'\setminus} \gneqq \rho_{F'\setminus}$, so $\mu_{E'\setminus}(b') \neq \beta, \rho_{E'\setminus}(b') \neq \beta$ and $\mu_{F'\setminus}(b') \neq \beta, \rho_{F'\setminus}(b') \neq \beta$. Hence

$$\beta = \mu_{K'\setminus}(b') = (\mu_{E'\setminus} \wedge \mu_{F'\setminus})\left(b'\right) = (\mu_{E'\setminus})\left(b'\right) \wedge (\mu_{F'\setminus})\left(b'\right) \neq \beta,$$

and

$$\beta = \rho_{K'\setminus}(b') = (\rho_{E'\setminus} \vee \rho_{F'\setminus})\left(b'\right) = (\rho_{E'\setminus})\left(b'\right) \vee (\rho_{F'\setminus})\left(b'\right) \neq \beta$$

contradiction. Hence $\mu_{K'\setminus} = \mu_{E'\setminus}$ *or* $\mu_{K'\setminus} = \mu_{F'\setminus}$ and $\rho_{K'\setminus} = \rho_{E'\setminus}$ *or* $\rho_{K'\setminus} = \rho_{F'\setminus}$. □

Theorem 4.3. *Every intuitionistic fuzzy right $k-$ideal in a hemiring $\underset{\leftrightarrow}{H}$ is contained in the intersection of all intuitionistic fuzzy right $k-$ideals of $\underset{\leftrightarrow}{H}$.*

Proof. Suppose that the intuitionistic fuzzy right $k-$ideal of $\underset{\leftrightarrow}{H}$ is $J'^{\setminus} = (\mu_{J'\setminus}, \rho_{J'\setminus})$ and assume that $\{J'^{\setminus}_{\alpha'} : \alpha' \in \Omega\}$ be the family of all intuitionistic fuzzy right $k-$ideals of $\underset{\leftrightarrow}{H}$. Clearly

$$\mu_{J'\setminus} \leqq \bigwedge_{\alpha' \in \Omega} \mu_{J'^{\setminus}_{\alpha'}}, \rho_{J'\setminus} \geqq \bigwedge_{\alpha' \in \Omega} \rho_{J'^{\setminus}_{\alpha'}}.$$

We show that

$$\mu_{J'\setminus} \geqq \bigwedge_{\alpha' \in \Omega} \mu_{J'^{\setminus}_{\alpha'}}, \rho_{J'\setminus} \leqq \bigwedge_{\alpha' \in \Omega} \rho_{J'^{\setminus}_{\alpha'}}.$$

Suppose a be any element of $\underset{\leftrightarrow}{H}$ $\exists$, $K'^{\setminus} = (\mu_{K'\setminus}, \rho_{K'\setminus})$

$$\mu_{J'\setminus} \leqq \mu_{K'\setminus},\ \rho_{J'\setminus} \geqq \rho_{K'\setminus}\ \ \&\ \ \mu_{J'\setminus}(a'') = \mu_{K'\setminus}(a''),\ \rho_{J'\setminus}(a'') = \rho_{K'\setminus}(a'').$$

Hence

$$k \in \{A_{\alpha'} : \alpha' \in \Omega\}.$$

Hence

$$\mu_{K'\setminus} \geqq \bigwedge_{\alpha' \in \Omega} \mu_{J'^{\setminus}_{\alpha'}},\ \rho_{K'\setminus} \leqq \bigwedge_{\alpha' \in \Omega} \rho_{J'^{\setminus}_{\alpha'}}$$

So

$$\mu_{J'\setminus}(a'') = \mu_{K'\setminus}(a'') \geqq \bigwedge_{\alpha' \in \Omega} \mu_{J'^{\setminus}_{\alpha'}}(a''),\ \rho_{J'\setminus}(a'') = \rho_{K'\setminus}(a'') \leqq \bigwedge_{\alpha' \in \Omega} \rho_{J'^{\setminus}_{\alpha'}}(a''),$$

i.e.,

$$\mu_{J'\setminus} \geqq \bigwedge_{\alpha' \in \Omega} \mu_{J'^{\setminus}_{\alpha'}},\ \rho_{J'\setminus} \leqq \bigwedge_{\alpha' \in \Omega} \rho_{J'^{\setminus}_{\alpha'}}.$$

Hence

$$\mu_{J'\setminus} = \bigwedge_{\alpha' \in \Omega} \mu_{J'^{\setminus}_{\alpha'}},\ \rho_{J'\setminus} = \bigwedge_{\alpha' \in \Omega} \rho_{J'^{\setminus}_{\alpha'}}.$$

□

Theorem 4.4. *For a hemiring $\underset{\leftrightarrow}{H}$ including indentity given statements are equivalent:*

$\mathbf{N}_1$ $\underset{\leftrightarrow}{H}$ is right $k-weaky$ regular hemiring,

$\mathbf{N}_2$ Intuitionistic fuzzy right $k-$ideals in a hemiring $\underset{\leftrightarrow}{H}$ are all k-idempotent,

$\mathbf{N}_3$ $\mu_{K'\setminus} \odot_k \mu_{L'\setminus} = \mu_{K'\setminus} \wedge \mu_{L'\setminus}$, $\rho_{K'\setminus} \odot_k \rho_{L'\setminus} = \rho_{K'\setminus} \vee \rho_{L'\setminus}$ for all intuitionistic fuzzy right $k-$ideals $K^{'\setminus} = (\mu_{K'\setminus}, \rho_{K'\setminus})$ and all intuitionistic fuzzy two sided $k-$ideals $L^{'\setminus} = (\mu_{L'\setminus}, \rho_{L'\setminus})$ of $\underset{\leftrightarrow}{H}$.

$\mathbf{N}_4$ Each intuitionistic fuzzy right $k-$ideal of $\underset{\leftrightarrow}{H}$ is also intuitionistic fuzzy $k-semiprime$.

Proof. By theorem 3.6, $(N_1), (N_2), (N_3)$ are equivalent. If $L^{'\setminus} = (\mu_{L'\setminus}, \rho_{L'\setminus})$ is an intuitionistic fuzzy right $k-$ideal of $\underset{\leftrightarrow}{H}$, then $\mu_{K'\setminus} \odot_k \mu_{K'\setminus} \leqq \mu_{L'\setminus}$, $\rho_{K'\setminus} \odot_k \rho_{K'\setminus} \geqq \rho_{L'\setminus}$, where $K^{'\setminus} = (\mu_{K'\setminus}, \rho_{K'\setminus})$ is an intuitionistic fuzzy right $k-$ideal of $\underset{\leftrightarrow}{H}$. By (2) $\mu_{K'\setminus} \odot_k \mu_{K'\setminus} = \mu_{K'\setminus}, \rho_{K'\setminus} \odot_k \rho_{K'\setminus} = \rho_{K'\setminus}$ therefore $\mu_{K'\setminus} \leqq \mu_{L'\setminus}, \rho_{K'\setminus} \geqq \rho_{L'\setminus}$. So $L^{'\setminus} = (\mu_{L'\setminus}, \rho_{L'\setminus})$.

Conversely, if $L^{'\setminus} = (\mu_{L'\setminus}, \rho_{L'\setminus})$ is an intuitionistic fuzzy right $k-$ideal of $\underset{\leftrightarrow}{H}$, $\mu_{L'\setminus} \odot_k \mu_{L'\setminus}$, $\rho_{L'\setminus} \odot_k \rho_{L'\setminus}$ are intuitionistic fuzzy right $k-$ideals of $\underset{\leftrightarrow}{H}$ and by (N_4) $\mu_{L'\setminus} \odot_k \mu_{L'\setminus}$, $\rho_{L'\setminus} \odot_k \rho_{L'\setminus}$ are intuitionistic fuzzy $k-semiprime$ right $k-$ideal of $\underset{\leftrightarrow}{H}$. As $\mu_{L'\setminus} \odot_k \mu_{L'\setminus} \leqq \mu_{L'\setminus} \odot_k \mu_{L'\setminus}$, $\rho_{L'\setminus} \odot_k \rho_{L'\setminus} \geqq \rho_{L'\setminus} \odot_k \rho_{L'\setminus}$ we have $\mu_{L'\setminus} \leqq \mu_{L'\setminus} \odot_k \mu_{L'\setminus}$, $\rho_{L'\setminus} \geqq \rho_{L'\setminus} \odot_k \rho_{L'\setminus}$. But $\mu_{L'\setminus} \odot_k \mu_{L'\setminus} \leqq \mu_{L'\setminus}$, $\rho_{L'\setminus} \odot_k \rho_{L'\setminus} \geqq \rho_{L'\setminus}$ always. Hence $\mu_{L'\setminus} \odot_k \mu_{L'\setminus} = \mu_{L'\setminus}$, $\rho_{L'\setminus} \odot_k \rho_{L'\setminus} = \rho_{L'\setminus}$. □

Theorem 4.5. *If every intuitionistic fuzzy right $k-$ideal of a hemiring $\underset{\leftrightarrow}{H}$ is an intuitionistic fuzzy $k-prime$ right $k-$ideal. Then $\underset{\leftrightarrow}{H}$ is a right $k-weakly$ regular hemiring and the set of intuitionistic fuzzy $k-$ideals of $\underset{\leftrightarrow}{H}$ is totally ordered.*

Proof. Let $\underset{\leftrightarrow}{H}$ be a hemiring in which each intuitionistic fuzzy right $k-$ideal is intuitionistic fuzzy prime. Suppose $L^{'\setminus} = (\mu_{L'\setminus}, \rho_{L'\setminus})$ be an intuitionistic fuzzy right $k-$ideal of $\underset{\leftrightarrow}{H}$. Then $\mu_{L'\setminus} \odot_k \mu_{L'\setminus}$, $\rho_{L'\setminus} \odot_k \rho_{L'\setminus}$ are also intuitionistic fuzzy right $k-$ideal of $\underset{\leftrightarrow}{H}$.

As

$$\mu_{L'\setminus} \odot_k \mu_{L'\setminus} \leqq \mu_{L'\setminus} \odot_k \mu_{L'\setminus} \implies \mu_{L'\setminus} \leqq \mu_{L'\setminus} \odot_k \mu_{L'\setminus},$$

$$\rho_{L'\setminus} \odot_k \rho_{L'\setminus} \geqq \rho_{L'\setminus} \odot_k \rho_{L'\setminus} \implies \rho_{L'\setminus} \geqq \rho_{L'\setminus} \odot_k \rho_{L'\setminus}$$

But

$$\mu_{L'\setminus} \geqq \mu_{L'\setminus} \odot_k \mu_{L'\setminus}, \rho_{L'\setminus} \geqq \rho_{L'\setminus} \odot_k \rho_{L'\setminus} \text{ always.}$$

Hence

$$\mu_{L'\setminus} = \mu_{L'\setminus} \odot_k \mu_{L'\setminus}, \rho_{L'\setminus} = \rho_{L'\setminus} \odot_k \rho_{L'\setminus}.$$

So $\underset{\leftrightarrow}{H}$ is a right $k-weakly$ regular hemiring.

Suppose $A = (\mu_{L'\setminus}, \rho_{L'\setminus})$, $M^{'\setminus} = (\mu_{M'\setminus}, \rho_{M'\setminus})$ are intuitionistic fuzzy $k-$ideals of $\underset{\leftrightarrow}{H}$. Subsequently

$$\mu_{L'\setminus} \odot_k \mu_{M'\setminus} \leqq \mu_{L'\setminus} \wedge \mu_{M'\setminus},\ \rho_{L'\setminus} \odot_k \rho_{M'\setminus} \geqq \rho_{L'\setminus} \vee \rho_{M'\setminus}.$$

As $\mu_{L'\setminus} \wedge \mu_{M'\setminus}$, $\rho_{L'\setminus} \vee \rho_{M'\setminus}$ are intuitionistic fuzzy $k-$ideal of $\underset{\leftrightarrow}{H}$. Consequently

$$\mu_{L'\setminus} \leqq \mu_{L'\setminus} \wedge \mu_{M'\setminus} \ or\ \mu_{M'\setminus} \leqq \mu_{L'\setminus} \wedge \mu_{M'\setminus},$$

$$\rho_{L'\setminus} \geqq \rho_{L'\setminus} \vee \rho_{M'\setminus} \ or\ \rho_{M'\setminus} \geqq \rho_{L'\setminus} \vee \rho_{M'\setminus}.$$

Hence

$$\mu_{L'\backslash} \leqq \mu_{M'\backslash} \ or\ \mu_{M'\backslash} \leqq \mu_{L'\backslash},$$
$$\rho_{L'\backslash} \geqq \rho_{M'\backslash} \ or\ \rho_{M'\backslash} \geqq \rho_{L'\backslash}.$$

□

Theorem 4.6. *If the set of all intuitionistic fuzzy right k−ideals of a right k − weakly regular hemiring $\underset{\leftrightarrow}{H}$ is totally ordered, then every intuitionistic fuzzy right k−ideal of $\underset{\leftrightarrow}{H}$ is an intuitionistic fuzzy k-prime right k−ideal of $\underset{\leftrightarrow}{H}$.*

Proof. Suppose $L^{'\backslash} = (\mu_{L'\backslash}, \rho_{L'\backslash})$, $M^{'\backslash} = (\mu_{M'\backslash}, \rho_{M'\backslash})$, $N^{'\backslash} = (\mu_{N'\backslash}, \rho_{N'\backslash})$ are intuitionistic fuzzy right k−ideals of $\underset{\leftrightarrow}{H}$ as $\mu_{L'\backslash} \odot_k \mu_{M'\backslash} \leqq \mu_{N'\backslash}, \rho_{M'\backslash} \odot_k \rho_{M'\backslash} \geqq \rho_{N'\backslash}$. Therefore the set of all intuitionistic fuzzy right k−ideals of a right k − *weakly* regular hemiring $\underset{\leftrightarrow}{H}$ is totally ordered, since $\mu_{L'\backslash} \leqq \mu_{M'\backslash}$ or $\mu_{M'\backslash} \leqq \mu_{L'\backslash}, \rho_{L'\backslash} \geqq \rho_{M'\backslash}$ or $\rho_{M'\backslash} \geqq \rho_{L'\backslash}$. Case 1, $\mu_{L'\backslash} \leqq \mu_{M'\backslash}$ then $\mu_{L'\backslash} = \mu_{L'\backslash} \odot_k \mu_{L'\backslash} \leqq \mu_{L'\backslash} \odot_k \mu_{M'\backslash} \leqq \mu_{N'\backslash}$ and $\rho_{L'\backslash} \geqq \rho_{M'\backslash}$ then $\rho_l = \rho_{L'\backslash} \odot_k \rho_{L'\backslash} \geqq \rho_{L'\backslash} \odot_k \rho_{M'\backslash} \geqq \rho_{N'\backslash}$.

Case 2, $\mu_{M'\backslash} \leqq \mu_{L'\backslash}$ then $\mu_{M'\backslash} = \mu_{M'\backslash} \odot_k \mu_{M'\backslash} \leqq \mu_{L'\backslash} \odot_k \mu_{M'\backslash} \leqq \mu_{N'\backslash}$ and $\rho_{M'\backslash} \geqq \rho_{L'\backslash}$ then $\rho_{M'\backslash} = \rho_{M'\backslash} \odot_k \rho_{M'\backslash} \geqq \rho_{L'\backslash} \odot_k \rho_{M'\backslash} \geqq \rho_{N'\backslash}$.

Hence $N^{'\backslash} = (\mu_{N'\backslash}, \rho_{N'\backslash})$ is an intuitionistic fuzzy k − *prime* right k − *ideal* of $\underset{\leftrightarrow}{H}$. □

5. Intuitionistic fuzzy right pure *k*-ideals

Definition 5.1. An intuitionistic fuzzy k − *ideal* $N^{'\backslash} = (\mu_{N'\backslash}, \rho_{N'\backslash})$ of a hemiring $\underset{\leftrightarrow}{H}$ is called right pure if and only if $\mu_{Y'\backslash} \wedge \mu_{N'\backslash} = \mu_{Y'\backslash} \odot_k \mu_{N'\backslash}$, $\rho_{Y'\backslash} \vee \rho_{N'\backslash} = \rho_{Y'\backslash} \odot_k \rho_{N'\backslash}$ for every intuitionistic fuzzy right k−ideal $Y^{'\backslash} = (\mu_{Y'\backslash}, \rho_{Y'\backslash})$ of $\underset{\leftrightarrow}{H}$.

Proposition 5.2. *The characteristic function of a nonempty subset $Z^{'\backslash}$ of a hemiring $\underset{\leftrightarrow}{H}$ is its right pure intuitutionistic fuzzy k−ideal if and only if $Z^{'\backslash}$ is a right pure k−ideal of $\underset{\leftrightarrow}{H}$.*

Proof. Suppose $Z^{'\backslash}$ be a right pure k−ideal of $\underset{\leftrightarrow}{H}$. Then $\chi_{Z'\backslash}$ is an intuitionistic fuzzy k−ideal of $\underset{\leftrightarrow}{H}$. To show that $\chi_{Z'\backslash}$ is a right pure we have to prove that for any intuitionistic fuzzy k−ideal $L^{'\backslash} = (\mu_{L'\backslash}, \rho_{L'\backslash})$ of $\underset{\leftrightarrow}{H}$,

$$\mu_{L'\backslash} \wedge \chi_{Z'\backslash} = \mu_A \odot_k \chi_{Z'\backslash},\ \rho_{L'\backslash} \vee \chi_{Z'\backslash} = \rho_{L'\backslash} \odot_k \chi_{Z'\backslash}.$$

Case (1), $u^{'''} \notin Z^{'\backslash}$, then

$$(\mu_{L'\backslash} \wedge \chi_{Z'\backslash})(u^{'''}) = \mu_{L'\backslash}(u^{'''}) \wedge \chi_{Z'\backslash}(u^{'''}) = 0 \leqq (\mu_{L'\backslash} \odot_k \chi_{Z'\backslash})(u^{'''})$$
$$(\rho_{L'\backslash} \vee \chi_{Z'\backslash})(u^{'''}) = \rho_{L'\backslash}(u^{'''}) \wedge \chi_{Z'\backslash}(u^{'''}) = 0 \geqq (\rho_{L'\backslash} \odot_k \chi_{Z'\backslash})(u^{'''})$$

Case (2), $u^{'''} \in Z^{'\backslash}$, since $Z^{'\backslash}$ is right pure k−ideal of $\underset{\leftrightarrow}{H}$, $\exists\, c, d \in Z^{'\backslash}$, such that $u^{'''} + u^{'''}c = u^{'''}d$. As $u^{'''}, c, d \in Z^{'\backslash}$, $\implies \chi_{Z'\backslash}(u^{'''}) = \chi_{Z'\backslash}(u^{'''}) = \chi_{Z'\backslash}(u^{'''}) = 1$.

$$(\mu_{L'\backslash} \odot_k \chi_{Z'\backslash})(u^{'''}) = \bigvee_{u^{'''} + \sum_{_*\sigma=1}^{\kappa} c_{_*\sigma} d_{_*\sigma} = \sum_{_*\varrho=1}^{\tau} c'_{_*\varrho} d'_{_*\varrho}} \left\{ \begin{array}{c} \bigwedge_{_*\sigma=1}^{\kappa} (\mu_{L'\backslash}(c_{_*\sigma}) \wedge \chi_{Z'\backslash}(d_{_*\sigma})) \\ \wedge \bigwedge_{_*\varrho=1}^{\tau} (\mu_{L'\backslash}(c'_{_*\varrho}) \wedge \chi_{Z'\backslash}(d'_{_*\varrho})) \end{array} \right\}$$

$$\geqq \min\left[\mu_{L'\backslash}(u''') \wedge \chi_{Z'\backslash}(c) \wedge \mu_{L'\backslash}(u''') \wedge \chi_{Z'\backslash}(d)\right]$$
$$\geqq \min\left[\mu_{L'\backslash}(u''') \wedge \chi_{Z'\backslash}(u''') \wedge \mu_{L'\backslash}(u''') \wedge \chi_{Z'\backslash}(u''')\right]$$
$$\geqq \mu_{L'\backslash}(u''') \wedge \chi_{Z'\backslash}(u''') = (\mu_{L'\backslash} \wedge \chi_{Z'\backslash})(u''')$$

and

$$(\rho_{L'\backslash} \odot_k \chi_{Z'\backslash})(u''') = \bigwedge_{u'''+\sum_{*\sigma=1}^{\kappa} c_{*\sigma}d_{*\sigma}=\sum_{*\varrho=1}^{b} c'_{*\varrho}d'_{*\varrho}} \left\{ \begin{array}{c} \bigvee_{*\sigma=1}^{\kappa} (\rho_{L'\backslash}(c_{*\sigma}) \vee \chi_{Z'\backslash}(d_{*\sigma})) \\ \vee \bigvee_{*\varrho=1}^{\tau} (\rho_{L'\backslash}(c'_{*\varrho}) \vee \chi_{Z'\backslash}(d'_{*\varrho})) \end{array} \right\}$$
$$\leqq \max\left[\rho_{L'\backslash}(u''') \vee \chi_{Z'\backslash}(c) \vee \rho_{L'\backslash}(u''') \vee \chi_{Z'\backslash}(d)\right]$$
$$\leqq \max\left[\rho_{L'\backslash}(u''') \vee \chi_{Z'\backslash}(u''') \vee \rho_{L'\backslash}(u''') \vee \chi_{Z'\backslash}(u''')\right]$$
$$\leqq \rho_{L'\backslash}(u''') \vee \chi_{Z'\backslash}(u''') = (\rho_{L'\backslash} \vee \chi_{Z'\backslash})(u''').$$

Now

$$\mu_{L'\backslash} \odot_k \chi_{Z'\backslash} \geqq \mu_{L'\backslash} \wedge \chi_{Z'\backslash},\ \rho_{L'\backslash} \vee \chi_{Z'\backslash} \geqq \rho_{L'\backslash} \odot_k \chi_{Z'\backslash}.$$

But

$$\mu_{L'\backslash} \odot_k \chi_{Z'\backslash} \leqq \mu_{L'\backslash} \wedge \chi_{Z'\backslash},\ \rho_{L'\backslash} \vee \chi_{Z'\backslash} \leqq \rho_{L'\backslash} \odot_k \chi_{Z'\backslash}$$

is always true. Then,

$$\mu_{L'\backslash} \odot_k \chi_{Z'\backslash} = \mu_{L'\backslash} \wedge \chi_{Z'\backslash},\ \rho_{L'\backslash} \vee \chi_{Z'\backslash} = \rho_{L'\backslash} \odot_k \chi_{Z'\backslash}.$$

Hence $\chi_{Z'\backslash}$ is right pure intuitionistic fuzzy $k-$ideal of $\underset{\leftrightarrow}{H}$. □

Proposition 5.3. *Let $L'^{\backslash} = (\mu_{L'\backslash}, \rho_{L'\backslash})$, $M'^{\backslash} = (\mu_{M'\backslash}, \rho_{M'\backslash})$ are right pure intuitionistic fuzzy $k-$ideals of $\underset{\leftrightarrow}{H}$, then so are $\mu_{L'\backslash} \wedge \mu_{M'\backslash}, \rho_{L'\backslash} \vee \rho_{M'\backslash}$.*

Proof. We know that $\mu_{L'\backslash} \wedge \mu_{M'\backslash}, \rho_{L'\backslash} \vee \rho_{M'\backslash}$ are intuitionistic fuzzy $k-$ideal of $\underset{\leftrightarrow}{H}$. We shall prove any intuitionistic fuzzy right $k-$ideal $N = (\mu_N, \rho_N)$ of $\underset{\leftrightarrow}{H}$,

$$\mu_N \odot_k (\mu_{L'\backslash} \wedge \mu_{M'\backslash}) = \mu_N \wedge (\mu_{L'\backslash} \wedge \mu_{M'\backslash}),$$

$$\rho_N \odot_k (\rho_{L'\backslash} \vee \rho_{M'\backslash}) = \rho_N \vee (\rho_{L'\backslash} \vee \rho_{M'\backslash}).$$

Since $M'^{\backslash} = (\mu_{M'\backslash}, \rho_{M'\backslash})$ is right pure intuitionistic fuzzy $k-$ideal of $\underset{\leftrightarrow}{H}$ therefore

$$\mu_{L'\backslash} \wedge \mu_{M'\backslash} = \mu_{L'\backslash} \odot_k \mu_{M'\backslash}, \rho_{L'\backslash} \vee \rho_{M'\backslash} = \rho_{L'\backslash} \odot_k \rho_{M'\backslash}.$$

Hence

$$\mu_N \odot_k (\mu_{L'\backslash} \odot_k \mu_{M'\backslash}) = \mu_N \odot_k (\mu_{L'\backslash} \wedge \mu_{M'\backslash}),$$
$$\rho_N \odot_k (\rho_{L'\backslash} \odot_k \rho_{M'\backslash}) = \rho_N \odot_k (\rho_{L'\backslash} \vee \rho_{M'\backslash}).$$

Also

$$\mu_N \wedge (\mu_{L'\backslash} \wedge \mu_{M'\backslash}) = (\mu_N \wedge \mu_{L'\backslash}) \wedge \mu_{M'\backslash} = (\mu_N \odot_k \mu_{L'\backslash}) \wedge \mu_{M'\backslash}$$

$$= (\mu_N \odot_k \mu_{L'\backslash}) \odot_k \mu_{M'\backslash} = \mu_N \odot_k (\mu_{L'\backslash} \odot_k \mu_{M'\backslash}),$$

$$\rho_N \vee (\rho_{L'\backslash} \vee \rho_{M'\backslash}) = (\rho_N \vee \rho_{L'\backslash}) \vee \rho_{M'\backslash} = (\rho_N \odot_k \rho_{L'\backslash}) \vee \rho_{M'\backslash} = (\rho_N \odot_k \rho_{L'\backslash}) \odot_k \rho_{M'\backslash}$$
$$= \rho_N \odot_k (\rho_{L'\backslash} \odot_k \rho_{M'\backslash}).$$

since $\mu_N \odot_k \mu_{L'\backslash}, \rho_N \odot_k \rho_{L'\backslash}$ are intuitionistic fuzzy right $k-$ideal of $\underset{\leftrightarrow}{H}$.
Thus

$$\mu_N \odot_k (\mu_{L'\backslash} \wedge \mu_{M'\backslash}) = \mu_N \wedge (\mu_{L'\backslash} \wedge \mu_{M'\backslash}),$$
$$\rho_N \odot_k (\rho_{L'\backslash} \vee \rho_{M'\backslash}) = \rho_N \vee (\rho_{L'\backslash} \vee \rho_{M'\backslash}).$$

□

Proposition 5.4. *For a hemiring $\underset{\leftrightarrow}{H}$ with identity the following statements are equivalent:*

N$_1$ $\underset{\leftrightarrow}{H}$ is right $k-$ *weaky* regular hemiring.

N$_2$ all right $k-$ideals of $\underset{\leftrightarrow}{H}$ are $k-$ *idempotent*.

N$_3$ every $k-$ideal of $\underset{\leftrightarrow}{H}$ is right pure.

N$_4$ all intuitionistic fuzzy right $k-$ideals of $\underset{\leftrightarrow}{H}$ are $k-$ *idempotent*.

N$_5$ every intuitionistic fuzzy $k-$ideal of $\underset{\leftrightarrow}{H}$ is right pure.

N$_6$ if $\underset{\leftrightarrow}{H}$ is commutative, then $\underset{\leftrightarrow}{H}$ is $k-$ *regular*.

Proof. $(N_1), (N_2), (N_3)$ are equivalent by Proposition 5.2 and $(N_1), (N_4)$ are equivalent by theorem 3.6.
$(N_4) \implies (N_5)$

Suppose $L^{'\backslash} = (\mu_{L'\backslash}, \rho_{L'\backslash})$ and $M^{'\backslash} = (\mu_{M'\backslash}, \rho_{M'\backslash})$ are intuitionistic fuzzy right and two sided $k-$ideals of $\underset{\leftrightarrow}{H}$.

Then $\mu_{L'\backslash} \wedge \mu_{M'\backslash},\ \rho_{L'\backslash} \vee \rho_{M'\backslash}$ are intuitionistic fuzzy $k-$ideals of $\underset{\leftrightarrow}{H}$.

$\mu_{L'\backslash} \odot_k \mu_{M'\backslash} \leqq \mu_{L'\backslash} \wedge \mu_{M'\backslash},\ \rho_{L'\backslash} \odot_k \rho_{M'\backslash} \geqq \rho_{L'\backslash} \vee \rho_{M'\backslash}$. By supposition, $\mu_{L'\backslash} \wedge \mu_{M'\backslash} = (\mu_{L'\backslash} \wedge \mu_{M'\backslash}) \odot_k (\mu_{L'\backslash} \wedge \mu_{M'\backslash}) \leqq \mu_{L'\backslash} \odot_k \mu_{M'\backslash},\ \rho_{L'\backslash} \vee \rho_{M'\backslash} = (\rho_{L'\backslash} \vee \rho_{M'\backslash}) \odot_k (\rho_{L'\backslash} \vee \rho_{M'\backslash}) \geqq \rho_{L'\backslash} \odot_k \rho_{M'\backslash}$. Hence $\mu_{L'\backslash} \odot_k \mu_{M'\backslash} = \mu_{L'\backslash} \wedge \mu_{M'\backslash},\ \rho_{L'\backslash} \odot_k \rho_{M'\backslash} = \rho_{L'\backslash} \vee \rho_{M'\backslash}$. Thus $M^{'\backslash} = (\mu_{M'\backslash}, \rho_{M'\backslash})$ is right pure.

$(N_5) \implies (N_1)$

$L^{'\backslash}$ be a right $k-$ *ideal* of $\underset{\leftrightarrow}{H}$ and $M^{'\backslash}$ be a two sided $k-$ *ideal* of $\underset{\leftrightarrow}{H}$ then the characteristic functions $\chi_{L'\backslash}$ is intuitionistic fuzzy two-sided $k-$ *ideal* of $\underset{\leftrightarrow}{H}$ and $\chi_{M'\backslash}$ is intuitionistic fuzzy $k-$ *ideal* of $\underset{\leftrightarrow}{H}$.

Hence $\chi_{L'\backslash} \odot_k \chi_{M'\backslash} = \chi_{L'\backslash} \wedge \chi_{M'\backslash}$.

(N_6) is obvious. □

6. Purely prime intuitionistic fuzzy k-ideal

Proposition 6.1. *Suppose $\underset{\leftrightarrow}{H}$ be a right $k-$ weakly regular hemiring. If $A^{'\backslash} = (\mu_{A'\backslash}, \rho_{A'\backslash})$ is a right pure intuitionistic fuzzy $k-$ideal of $\underset{\leftrightarrow}{H}$ with $\mu_{A'\backslash}(a^{'}) = s, \rho_{A'\backslash}(a^{'}) = s$ where $a^{'} \in \underset{\leftrightarrow}{H}$ and $s \in [0, 1]$, then $\exists$ a purely prime intuitionistic fuzzy $k-$ideal $B^{'\backslash} = (\mu_{B'\backslash}, \rho_{B'\backslash})$ of $\underset{\leftrightarrow}{H}$ such that $\mu_{A'\backslash} \leqq \mu_{B'\backslash},\ \rho_{A'\backslash} \geqq \rho_{B'\backslash}$ and $\mu_{A'\backslash}(a^{'}) = s, \rho_{A'\backslash}(a^{'}) = s$.*

Proof. Suppose $Y = \{C^{'\backslash} = (\mu_{C'\backslash}, \rho_{C'\backslash}) : \mu_{C'\backslash}(b) = \beta,\ \rho_{C'\backslash}(b) = \beta\ \ \&\ \ \mu_{A'\backslash} \leqq \mu_{C'\backslash},\ \rho_{A'\backslash} \geqq \rho_{C'\backslash}\}$, where $C^{'\backslash}$ is a right pure intuitionistic fuzzy $k-$ideal of $\underset{\leftrightarrow}{H}$. Then $Y \neq \{\}$, since $A^{'\backslash} \in Y$. Suppose $F \subseteq Y$, F be a totally odered set, say $F = \{A^{'\backslash}_{\xi''} = (\mu_{A'\backslash_{\xi''}}, \rho_{A'\backslash_{\xi''}}) : \xi'' \in F\}$. First we prove $\bigvee_{\xi'' \in F} \mu_{A'\backslash_{\xi''}},\ \bigwedge_{\xi'' \in F} \rho_{A'\backslash_{\xi''}}$ is an intuitionistic fuzzy right $k-$ideal of $\underset{\leftrightarrow}{H}$. For any $y''',\ s''' \in \underset{\leftrightarrow}{H}$, we have

$$(\bigvee_{\xi'' \in F} \mu_{A'\backslash_{\xi''}})(y''') = \bigvee_{\xi'' \in F} (\mu_{A'\backslash_{\xi''}}(y''')) \leqq \bigvee_{\xi'' \in F} (\mu_{A'\backslash_{\xi''}}(y''' s''')) = (\bigvee_{\xi'' \in F} \mu_{A'\backslash_{\xi''}})(y''' s'''),$$

$$(\bigwedge_{\xi'' \in F} \rho_{A'\backslash_{\xi''}})(y''') = \bigwedge_{\xi'' \in F} (\rho_{A'\backslash_{\xi''}}(y''')) \geqq \bigwedge_{\xi'' \in F} (\rho_{A'\backslash_{\xi''}}(y''' s''')) = (\bigwedge_{\xi'' \in F} \rho_{A'\backslash_{\xi''}})(y''' s''').$$

Suppose $y''',\ z^{\upsilon'''} \in \underset{\leftrightarrow}{H}$,

$$\begin{aligned}
(\bigvee_{\xi'' \in F} \mu_{A'\backslash_{\xi''}})(y''') \wedge (\bigvee_{\varrho'' \in F} \mu_{A'\backslash_{\varrho''}})(z^{\upsilon'''}) &= (\bigvee_{\xi'' \in F} \mu_{A'\backslash_{\xi''}}(y''') \wedge (\bigvee_{\varrho'' \in F} \mu_{A'\backslash_{\varrho''}}(z^{\upsilon'''})) \\
&= \bigvee_{\varrho'' \in F} (\bigvee_{\xi'' \in F} (\mu_{A'\backslash_{\xi''}}(y''') \wedge \mu_{A'\backslash_{\varrho''}}(z^{\upsilon'''}))) \\
&\leqq \bigvee_{\varrho'' \in F} (\bigvee_{\xi'' \in F} (\max\{\mu_{A'\backslash_{\xi''}}(y'''), \mu_{A'\backslash_{\varrho''}}(y''')\} \\
&\wedge \max\{\mu_{A'\backslash_{\xi''}}(z^{\upsilon'''}), \mu_{A'\backslash_{\varrho''}}(z^{\upsilon'''})\}) \\
&\leqq \bigvee_{\varrho'' \in F} (\bigvee_{\xi'' \in F} (\max\{\mu_{A'\backslash_{\xi''}}(y''' + z^{\upsilon'''}), \mu_{A'\backslash_{\varrho''}}(y''' + z^{\upsilon'''})\}) \\
&\leqq \bigvee_{\xi'', \varrho'' \in F} \max\{\mu_{A'\backslash_{\xi''}}(y''' + z^{\upsilon'''}), \mu_{A'\backslash_{\varrho''}}(y''' + z^{\upsilon'''})\} \\
&\leqq (\bigvee_{\xi'' \in F} \mu_{A'\backslash_{\xi''}})(y''' + z^{\upsilon'''})
\end{aligned}$$

and

$$\begin{aligned}
(\bigwedge_{\xi'' \in F} \rho_{A'\backslash_{\xi''}})(y''') \vee (\bigwedge_{\varrho'' \in F} \rho_{A'\backslash_{\varrho''}})(z^{\upsilon'''}) &= (\bigwedge_{\xi'' \in F} \rho_{A'\backslash_{\xi''}}(y''')) \vee (\bigwedge_{\varrho'' \in F} \rho_{A'\backslash_{\varrho''}}(z^{\upsilon'''})) \\
&= \bigwedge_{\varrho'' \in F} (\bigwedge_{\xi'' \in F} (\rho_{A'\backslash_{\xi''}}(y''') \vee \rho_{A'\backslash_{\varrho''}}(z^{\upsilon'''}))) \\
&\geqq \bigwedge_{\varrho'' \in F} (\bigwedge_{\xi'' \in F} (\min\{\rho_{A'\backslash_{\xi''}}(y'''), \rho_{A'\backslash_{\varrho''}}(y''')\} \\
&\vee \min\{\rho_{A'\backslash_{\xi''}}(z^{\upsilon'''}), \rho_{A'\backslash_{\varrho''}}(z^{\upsilon'''})\}) \\
&\geqq \bigwedge_{\varrho'' \in F} (\bigwedge_{\xi'' \in F} (\min\{\rho_{A'\backslash_{\xi''}}(y''' + z^{\upsilon'''}), \rho_{A'\backslash_{\varrho''}}(y''' + z^{\upsilon'''})\}) \\
&\geqq \bigwedge_{\xi'', \varrho'' \in F} \min\{\rho_{A'\backslash_{\xi''}}(y''' + z^{\upsilon'''}), \rho_{A'\backslash_{\varrho''}}(y''' + z^{\upsilon'''})\}
\end{aligned}$$

$$\geqq (\bigwedge_{\xi'' \in F} \rho_{A'^{\setminus}_{\xi''}})(y''' + z^{\upsilon'''}).$$

Now

Let $y''' + c''' = d'''$ where $c''', d''' \in \underset{\leftrightarrow}{H}$ then

$$\begin{aligned}
(\bigvee_{\xi'' \in F} \mu_{A'^{\setminus}_{\xi''}})(c''') \wedge (\bigvee_{\varrho'' \in F} \mu_{A'^{\setminus}_{\varrho''}})(d''') &= (\bigvee_{\xi'' \in F} \mu_{A'^{\setminus}_{\xi''}}(c''')) \wedge (\bigvee_{\varrho'' \in F} \mu_{A'^{\setminus}_{\varrho''}}(d''')) \\
&= \bigvee_{\varrho'' \in F} (\bigvee_{\xi'' \in F} (\mu_{A'^{\setminus}_{\xi''}}(c''') \wedge \mu_{A'^{\setminus}_{\varrho''}}(d'''))) \\
&\leqq \bigvee_{\varrho'' \in F} (\bigvee_{\xi'' \in F} (\max\{\mu_{A'^{\setminus}_{\xi''}}(c'''), \mu_{A'^{\setminus}_{\varrho''}}(c''')\} \\
&\wedge \max\{\mu_{A'^{\setminus}_{\xi''}}(d'''), \mu_{A'^{\setminus}_{\varrho''}}(d''')\}) \\
&\leqq \bigvee_{\xi'', \varrho'' \in F} \max\{\mu_{A'^{\setminus}_{\xi''}}(y'''), \mu_{A'^{\setminus}_{\varrho''}}(y''')\} \\
&\leqq \bigvee_{\xi'' \in F} \mu_{A'^{\setminus}_{\xi''}}(y''')
\end{aligned}$$

and

$$\begin{aligned}
(\bigwedge_{\xi'' \in F} \rho_{A'^{\setminus}_{\xi''}})(c''') \vee (\bigwedge_{\varrho'' \in F} \rho_{A'^{\setminus}_{\varrho''}})(d''') &= (\bigwedge_{\xi'' \in F} \rho_{A'^{\setminus}_{\xi''}}(c''')) \vee (\bigwedge_{\varrho'' \in F} \rho_{A'^{\setminus}_{\varrho''}}(d''')) \\
&= \bigwedge_{\varrho'' \in F} (\bigwedge_{\xi'' \in F} (\rho_{A'^{\setminus}_{\xi''}}(c''') \vee \rho_{A'^{\setminus}_{\varrho''}}(d'''))) \\
&\geqq \bigwedge_{\varrho'' \in F} (\bigwedge_{\xi'' \in F} (\min\{\rho_{A'^{\setminus}_{\xi''}}(c'''), \rho_{A'^{\setminus}_{\varrho''}}(c''')\} \\
&\vee \min\{\rho_{A'^{\setminus}_{\xi''}}(d'''), \rho_{A'^{\setminus}_{\varrho''}}(d''')\}) \\
&\geqq \bigwedge_{\xi'', \varrho'' \in F} \min\{\rho_{A'^{\setminus}_{\xi''}}(y'''), \rho_{A'^{\setminus}_{\varrho''}}(y''')\} \\
&\geqq \bigwedge_{\xi'' \in F} \rho_{A'^{\setminus}_{\xi''}})(y''').
\end{aligned}$$

Thus $\bigvee_{\xi'' \in F} \mu_{A'^{\setminus}_{\xi''}}$, $\bigwedge_{\xi'' \in F} \rho_{A'^{\setminus}_{\xi''}}$ is an intuitionistic fuzzy right $k-$ideal of $\underset{\leftrightarrow}{H}$.

Clearly $\mu_{A'^{\setminus}} \leqslant \bigvee_{\xi'' \in F} \mu_{A'^{\setminus}_{\xi''}}$, $\rho_{A'^{\setminus}} \geqq \bigwedge_{\xi'' \in F} \rho_{A'^{\setminus}_{\xi''}}$ and $\bigvee_{\xi'' \in F} \mu_{A'^{\setminus}_{\xi''}}(a) = \alpha$, $\bigwedge_{\xi'' \in F} \rho_{A'^{\setminus}_{\xi''}}(a) = \alpha$.

Thus $\bigvee_{\xi'' \in F} \mu_{A'^{\setminus}_{\xi''}}$ is the least upper bound, $\bigwedge_{\xi'' \in F} \rho_{A'^{\setminus}_{\xi''}}$ is the greatest lower bound. Hence by Zorn's lemma there exists a right pure intuitionistic fuzzy $k-$ideal $B'^{\setminus} = (\mu_{B'^{\setminus}}, \rho_{B'^{\setminus}})$ of $\underset{\leftrightarrow}{H}$. By the property that $\mu_{A'^{\setminus}} \leqq \mu_{B'^{\setminus}}$, $\rho_{A'^{\setminus}} \geqq \rho_{B'^{\setminus}}$ and $\mu_{B'^{\setminus}}(b') = \beta$, $\rho_{B'^{\setminus}}(b') = \beta$, $B'^{\setminus} = (\mu_{B'^{\setminus}}, \rho_{B'^{\setminus}})$ is maximal.

We shall prove that $B'^{\setminus} = (\mu_{B'^{\setminus}}, \rho_{B'^{\setminus}})$ is purely prime intuitionistic fuzzy $k-$ideal of $\underset{\leftrightarrow}{H}$. Suppose $C'^{\setminus} = (\mu_{C'^{\setminus}}, \rho_{C'^{\setminus}})$ and $D'^{\setminus} = (\mu_{D'^{\setminus}}, \rho_{D'^{\setminus}})$ are right pure intuitionistic fuzzy $k-$ideals$'''$ of $\underset{\leftrightarrow}{H}$. Let $\mu_{B'^{\setminus}} = \mu_{C'^{\setminus}} \wedge \mu_{D'^{\setminus}}$ and $\rho_{B'^{\setminus}} = \rho_{C'^{\setminus}} \vee \rho_{D'^{\setminus}}$. Thus $\mu_{B'^{\setminus}} \leqq \mu_{C'^{\setminus}}$, $\mu_{B'^{\setminus}} = \mu_{D'^{\setminus}}$ and $\rho_{B'^{\setminus}} \geqq \rho_{C'^{\setminus}}$, $\rho_{B'^{\setminus}} \geqq \rho_{D'^{\setminus}}$ .We claim that either $\mu_{B'\backslash} = \mu_{C'\backslash}$ or $\mu_{B'\backslash} = \mu_{D'\backslash}$ and $\rho_{B'\backslash} = \rho_{C'\backslash}$ or $\rho_{B'\backslash} = \rho_{D'\backslash}$. Suppose $\mu_{B'\backslash} \neq \mu_{C'\backslash}$ & $\mu_{B'\backslash} \neq \mu_{D'\backslash}$ and $\rho_{B'\backslash} \neq \rho_{C'\backslash}$ & $\rho_{B'\backslash} \neq \rho_{D'\backslash}$. By the property that $\mu_{A'\backslash}(b^{'}) = \beta,\ \rho_{A'\backslash}(b^{'}) = \beta$, $B^{'\backslash}$ is maximal and since $\mu_{B'\backslash} \lneqq \mu_{C'\backslash}$, $\mu_{B'\backslash} \lneqq \mu_{D'\backslash}$ and $\rho_{B'\backslash} \gneqq \rho_{C'\backslash}$, $\rho_{B'\backslash} \gneqq \rho_{D'\backslash}$, so $\mu_{C'\backslash}(b^{'}) \neq \beta,\ \rho_{C'\backslash}(b^{'}) \neq \beta$ and $\mu_{D'\backslash}(b^{'}) \neq \beta,\ \rho_{D'\backslash}(b^{'}) \neq \beta$. Hence

$$\beta = \mu_{B'\backslash}(b^{'}) = (\mu_{C'\backslash} \wedge \mu_{D'\backslash})\left(b^{'}\right) = (\mu_{C'\backslash})\left(b^{'}\right) \wedge (\mu_{D'\backslash})\left(b^{'}\right) \neq \beta$$

and

$$\beta = \rho_{B'\backslash}(b^{'}) = (\rho_{C'\backslash} \vee \rho_{D'\backslash})\left(b^{'}\right) = (\rho_{C'\backslash})\left(b^{'}\right) \vee (\rho_{D'\backslash})(b') \neq \beta$$

contrdiction. Hence $\mu_{B'\backslash} = \mu_{C'\backslash}$ or $\mu_{B'\backslash} = \mu_{D'\backslash}$ and $\rho_{B'\backslash} = \rho_{C'\backslash}$ or $\rho_{B'\backslash} = \rho_{D'\backslash}$. Thus $B^{'\backslash} = (\mu_{B'\backslash}, \rho_{B'\backslash})$ is purely prime intutionistic fuzzy $k-$ideal of $\underset{\leftrightarrow}{H}$. □

Proposition 6.2. *Suppose $\underset{\leftrightarrow}{H}$ be a right $k-$weakly regular hemiring.Then each proper intuitionistic fuzzy right pure $k-$ideal is the intersection of all purely prime intuitionistic fuzzy $k-$ideals of $\underset{\leftrightarrow}{H}$ which contain it.*

Proof. Suppose $A^{'\backslash} = (\mu_{A'\backslash}, \rho_{A'\backslash})$ be the intuitionistic fuzzy right right pure $k-$ideal of $\underset{\leftrightarrow}{H}$ and let $\{A_{\alpha}^{'\backslash} : \alpha \in \Omega\}$ be the family of all intuitionistic fuzzy purely prime $k-$ideals of $\underset{\leftrightarrow}{H}$ which contain $A^{'\backslash}$. Clearly

$$\mu_{A'\backslash} \leqq \bigwedge_{\alpha\in\Omega} \mu_{A_{\alpha}^{'\backslash}}, \rho_{A'\backslash} \geqq \bigwedge_{\alpha\in\Omega} \rho_{A_{\alpha}^{'\backslash}}.$$

We show that

$$\mu_{A'\backslash} \geqq \bigwedge_{\alpha\in\Omega} \mu_{A_{\alpha}^{'\backslash}}, \rho_{A'\backslash} \leqq \bigwedge_{\alpha\in\Omega} \rho_{A_{\alpha}^{'\backslash}}.$$

Suppose a be any element of $\underset{\leftrightarrow}{H}$, $\exists$ an intuitionistic fuzzy purely prime $k-$ideal $B^{'\backslash} = (\mu_{B'\backslash}, \rho_{B'\backslash})$ such that

$$\mu_{A'\backslash} \leqq \mu_{B'\backslash}, \rho_{A'\backslash} \geqq \rho_{B'\backslash} \ \ \& \ \ \mu_{A'\backslash}(a) = \mu_{B'\backslash}(a), \rho_{A'\backslash}(a) = \rho_{B'\backslash}(a).$$

$$B^{'\backslash} \in \{A_{\alpha}^{'\backslash} : \alpha \in \Omega\}.$$

Hence

$$\mu_{B'\backslash} \geqq \bigwedge_{\alpha\in\Omega} \mu_{A_{\alpha}^{'\backslash}}, \rho_{B'\backslash} \leqq \bigwedge_{\alpha\in\Omega} \rho_{A_{\alpha}^{'\backslash}}$$

So

$$\mu_{A'\backslash}(a) = \mu_{B'\backslash}(a) \geqq \bigwedge_{\alpha\in\Omega} \mu_{A_{\alpha}^{'\backslash}}(a), \rho_{A'\backslash}(a) = \rho_{B'\backslash}(a) \leqq \bigwedge_{\alpha\in\Omega} \rho_{A_{\alpha}^{'\backslash}}(a),$$

i.e.,

$$\mu_{A'\backslash} \geqq \bigwedge_{\alpha\in\Omega} \mu_{A_{\alpha}^{'\backslash}}, \rho_{A'\backslash} \leqq \bigwedge_{\alpha\in\Omega} \rho_{A_{\alpha}^{'\backslash}}.$$

Hence

$$\mu_{A'\backslash} = \bigwedge_{\alpha\in\Omega} \mu_{A_{\alpha}^{'\backslash}}, \rho_{A'\backslash} = \bigwedge_{\alpha\in\Omega} \rho_{A_{\alpha}^{'\backslash}}.$$

□

7. Conclusions

Intuitionistic fuzzy ideals with specific properties are important in the study of fuzzy algebraic systems. In this research, we investigate hemirings in which every intuitionistic fuzzy k-ideal is idempotent and characterize these hemirings in terms of prime and semiprime intuitionistic fuzzy k-ideals. In our opinion the future study of (intuitionistic) fuzzy ideals of semirings and hemirings can be connected with (1) investigating semiprime and prime (intuitionistic) fuzzy one-sided k-ideal and every intuitionistic fuzzy k-bi-ideal is idempotent. (2) Finding intuitionistic and/or interval valued fuzzy sets and triangular norms. The obtained results can be used to solve some social networks problems and decide whether the corresponding graph is balanced or clusterable.

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Conflict of interest

The authors declare that they have no competing interests.

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