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Research article

Multi-assets Asian rainbow options pricing with stochastic interest rates obeying the Vasicek model

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Abstract: Asian rainbow options provide investors with a new option solution as an effective tool for asset allocation and risk management. In this paper, we address the pricing problem of Asian rainbow options with stochastic interest rates that obey the Vasicek model. By introducing the Vasicek model as the change process of the stochastic interest rate, based on the non-arbitrage principle and the stochastic differential equation, the number of assets of the Asian rainbow option is expanded to n dimensions, and the pricing formulas of the Asian rainbow option with multiple (n) assets under the Vasicek interest rate model are obtained. The multi-asset pricing results under stochastic interest rates provide more possibilities for Asian rainbow options. Furthermore, Monte Carlo simulation experiments show that the pricing formula is accurate and efficient under double stochastic errors. Finally, we perform parameter sensitivity analysis to further justify the pricing model.

Keywords: Vasicek model; rainbow options; Asian options; Monte Carlo simulation; multi-asset option
Mathematics Subject Classification: 91B70, 91G30, 91G60

1. Introduction

Asian options are path-dependent options, meaning that the payoffs are based on the average price of the underlying asset over a specific period of time [1]. Asian options can lessen their sensitivity to asset price volatility thanks to this averaging feature, which also lowers trading risk [2]. As a result, it is frequently utilized for risk management. Multi-asset options, such as rainbow options, can be used in multi-asset allocation to lower the risks posed by multiple assets. In incentive contract design and risk management, Asian rainbow options, a hybrid of Asian and rainbow options, are frequently used [3]. Asian rainbow options combine the characteristics of Asian and rainbow options. They are a multi-asset, path-dependent option. Therefore, the maximum or minimum average price of the underlying assets determines whether Asian rainbow options will pay off [4]. However, the price of multi-asset options is rarely mentioned in the existing literature, and two- or three-asset options are frequently employed as a substitute for multi-asset options [5]. To derive analytical pricing formulas for Asian rainbow call options on two geometric mean assets, the authors of [6] expanded the methodology for pricing Asian and Rainbow options using PDE. Before and after taking into account fractional and mixed fractional Brownian motion, Wang et al. [4] and Ahmadian et al. [7, 8] analyze the pricing of Asian rainbow options, respectively. Despite the fact that some researchers [5, 7, 8] attempted to increase the number of assets to n-dimensional, they weren't taking into account the randomness of real interest rates.

According to the efficient market hypothesis, the price of the underlying asset is currently determined by a widely understood stochastic process [9], and this stochastic process can be summarized as:

$$dS(t) = \mathbf{r}(t, S(t))dt + \boldsymbol{\sigma}(t, S(t))dW(t), \tag{1.1}$$

where according to different random processes, the interest rate $\mathbf{r}$ and volatility rate $\boldsymbol{\sigma}$ can be constants or specific functions. To reflect realism, the effects of stochastic volatility and stochastic interest rates have been incorporated into the model in several ways to reflect reality [10–16]. For example, Stein et al. [12] investigated the distributions of stock prices that result from diffusion processes with stochastically varying volatility parameters. In [13], the authors used a hybrid model that combines the CIR stochastic interest rate model and the Heston stochastic volatility model to get a closed-form pricing formula for European options. He and Lin [14] formed a rough Heston-CIR model to capture both the rough behavior of the volatility and the stochastic nature of the interest rate and present a semi-analytical pricing formula for European options. The pricing of volatility and variance swaps using a hybrid model with Markov-modulated jump-diffusion with discrete sampling times is the subject of [16].

Furthermore, given that some short-term government bonds have negative interest rates, some researchers have demonstrated that models like the Vasicek model that incorporate negative interest rates can improve option pricing and implied volatility forecasting [17–24]. Guo [20] used an assumption to derive the pricing formula for European call options: the price of the underlying asset will follow the Heston stochastic volatility model, while the interest will follow the Vasicek stochastic interest model. In [21], the issue of Bermuda option pricing on zero-coupon bonds was examined. In this case, the model's interest rate dynamics are based on the mixed fractional Vasicek model. Mehrdoust and Najafi [22] obtained the value of the European option on the zero-coupon bond with transaction cost by using the fractional Vasicek model to predict the interest rates in France and Australia. Under the Vasicek interest rate model, Zhao and Xu [23] calibrated the time-dependent volatility function for European options. Kharrat and Arfaoui [24] disentangled the Vasicek model for European options to verify the stability and reliability of the model with option theory.

Based on the above, in this paper, we adopt the Vasicek model as a stochastic interest rate and expand the number of assets of Asian rainbow options to n-dimensional cases to aim to present the pricing formula of geometric mean Asian rainbow options. The rest of the paper is organized as follows. In Section 2, we present the Vasicek and underlying asset price models and introduce the payoff of the Geometric Asian Rainbow options. Four explicit formulas for the geometric Asian rainbow options are derived in Section 3. In Section 4, the call option of the two and three assets are obtained by applying the Monte Carlo simulation method and different values of the model's parameters. Section 5 carries out a sensitivity analysis of the parameters in the pricing formula. In Section 6, conclusions are drawn.

2. Asian rainbow options under the Vasicek interest rate model

In this paper, we use the Vasicek model [19] as the stochastic interest rate model, which enables negative interest rates and follows the actual requirements of the financial market. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbf{Q})$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions (i.e., it is increasing and right-continuous when $\mathcal{F}_0$ contains all $\mathbf{Q}$-null sets), where $\mathbf{Q}$ is a risk-neutral probability measure. For the risk-neutral measure $\mathbf{Q}$, consider the change process of the risk-free interest rate satisfying the Vasicek model; that is, $\{r(t)\colon t \ge 0\}$ is the dynamic process of the risk-free interest rate change at time t that follows the model

$$\mathrm{d}r(t) = (\alpha - \beta r(t))\mathrm{d}t + \sigma_r \mathrm{d}W_r(t), \quad 0 \le t \le T, \tag{2.1}$$

where α and $\beta \neq 0$ are positive constants, the parameter $\sigma_r > 0$ is the volatility of the risk-free interest rate, and $W_r(t)$ is a standard Wiener process. Next, in the risk-neutral measure $\mathbf{Q}$, we introduce the Vasicek stochastic interest rate model to the process of change in the underlying asset and investigate the dynamic process of underlying asset price change $\{S_i(t)\colon t \ge 0\}$ at time t, which follows

$$\mathrm{d}S_i(t) = r(t)S_i(t)\mathrm{d}t + \sigma_i S_i(t)\mathrm{d}W_i(t), \quad i = 1, 2, \cdots, n, \tag{2.2}$$

where the constant $\sigma_i > 0$ is the volatility of the underlying asset. $W_i(t)$ is a standard Wiener process and any two dependent standard Brownian motions have a correlation coefficient $\rho^{ij} \in [-1, 1]$, i.e., $\mathrm{d}W_i(t)\mathrm{d}W_j(t) = \rho^{ij}\mathrm{d}t$ $(j \neq i)$. In addition, we assume that the Brownian motion $W_i(t)$ and $W_r(t)$ are independent of each other, i.e., $\mathrm{d}W_i(t)\mathrm{d}W_r(t) = 0$, and present the following assumptions:

(A1) The underlying asset price dynamics follow a log-normal distribution, and price volatility is constant.
(A2) The risk-free interest rate is not constant, and its change follows the Vasicek model.
(A3) Securities trading is continuous; there are no riskless arbitrage opportunities; no dividends will be paid during the option's validity period; there are no transaction costs or taxes; all securities are perfectly divisible; investors can borrow or lend funds at the same interest rate.
(A4) The option can only be exercised at maturity.

Asian rainbow options have the same characteristics as Asian options because they are a combination of the two, i.e., the option price is determined by the average price of the underlying asset from the start date to the expiration date [7]. Considering that the price of the underlying asset (2.2) is followed by a geometric Brownian motion, we let

$$G_i(T, S_i) = \mathrm{e}^{\frac{1}{T}\int_0^T \ln S_i(t)\mathrm{d}t}, \quad i = 1, 2, \cdots, n, \tag{2.3}$$

be the geometric mean of the price S_i $(i = 1, 2, \cdots, n)$ in $[0, T]$. The four geometric Asian rainbow options contract that we can take into consideration are call on max, call on min, put on max, and put on min [3]. The payoffs of geometric Asian rainbow call on max and min options with the strike price K and maturity T on n underlying assets are given by

$$\max\left(\max\left(G_1(T, S_1), G_2(T, S_2), \cdots, G_n(T, S_n)\right) - K, 0\right) \tag{2.4}$$

and

$$\max\left(\min\left(G_1(T,S_1),\ G_2(T,S_2),\cdots,G_n(T,S_n)\right)-K,0\right), \tag{2.5}$$

respectively. The payoffs of the geometric Asian rainbow put on max and min options [7] are given by

$$\max\left(K-\max\left(G_1(T,S_1),\ G_2(T,S_2),\cdots,G_n(T,S_n)\right),0\right) \tag{2.6}$$

and

$$\max\left(K-\min\left(G_1(T,S_1),\ G_2(T,S_2),\cdots,G_n(T,S_n)\right),0\right), \tag{2.7}$$

respectively.

Combining the above models and assumptions, the pricing of an Asian rainbow option $\mathbf{op}(T,r,S_i,V)$ under the $\mathbf{Q}$-measure can be written as follows:

$$\mathbf{op}(T,r,S_i,V)=\mathbb{E}^{\mathbf{Q}}\left[\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}V(T,S_i)|\mathcal{F}_0\right], \tag{2.8}$$

where $V(T,S_i)$ is the payoff of the option and is a function of T and $S_i(T)$. The conditional expectation under the $\mathcal{F}_0$ condition refers to valuing the expected payoff of the option when the initial time $t=0$.

The stochastic interest rate function and the logarithm of the underlying asset price functions have definite integrals from time 0 to T, according to Eqs (2.3) and (2.8). The stochastic interest rate function $r(t)$ is integrated as follows:

$$\int_0^T r(t)\mathrm{d}t=\frac{1}{\beta}\left(r(0)-\frac{\alpha}{\beta}\right)\left(1-\mathrm{e}^{-\beta T}\right)+\frac{\alpha}{\beta}T-\frac{\sigma_r}{\beta}\int_0^T\left(\mathrm{e}^{-\beta T}\mathrm{e}^{\beta u}-1\right)\mathrm{d}W_r(u). \tag{2.9}$$

Let

$$\begin{aligned}C_0(T)&=\frac{1}{\beta}\left(r(0)-\frac{\alpha}{\beta}\right)\left(1-\mathrm{e}^{-\beta T}\right)+\frac{\alpha}{\beta}T,\\ m(u,T)&=-\frac{\sigma_r}{\beta}\left(\mathrm{e}^{-\beta T}\mathrm{e}^{\beta u}-1\right),\end{aligned} \tag{2.10}$$

which are the constant and integral terms of $\int_0^T r(t)\mathrm{d}t$, respectively. The logarithm of the underlying asset price function $\ln S_i(t)$ is integrated as follows:

$$\begin{aligned}\frac{1}{T}\int_0^T \ln S_i(t)\mathrm{d}t=\ln S_i(0)&+\frac{1}{\beta}\left(r(0)-\frac{\alpha}{\beta}\right)\left[1-\frac{1}{\beta T}(1-e^{-\beta T})\right]+\frac{1}{2}\left(\frac{\alpha}{\beta}-\frac{1}{2}\sigma_i^2\right)T\\ &+\frac{1}{T}\int_0^T\int_u^T m(u,t)\mathrm{d}t\mathrm{d}W_r(u)+\frac{1}{T}\int_0^T\sigma_i(T-u)\mathrm{d}W_i(u).\end{aligned} \tag{2.11}$$

And let

$$\begin{aligned}C_i(T)&=\ln S_i(0)+\frac{1}{\beta}\left(r(0)-\frac{\alpha}{\beta}\right)\left[1-\frac{1}{\beta T}\left(1-\mathrm{e}^{-\beta T}\right)\right]+\frac{1}{2}\left(\frac{\alpha}{\beta}-\frac{1}{2}\sigma_i^2\right)T,\\ f(u,T)&=\frac{\sigma_r}{\beta^2 T}\left(\mathrm{e}^{-\beta T}\mathrm{e}^{\beta u}-1\right)+\frac{\sigma_r}{\beta}\left(1-\frac{u}{T}\right),\\ g_i(u,T)&=\frac{1}{T}\sigma_i(T-u)\end{aligned} \tag{2.12}$$

which are a constant term and two integral terms of $\frac{1}{T}\int_0^T\ln S_i(t)\mathrm{d}t$, respectively. For the sake of concise writing, we abbreviate the above functions respectively as C_0, m, C_i, f, and g_i.

Based on the above model, we will analyze the pricing formulas of geometric Asian rainbow options for multiple assets.

3. Pricing formula of multi-asset

In this section, we consider geometric Asian rainbow option prices for multiple assets (n), including call options, put options, and the parity relationship.

3.1. Call options

Referring to [25], we derive the following lemma first to determine the precise expression for this option's price.

Lemma 3.1. *Let $\boldsymbol{Z} = (Z_1, Z_2, Z_3)^{\mathrm{T}}$ be standard normal random variables, follow the three-dimensional standard normal distribution $N(\mathbf{0}, \boldsymbol{\Lambda})$ with the covariance matrix $\boldsymbol{\Lambda} = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}$. Then, for arbitrary constants a, b, c, k and l,*

$$\mathbb{E}\left[\mathrm{e}^{aZ_1}\mathbf{I}_{\{bZ_2\geq k,\, cZ_3\geq l\}}\right] = \mathrm{e}^{\frac{a^2}{2}}\mathbf{N}\left(\frac{\rho_{12}ab-k}{b}, \frac{\rho_{13}ac-l}{c}; \rho_{23}\right), \tag{3.1}$$

where $\mathbf{I}(\cdot)$ is the indicator function that takes the value 1 if the expression in parentheses is true and 0 otherwise, and $\mathbf{N}(\cdot)$ is the two-dimensional cumulative standard normal distribution.

Proof. Let $\boldsymbol{z} = (z_1, z_2, z_3)^{\mathrm{T}}$, $t = \frac{1}{\sqrt{|\boldsymbol{\Lambda}|}}\left(\sqrt{1-\rho_{23}^2}z_1 + \frac{(\rho_{13}\rho_{23}-\rho_{12})z_2+(\rho_{12}\rho_{23}-\rho_{13})z_3-a|\boldsymbol{\Lambda}|}{\sqrt{1-\rho_{23}^2}}\right)$, $\xi = z_2 - \rho_{12}a$, $\eta = z_3 - \rho_{13}a$, we then obtain

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{aZ_1}\mathbf{I}_{\{bZ_2\geq k, cZ_3\geq l\}}\middle|Z_1 = z_1, Z_2 = z_2, Z_3 = z_3\right] = \int_{\frac{l}{c}}^{+\infty}\int_{\frac{k}{b}}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{az_1}\frac{1}{(2\pi)^{\frac{3}{2}}|\boldsymbol{\Lambda}|^{\frac{1}{2}}}\mathrm{e}^{-\frac{1}{2}\boldsymbol{z}^{\mathrm{T}}\boldsymbol{\Lambda}^{-1}\boldsymbol{z}}\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3 \\
&= \int_{\frac{l}{c}}^{+\infty}\int_{\frac{k}{b}}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{|\boldsymbol{\Lambda}|}}\mathrm{e}^{-\frac{1}{2|\boldsymbol{\Lambda}|}\left(\sqrt{1-\rho_{23}^2}z_1+\frac{(\rho_{13}\rho_{23}-\rho_{12})z_2+(\rho_{12}\rho_{23}-\rho_{13})z_3-a|\boldsymbol{\Lambda}|}{\sqrt{1-\rho_{23}^2}}\right)^2} \\
&\times\mathrm{e}^{-\frac{1}{2|\boldsymbol{\Lambda}|}\left((1-\rho_{13}^2)z_2^2+(1-\rho_{12}^2)z_3^2+2(\rho_{12}\rho_{13}-\rho_{23})z_2z_3-\left(\frac{(\rho_{13}\rho_{23}-\rho_{12})z_2+(\rho_{12}\rho_{23}-\rho_{13})z_3-a|\boldsymbol{\Lambda}|}{\sqrt{1-\rho_{23}^2}}\right)^2\right)}\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3 \\
&= \int_{\frac{l}{c}}^{+\infty}\int_{\frac{k}{b}}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{|\boldsymbol{\Lambda}|}}\mathrm{e}^{-\frac{1}{2|\boldsymbol{\Lambda}|}\left((1-\rho_{13}^2)z_2^2+(1-\rho_{12}^2)z_3^2+2(\rho_{12}\rho_{13}-\rho_{23})z_2z_3-\left(\frac{(\rho_{13}\rho_{23}-\rho_{12})z_2+(\rho_{12}\rho_{23}-\rho_{13})z_3-a|\boldsymbol{\Lambda}|}{\sqrt{1-\rho_{23}^2}}\right)^2\right)} \\
&\times\mathrm{e}^{-\frac{t^2}{2}}\frac{\sqrt{|\boldsymbol{\Lambda}|}}{\sqrt{1-\rho_{23}^2}}\mathrm{d}t\mathrm{d}z_2\mathrm{d}z_3 \\
&= \int_{\frac{l}{c}}^{+\infty}\int_{\frac{k}{b}}^{+\infty}\frac{1}{2\pi\sqrt{1-\rho_{23}^2}}\mathrm{e}^{-\frac{1}{2|\boldsymbol{\Lambda}|}\left((1-\rho_{13}^2)z_2^2+(1-\rho_{12}^2)z_3^2+2(\rho_{12}\rho_{13}-\rho_{23})z_2z_3-\left(\frac{(\rho_{13}\rho_{23}-\rho_{12})z_2+(\rho_{12}\rho_{23}-\rho_{13})z_3-a|\boldsymbol{\Lambda}|}{\sqrt{1-\rho_{23}^2}}\right)^2\right)}\mathrm{d}z_2\mathrm{d}z_3 \\
&= \int_{\frac{l}{c}}^{+\infty}\int_{\frac{k}{b}}^{+\infty}\frac{1}{2\pi\sqrt{1-\rho_{23}^2}}\mathrm{e}^{-\frac{1}{2(1-\rho_{23}^2)}\left((z_2-\rho_{12}a)^2+(z_3-\rho_{13}a)^2+2\rho_{13}\rho_{23}az_2+2\rho_{13}\rho_{23}az_3-2\rho_{23}z_2z_3-a^2(|\boldsymbol{\Lambda}|+\rho_{12}^2+\rho_{13}^2)\right)}\mathrm{d}z_2\mathrm{d}z_3 \\
&= \int_{\frac{l}{c}-\rho_{13}a}^{+\infty}\int_{\frac{k}{b}-\rho_{12}a}^{+\infty}\frac{1}{2\pi\sqrt{1-\rho_{23}^2}}\mathrm{e}^{-\frac{1}{2(1-\rho_{23}^2)}(\xi^2+\eta^2-2\rho_{23}\xi\eta)}\mathrm{e}^{\frac{a^2}{2}}\mathrm{d}\xi\mathrm{d}\eta \\
&= \mathrm{e}^{\frac{a^2}{2}}\mathbf{N}\left(\frac{\rho_{12}ab-k}{b}, \frac{\rho_{13}ac-l}{c}; \rho_{23}\right).
\end{aligned} \tag{3.2}$$

This completes the proof of Lemma 3.1. □

From Lemma 3.1, we extend to the n-dimensional case.

Corollary 3.1. *Let* $\boldsymbol{Z} = (Z_0, Z_1, \cdots, Z_n)^{\mathrm{T}}$ *be standard normal random variables, follow the* $n+1$ *dimensional standard normal distribution* $N(\boldsymbol{0}, \boldsymbol{\Lambda})$ *with the covariance matrix* $\boldsymbol{\Lambda} = \begin{pmatrix} 1 & \rho_{01} & \cdots & \rho_{0n} \\ \rho_{01} & 1 & \cdots & \rho_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{0n} & \rho_{1n} & \cdots & 1 \end{pmatrix}$. *Then, for arbitrary constants* a_0, a_1, $\cdots$, a_n, k_1, $\cdots$, k_n,

$$\mathbb{E}\left[\mathrm{e}^{a_0 Z_0} \mathbf{I}_{\{a_1 Z_1 \geq k_1, \cdots, a_n Z_n \geq k_n\}}\right] = \mathrm{e}^{\frac{a_0^2}{2}} \mathbf{N}\left(\frac{\rho_{01} a_0 a_1 - k_1}{a_1}, \cdots, \frac{\rho_{0n} a_0 a_n - k_n}{a_n}\right). \tag{3.3}$$

where $\mathbf{I}(\cdot)$ *is the indicator function that takes the value* 1 *if the expression in parentheses is true and* 0 *otherwise, and* $\mathbf{N}(\cdot)$ *is the n-dimensional cumulative standard normal distribution.*

Set the initial time and expiration time as 0 and T, respectively, we then calculate the geometric mean of the time period from 0 to T for S_1, S_2, $\cdots$, S_n. The strike price of the option is set to K, and the payoff $V^c_{\max}(T, S_1, S_2, \cdots, S_n)$ for this option is:

$$V^c_{\max}(T, S_1, S_2, \cdots, S_n) = \max\left(\max\left(G_1(T, S_1), G_2(T, S_2), \cdots, G_n(T, S_n)\right) - K, 0\right), \tag{3.4}$$

which is the n-dimensional case of Eq (2.4). It denotes that the price of a call on max option is determined by the geometric mean of n underlying assets at their maximum. We use Eqs (2.8) and (3.4) to calculate the price of a call on max option $C_{\max}(T, r, S_1, S_2, \cdots, S_n)$, that is,

$$C_{\max}(T, r, S_1, S_2, \cdots, S_n) = \mathbf{op}(T, r, S_1, S_2, \cdots, S_n, V^c_{\max}). \tag{3.5}$$

Through direct calculation, the following theorem provides the expression for the price of a call on the max option.

Theorem 3.1. (Call on max option) *Assume that the underlying assets* S_i $(i = 1, 2, \cdots, n)$ *obey generalized geometric Brownian motion, the stochastic interest rate follows the Vasicek model* (2.1), *and the initial to expiration time is* 0 *to* T, *then the price of the geometric Asian rainbow call on max option* $C_{\max}(T, r, S_1, S_2, \cdots, S_n)$ *is given as follows:*

$$\begin{aligned} C_{\max}(T, r, S_1, S_2, \cdots, S_n) = {} & \sum_{i=1}^{n} \mathrm{e}^{C_i - C_0 + \frac{1}{2}\int_0^T ((f-m)^2 + g_i^2)\mathrm{d}u} \mathbf{N}_i\Bigg(\frac{\int_0^T (f(f-m)+g_i^2)\mathrm{d}u + C_i - \ln K}{\sqrt{\int_0^T (f^2+g_i^2)\mathrm{d}u}}, \underbrace{\cdots, \frac{\int_0^T g_i\left(g_i - \rho^{ij} g_j\right)\mathrm{d}u + C_i - C_j}{\sqrt{\int_0^T \left(g_i^2 + g_j^2 - 2\rho^{ij} g_i g_j\right)\mathrm{d}u}}, \cdots}_{n-1}\Bigg) \\ & - K\mathrm{e}^{-C_0 + \frac{1}{2}\int_0^T m^2 \mathrm{d}u} \sum_{i=1}^{n} \mathbf{N}_i\Bigg(\frac{-\int_0^T fm\mathrm{d}u + C_i - \ln K}{\sqrt{\int_0^T (f^2+g_i^2)\mathrm{d}u}}, \underbrace{\cdots, \frac{C_i - C_j}{\sqrt{\int_0^T \left(g_i^2 + g_j^2 - 2\rho^{ij} g_i g_j\right)\mathrm{d}u}}, \cdots}_{n-1}\Bigg), \end{aligned} \tag{3.6}$$

where $i, j = 1, 2, \cdots, n$, $i \neq j$, C_0, m, C_i, f *and* g_i *are defined in Eqs* (2.10) *and* (2.12). $\mathbf{N}_i(\cdot)$ *is the n-dimensional cumulative standard normal distribution with the covariance matrix* $\boldsymbol{\Lambda_i} = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{12} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1n} & \rho_{2n} & \cdots & 1 \end{pmatrix}$ *as follows:*

$$\rho_{1a} = \frac{\int_0^T g_i\left(g_i - \rho^{ib} g_b\right)\mathrm{d}u}{\int_0^T \sqrt{\left(f^2+g_i^2\right)\left(g_i^2+g_b^2-2\rho^{ib} g_i g_b\right)}\mathrm{d}u}, \quad \textit{with} \quad a = 2, \cdots, n, \quad \begin{cases} b = a-1, & a \leq i \\ b = a, & a > i \end{cases},$$

and

$$\rho_{cd}=\frac{\int_0^T\left(g_i^2-\rho^{ie}g_ig_e-\rho^{if}g_ig_f+\rho^{ef}g_eg_f\right)\mathrm{d}u}{\int_0^T\sqrt{\left(g_i^2+g_e^2-2\rho^{ie}g_ig_e\right)\left(g_i^2+g_f^2-2\rho^{if}g_ig_f\right)}\mathrm{d}u},\quad \textit{with}\quad c=2,\cdots,n-1,\quad \begin{cases}e=c-1, & c\le i\\ e=c, & c>i\end{cases},$$

$$d=c+1,\cdots,n,\quad \begin{cases}f=d-1, & d\le i\\ f=d, & d>i\end{cases}.$$

Proof. From Eq (3.4), the payoff of the call on max option $V^c_{\max}(T,S_1,S_2,\cdots,S_n)$ can be expressed by the following equation

$$\begin{aligned}V^c_{\max}(T,S_1,S_2,\cdots,S_n)=&(G_1(T,S_1)-K)\mathbf{I}_{\{G_1(T,S_1)\ge K,\,G_1(T,S_1)\ge G_2(T,S_2),\cdots,\,G_1(T,S_1)\ge G_n(T,S_n)\}}\\&+(G_2(T,S_2)-K)\mathbf{I}_{\{G_2(T,S_2)\ge K,\,G_2(T,S_2)\ge G_1(T,S_1),\cdots,\,G_2(T,S_2)\ge G_n(T,S_n)\}}\\&+\cdots\cdots\\&+(G_n(T,S_n)-K)\mathbf{I}_{\{G_n(T,S_n)\ge K,\,G_n(T,S_n)\ge G_1(T,S_1),\cdots,G_n(T,S_n)\ge G_{n-1}(T,S_{n-1})\}}.\end{aligned}\tag{3.7}$$

Set $A_i=\{G_i(T,S_i)\ge K,\cdots,G_i(T,S_i)\ge G_j(T,S_j),\cdots\}$ $(i,j=1,2,\cdots,n,\ i\ne j)$, based on Eqs (2.3) and (2.11), we can obtain

$$A_i=\left\{\int_0^T f\mathrm{d}W_r(u)+\int_0^T g_i\mathrm{d}W_i(u)\ge \ln K-C_i,\cdots,\int_0^T g_i\mathrm{d}W_i(u)-\int_0^T g_j\mathrm{d}W_j(u)\ge C_j-C_i,\cdots\right\}.\tag{3.8}$$

It follows from Eq (3.5), we derive the call on the max option price as

$$\begin{aligned}C_{\max}(T,r,S_1,S_2,\cdots,S_n)&=\sum_{i=1}^n\mathbb{E}\left[\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t+\frac{1}{T}\int_0^T\ln S_i(t)\mathrm{d}t}\mathbf{I}_{A_i}\right]-\sum_{i=1}^n K\mathbb{E}\left[\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}\mathbf{I}_{A_i}\right]\\&=\sum_{i=1}^n I_i-\sum_{i=1}^n I_{n+i}.\end{aligned}\tag{3.9}$$

From Eqs (2.9), (2.11) and (3.8), it is easy to see that

$$I_i=\mathrm{e}^{C_i-C_0}\mathbb{E}\left[\mathrm{e}^{\int_0^T(f-m)\mathrm{d}W_r(u)+\int_0^T g_i\mathrm{d}W_i(u)}\mathbf{I}_{\left\{\int_0^T f\mathrm{d}W_r(u)+\int_0^T g_i\mathrm{d}W_i(u)\ge\ln K-C_i,\cdots,\int_0^T g_i\mathrm{d}W_i(u)-\int_0^T g_j\mathrm{d}W_j(u)\ge C_j-C_i,\cdots\right\}}\right],\tag{3.10}$$

where $i,j=1,2,\cdots,n$ and $i\ne j$. Let

$$\begin{aligned}Z_0&=\int_0^T(f-m)\,\mathrm{d}W_r(u)+\int_0^T g_i\mathrm{d}W_i(u),\quad i=1,2,\cdots,n,\\Z_1&=\int_0^T f\mathrm{d}W_r(u)+\int_0^T g_i\mathrm{d}W_i(u),\quad i=1,2,\cdots,n,\\Z_a&=\int_0^T g_i\mathrm{d}W_i(u)-\int_0^T g_j\mathrm{d}W_j(u),\quad a=2,\cdots,n,\quad i,j=1,2,\cdots,n,\ i\ne j.\end{aligned}\tag{3.11}$$

Since $\mathrm{d}W(u)=\varepsilon\mathrm{d}\sqrt{u}$ and $\varepsilon\sim N(0,1)$ is a standard normal random variable, Z_0 can be expressed as

$$Z_0=\varepsilon_r\int_0^T(f-m)\,\mathrm{d}\sqrt{u}+\varepsilon_i\int_0^T g_i\mathrm{d}\sqrt{u},\quad i=1,2,\cdots,n,\tag{3.12}$$

then the variance of the random variable Z_0 is

$$\sigma_0^2=\int_0^T\left((f-m)^2+g_i^2\right)\mathrm{d}u,\quad i=1,2,\cdots,n.\tag{3.13}$$

Similarly, the variance of Z_1 and Z_a can be obtained as

$$\sigma_1^2 = \int_0^T \left(f^2 + g_i^2\right) \mathrm{d}u, \quad i = 1, 2, \cdots, n, \tag{3.14}$$

and

$$\sigma_a^2 = \int_0^T \left(g_i^2 + g_j^2 - 2\rho^{ij} g_i g_j\right) \mathrm{d}u, \quad a = 2, \cdots, n, \quad i, j = 1, 2, \cdots, n, \ i \neq j, \tag{3.15}$$

respectively. The correlation coefficients between the $n+1$ random variables Z_0, Z_1, Z_a $(a = 2, \cdots, n)$ with respect to

$$\rho_{01} = \frac{\int_0^T \left(f(f-m) + g_i^2\right) \mathrm{d}u}{\sigma_0 \sigma_i}, \quad i = 1, 2, \cdots, n, \tag{3.16}$$

$$\rho_{0a} = \frac{\int_0^T g_i\left(g_i - \rho^{ib} g_b\right) \mathrm{d}u}{\sigma_0 \sigma_a}, \quad a = 2, \cdots, n, \quad \begin{cases} b = a-1, & a \leq i \\ b = a, & a > i \end{cases}, \ i = 1, 2, \cdots, n, \tag{3.17}$$

$$\rho_{1a} = \frac{\int_0^T g_i\left(g_i - \rho^{ib} g_b\right) \mathrm{d}u}{\sigma_1 \sigma_b}, \quad a = 2, \cdots, n, \quad \begin{cases} b = a-1, & a \leq i \\ b = a, & a > i \end{cases}, \ i = 1, 2, \cdots, n, \tag{3.18}$$

and

$$\begin{aligned} \rho_{cd} = \frac{\int_0^T \left(g_i^2 - \rho^{ie} g_i g_e - \rho^{if} g_i g_f + \rho^{ef} g_e g_f\right) \mathrm{d}u}{\int_0^T \sqrt{\left(g_i^2 + g_e^2 - 2\rho^{ie} g_i g_e\right)\left(g_i^2 + g_f^2 - 2\rho^{if} g_i g_f\right)} \mathrm{d}u}, \quad & c = 2, \cdots, n-1, \quad \begin{cases} e = c-1, & c \leq i \\ e = c, & c > i \end{cases}, \\ & d = c+1, \cdots, n, \quad \begin{cases} f = d-1, & d \leq i \\ f = d, & d > i \end{cases}, \\ & i = 1, 2, \cdots, n. \end{aligned} \tag{3.19}$$

Given the above, $(Z_0, Z_1, \cdots, Z_n)^{\mathrm{T}} \sim N(\mathbf{0}, \tilde{\mathbf{\Lambda}}_i)$ with $\tilde{\mathbf{\Lambda}}_i = \begin{pmatrix} \sigma_0^2 & \rho_{01}\sigma_0\sigma_1 & \cdots & \rho_{0n}\sigma_0\sigma_n \\ \rho_{01}\sigma_0\sigma_1 & \sigma_1^2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{0n}\sigma_0\sigma_n & \rho_{1n}\sigma_1\sigma_n & \cdots & \sigma_n^2 \end{pmatrix}$, $i = 1, 2, \cdots, n$. According to Corollary 3.1, we derive the following result

$$\begin{aligned} I_i &= \mathrm{e}^{C_i - C_0} \mathbb{E}\left[\mathrm{e}^{\sigma_0 \frac{Z_0}{\sigma_0}} \mathbf{I}_{\left\{\sigma_1 \frac{Z_1}{\sigma_1} \geq \ln K - C_i, \cdots, \sigma_a \frac{Z_a}{\sigma_a} \geq C_j - C_i, \cdots\right\}}\right] \\ &= \mathrm{e}^{C_i - C_0 + \frac{1}{2}\sigma_0^2} \mathbf{N}_i\left(\frac{\rho_{01}\sigma_0\sigma_1 - \ln K + C_i}{\sigma_1}, \cdots, \frac{\rho_{0a}\sigma_0\sigma_a + C_i - C_j}{\sigma_a}, \cdots\right) \\ &= \mathrm{e}^{C_i - C_0 + \frac{1}{2}\int_0^T \left((f-m)^2 + g_i^2\right) \mathrm{d}u} \mathbf{N}_i\left(\frac{\int_0^T \left(f(f-m) + g_i^2\right) \mathrm{d}u + C_i - \ln K}{\sqrt{\int_0^T \left(f^2 + g_i^2\right) \mathrm{d}u}}, \cdots, \frac{\int_0^T g_i\left(g_i - \rho^{ij} g_j\right) \mathrm{d}u + C_i - C_j}{\sqrt{\int_0^T \left(g_i^2 + g_j^2 - 2\rho^{ij} g_i g_j\right) \mathrm{d}u}}, \cdots\right), \end{aligned} \tag{3.20}$$

where $i, j = 1, 2, \cdots, n$, $i \neq j$, C_0, m, C_i, f and g_i are defined in Eqs (2.10) and (2.12). $\mathbf{N}_i(\cdot)$ is the n-dimensional cumulative standard normal distribution with the covariance matrix $\mathbf{\Lambda}_i = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{12} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1n} & \rho_{2n} & \cdots & 1 \end{pmatrix}$, where the correlation coefficients ρ_{1a} and ρ_{cd} are defined in Eqs (3.18) and (3.19).

Moreover, $I_{n+i}\ (i, j = 1, 2, \cdots, n,\ i \neq j)$ follows

$$I_{n+i} = K\mathrm{e}^{-C_0+\frac{1}{2}\int_0^T m^2\mathrm{d}u}\sum_{i=1}^{n}\mathbf{N}_i\left(\frac{-\int_0^T fm\mathrm{d}u+C_i-\ln K}{\sqrt{\int_0^T(f^2+g_i^2)\mathrm{d}u}},\cdots,\frac{C_i-C_j}{\sqrt{\int_0^T\left(g_i^2+g_j^2-2\rho^{ij}g_ig_j\right)\mathrm{d}u}},\cdots\right). \tag{3.21}$$

Hence, the proof of Theorem 3.1 is confirmed. □

Remark 3.1. *Equation* (3.6) *displays the geometric Asian rainbow call on max option, which supports the investor in choosing multiple (n) underlying assets to diversify risk. Suppose that an investor purchases an option at time* 0 *and chooses whether to trade it at time T based on market conditions. During this period, the price of the underlying asset is fluctuating randomly. In the absence of market-based arbitrage opportunities, an option purchased at $t = 0$ takes into account the expectation of the future asset price. Due to the path dependence of Asian rainbow options, it is necessary to determine the mean value of the n assets being chosen for the period between purchase and execution. We employ a geometric mean approach, which enables the option price to depend on both the asset price at expiration and the average price over the time between purchase and execution. As a result, the risk of market conditions changes can be reduced. The option price given in Theorem 3.1 supports both of these risk reduction options.*

The following Theorem 3.2 provides an analytical solution for the call option with the minimum geometric mean of n underlying assets, where the option has the payoff $V^c_{\min}(T, S_1, S_2, \cdots, S_n)$ as:

$$V^c_{\min}(T, S_1, S_2, \cdots, S_n) = \max\left(\min\left(G_1(T, S_1), G_2(T, S_2), \ldots, G_n(T, S_n)\right) - K, 0\right). \tag{3.22}$$

which is the n-dimensional case of Eq (2.5). That means that the price of a call on min option is determined by the minimum of the geometric mean of the n underlying assets. It implies by (2.8) and (3.22) that the price of a call on min option is

$$C_{\min}(T, r, S_1, S_2, \cdots, S_n) = \mathbf{op}(T, r, S_1, S_2, \cdots, S_n, V^c_{\min}). \tag{3.23}$$

Theorem 3.2. (Call on min option) *Assume that the underlying assets S_i $(i = 1, 2, \cdots, n)$ obey generalized geometric Brownian motion, the stochastic interest rate follows the Vasicek model* (2.1)*, and the initial to expiration time range is* 0 *to T, then the price of the geometric Asian rainbow call on min option $C_{\min}(T, r, S_1, S_2, \cdots, S_n)$ is determined by the following equation*

$$\begin{aligned} C_{\min}(T, r, S_1, S_2, \cdots, S_n) = {} & \sum_{i=1}^{n}\mathrm{e}^{C_i-C_0+\frac{1}{2}\int_0^T\left((f-m)^2+g_i^2\right)\mathrm{d}u}\mathbf{N}_i\left(\frac{\int_0^T\left(f(f-m)+g_i^2\right)\mathrm{d}u+C_i-\ln K}{\sqrt{\int_0^T(f^2+g_i^2)\mathrm{d}u}},\underbrace{\cdots,\frac{\int_0^T g_i\left(\rho^{ij}g_j-g_i\right)\mathrm{d}u-C_i+C_j}{\sqrt{\int_0^T\left(g_i^2+g_j^2-2\rho^{ij}g_ig_j\right)\mathrm{d}u}},\cdots}_{n-1}\right) \\ & -K\mathrm{e}^{-C_0+\frac{1}{2}\int_0^T m^2\mathrm{d}u}\sum_{i=1}^{n}\mathbf{N}_i\left(\frac{-\int_0^T fm\mathrm{d}u+C_i-\ln K}{\sqrt{\int_0^T(f^2+g_i^2)\mathrm{d}u}},\underbrace{\cdots,\frac{C_j-C_i}{\sqrt{\int_0^T\left(g_i^2+g_j^2-2\rho^{ij}g_ig_j\right)\mathrm{d}u}},\cdots}_{n-1}\right), \end{aligned} \tag{3.24}$$

where $i, j \in \{1, 2, \cdots, n\}$ *(*$i \neq j$*),* C_0, m, C_i, f *and* g_i *are given in Eqs* (2.10) *and* (2.12). $\mathbf{N}_i(\cdot)$ *is the n-dimensional cumulative standard normal distribution with the covariance matrix* $\mathbf{\Lambda_i} = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{12} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1n} & \rho_{2n} & \cdots & 1 \end{pmatrix}$ *and*

$$\rho_{1a} = \frac{\int_0^T g_i\left(\rho^{ib} g_b - g_i\right) \mathrm{d}u}{\int_0^T \sqrt{\left(f^2+g_i^2\right)\left(g_i^2+g_b^2-2\rho^{ib} g_i g_b\right)} \mathrm{d}u}, \qquad \text{with} \quad a = 2, \cdots, n, \quad \begin{cases} b = a-1, & a \leq i \\ b = a, & a > i \end{cases},$$

$$\rho_{cd} = \frac{\int_0^T \left(g_i^2 - \rho^{ie} g_i g_e - \rho^{if} g_i g_f + \rho^{ef} g_e g_f\right) \mathrm{d}u}{\int_0^T \sqrt{\left(g_i^2+g_e^2-2\rho^{ie} g_i g_e\right)\left(g_i^2+g_f^2-2\rho^{if} g_i g_f\right)} \mathrm{d}u}, \quad \text{with} \quad c = 2, \cdots, n-1, \quad \begin{cases} e = c-1, & c \leq i \\ e = c, & c > i \end{cases},$$

$$d = c+1, \cdots, n, \quad \begin{cases} f = d-1, & d \leq i \\ f = d, & d > i \end{cases}.$$

The proof of Theorem 3.2 is attached in A.

3.2. Put options

In order to price put options, the following Theorem 3.3 provides an analytical solution for the put option with the maximum geometric mean of n underlying assets, where the option has the payoff $V_{\max}^p(T, S_1, S_2, \cdots, S_n)$ as

$$V_{\max}^p(T, S_1, S_2, \cdots, S_n) = \max\left(K - \max\left(G_1(T, S_1), G_2(T, S_2), \cdots, G_n(T, S_n)\right), 0\right). \tag{3.25}$$

This means that the price of a put on max option is determined by the maximum of the geometric mean of the n underlying assets. It follows from Eqs (2.8) and (3.25) that the price of a put on max option

$$P_{\max}(T, r, S_1, S_2, \cdots, S_n) = \mathbf{op}(T, r, S_1, S_2, \cdots, S_n, V_{\max}^p). \tag{3.26}$$

Through direct calculation, the following theorem provides the expression for the price of a put on max option.

Theorem 3.3. (Put on max option) *Assume that the underlying assets* S_i $(i = 1, 2, \cdots, n)$ *obey generalized geometric Brownian motion, the stochastic interest rate follows the Vasicek model* (2.1)*, and the initial to expiration time is* 0 *to* T*, then the price of the geometric Asian rainbow put on max option* $P_{\max}(T, r, S_1, S_2, \cdots, S_n)$ *is described by*

$$\begin{aligned} P_{\max}(T, r, S_1, S_2, \cdots, S_n) = & -\sum_{i=1}^{n} \mathrm{e}^{C_i - C_0 + \frac{1}{2}\int_0^T \left((f-m)^2 + g_i^2\right) \mathrm{d}u} \mathbf{N}_i\left(\frac{-\int_0^T \left(f(f-m)+g_i^2\right)\mathrm{d}u - C_i + \ln K}{\sqrt{\int_0^T \left(f^2+g_i^2\right)\mathrm{d}u}}, \underbrace{\cdots, \frac{\int_0^T g_i\left(g_i - \rho^{ij} g_j\right)\mathrm{d}u + C_i - C_j}{\sqrt{\int_0^T \left(g_i^2 + g_j^2 - 2\rho^{ij} g_i g_j\right)\mathrm{d}u}}, \cdots}_{n-1} \right) \\ & + K\mathrm{e}^{-C_0 + \frac{1}{2}\int_0^T m^2 \mathrm{d}u} \sum_{i=1}^{n} \mathbf{N}_i\left(\frac{\int_0^T fm\mathrm{d}u - C_i + \ln K}{\sqrt{\int_0^T \left(f^2+g_i^2\right)\mathrm{d}u}}, \underbrace{\cdots, \frac{C_i - C_j}{\sqrt{\int_0^T \left(g_i^2 + g_j^2 - 2\rho^{ij} g_i g_j\right)\mathrm{d}u}}, \cdots}_{n-1} \right), \end{aligned} \tag{3.27}$$

where $i, j \in \{1, \ldots, n\}$ $(i \neq j)$, C_0, m, C_i, f and g_i are taken as (2.10) *and* (2.12). $\mathbf{N}_i(\cdot)$ *is the n-dimensional cumulative standard normal distribution with the covariance matrix* $\mathbf{\Lambda_i} = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{12} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1n} & \rho_{2n} & \cdots & 1 \end{pmatrix}$ *and*

$$\rho_{1a} = \frac{\int_0^T g_i\left(\rho^{ib}g_b - g_i\right)\mathrm{d}u}{\int_0^T \sqrt{\left(f^2+g_i^2\right)\left(g_i^2+g_b^2-2\rho^{ib}g_ig_b\right)}\mathrm{d}u}, \quad \text{with} \quad a = 2, \cdots, n, \quad \begin{cases} b = a-1, & a \leq i \\ b = a, & a > i \end{cases},$$

$$\rho_{cd} = \frac{\int_0^T \left(g_i^2 - \rho^{ie}g_ig_e - \rho^{if}g_ig_f + \rho^{ef}g_eg_f\right)\mathrm{d}u}{\int_0^T \sqrt{\left(g_i^2+g_e^2-2\rho^{ie}g_ig_e\right)\left(g_i^2+g_f^2-2\rho^{if}g_ig_f\right)}\mathrm{d}u}, \quad \text{with} \quad c = 2, \cdots, n-1, \quad \begin{cases} e = c-1, & c \leq i \\ e = c, & c > i \end{cases},$$

$$d = c+1, \cdots, n, \quad \begin{cases} f = d-1, & d \leq i \\ f = d, & d > i \end{cases}.$$

The proof of Theorem 3.3 is similar to the proof shown in Theorem 3.1, thus it is omitted here.

The put option with the minimum geometric mean of n underlying assets is summarized in Theorem 3.4, where the option has the payoff

$$V_{\min}^p(T, S_1, S_2, \cdots, S_n) = \max\left(K - \min\left(G_1(T, S_1), G_2(T, S_2), \cdots, G_n(T, S_n)\right), 0\right). \tag{3.28}$$

Combining (2.8) and (3.28), we have the price of a put on min option

$$P_{\min}(T, r, S_1, S_2, \cdots, S_n) = \mathbf{op}(T, r, S_1, S_2, \cdots, S_n, V_{\min}^p). \tag{3.29}$$

Straight forward computation yields that the following theorem provides the expression for the price of a put on min option.

Theorem 3.4. (Put on min option) *Assume that the underlying assets S_i $(i = 1, 2, \cdots, n)$ obey generalized geometric Brownian motion, the stochastic interest rate follows the Vasicek model* (2.1)*, and the initial to expiration time is* 0 *to T, then the price of the geometric Asian rainbow put on min option $P_{\min}(T, r, S_1, S_2, \cdots, S_n)$ is given by*

$$\begin{aligned} P_{\min}(T, r, S_1, S_2, \cdots, S_n) = &-\sum_{i=1}^n \mathrm{e}^{C_i - C_0 + \frac{1}{2}\int_0^T \left((f-m)^2 + g_i^2\right)\mathrm{d}u} \mathbf{N}_i\left(\frac{-\int_0^T \left(f(f-m)+g_i^2\right)\mathrm{d}u - C_i + \ln K}{\sqrt{\int_0^T \left(f^2+g_i^2\right)\mathrm{d}u}}, \underbrace{\cdots, \frac{\int_0^T g_i\left(\rho^{ij}g_j - g_i\right)\mathrm{d}u - C_i + C_j}{\sqrt{\int_0^T \left(g_i^2 + g_j^2 - 2\rho^{ij}g_ig_j\right)\mathrm{d}u}}, \cdots}_{n-1}\right) \\ &+ K\mathrm{e}^{-C_0 + \frac{1}{2}\int_0^T m^2 \mathrm{d}u} \sum_{i=1}^n \mathbf{N}_i\left(\frac{\int_0^T f m \mathrm{d}u - C_i + \ln K}{\sqrt{\int_0^T \left(f^2+g_i^2\right)\mathrm{d}u}}, \underbrace{\cdots, \frac{C_j - C_i}{\sqrt{\int_0^T \left(g_i^2 + g_j^2 - 2\rho^{ij}g_ig_j\right)\mathrm{d}u}}, \cdots}_{n-1}\right), \end{aligned} \tag{3.30}$$

where $i, j \in \{1, \ldots, n\}$ $(i \neq j)$, C_0, m, C_i, f and g_i are stated in (2.10) *and* (2.12). $\mathbf{N}_i(\cdot)$ *is the n dimensional cumulative standard normal distribution with the covariance matrix* $\mathbf{\Lambda_i} =$

$$\begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{12} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1n} & \rho_{2n} & \cdots & 1 \end{pmatrix} \textit{and}$$

$$\rho_{1a} = \frac{\int_0^T g_i\big(g_i - \rho^{ib} g_b\big)\mathrm{d}u}{\int_0^T \sqrt{\big(f^2+g_i^2\big)\big(g_i^2+g_b^2-2\rho^{ib}g_i g_b\big)}\mathrm{d}u}, \quad \textit{with} \quad a = 2,\cdots,n, \quad \begin{cases} b = a-1, & a \le i \\ b = a, & a > i \end{cases},$$

$$\rho_{cd} = \frac{\int_0^T \big(g_i^2 - \rho^{ie} g_i g_e - \rho^{if} g_i g_f + \rho^{ef} g_e g_f\big)\mathrm{d}u}{\int_0^T \sqrt{\big(g_i^2+g_e^2-2\rho^{ie}g_i g_e\big)\big(g_i^2+g_f^2-2\rho^{if}g_i g_f\big)}\mathrm{d}u}, \quad \textit{with} \quad c = 2,\cdots,n-1, \quad \begin{cases} e = c-1, & c \le i \\ e = c, & c > i \end{cases},$$

$$d = c+1,\cdots,n, \quad \begin{cases} f = d-1, & d \le i \\ f = d, & d > i \end{cases}.$$

Since the proof of this Theorem is similar to Theorem 3.2, we omit the proof here.

3.3. Parity relationship

For multi-asset options, put-call and min-max combined parity describe the relationship between the prices of call on max, call on min, put on max, and put on min options with the same underlying portfolio, strike price, and expiration date. This section presents this relationship for Asian rainbow options based on stochastic interest rate dynamics equations obeying the Vasicek model (2.1). Here, we fix $n = 2$ and summarize the relationship as the following theorem.

Theorem 3.5. *The put-call and min-max combined parity relationship for two-asset geometric Asian rainbow options under the Vasicek interest rate model* (2.1) *with a fixed strike price of K at maturity T can be given by*

$$\begin{aligned} & C_{\max}(T,r,S_1,S_2) + C_{\min}(T,r,S_1,S_2) - P_{\max}(T,r,S_1,S_2) - P_{\min}(T,r,S_1,S_2) \\ & = \mathrm{e}^{-C_0+\frac{1}{2}\int_0^T (f-m)^2 \mathrm{d}u}\left(\mathrm{e}^{C_1+\frac{1}{2}\int_0^T g_1^2 \mathrm{d}u} + \mathrm{e}^{C_2+\frac{1}{2}\int_0^T g_2^2 \mathrm{d}u}\right) - 2K\mathrm{e}^{-C_0+\frac{1}{2}\int_0^T m^2 \mathrm{d}u}, \end{aligned} \tag{3.31}$$

where C_0, m, C_1, C_2, f, g_1 and g_2 are respectively taken as (2.10) *and* (2.12).

Proof. For the sake of simplicity, we set

$$\begin{aligned} A_1 &= \frac{\int_0^T \big(f(f-m)+g_1^2\big)\mathrm{d}u - \ln K + C_1}{\sqrt{\int_0^T \big(f^2+g_1^2\big)\mathrm{d}u}}, & B_1 &= \frac{\int_0^T g_1(g_1-\rho g_2)\mathrm{d}u + C_1 - C_2}{\sqrt{\int_0^T \big(g_1^2+g_2^2-2\rho g_1 g_2\big)\mathrm{d}u}}, \\ A_2 &= \frac{\int_0^T \big(f(f-m)+g_2^2\big)\mathrm{d}u - \ln K + C_2}{\sqrt{\int_0^T \big(f^2+g_2^2\big)\mathrm{d}u}}, & B_2 &= \frac{\int_0^T g_2(g_2-\rho g_1)\mathrm{d}u - C_1 + C_2}{\sqrt{\int_0^T \big(g_1^2+g_2^2-2\rho g_1 g_2\big)\mathrm{d}u}}, \\ A_3 &= \frac{-\int_0^T fm\mathrm{d}u - \ln K + C_1}{\sqrt{\int_0^T \big(f^2+g_1^2\big)\mathrm{d}u}}, & B_3 &= \frac{C_1 - C_2}{\sqrt{\int_0^T \big(g_1^2+g_2^2-2\rho g_1 g_2\big)\mathrm{d}u}}, \\ A_4 &= \frac{-\int_0^T fm\mathrm{d}u - \ln K + C_2}{\sqrt{\int_0^T \big(f^2+g_2^2\big)\mathrm{d}u}}, & B_4 &= \frac{C_2 - C_1}{\sqrt{\int_0^T \big(g_1^2+g_2^2-2\rho g_1 g_2\big)\mathrm{d}u}}, \\ \rho_1 &= \frac{\int_0^T g_1(g_1-\rho g_2)\mathrm{d}u}{\int_0^T \sqrt{\big(f^2+g_1^2\big)\big(g_1^2+g_2^2-2\rho g_1 g_2\big)}\mathrm{d}u}, & \rho_2 &= \frac{\int_0^T g_2(g_2-\rho g_1)\mathrm{d}u}{\int_0^T \sqrt{\big(f^2+g_2^2\big)\big(g_1^2+g_2^2-2\rho g_1 g_2\big)}\mathrm{d}u}, \\ e_1 &= \mathrm{e}^{C_1-C_0+\frac{1}{2}\int_0^T \big((f-m)^2+g_1^2\big)\mathrm{d}u}, & e_2 &= \mathrm{e}^{C_2-C_0+\frac{1}{2}\int_0^T \big((f-m)^2+g_2^2\big)\mathrm{d}u}, \quad e_3 = \mathrm{e}^{-C_0+\frac{1}{2}\int_0^T m^2 \mathrm{d}u}. \end{aligned}$$

Then, by direct calculating, the sum of the call on max and min option prices is given by

$$\begin{aligned} & C_{\max}(T, r, S_1, S_2) + C_{\min}(T, r, S_1, S_2) \\ = & \, e_1 \left(\mathbf{N}(A_1, B_1; \rho_1) + \mathbf{N}(A_1, -B_1; -\rho_1)\right) + e_2 \left(\mathbf{N}(A_2, B_2; \rho_2) + \mathbf{N}(A_2, -B_2; -\rho_2)\right) \\ & -Ke_3 \left(\mathbf{N}(A_3, B_3; \rho_1) + \mathbf{N}(A_3, -B_3; -\rho_1)\right) - Ke_3 \left(\mathbf{N}(A_4, B_4; \rho_2) + \mathbf{N}(A_4, -B_4; -\rho_2)\right). \end{aligned} \tag{3.32}$$

According to the properties of two-dimensional normal distribution, we have

$$\begin{aligned} & \mathbf{N}(A, B; \rho) + \mathbf{N}(A, -B; -\rho) \\ = & \int_{-\infty}^{A} \int_{-\infty}^{B} \frac{1}{2\pi \sqrt{1-\rho^2}} \mathrm{e}^{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}} \mathrm{d}y\mathrm{d}x + \int_{-\infty}^{A} \int_{-\infty}^{-B} \frac{1}{2\pi \sqrt{1-\rho^2}} \mathrm{e}^{-\frac{x^2+y^2+2\rho xy}{2(1-\rho^2)}} \mathrm{d}y\mathrm{d}x \\ = & \int_{-\infty}^{A} \frac{1}{2\pi \sqrt{1-\rho^2}} \left(\int_{-\infty}^{B} \mathrm{e}^{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}} \mathrm{d}y + \int_{-\infty}^{-B} \mathrm{e}^{-\frac{x^2+y^2+2\rho xy}{2(1-\rho^2)}} \mathrm{d}y \right) \mathrm{d}x \\ = & \int_{-\infty}^{A} \frac{1}{2\pi \sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}} \mathrm{d}y\mathrm{d}x \\ = & \, \mathbf{N}(A, +\infty; \rho). \end{aligned} \tag{3.33}$$

Therefore, we can get

$$\begin{aligned} & C_{\max}(T, r, S_1, S_2) + C_{\min}(T, r, S_1, S_2) \\ = & \, e_1 \mathbf{N}(A_1, +\infty; \rho_1) + e_2 \mathbf{N}(A_2, +\infty; \rho_2) - Ke_3 \left(\mathbf{N}(A_3, +\infty; \rho_1) + \mathbf{N}(A_4, +\infty; \rho_2)\right). \end{aligned} \tag{3.34}$$

Similarly, we sum the put on max and min option prices

$$\begin{aligned} & P_{\max}(T, r, S_1, S_2) + P_{\min}(T, r, S_1, S_2) \\ = & -e_1 \left(\mathbf{N}(-A_1, B_1; -\rho_1) + \mathbf{N}(-A_1, -B_1; \rho_1)\right) - e_2 \left(\mathbf{N}(-A_2, B_2; -\rho_2) + \mathbf{N}(-A_2, -B_2; \rho_2)\right) \\ & +Ke_3 \left(\mathbf{N}(-A_3, B_3; -\rho_1) + \mathbf{N}(-A_3, -B_3; \rho_1)\right) + Ke_3 \left(\mathbf{N}(-A_4, B_4; -\rho_2) + \mathbf{N}(-A_4, -B_4; \rho_2)\right). \end{aligned} \tag{3.35}$$

In view of (3.33), we obtain that

$$\begin{aligned} & P_{\max}(T, r, S_1, S_2) + P_{\min}(T, r, S_1, S_2) \\ = & -e_1 \mathbf{N}(-A_1, +\infty; -\rho_1) - e_2 \mathbf{N}(-A_2, +\infty; -\rho_2) + Ke_3 \left(\mathbf{N}(-A_3, +\infty; -\rho_1) + \mathbf{N}(-A_4, +\infty; -\rho_2)\right). \end{aligned} \tag{3.36}$$

As a result, we deduce that

$$\begin{aligned} & C_{\max}(T, r, S_1, S_2) + C_{\min}(T, r, S_1, S_2) - P_{\max}(T, r, S_1, S_2) - P_{\min}(T, r, S_1, S_2) \\ = & \, e_1 \left(\mathbf{N}(A_1, +\infty; \rho_1) + \mathbf{N}(-A_1, +\infty; -\rho_1)\right) + e_2 \left(\mathbf{N}(A_2, +\infty; \rho_2) + \mathbf{N}(-A_2, +\infty; -\rho_2)\right) \\ & -Ke_3 \left(\mathbf{N}(A_3, +\infty; \rho_1) + \mathbf{N}(-A_3, +\infty; -\rho_1)\right) - Ke_3 \left(\mathbf{N}(A_4, +\infty; \rho_2) + \mathbf{N}(-A_4, +\infty; -\rho_2)\right) \\ = & \, e_1 + e_2 - 2Ke_3. \end{aligned} \tag{3.37}$$

The proof is hence complete. □

Remark 3.2. *For the geometric Asian rainbow option using the stochastic interest rate obeying the Vasicek model, the put-call and min-max combined parity formulas are provided in Eq* (3.31). *It means that the sum of call on max, call on min, put on max, and put on min option prices is a fixed value. This fixed value ensures that there is no arbitrage opportunity in the pricing formula of Theorems 3.1–3.4 and that the pricing formula is realistic and consistent with market conditions. Investors can construct a portfolio of long and short positions to obtain additional returns from the portfolio.*

4. Monte Carlo simulations

This section discusses the simulated values from the Monte Carlo simulations and analytical values from (3.6) for the multi-asset Asian rainbow call option.

4.1. Two-asset case

For convenience, we choose $n = 2$ and assume that the initial prices of two assets S_1 and S_2 are both 40, and the option expiry period is 6 months, i.e., $T = 0.5$. The stochastic volatility values $\sigma_r = 0.1$, $\sigma_1 = 0.1$ and $\sigma_2 = 0.2$ are adopted, and other parameters are shown in Table 1.

Assume that the initial time is 0 and the expiration time is T, we divide the option period $[0, T]$ into m equal intervals, namely, $0 = t_0 < t_1 < \cdots < t_m = T$ and the time interval $\Delta t = \frac{T}{m}$, where m can be understood as the step size of the Monte Carlo simulation under a path. $\frac{S_i(t_{j+1})}{S_i(t_j)}$ $(i = 1, 2;\ j = 0, 1, \cdots, m-1)$ follows the log-normal distribution with a mean and variance of $\left[r(t_{j-1})\mathrm{e}^{-\beta\Delta t} + \frac{\alpha}{\beta}\left(1 - \mathrm{e}^{-\beta\Delta t}\right) - \frac{1}{2}\sigma_i^2\right]\Delta t$ and $\left(\sigma_r \mathrm{e}^{\beta t_{j-1}}\Delta t + \sigma_i\right)^2 \Delta t$, respectively. Then we can classify the simulation procedure in the following four steps.

Step 1. Simulating the path of the underlying asset price and stochastic risk-free interest rate changes. Combining time division and Brownian motion $\mathrm{d}W(t) = \varepsilon \mathrm{d}\sqrt{\Delta t}$, where $\varepsilon \sim N(0, 1)$ is a standard normal random variable, we derive

$$\begin{cases} S_1(t_1) = S_1(t_0) + r(t_0)S_1(t_0)\Delta t + \sigma_1 S_1(t_0)\varepsilon\sqrt{\Delta t}, \\ S_2(t_1) = S_2(t_0) + r(t_0)S_2(t_0)\Delta t + \sigma_2 S_2(t_0)\varepsilon\sqrt{\Delta t}, \\ r(t_1) = r(t_0) + (\alpha + \beta r(t_0))\Delta t + \sigma_r\varepsilon\sqrt{\Delta t}, \\ \qquad \vdots \\ S_1(t_m) = S_1(t_{m-1}) + r(t_{m-1})S_1(t_{m-1})\Delta t + \sigma_1 S_1(t_{m-1})\varepsilon\sqrt{\Delta t} = S_1(T), \\ S_2(t_m) = S_2(t_{m-1}) + r(t_{m-1})S_2(t_{m-1})\Delta t + \sigma_2 S_2(t_{m-1})\varepsilon\sqrt{\Delta t} = S_2(T), \\ r(t_m) = r(t_{m-1}) + (\alpha + \beta r(t_{m-1}))\Delta t + \sigma_r\varepsilon\sqrt{\Delta t} = r(T). \end{cases} \tag{4.1}$$

Step 2. Iterating κ times in the first step, we obtain the underlying asset price and the risk-free rate under the κ paths, namely,

$$S_1^1(t_j),\ S_2^1(t_j),\ r^1(t_j),\ \cdots,\ S_1^\kappa(t_j),\ S_2^\kappa(t_j),\ r^\kappa(t_j), \quad j = 0, 1, \cdots, m.$$

Step 3. Calculate the κ samples of call on max option payoff for the given κ paths

$$V_{\max}^c\left(T, S_1^\hbar, S_2^\hbar\right) = \max\left(\mathrm{e}^{\frac{1}{m+1}\sum_{j=0}^{m}\ln S_1^\hbar(t_j)} - K,\ \mathrm{e}^{\frac{1}{m+1}\sum_{j=0}^{m}\ln S_2^\hbar(t_j)} - K,\ 0\right),\ j = 0, 1, \cdots, m,\ \hbar = 1, 2, \cdots, \kappa. \tag{4.2}$$

Step 4. Calculate the sample mean to obtain the Monte Carlo simulation of the two-asset geometric Asian rainbow call on max option price

$$C_{\max}(T, r, S_1, S_2) = \frac{1}{\kappa}\sum_{\hbar=1}^{\kappa} V_{\max}^c\left(T, S_1^\hbar, S_2^\hbar\right)\mathrm{e}^{-\frac{T}{m+1}\sum_{j=0}^{m} r^\hbar(t_j)}, \quad j = 0, 1, \cdots, m,\ \hbar = 1, 2, \cdots, \kappa. \tag{4.3}$$

Table 1. Two-asset Asian rainbow call option simulation and analytical values.

α	β	K	$r(0) = 0.03$			$r(0) = 0.05$			$r(0) = 0.07$		
			S-value	A-value	R-error(%)	S-value	A-value	R-error(%)	S-value	A-value	R-error(%)
$\rho^{12} = -0.3$											
0.005	0.1	35	6.6460	6.7868	2.0745	6.7325	6.9223	2.7418	6.9021	7.0556	2.1760
0.005	0.1	40	1.9041	1.9943	4.5239	2.0572	2.1470	4.1842	2.2202	2.3030	3.5938
0.005	0.1	45	0.1435	0.1380	−4.0077	0.1497	0.1565	4.3391	0.1772	0.1772	−0.0330
0.005	0.2	35	6.6169	6.7843	2.4680	6.7384	6.9183	2.6009	6.9118	7.0501	1.9619
0.005	0.2	40	1.9332	1.9907	2.8885	2.0610	2.1410	3.7379	2.1878	2.2945	4.6526
0.005	0.2	45	0.1383	0.1375	−0.5979	0.1451	0.1557	6.7693	0.1676	0.1759	4.7166
0.01	0.2	35	6.6079	6.7886	2.6619	6.7140	6.9225	3.0111	6.8722	7.0541	2.5797
0.01	0.2	40	1.8893	1.9965	5.3676	2.0527	2.1470	4.3926	2.2250	2.3006	3.2840
0.01	0.2	45	0.1278	0.1382	7.4851	0.1488	0.1564	4.8857	0.1783	0.1768	−0.8617
$\rho^{12} = 0.1$											
0.005	0.1	35	6.5672	6.5641	−0.0483	6.7499	6.7007	−0.7334	6.9295	6.8351	−1.3812
0.005	0.1	40	1.9170	1.8665	−2.7032	2.0299	2.0081	−1.0833	2.2194	2.1535	−3.0591
0.005	0.1	45	0.1239	0.1377	10.0102	0.1541	0.1561	1.2815	0.1821	0.1766	−3.0890
0.005	0.2	35	6.5911	6.5616	−0.4509	6.7319	6.6966	−0.5266	6.9053	6.8294	−1.1116
0.005	0.2	40	1.8745	1.8631	−0.6122	2.0261	2.0025	−1.1796	2.1733	2.1456	−1.2931
0.005	0.2	45	0.1377	0.1373	−0.2970	0.1521	0.1553	2.0321	0.1817	0.1754	−3.6172
0.01	0.2	35	6.5805	6.5659	−0.2214	6.7531	6.7008	−0.7797	6.8811	6.8335	−0.6955
0.01	0.2	40	1.8949	1.8685	−1.4097	2.0498	2.0080	−2.0806	2.1720	2.1512	−0.9669
0.01	0.2	45	0.1311	0.1380	5.0011	0.1609	0.1561	−3.1260	0.1706	0.1762	3.1849
$\rho^{12} = 0.5$											
0.005	0.1	35	6.5993	6.2978	−4.7870	6.7393	6.4357	−4.7167	6.9443	6.5713	−5.6759
0.005	0.1	40	1.8975	1.7139	−10.7117	1.9855	1.8438	−7.6846	2.2101	1.9781	−11.7302
0.005	0.1	45	0.1381	0.1369	−0.9423	0.1617	0.1549	−4.4270	0.1732	0.1749	0.9325
0.005	0.2	35	6.6062	6.2952	−4.9394	6.7847	6.4315	−5.4926	6.8673	6.5655	−4.5961
0.005	0.2	40	1.8990	1.7108	−10.9983	2.0482	1.8386	−11.3977	2.1742	1.9707	−10.3279
0.005	0.2	45	0.1431	0.1364	−4.8938	0.1462	0.1541	5.0882	0.1828	0.1736	−5.2513
0.01	0.2	35	6.5984	6.2997	−4.7413	6.7487	6.4359	−4.8609	6.8823	6.5698	−4.7574
0.01	0.2	40	1.8948	1.7157	−10.4396	2.0241	1.8437	−9.7833	2.1892	1.9759	−10.7921
0.01	0.2	45	0.1307	0.1371	4.6602	0.1494	0.1548	3.5128	0.1742	0.1745	0.1704

This table displays the prices of two-asset geometric Asian rainbow call on max options, where $S_1(0) = 40$, $S_2(0) = 40$, $T = 0.5$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, $\sigma_r = 0.1$, $m = 100$ and $\kappa = 10000$. "A-value", "S-value" and "R-error" represent "analytical value", "simulated value" and "relative error", respectively. Furthermore, (analytical value - simulated value)/analytical value is the formula for the relative error.

Using the above four steps, we calculate the Monte Carlo simulation values (see S-value in Table 1). The strike price is set at $K = 35, 40, 45$, the initial risk-free rate is set at $r(0) = 0.03, 0.05, 0.07$, the drift term is set at $\alpha = 0.005, 0.01$, and the diffusion term is set at $\beta = 0.1, 0.2$, and the correlation coefficient is set at $\rho^{12} = -0.3, 0.1, 0.5$, respectively. We separately combine each parameter. As an example, when $K = 35$, $r(0) = 0.03$, $\alpha = 0.005$, and $\beta = 0.1$, ρ is equal to -0.3, 0.1, and 0.5, respectively. The pricing results of Eq (3.6) are calculated using the same inputs; for details, see A-values in Table 1. R-error in Table 1 shows the results of our comparison of the errors between the simulated and analyzed values.

Table 1 demonstrates how the impact of changing parameters on the option price can have a significant economic impact when the strike price K is allowed to rise. Take $r(0) = 0.03$, $\alpha = 0.005$, $\beta = 0.1$, and $\rho^{12} = -0.3$ as an example, the simulated prices for K equal to 35, 40, and 45 are 6.6460, 1.9041, and 0.1435, respectively. The option price may decrease significantly with an increase in the strike price of K. The same phenomenon is noted for the analytical values, 6.7868, 1.9943, and 0.1380, respectively. When $K = 35$, the analytical value is greater than the simulated value, and the relative error between the two values is 2.0745%. The simulated value, analytical value, and relative errors are 6.7325, 6.9223, and 2.7418%, respectively, when the other parameters are held constant and the initial risk-free rate, $r(0)$, is raised to 0.05. The errors do not significantly rise as $r(0)$ increases. This indicates that our pricing results are reasonable. The simulated, analyzed, and relative errors are, respectively, 6.5672, 6.5641, and -0.0483% when all other parameters remain the same and the correlation coefficient ρ is increased to 0.1. There is no discernible difference in the errors after changing ρ. Similar results are discovered for the other parameters as well. As can be seen, changing one of the parameters does not affect the relative errors but changes the simulated or analytical values. In other words, the relative errors do not significantly change when a particular parameter is changed. The results of the pricing formula are accurate and reasonable following the verification of the Monte Carlo simulation.

4.2. *Three-asset case*

Taking $n = 3$, we set the initial prices of three assets S_1, S_2 and S_3 are all 40, and the option expiry period is still 6 months, i.e., $T = 0.5$. The stochastic volatility values are respectively $\sigma_r = 0.1$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$ and $\sigma_3 = 0.3$, and other parameters are shown in Table 2. Since the steps of the simulation procedure are comparable to those in the case of the two assets, we omit the details.

The correlation coefficients $\left(\rho^{12}, \rho^{13}, \rho^{23}\right)$ are chosen to be $(-0.5, -0.1, 0.3)$; $(-0.4,\ 0, 0.4)$; and $(-0.3, 0.1, 0.5)$. The other parameters are the same as those in Table 1. The simulated prices for K equal to 35, 40, and 45 in Table 2 at $r(0) = 0.03$, $\alpha = 0.005$, $\beta = 0.1$, $\left(\rho^{12}, \rho^{13}, \rho^{23}\right) = (-0.5, -0.1, 0.3)$ are 8.0668, 3.2050, and 0.5974, respectively. The change in the strike price K will significantly lower the option price, just as it did in the case of $n = 2$. The same phenomenon is seen for the analytical values, which are 7.9958, 3.1468, and 0.6188, respectively. When $K = 35$, the relative error—the analytical value being lower than the simulated value—is calculated as 0.8886%. The simulated value, analytical value, and relative errors are 8.1916, 8.124, and -0.8260%, respectively, when the initial risk-free rate, $r(0)$, is increased to 0.05 while the other parameters are held constant. With an increase in $r(0)$, the errors do not significantly change. This matches the finding in Table 1's results. The relative error does not grow (or possibly even shrink) in proportion to the increase in the number of underlying assets n when comparing the results of Table 1 with 2.7418%. This shows that the pricing formula also works in the multi-asset case and is not just applicable in the two-asset case. The accuracy of the pricing formula is confirmed following Monte Carlo simulations for the $n = 2$ and $n = 3$ cases.

Table 2. Three-asset Asian rainbow call option simulation and analytical values.

α	β	K	$r(0) = 0.03$			$r(0) = 0.05$			$r(0) = 0.07$		
			S-value	A-value	R-error(%)	S-value	A-value	R-error(%)	S-value	A-value	R-error(%)
$\rho^{12} = -0.5, \rho^{13} = -0.1, \rho^{23} = 0.3$											
0.005	0.1	35	8.0668	7.9958	−0.8886	8.1916	8.1245	−0.8260	8.3443	8.2508	−1.1325
0.005	0.1	40	3.2050	3.1468	−1.8492	3.3765	3.3048	−2.1697	3.5716	3.4641	−3.1038
0.005	0.1	45	0.5974	0.6188	3.4508	0.6497	0.6647	2.2613	0.6926	0.7134	2.9154
0.005	0.2	35	8.1270	7.9937	−1.6681	8.1799	8.1210	−0.7250	8.2983	8.2461	−0.6329
0.005	0.2	40	3.2335	3.1434	−2.8667	3.3940	3.2989	−2.8819	3.5601	3.4559	−3.0151
0.005	0.2	45	0.5970	0.6176	3.3479	0.6290	0.6628	5.0959	0.6887	0.7106	3.0822
0.01	0.2	35	8.0731	7.9974	−0.9458	8.2268	8.1246	−1.2569	8.3148	8.2496	−0.7906
0.01	0.2	40	3.2639	3.1492	−3.6419	3.4201	3.3048	−3.4894	3.5484	3.4618	−2.4998
0.01	0.2	45	0.5957	0.6193	3.8144	0.6507	0.6646	2.0942	0.7227	0.7126	−1.4261
$\rho^{12} = -0.4, \rho^{13} = 0, \rho^{23} = 0.4$											
0.005	0.1	35	8.0462	7.9817	−0.8085	8.1941	8.1149	−0.9763	8.3076	8.2456	−0.7526
0.005	0.1	40	3.2615	3.0574	−6.6747	3.3858	3.2134	−5.3662	3.5669	3.3713	−5.7990
0.005	0.1	45	0.6004	0.6102	1.5977	0.6512	0.6545	0.5136	0.7065	0.7016	−0.7007
0.005	0.2	35	8.0539	7.9795	−0.9321	8.1526	8.1112	−0.5099	8.3338	8.2406	−1.1314
0.005	0.2	40	3.2010	3.0539	−4.8160	3.3980	3.2076	−5.9378	3.5394	3.3632	−5.2397
0.005	0.2	45	0.5934	0.6091	2.5736	0.6957	0.6527	−6.5867	0.7202	0.6989	−3.0419
0.01	0.2	35	8.0555	7.9834	−0.9027	8.1895	8.1150	−0.9181	8.2868	8.2443	−0.5154
0.01	0.2	40	3.2408	3.0597	−5.9185	3.3733	3.2134	−4.9746	3.5800	3.3691	−6.2619
0.01	0.2	45	0.6312	0.6107	−3.3471	0.6685	0.6545	−2.1391	0.6844	0.7008	2.3299
$\rho^{12} = -0.3, \rho^{13} = 0.1, \rho^{23} = 0.5$											
0.005	0.1	35	8.0715	7.9795	−1.1522	8.1421	8.1177	−0.3003	8.3594	8.2534	−1.2843
0.005	0.1	40	3.2284	2.9705	−8.6823	3.4442	3.1248	−10.2194	3.5793	3.2818	−9.0653
0.005	0.1	45	0.6002	0.5991	−0.1907	0.6357	0.6418	0.9551	0.6824	0.6871	0.6861
0.005	0.2	35	8.0552	7.9772	−0.9781	8.2085	8.1138	−1.1662	8.3083	8.2481	−0.7303
0.005	0.2	40	3.2099	2.9670	−8.1873	3.4146	3.1190	−9.4760	3.4833	3.2736	−6.4050
0.005	0.2	45	0.6086	0.5980	−1.7708	0.6541	0.6401	−2.1861	0.7050	0.6846	−2.9808
0.01	0.2	35	8.0563	7.9813	−0.9393	8.2468	8.1179	−1.5880	8.3783	8.2520	−1.5312
0.01	0.2	40	3.2387	2.9727	−8.9485	3.4103	3.1248	−9.1362	3.5475	3.2795	−8.1743
0.01	0.2	45	0.6195	0.5996	−3.3154	0.6466	0.6417	−0.7571	0.7249	0.6863	−5.6208

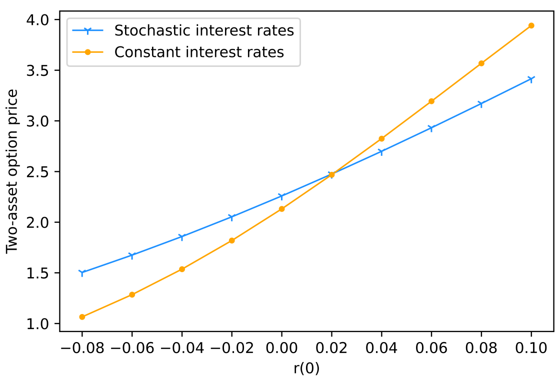
This table shows the prices of three-asset geometric Asian rainbow call on max options, where $S_1(0) = S_2(0) = S_3(0) = 40$, $T = 0.5$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, $\sigma_3 = 0.3$, $\sigma_r = 0.1$, $m = 100$ and $\kappa = 10000$. "A-value", "S-value" and "R-error" represent "analytical value", "simulated value" and "relative error", respectively. Furthermore, (analytical value - simulated value)/analytical value is the formula for the relative error.

5. Sensitivity analysis

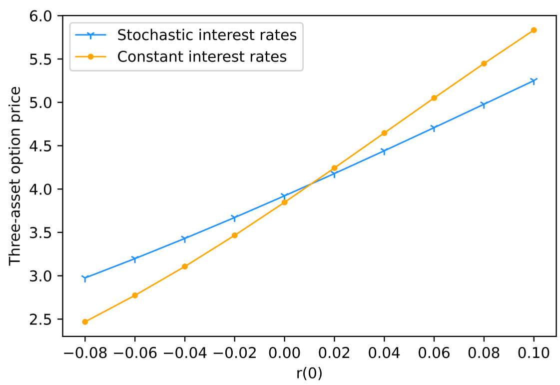
To assess the effectiveness of the model, we look at the effects of a few key parameters on the analysis findings in this section. The strike price K, the correlation of the underlying assets ρ^{ij} ($i = 1,2; j = 2,3; i < j$), and the initial value of the risk-free interest rate $r(0)$ were the main factors we took into account. As a result, we can see how these parameters influence how much the option price changes.

5.1. Sensitivity analysis of interest rate randomness

The effects of stochastic and constant interest rates on option prices are contrasted in this subsection. In Figure 1, we depict the changes in Asian rainbow call on max option prices at two assets ($n = 2$) and three assets ($n = 3$) with the stochastic or constant interest rates, respectively. The initial risk-free interest rate is $r(0)$, which ranges from -0.1 to 0.1. When the interest rate is stochastic, the drift term, diffusion term, and volatility are all set to $\alpha = 0.01$, $\beta = 0.2$, and $\sigma_r = 0.2$, respectively. The drift term, diffusion term, and volatility are defined by $\alpha = 0$, $\beta = -1$, and $\sigma_r = 0$ when the interest rate is constant. From Figure (1a) and (1b), we can see that an increase in the risk-free interest rate causes the option's price to rise, but the option price increases more slowly at a stochastic rate than it does at a constant interest rate. This is a result of the stochastic interest rate's volatility, which affects a portion of the option price and makes it less sensitive than it would be if the interest rate were constant. In other words, under the stochastic interest rate, the option price is less influenced by the risk-free interest rate's size, reducing the dependence of the option price on the interest rate.



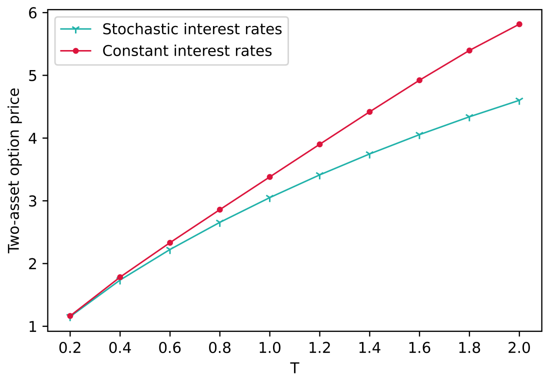
(a) Two-asset Asian rainbow call on max option price v.s. the initial risk-free interest rate $r(0)$, when $S_1 = S_2 = K = 40$ and $\rho^{12} = 0.5$



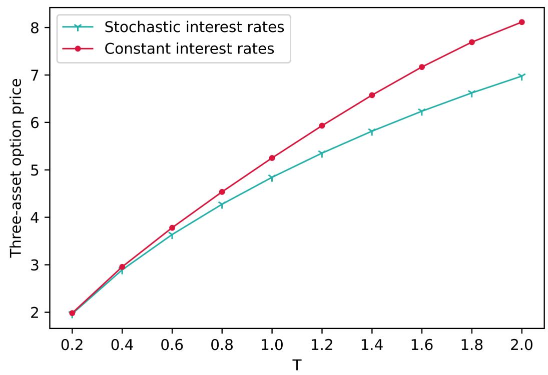
(b) Three-asset Asian rainbow call on max option price v.s. the initial risk-free interest rate $r(0)$, when $S_1 = S_2 = S_3 = K = 40$ and $\rho^{12} = -0.3$, $\rho^{13} = 0.1$, $\rho^{23} = 0.5$

Figure 1. Comparison of two/three-asset option price with the constant or stochastic risk-free interest rate, where $T = 1$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, and $\sigma_3 = 0.3$.

Figure 2 shows how the risk-free interest rate changes as expiration T increases, where the range of the option's expiration period, T, is 0 to 2. The other parameters of interest rates are the same as in Figure 1. Figure (2a) and (2b) illustrate that an extension of the expiration period causes the option's price to rise. The impact of the expiration period on the option price will, however, be diminished and will rise logarithmically at the stochastic interest rate. The price of options rises more quickly over time with a constant interest rate. The stochastic interest rate will not produce an absolute return and can be viewed as a risk if the interest rate changes at random. Options will not experience the same absolute return as a constant rate when this risk is present. Therefore, with a stochastic interest rate, the option's price must be reduced in order to offset the risk that the interest rate poses, preventing the price from rising too quickly as the time period is extended.



(a) Two-asset Asian rainbow call on max option price v.s. the maturities T, when $S_1 = S_2 = K = 40$ and $\rho^{12} = 0.5$



(b) Three-asset Asian rainbow call on max option price v.s. the maturities T, when $S_1 = S_2 = S_3 = K = 40$ and $\rho^{12} = -0.3$, $\rho^{13} = 0.1$, $\rho^{23} = 0.5$

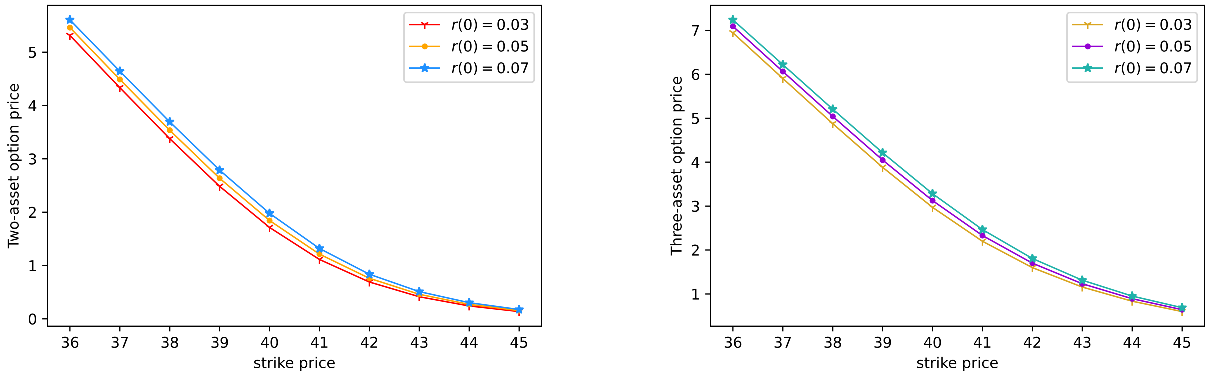
Figure 2. Comparison of two/three-asset option price with the constant or stochastic risk-free interest rate, where $r(0) = 0.7$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, and $\sigma_3 = 0.3$.

5.2. Sensitivity analysis of the initial risk-free interest rate

Figure (3a) and (3b) respectively depict the changes in Asian rainbow call on max option prices at two-assets ($n = 2$) and three-assets ($n = 3$) with the strike price K ranging from 35 to 45 for the initial risk-free interest rate $r(0)$ is equal to 0.03, 0.05, and 0.07. For two assets $n = 2$ and three assets $n = 3$, we set the initial prices S_i $(i = 1, 2)$ and S_j $(j = 1, 2, 3)$ to be all 40 respectively, and the option expiry period is still 6 months, i.e., $T = 0.5$. The stochastic volatility values are $\sigma_r = 0.1$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, and $\sigma_3 = 0.3$, respectively. The correlation of the two and three underlying assets are respectively $\rho^{12} = 0.5$ and $\rho^{12} = -0.3$, $\rho^{13} = 0.1$, $\rho^{23} = 0.5$.

The option price rises with the initial risk-free interest rate $r(0)$ at the same strike price K, as shown in Figure 3. The expected future return of the underlying asset will rise along with the rise in the risk-free interest rate when the price of the underlying asset remains constant. When discounted time prices are taken into account, a rise in the return on asset value appreciation, however, results in a fall in the value of the discount. Investors who purchase call options will profit when the price of options increases due to the higher risk-free rate. Furthermore, investing in stocks will cost more than purchasing the same number of call options. In the end, when interest rates are high, investors prefer to purchase call options because it is more expensive to buy stocks and hold them until they mature. Figure 3 also demonstrates that as the strike price K rises, the option price gradually converges to 0. It is more challenging for investors to profit from the options trade at expiration when the strike price of K is very high. As a result of discounting to a very short initial time, the option price converges to zero.

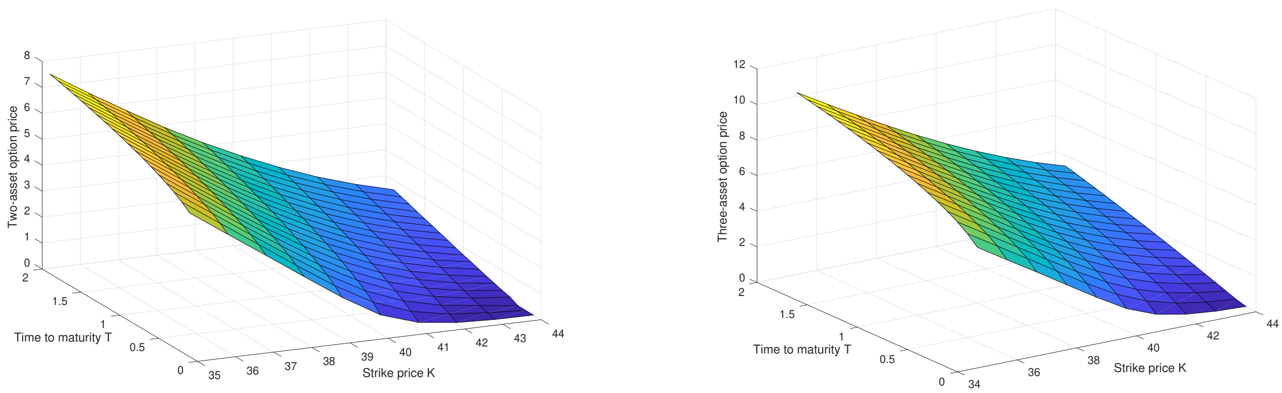
Additionally, as can be seen in Figure 4, the option price of the two/three-asset Asian rainbow increases as the time to maturity T increases and decreases as the strike price K increases. For call options, the underlying asset price continues to rise over time. Therefore, extending the strike time is beneficial for investors to meet their desired expectations. The longer the term of the option, the greater the option's time value, which in turn increases the option price.



(a) The prices of the two-asset Asian rainbow call on max option v.s. the strike price K, when the initial value of the risk-free interest rate $r(0) = 0.03, 0.05$, and 0.07, $S_1 = S_2 = 40$ and $\rho^{12} = 0.5$

(b) The prices of the three-asset Asian rainbow call on max option v.s. the strike price K, when the initial value of the risk-free interest rate $r(0) = 0.03, 0.05$, and 0.07, $S_1 = S_2 = S_3 = 40$ and $\rho^{12} = -0.3$, $\rho^{13} = 0.1$, $\rho^{23} = 0.5$

Figure 3. The option's price at various strike prices (K) under various ($r(0)$), where $T = 0.5$, $\sigma_r = 0.1$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, and $\sigma_3 = 0.3$.



(a) Two-asset Asian rainbow call on max option price v.s. the strike price K and maturities T, when the initial value of the risk-free interest rate $r(0) = 0.03$, $S_1 = S_2 = 40$ and $\rho^{12} = 0.5$

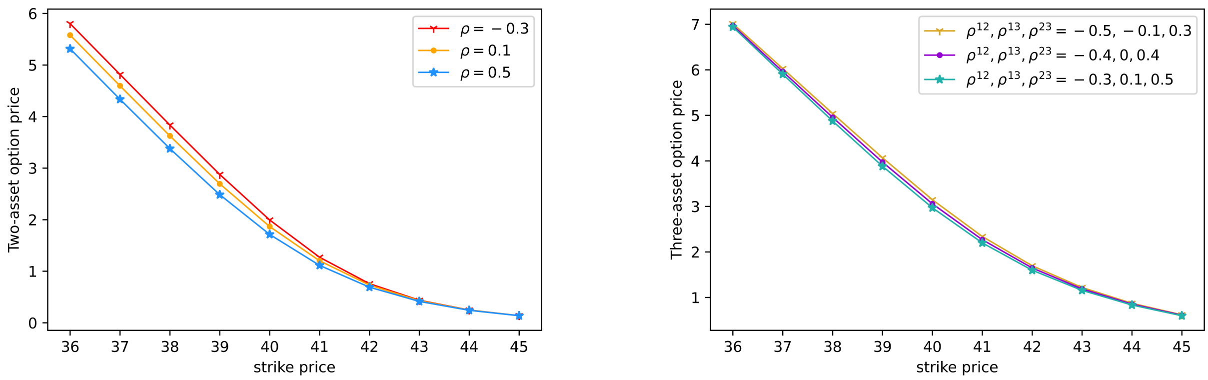
(b) Three-asset Asian rainbow call on max option price v.s. the strike price K and maturities T, when the initial value of the risk-free interest rate $r(0) = 0.03$, $S_1 = S_2 = S_3 = 40$ and $\rho^{12} = -0.3$, $\rho^{13} = 0.1$, $\rho^{23} = 0.5$

Figure 4. Comparison of two/three-asset option price with various strike prices (K) and maturities (T), where $\sigma_r = 0.1$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, and $\sigma_3 = 0.3$.

5.3. Sensitivity analysis of the correlation coefficient of underlying assets

Asian rainbow call option prices at $n = 2$ and $n = 3$ are displayed in Figure (5a) and (5b), respectively, with strike prices K ranging from 35 to 45 and various correlation coefficients of the underlying asset ρ^{ij} $(i = 1, 2; j = 2, 3; i < j)$.

As can be seen in Figure 5, the higher correlation coefficient of the underlying assets, ρ^{ij}, increases the likelihood that the underlying assets S_i and S_j will rise or fall together. The implied volatility of the option is higher because the Asian Rainbow call on max option is based on the higher price of the underlying asset. The price of options has increased as implied volatility has increased. Implied volatility, however, only has a modest impact on option prices. As a result, the variations in option prices under the various correlation coefficients are not significant when the strike price K is high.



(a) The prices of the two-asset Asian rainbow call on max option v.s. the strike price K, when the correlation coefficient of underlying assets $\rho = \rho^{12} = -0.3, 0.1$ and 0.5

(b) The prices of the three-asset Asian rainbow call on max option v.s. the strike price K with different correlation coefficients ρ^{ij} $(i = 1, 2; j = 2, 3; i < j)$

Figure 5. The option's price at various strike prices (K) under various (ρ^{ij}).

6. Discussion

In this study, we take into account the pricing of geometric Asian rainbow options for multiple assets (n) when the stochastic interest rate followed the Vasicek model. Our study focuses not only on the use of stochastic interest rates but also more importantly on the expansion of the asset dimension. This is also the significance of our study on rainbow options. The relevance of multi-asset options in diversifying risk is obvious and is one of the purposes of this work. In detail, we propose an option payoff based on the geometric mean using the Vasicek model and the underlying asset price model combined. Four Asian rainbow option pricing formulas, including a call on max, call on min, put on max, and put on min option, are derived through rigorous derivation. These four options' pricing outcomes all satisfy the option parity relationship. Put-call and min-max combination parity theories are suggested based on this, and justifications are provided. Utilizing Monte Carlo simulations, the option prices were determined, and the validity of the pricing formulas was confirmed. In order to further demonstrate the compatibility of the pricing formulas with the financial derivatives market, a sensitivity analysis of variables including strike price, the initial value of risk-free rate, correlation coefficient, and strike time is carried out.

Certain limitations still exist in our model (2.2). To define time-varying interest rates, some stochastic processes are generally used: the Vasicek model, the CIR model, and the Hull-White model, which is an expansion of the Vasicek and CIR interest-rate models. In this study, we are only concerned with the former. However, it is reasonable to speculate that in some situations, the market's predictions about future interest rates include time-dependent parameters. The results in the Vasicek model might not be the same as those of the CIR model and the Hull-White model. Additionally, it might be valuable to search for a more inclusive and liberal result when choosing a stochastic interest rate model. We shall deploy these in future work.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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A. Proof of Theorem 3.2

Proof. From Eq (3.22), the payoff of the call on min option $V^c_{\min}(T, S_1, S_2, \cdots, S_n)$ can be expressed by the following equation

$$\begin{aligned} V^c_{\min}(T, S_1, S_2, \cdots, S_n) = &(G_1(T, S_1) - K)\mathbf{I}_{\{G_1(T,S_1)\geq K, G_2(T,S_2)\geq G_1(T,S_1),\cdots,G_n(T,S_n)\geq G_1(T,S_1)\}} \\ &+(G_2(T, S_2) - K)\mathbf{I}_{\{G_2(T,S_2)\geq K, G_1(T,S_1)\geq G_2(T,S_2),\cdots,G_n(T,S_n)\geq G_2(T,S_2)\}} \\ &+\cdots\cdots \\ &+(G_n(T, S_n) - K)\mathbf{I}_{\{G_n(T,S_n)\geq K, G_1(T,S_1)\geq G_n(T,S_n),\cdots,G_{n-1}(T,S_{n-1})\geq G_n(T,S_n)\}}. \end{aligned} \tag{A.1}$$

Set $A_i = \{G_i(T, S_i) \geq K, \cdots, G_j(T, S_j) \geq G_i(T, S_i), \cdots\}$ $(i, j = 1, 2, \cdots, n,\ i \neq j)$, based on Eqs (2.3) and (2.11), we can obtain

$$A_i = \left\{\int_0^T f\mathrm{d}W_r(u) + \int_0^T g_i\mathrm{d}W_i(u) \geq \ln K - C_i, \cdots, \int_0^T g_j\mathrm{d}W_j(u) - \int_0^T g_i\mathrm{d}W_i(u) \geq C_i - C_j, \cdots\right\}. \tag{A.2}$$

It follows from Eq (3.23), we derive the call on min option price as

$$\begin{aligned} C_{\min}(T, r, S_1, S_2, \cdots, S_n) &= \sum_{i=1}^n \mathbb{E}\left[e^{-\int_0^T r(t)\mathrm{d}t + \frac{1}{T}\int_0^T \ln S_i(t)\mathrm{d}t}\mathbf{I}_{A_i}\right] - K\sum_{i=1}^n \mathbb{E}\left[e^{-\int_0^T r(t)\mathrm{d}t}\mathbf{I}_{A_i}\right] \\ &= \sum_{i=1}^n I_i - \sum_{i=1}^n I_{n+i}. \end{aligned} \tag{A.3}$$

It follows from (2.9), (2.11) and (A.2) that we compute

$$I_i = \mathrm{e}^{C_i - C_0}\mathbb{E}\left[\mathrm{e}^{\int_0^T (f-m)\mathrm{d}W_r(u) + \int_0^T g_i\mathrm{d}W_i(u)}\mathbf{I}_{\left\{\int_0^T f\mathrm{d}W_r(u)+\int_0^T g_i\mathrm{d}W_i(u)\geq \ln K - C_i,\cdots,\int_0^T g_j\mathrm{d}W_j(u)-\int_0^T g_i\mathrm{d}W_i(u)\geq C_i-C_j,\cdots\right\}}\right], \tag{A.4}$$

where $i, j = 1, 2, \cdots, n$ and $i \neq j$. Taking

$$\begin{aligned} Z_0 &= \int_0^T (f - m)\,\mathrm{d}W_r(u) + \int_0^T g_i\mathrm{d}W_i(u), \quad i = 1, 2, \cdots, n, \\ Z_1 &= \int_0^T f\mathrm{d}W_r(u) + \int_0^T g_i\mathrm{d}W_i(u), \quad i = 1, 2, \cdots, n, \\ Z_a &= \int_0^T g_j\mathrm{d}W_j(u) - \int_0^T g_i\mathrm{d}W_i(u), \quad a = 2, \cdots, n, \quad i, j = 1, 2, \cdots, n,\ i \neq j, \end{aligned} \tag{A.5}$$

due to $\mathrm{d}W(u) = \varepsilon\mathrm{d}\sqrt{u}$ and $\varepsilon \sim N(0, 1)$ is a standard normal random variable, then Z_0 is rewritten as

$$Z_0 = \varepsilon_r\int_0^T (f - m)\,\mathrm{d}\sqrt{u} + \varepsilon_i\int_0^T g_i\mathrm{d}\sqrt{u}, \quad i = 1, 2, \cdots, n, \tag{A.6}$$

and the variance of the random variable Z_0 is

$$\sigma_0^2 = \int_0^T \left((f-m)^2 + g_i^2\right) \mathrm{d}u, \quad i = 1, 2, \cdots, n. \tag{A.7}$$

Similarly, the variances of Z_1 and Z_a take the form of

$$\sigma_1^2 = \int_0^T \left(f^2 + g_i^2\right) \mathrm{d}u, \quad i = 1, 2, \cdots, n, \tag{A.8}$$

and

$$\sigma_a^2 = \int_0^T \left(g_i^2 + g_j^2 - 2\rho^{ij} g_i g_j\right) \mathrm{d}u, \quad a = 2, \cdots, n, \quad i, j = 1, 2, \cdots, n, \; i \neq j, \tag{A.9}$$

respectively. The correlation coefficients between the $n+1$ random variables Z_0, Z_1, Z_a $(a = 2, \cdots, n)$ satisfy

$$\rho_{01} = \frac{\int_0^T \left(f(f-m) + g_i^2\right) \mathrm{d}u}{\sigma_0 \sigma_i}, \quad i = 1, 2, \cdots, n, \tag{A.10}$$

$$\rho_{0a} = \frac{\int_0^T g_i\left(\rho^{ib} g_b - g_i\right) \mathrm{d}u}{\sigma_0 \sigma_a}, \quad a = 2, \cdots, n, \begin{cases} b = a-1, & a \leq i \\ b = a, & a > i \end{cases}, \; i = 1, 2, \cdots, n, \tag{A.11}$$

$$\rho_{1a} = \frac{\int_0^T g_i\left(\rho^{ib} g_b - g_i\right) \mathrm{d}u}{\sigma_1 \sigma_b}, \quad a = 2, \cdots, n, \begin{cases} b = a-1, & a \leq i \\ b = a, & a > i \end{cases}, \; i = 1, 2, \cdots, n, \tag{A.12}$$

and

$$\rho_{cd} = \frac{\int_0^T \left(g_i^2 - \rho^{ie} g_i g_e - \rho^{if} g_i g_f + \rho^{ef} g_e g_f\right) \mathrm{d}u}{\int_0^T \sqrt{\left(g_i^2 + g_e^2 - 2\rho^{ie} g_i g_e\right)\left(g_i^2 + g_f^2 - 2\rho^{if} g_i g_f\right)} \mathrm{d}u}, \quad \begin{aligned} & c = 2, \cdots, n-1, \begin{cases} e = c-1, & c \leq i \\ e = c, & c > i \end{cases}, \\ & d = c+1, \cdots, n, \begin{cases} f = d-1, & d \leq i \\ f = d, & d > i \end{cases}, \\ & i = 1, 2, \cdots, n. \end{aligned} \tag{A.13}$$

Summing up the above, $(Z_0, Z_1, \cdots, Z_n)^{\mathrm{T}} \sim N(\mathbf{0}, \tilde{\mathbf{\Lambda}}_i)$ with $\tilde{\mathbf{\Lambda}}_i = \begin{pmatrix} \sigma_0^2 & \rho_{01}\sigma_0\sigma_1 & \cdots & \rho_{0n}\sigma_0\sigma_n \\ \rho_{01}\sigma_0\sigma_1 & \sigma_1^2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{0n}\sigma_0\sigma_n & \rho_{1n}\sigma_1\sigma_n & \cdots & \sigma_n^2 \end{pmatrix}$, $i = 1, 2, \cdots, n$. In view of Corollary 3.1, we can obtain

$$\begin{aligned} I_i &= \mathrm{e}^{C_i - C_0} \mathbb{E}\left[\mathrm{e}^{\sigma_0 \frac{Z_0}{\sigma_0}} \mathbf{I}_{\left\{\sigma_1 \frac{Z_1}{\sigma_1} \geq \ln K - C_i, \cdots, \sigma_a \frac{Z_a}{\sigma_a} \geq C_i - C_j, \cdots\right\}}\right] \\ &= \mathrm{e}^{C_i - C_0 + \frac{1}{2}\sigma_0^2} N_i\left(\frac{\rho_{01}\sigma_0\sigma_1 - \ln K + C_i}{\sigma_1}, \cdots, \frac{\rho_{0a}\sigma_0\sigma_a - C_i + C_j}{\sigma_a}, \cdots\right) \\ &= \mathrm{e}^{C_i - C_0 + \frac{1}{2}\int_0^T \left((f-m)^2 + g_i^2\right) \mathrm{d}u} \mathbf{N}_i\left(\frac{\int_0^T \left(f(f-m) + g_i^2\right) \mathrm{d}u + C_i - \ln K}{\sqrt{\int_0^T \left(f^2 + g_i^2\right) \mathrm{d}u}}, \cdots, \frac{\int_0^T g_i\left(\rho^{ij} g_j - g_i\right) \mathrm{d}u - C_i + C_j}{\sqrt{\int_0^T \left(g_i^2 + g_j^2 - 2\rho^{ij} g_i g_j\right) \mathrm{d}u}}, \cdots\right), \end{aligned} \tag{A.14}$$

where $i, j = 1, 2, \cdots, n$, $i \neq j$, C_0, m, C_i, f and g_i are defined in Eqs (2.10) and (2.12). $\mathbf{N}_i(\cdot)$ is the n-dimensional cumulative standard normal distribution with the covariance matrix $\mathbf{\Lambda}_i = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{12} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1n} & \rho_{2n} & \cdots & 1 \end{pmatrix}$, where the correlation coefficients ρ_{1a} and ρ_{cd} are defined in Eqs (A.12) and (A.13).

In addition, $I_{n+i}\,(i, j = 1, 2, \cdots, n,\ i \neq j)$ satisfies

$$I_{n+i} = K\mathrm{e}^{-C_0+\frac{1}{2}\int_0^T m^2\mathrm{d}u}\sum_{i=1}^{n}\mathbf{N}_i\left(\frac{-\int_0^T fm\mathrm{d}u+C_i-\ln K}{\sqrt{\int_0^T\left(f^2+g_i^2\right)\mathrm{d}u}},\cdots,\frac{C_j-C_i}{\sqrt{\int_0^T\left(g_i^2+g_j^2-2\rho^{ij}g_ig_j\right)\mathrm{d}u}},\cdots\right). \tag{A.15}$$

This completes the proof of Theorem 3.2. □

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