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Research article

A study on the existence of numerical and analytical solutions for fractional integrodifferential equations in Hilfer type with simulation

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Abstract: Previous studies have shown that fractional derivative operators have become an integral part of modeling natural and physical phenomena. During the progress and evolution of these operators, it has become clear to researchers that each of these operators has special capacities for investigating phenomena in engineering sciences, physics, biological mathematics, etc. Fixed point theory and its famous contractions have always served as useful tools in these studies. In this regard, in this work, we considered the Hilfer-type fractional operator to study the proposed integrodifferential equation. We have used the capabilities of measure theory and fixed point techniques to provide the required space to guarantee the existence of the solution. The Schauder and Arzela-Ascoli theorems play a fundamental role in the existence of solutions. Finally, we provided two examples with some graphical and numerical simulation to make our results more objective.

Keywords: fractional integrodifferential equation; Hilfer derivative; fixed point; generalized Riemann-Liouville fractional derivative
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1. Introduction

Today, the use of fractional calculus in modeling natural phenomena has caused significant growth and has been to the attention of researchers in various fields of engineering [1], mathematics [2, 3], physics [4, 5]. As one of the most prominent features of fractional operators, we can mention their non-locality. Based on the available results and evidence, modeling by ordinary calculus is not capable of describing the real behavior of phenomena and is often associated with the error of estimating the phenomenon [6]. Researchers in the fields of science and engineering have different approaches to the non-local character of fractional calculus. Physicists' approach to this issue led to interesting modeling for physical phenomena such as heat flow, hereditary polarization in dielectrics, viscoelasticity and so on [7]. Such phenomena were modeled with equations which are influenced by the past values of one or more variables and were called equation with memory in the literature. On the other hand, we know that the history of mathematics has always been associated with the generalization of different concepts, so it is worth mentioning that during the entry of fractional calculus into various fields of science, some researchers took steps willingly in the field of generalization and introduction of new fractional operators. We can refer to the fractional operators of Riemann-Liouville (RL), Caputo, Atangana-Baleanu (AB), Hadamard, fractal fractional, Caputo-Fabrizio, Hilfer, fractional q-derivative, etc. To get information about some of the works done on the mentioned operators, the reader can refer to references [8–15]. Certainly, the developments that directly lead to the improvement of human life are investigated more. As an example, the efforts that have been made recently in the field of modeling can be mentioned as follows. In Biomath: COVID-19 [16–18], Mump Virus [19], hepatitis B [20, 21], human liver [22], an immunogenetic tumor model [23]. In thermodynamics and physics, we can refer to [24–34].

In 2000, Rudolf Hilfer published a book titled Applications of Fractional Calculus in Physics and presented a new definition of fractional derivative [3]. In this book, he called his new fractional operator Right-Sided (Left-Sided) Generalized RL derivative. This new derivative, which was often called from the fractional order $\xi \in (0, 1)$ and $\nu \in [0, 1]$ type, and represented by $\mathfrak{D}^{\xi,\nu}$, was fractional operator between the Riemann-Liouville ($\nu = 0$), and Caputo ($\nu = 1$) operators. However, this type of operator quickly attracted the attention of researchers and is often referred to as Hilfer fractional derivative. In 2016, Rafal Kamocki presented a new formula of this type of derivative [35]. For more information about this fractional operator see [36–38]. In the last half century, Sectorial operators have been widely investigated. In 2002, Francisco Periago and Straub constructed functional calculus for Almost Sectorial Operators (ASO) [39]. In addition to formulating the analysis of these operators, they also described its applications in solving differential equations. After that, several articles were published focusing on providing mild solutions for fractional differential equations using ASO [40–43].

In 2012 [38], Furati et al. studied the following initial value problem

$$\begin{cases} \mathfrak{D}^{\xi,\nu}\wp(\ell) = \mathscr{F}(\ell, \wp), \\ I^{1-\lambda}\wp(k^+) = \wp_k, \end{cases}$$

where $\mathfrak{D}^{\xi,\nu}$ is the Hilfer derivative, $\ell > k$, $\xi \in (0, 1)$, $\nu \in [0, 1]$, $\lambda = \xi + \nu - \xi\nu$, and $I^{1-\lambda}$ is Riemann-Liouville integral of fraction order $1 - \lambda$.

In 2013 [42], Fang Li, investigated the existence of mild solution to the following problem

$$\begin{cases} {}^{c}\mathfrak{D}^{\xi}\wp(\ell) = \mathcal{A}\wp(\ell) + \mathscr{F}(\ell, \wp(\ell), \wp_{\ell}), \ \ell \in [0, \mathcal{L}], \\ \wp_0 = \phi \in \Phi, \end{cases}$$

such that ${}^{c}\mathfrak{D}^{\xi}$ is the Caputo derivative, $\xi \in (0,1)$, $\mathcal{A}$ is an ASO, and $\wp_{\ell}(x) = \wp(\ell + x)$ for $x \in (-\infty, 0]$.

In 2015 [39], Gu and Trujillo, examined the existence of mild solution for the following evolution problem

$$\begin{cases} \mathfrak{D}^{\xi,\nu}\wp(\ell) = G\wp(\ell) + \mathscr{F}(\ell, \wp(\ell)), \ \ell \in [0, \mathcal{L}], \\ I^{(1-\xi)(1-\nu)}\wp(0) = \wp_0, \end{cases}$$

which $\mathfrak{D}^{\xi,\nu}$ is Hilfer fractional derivative of order $\xi \in [0,1]$, $\nu \in (0,1)$, G is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators in Banach space $\mathscr{H}$, and $\wp_0 \in \mathscr{H}$.

In this work, with motivation from the history mentioned above and previous works, especially [42] and [39], we intend to prepare suitable space for the existence of mild solution for the following fractional problem

$$\begin{cases} \mathfrak{D}^{\xi,\nu}\wp(\ell) + \mathcal{A}\wp(\ell) = \mathscr{F}(\ell, \wp(\ell), \wp'(\ell), \int_0^{\ell} \varsigma(\ell, s)\vartheta(s, \wp(s), \wp'(s))ds), \ \ell \in (0, L] = \mathcal{L}, \\ I^{(1-\xi)(1-\nu)}\wp(0) = \wp_0, \end{cases} \tag{1.1}$$

where $\mathfrak{D}^{\xi,\nu}$ is Hilfer fractional derivative of order $\xi \in (0,1)$, $\nu \in [0,1]$, $\mathcal{A}$ is an ASO in the Banach space $\mathscr{H}$ having norm $\|\cdot\|$, $\mathscr{F} : \mathcal{J} \times \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ is a function that will be defined later, $I^{(1-\xi)(1-\nu)}$ is Riemann-Liouville integral of order $(1-\xi)(1-\nu)$, and $\wp_0 \in \mathscr{H}$. Here, for the uniqueness of the solution, it is necessary to consider an initial condition according to the fractional order $\xi \in (0,1)$. We have considered to be non-zero so that the initial condition is not independent of the type of fractional derivative. Also, inspired by Podlubny and Heymans's approach to the physical interpretation of the initial condition [44], that is, the concept of "inseparable twins", the two sides of the initial condition indicate the relationship between two functions that are related to the basic laws of physics. Note that the fractional integral $I^{(1-\xi)(1-\nu)}\wp(0) = \wp_0$, remains conserved and constant while ξ, ν varies.

2. Preliminaries

Definition 2.1. *[10] The Riemann-Liouville fractional integral of order $\xi > 0$, defined by*

$$I^{\xi}\wp(\ell) = \frac{1}{\Gamma(\xi)} \int_0^{\ell} (\ell - r)^{\xi-1}\wp(r)dr,$$

such that $\wp : [0,\infty) \to \mathbb{R}$, and the right side of integral exists.

Definition 2.2. *[10] Let $0 < \xi < 1$, then the Riemann-Liouville and Caputo fractional derivatives of order ξ, for a function $\wp$, are defined as follows respectively*

$$\mathfrak{D}^{\xi}\wp(\ell) = \frac{1}{\Gamma(1-\xi)} \frac{d}{d\ell} \int_0^{\ell} \frac{\wp(r)}{(\ell - r)^{\xi}}dr,$$

and

$${}^{c}\mathfrak{D}^{\xi}\wp(\ell) = \frac{1}{\Gamma(1-\xi)} \int_0^{\ell} \frac{f'(r)}{(\ell - r)^{\xi}}dr.$$

Definition 2.3. *[3] Rudolf Hilfer proposed a generalization of the Riemann-Liouville and Caputo fraction derivative of order* $0 < \xi < 1$ *and type* ν *which reads as follows*

$$\mathfrak{D}^{\xi,\nu}\wp(\ell) = I^{\nu(1-\xi)}\frac{d}{d\ell}I^{1-\nu(1-\xi)}\wp(\ell).$$

Here, we recall two important examples of the measure of noncompactness, namely Hausdorff and Kuratowski.

Definition 2.4. *[45] Assume that* $\mathscr{G}$ *be a bounded subset of* $\mathscr{H}$*, then the Hausdorff and Kuratowski measure of noncompactness are defined as follows, respectively*

$$\boldsymbol{\mu}(\mathscr{G}) = \inf\Big\{\mathfrak{q} > 0 : \quad \mathscr{G} \subset \bigcup_{i=1}^{z}\mathcal{B}_{\mathfrak{q}}(h_i) \quad \text{and} \quad h_i \in \mathscr{H}\Big\},$$

and

$$\boldsymbol{\mu}^*(\mathscr{G}) = \inf\Big\{\mathfrak{q} > 0 : \quad \mathscr{G} \subset \bigcup_{i=1}^{z}\mathcal{M}_i \quad \text{and} \quad \sup\{\|m-n\| : m,n \in \mathcal{M}_i\} \leq \mathfrak{q}\Big\},$$

such that $\mathcal{B}_{\mathfrak{q}}(h_i)$ *represents the balls with centers* h_i *and radius* $\leq \mathfrak{q}$*. The reader can see these two measures enjoy some properties in [45–48].*

Definition 2.5. *[48] Let* $\mathcal{G}$ *be a subset of the Banach space* $C(\mathcal{L},\mathscr{H})$ *and* $\mathcal{G}(r) = \{g(r) \in \mathscr{H} : g \in \mathcal{G}\}$*, then we define*

$$\int_0^{\ell}\mathcal{G}(r)dr = \Big\{\int_0^{\ell}g(r)dr : g \in \mathcal{G}\Big\},\ \ell \in \mathcal{L}.$$

Definition 2.6. *[40] Assume that* $-1 < \mathfrak{a} < 0$ *and* $0 < \boldsymbol{\kappa} < \frac{\pi}{2}$*, then we define a closed linear operator* $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathscr{H} \to \mathscr{H}$*, and we represent the family of these operators by* $\Psi_{\boldsymbol{\kappa}}^{\mathfrak{a}}$*, such that following statement are hold true:*

- $\sigma(\mathcal{A}) \subset \mathcal{S}_{\boldsymbol{\kappa}}$*, which* $\sigma(\mathcal{A})$ *is the specturm of* $\mathcal{A}$ *and* $\mathcal{S}_{\boldsymbol{\kappa}} = \{z \in \mathbb{C} - \{0\} : |\arg z| \leq \boldsymbol{\kappa}\} \cup \{0\}$.
- $\forall\boldsymbol{\lambda} \in (\boldsymbol{\kappa},\pi)$*, there exist positive constant* $\mathfrak{b}_{\lambda}$ *which*

$$\|(z-\mathcal{A})^{-1}\| \leq \mathfrak{b}_{\lambda}|z|^{\mathfrak{a}},\ \forall z \notin \mathcal{S}_{\boldsymbol{\kappa}}.$$

Then the operator $\mathcal{A}$ *is called an almost sectorial operator (ASO).*

Definition 2.7. *[40] Suppose that* $\mathcal{A} \in \Psi_{\boldsymbol{\kappa}}^{\mathfrak{a}}$*, then we define the semigroup* $\{\mathcal{T}(\ell)\}_{\ell\geq 0}$ *associated with* $\mathcal{A}$*, as follows*

$$\mathcal{T}(\ell) = e^{-\ell z}(\mathcal{A}) = \frac{1}{2\pi i}\int_{\Gamma_\mu}e^{-\ell z}(z-\mathcal{A})^{-1}dz,\ \ell \in \mathcal{S}_{\frac{\pi}{2}-\boldsymbol{\kappa}},$$

where $\Gamma_\mu = \{\mathbb{R}^+e^{i\mu}\} \cup \{\mathbb{R}^+e^{-i\mu}\}$*, such that* $\kappa < \mu < \frac{\pi}{2} - |\arg \ell|$.

Notation 2.1. *[49] Throughout this work* $\mathfrak{R}_{\xi}(z)$ *denotes the following wright function*

$$\mathfrak{R}_{\xi}(z) = \sum_{n\in\mathbb{N}}\frac{(-z)^{n-1}}{\Gamma(1-\xi n)(n-1)!},\ z \in \mathbb{C},\ \xi \in (0,1). \tag{2.1}$$

Definition 2.8. *Let $\Re_\xi(z)$ is the same as in* (2.1) *and $\ell \in \mathcal{S}_{\frac{\pi}{2}-\kappa}$, then we define the following two operator*

$$\mathcal{U}_\xi(\ell) = \int_0^\infty \Re_\xi(z)\mathcal{T}(\ell^\xi z)dz, \tag{2.2}$$

$$\mathcal{V}_\xi(\ell) = \int_0^\infty \xi z\Re_\xi(z)\mathcal{T}(\ell^\xi z)dz. \tag{2.3}$$

Remark 2.1. *[50] The operators defined in* (2.2) *and* (2.3) *are linear, bounded and also $\forall \ell \in \mathcal{S}_{\frac{\pi}{2}-\kappa}$ the following inequalities hold true*

$$|\mathcal{U}_\xi(\ell)| \leq j_1 \ell^{-\xi(1+\mathfrak{a})},$$

and

$$|\mathcal{V}_\xi(\ell)| \leq j_2 \ell^{-\xi(1+\mathfrak{a})},$$

such that j_1, j_2 are constant.

Lemma 2.1. *[51] The problem mentioned in* (1.1)*, is equivalent to the following equation*

$$\begin{aligned}\wp(\ell) &= \frac{\wp_0}{\Gamma(\nu(1-\xi)+\xi)}\ell^{(1-\xi)(\nu-1)} \\ &+ \frac{1}{\Gamma(\xi)}\int_0^\ell (\ell-s)^{\xi-1}[\mathscr{F}(\ell,\wp(\ell),\wp'(\ell),\int_0^\ell \varsigma(\ell,s)\vartheta(s,\wp(s),\wp'(s))) - \mathscr{E}\wp(s)]ds, \quad \ell \in \mathcal{L}.\end{aligned} \tag{2.4}$$

For simplicity in writing, we set $\hbar(\ell) = \int_0^\ell \varsigma(\ell,s)\vartheta(s,\wp(s),\wp'(s))$, which $\hbar(\ell)$ is a function in terms of variable ℓ.

Lemma 2.2. *[51] If $\wp(\ell)$ satisfied in* (2.4)*, then we have*

$$\wp(\ell) = \mathcal{U}_{\xi,\nu}(\ell)\wp_0 + \int_0^\ell \mathcal{V}^*_\xi(\ell-s)\mathscr{F}(\ell,\wp(\ell),\wp'(\ell),\hbar(s))ds, \tag{2.5}$$

such that $\mathcal{U}_{\xi,\nu}(\ell) = \mathcal{I}^{(1-\xi)\nu}\mathcal{V}^*_\xi(\ell)$ and $\mathcal{V}^*_\xi(\ell) = \ell^{\xi-1}\mathcal{V}_\xi(\ell)$.

Definition 2.9. *The function $\wp(\ell) \in C^1(\mathcal{L}_*, \mathscr{H})$ which satisfied in* (2.5) *is called a mild solution of the Eq* (1.1)*. Now, according to the definition of mild solution, we define the operator $\eth : \mathcal{B}_r(\mathcal{L}) \to \mathcal{B}_r(\mathcal{L})$ via*

$$\eth\wp(\ell) = \mathcal{U}_{\xi,\nu}(\ell)\wp_0 + \int_0^\ell \mathcal{V}_\xi(\ell-s)^{\xi-1}\mathscr{F}(\ell,\wp(\ell),\wp'(\ell),\hbar(s))ds,$$

where $\mathcal{B}_r(\mathcal{L}) = \{\mathfrak{m} \in C^1(\mathcal{L},\mathscr{H}) : \|\mathfrak{m}\| \leq r\}$.

Theorem 2.1. *[40] For each $\ell > 0$, the operators $\mathcal{U}_{\xi,\nu}(\ell)$ and $\mathcal{V}_\xi(\ell)$ are linear, bounded and strongly continuous. Also these operators satisfy the following inequalities*

$$\|\mathcal{U}_{\xi,\nu}(\ell)\mathfrak{n}\| \leq j_3 \frac{\Gamma(-\xi\mathfrak{a})}{\Gamma(\nu(1-\xi)-\xi\mathfrak{a})}\ell^{\nu(1-\xi)-1-\xi\mathfrak{a}}\|\mathfrak{n}\| \quad \text{and} \quad \|\mathcal{V}_\xi(\ell)\mathfrak{n}\| \leq j_3\ell^{-1-\xi\mathfrak{a}}.$$

Lemma 2.3. *[52] Let $\mathcal{G} \subset C(\mathcal{L},\mathscr{H})$ be bounded and continuous, then the following statements are hold true*

- $\overline{co}\mathcal{G} \subset C(\mathcal{L}, \mathscr{H})$ *is bounded and equicontinuous.*
- $\ell \to \boldsymbol{\mu}(\mathcal{G}(\ell))$ *is continuous on* $\mathcal{L}$*, and* $\forall \ell \in \mathcal{L}$*, we have*

$$\boldsymbol{\mu}(\mathcal{G}) = \max_{\ell \in \mathcal{L}} \boldsymbol{\mu}(\mathcal{G}(\ell)), \quad \boldsymbol{\mu}\Big(\int_0^\ell \mathcal{G}(r)dr \Big) \leq \int_0^\ell \boldsymbol{\mu}(\mathcal{G}(r))dr.$$

- $\forall \epsilon > 0$*, there is a sequence* $\{\wp_k\}_{k=1}^\infty \subset \mathcal{G}$*, such that* $\boldsymbol{\mu}(\mathcal{G}) \leq 2\boldsymbol{\mu}(\{\wp_k\}_{k=1}^\infty)$.

Lemma 2.4. *[53] If for a family of continuous function* $\{\wp_k\}_{k=1}^\infty$*, there exists* $\mathfrak{f} \in L^1(\mathcal{L}, \mathbb{R}^+)$*, such that* $|\wp_k(\ell)| \leq \mathfrak{f}(\ell)$*, then* $\boldsymbol{\mu}(\{\wp_k(\ell)\}_{k=1}^\infty)$ *is integrable on* $\mathcal{L}$*, and*

$$\boldsymbol{\mu}\Big(\Big\{ \int_0^\ell \wp_k(r)dr \Big\}_{k=1}^\infty\Big) \leq 2 \int_0^\ell \boldsymbol{\mu}(\{\wp_k(r)\}_{k=1}^\infty)dr.$$

3. Main results

In this section, first, four hypotheses are proposed, then we will deduce our main results by proving two auxiliary theorems.

($\mathscr{B}_1$) $\forall \ell \in \mathcal{L}_*$, the function $\mathscr{F}(\ell, ., ., .) : \mathscr{H} \times \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ is continuous and $\forall \wp \in C^1(\mathcal{L}_*, \mathscr{H})$, the function $\mathscr{F}(., \wp, \wp', \hbar) : \mathcal{L}_* \to \mathscr{H}$ is strongly measurable.

($\mathscr{B}_2$) There is a function $\delta \in L^1(\mathcal{L}_*, \mathbb{R}^+)$ which

$$\mathcal{I}^{-\mathfrak{a}\xi}\delta \in C^1(\mathcal{L}_*, \mathscr{H}), \quad \text{and} \quad \lim \ell^{(1+\mathfrak{a}\xi)(1-\nu)}\mathcal{I}^{-\mathfrak{a}\xi}\delta(\ell) = 0.$$

($\mathscr{B}_3$) $\forall w \in \mathcal{D}(\mathcal{A}^\theta)$, $\exists \eta > 0$, where

$$\sup_{\ell \in \mathcal{L}_*} \Big(\ell^{(1+\mathfrak{a}\xi)(1-\nu)} \|\mathcal{U}_{\xi,\nu}(\ell)w\| + \ell^{(1+\mathfrak{a}\xi)(1-\nu)} \int_0^\ell (\ell - s)^{-\mathfrak{a}\xi-1}\delta(s)ds \Big) \leq \eta,$$

such that $\theta > \mathfrak{a} + 1$.

($\mathscr{B}_4$) Let $\{\wp_k\}_{k=1}^\infty$ be a sequence of functions such that are differentiable on $\mathcal{L}$, and $\exists \ell_0 \in \mathcal{L}$, where $\{\wp_k(\ell_0)\}$ is convergent. If $\{\wp'_k\}$ be uniformly convergent on $\mathcal{L}$, then $\{\wp_k\}$ is uniformly convergent to function $\wp$ and $\lim_{k\to\infty} \wp'_k(\ell) = \wp'(\ell)$.

Theorem 3.1. *Suppose that the conditions* ($\mathscr{B}_1 - \mathscr{B}_3$) *are hold true and* $\mathcal{A} \in \Psi_{\boldsymbol{\kappa}}^{\mathfrak{a}}$*. Then the element of* $\{\mathfrak{D}\mathfrak{m} : \mathfrak{m} \in \mathcal{B}_r(\mathcal{L})\}$*, are equicontinuous and* $\wp_0 \in \mathcal{D}(\mathcal{A}^\theta)$*, such that* $\theta > \mathfrak{a} + 1$.

Proof. Let $\mathfrak{m} \in \mathcal{B}_r(\mathcal{L})$ and $\ell_1 = 0 < \ell_2 \leq \mathcal{L}$, we can write

$$\begin{aligned}
\Big\| \mathfrak{D}\mathfrak{m}(\ell_2) - \mathfrak{D}\mathfrak{m}(0) \Big\| &= \Big\| {\ell_2}^{(1+\xi\mathfrak{a})(1-\nu)} \Big(\mathcal{U}_{\xi,\nu}(\ell_2)\wp_0 + \int_0^{\ell_2} (\ell_2 - s)^{\xi-1} \mathcal{V}_\xi(\ell_2 - s)\mathscr{F}(s, \wp(s), \wp'(s), \hbar(s)ds \Big) \Big\| \\
&\leq \Big\| \ell_2^{(1+\xi\mathfrak{a})(1-\nu)} \mathcal{U}_{\xi,\nu}(\ell_2)\wp_0 \Big\| \\
&+ \Big\| \ell_2^{(1+\xi\mathfrak{a})(1-\nu)} \int_0^{\ell_2} (\ell_2 - s)^{\xi-1} \mathcal{V}_\xi(\ell_2 - s)\mathscr{F}(s, \wp(s), \wp'(s), \hbar(s))ds \Big\| \to 0, \\
&\mathit{as} \quad \ell_2 \to 0.
\end{aligned}$$

Suppose this time, $0 < \ell_1 < \ell_2 \leq \mathcal{L}$, then

$$\begin{aligned}\Big\|\eth\mathfrak{m}(\ell_2) - \eth\mathfrak{m}(\ell_1)\Big\| = {} & \Big\|{\ell_2}^{(1+\xi\mathfrak{a})(1-\nu)}\mathcal{U}_{\xi,\nu}(\ell_2)\wp_0 - {\ell_1}^{(1+\xi\mathfrak{a})(1-\nu)}\mathcal{U}_{\xi,\nu}(\ell_1)\wp_0\Big\| \\ & + \Big\|\ell_2^{(1+\xi\mathfrak{a})(1-\nu)}\int_0^{\ell_2}(\ell_2 - s)^{\xi-1}\mathcal{V}_\xi(\ell_2 - s)\mathscr{F}(s, \wp(s), \wp'(s), \hbar(s))ds \\ & - \ell_1^{(1+\xi\mathfrak{a})(1-\nu)}\int_0^{\ell_1}(\ell_1 - s)^{\xi-1}\mathcal{V}_\xi(\ell_1 - s)\mathscr{F}(s, \wp(s), \wp'(s), \hbar(s))ds\Big\|.\end{aligned}$$

In view of the triangle inequality, we get

$$\begin{aligned}\Big\|\eth\mathfrak{m}(\ell_2) - \eth\mathfrak{m}(\ell_1)\Big\| = {} & \Big\|{\ell_2}^{(1+\xi\mathfrak{a})(1-\nu)}\mathcal{U}_{\xi,\nu}(\ell_2)\wp_0 - {\ell_1}^{(1+\xi\mathfrak{a})(1-\nu)}\mathcal{U}_{\xi,\nu}(\ell_1)\wp_0\Big\| \\ & + \Big\|\ell_2^{(1+\xi\mathfrak{a})(1-\nu)}\int_0^{\ell_2}(\ell_2 - s)^{\xi-1}\mathcal{V}_\xi(\ell_2 - s)\mathscr{F}(s, \wp(s), \wp'(s), \hbar(s))ds\Big\| \\ & + \Big\|\ell_2^{(1+\xi\mathfrak{a})(1-\nu)}\int_0^{\ell_2}(\ell_2 - s)^{\xi-1}\mathcal{V}_\xi(\ell_2 - s)\mathscr{F}(s, \wp(s), \wp'(s), \hbar(s))ds \\ & - \ell_1^{(1+\xi\mathfrak{a})(1-\nu)}\int_0^{\ell_1}(\ell_1 - s)^{\xi-1}\mathcal{V}_\xi(\ell_2 - s)\mathscr{F}(s, \wp(s), \wp'(s), \hbar(s))ds\Big\| \\ & + \Big\|\ell_1^{(1+\xi\mathfrak{a})(1-\nu)}\int_0^{\ell_1}(\ell_1 - s)^{\xi-1}\mathcal{V}_\xi(\ell_2 - s)\mathscr{F}(s, \wp(s), \wp'(s), \hbar(s))ds \\ & - \ell_1^{(1+\xi\mathfrak{a})(1-\nu)}\int_0^{\ell_1}(\ell_1 - s)^{\xi-1}\mathcal{V}_\xi(\ell_1 - s)\mathscr{F}(s, \wp(s), \wp'(s), \hbar(s))ds\Big\| \\ & := \mathfrak{b}_1 + \mathfrak{b}_2 + \mathfrak{b}_3 + \mathfrak{b}_4.\end{aligned}$$

Since according to Theorem 2.1, $\mathcal{U}_{\xi,\nu}(\ell)$ is strongly continuous, we conclude that $\mathfrak{b}_1 \to 0$, as $\ell_2 \to \ell_1$. Now, about $\mathfrak{b}_2$, we have

$$\begin{aligned}\mathfrak{b}_2 & \leq j_3\ell_2^{(1+\xi\mathfrak{a})(1-\nu)}\int_{\ell_1}^{\ell_2}(\ell_2 - s)^{-1-\xi\mathfrak{a}}\delta(s)ds \\ & \leq j_3\Big|\int_0^{\ell_2}(\ell_2 - s)^{-1-\xi\mathfrak{a}}\delta(s)ds - \ell_2^{(1+\xi\mathfrak{a})(1-\nu)}\int_0^{\ell_1}(\ell_1 - s)^{-1-\xi\mathfrak{a}}\delta(s)ds\Big| \\ & \leq j_3\int_0^{\ell_1}\Big|\ell_1^{(1+\xi\mathfrak{a})(1-\nu)}(\ell_1 - s)^{-1-\xi\mathfrak{a}} - \ell_2^{(1+\xi\mathfrak{a})(1-\nu)}(\ell_2 - s)^{-1-\xi\mathfrak{a}}\Big|\delta(s)ds,\end{aligned}$$

Now, according to the dominated convergence theorem (DCT) and condition $\mathscr{B}_\mathbf{2}$, we obtain $\mathfrak{b}_2 \to 0$ as $\ell_2 \to \ell_1$. For $\mathfrak{b}_3$, we have

$$\mathfrak{b}_3 \leq j_3\int_0^{\ell_1}(\ell_2 - s)^{-\xi-\xi\mathfrak{a}}\Big|\ell_2^{(1+\xi\mathfrak{a})(1-\nu)}(\ell_2 - s)^{\xi-1} - \ell_1^{(1+\xi\mathfrak{a})(1-\nu)}(\ell_1 - s)^{\xi-1}\Big|\delta(s)ds,$$

where

$$\begin{aligned}& (\ell_2 - s)^{-\xi-\xi\mathfrak{a}}\Big|\ell_2^{(1+\xi\mathfrak{a})(1-\nu)}(\ell_2 - s)^{\xi-1} - \ell_1^{(1+\xi\mathfrak{a})(1-\nu)}(\ell_1 - s)^{\xi-1}\Big|\delta(s)ds \\ & \leq \ell_2^{(1+\xi\mathfrak{a})(1-\nu)}(\ell_2 - s)^{\xi-1}\delta(s) + \ell_1^{(1+\xi\mathfrak{a})(1-\nu)}(\ell_1 - s)^{\xi-1}\delta(s)\end{aligned}$$

$$\leq 2\ell_1^{(1+\xi\mathfrak{a})(1-\nu)}(\ell_1 - s)^{\xi-1}\delta(s),$$

but $\int_0^{\ell_1} \leq 2\ell_1^{(1+\xi\mathfrak{a})(1-\nu)}(\ell_1 - s)^{\xi-1}\delta(s)$ exist, namely, $\mathfrak{b}_3 \to 0$, as $\ell_2 \to \ell_1$. Finally, $\forall\epsilon > 0$, for $\mathfrak{b}_4$, we can write

$$\begin{aligned}
\mathfrak{b}_4 &= \left\| \int_0^{\ell_1} \ell_1^{(1+\xi\mathfrak{a})(1-\nu)}[\mathcal{V}_\xi(\ell_2 - s) - \mathcal{V}_\xi(\ell_1 - s)](\ell_1 - s)^{\xi-1}\mathscr{F}(s, \wp(s), \wp'(s), \hbar(s))ds \right\| \\
&\leq \int_0^{\ell_1-\epsilon} \ell_1^{(1+\xi\mathfrak{a})(1-\nu)}\|\mathcal{V}_\xi(\ell_2 - s) - \mathcal{V}_\xi(\ell_1 - s)\|(\ell_1 - s)^{\xi-1}\delta(s) \\
&+ \int_{\ell_1-\epsilon}^{\ell_1} \ell_1^{(1+\xi\mathfrak{a})(1-\nu)}\|\mathcal{V}_\xi(\ell_2 - s) - \mathcal{V}_\xi(\ell_1 - s)\|(\ell_1 - s)^{\xi-1}\delta(s) \\
&\leq \ell_1^{(1+\xi\mathfrak{a})(1-\nu)} \int_0^{\ell_1} (\ell_1 - s)^{\xi-1}\delta(s)ds \sup_{s\in[0,\ell_1-\epsilon]} \|\mathcal{V}_\xi(\ell_2 - s) - \mathcal{V}_\xi(\ell_1 - s)\| \\
&+ j_3 \int_{\ell_1-\epsilon}^{\ell_1} \ell_1^{(1+\xi\mathfrak{a})(1-\nu)}((\ell_2 - s)^{-\xi-\xi\mathfrak{a}} + (\ell_1 - s)^{-\xi-\xi\mathfrak{a}})(\ell_1 - s)^{\xi-1}\delta(s)ds \\
&\leq \ell_1^{(1+\xi\mathfrak{a})(1-\nu)+\xi(1+\mathfrak{a})} \int_0^{\ell_1} (\ell_1 - s)^{-1-\xi\mathfrak{a}}\delta(s)ds \sup_{s\in[0,\ell_1-\epsilon]} \|\mathcal{V}_\xi(\ell_2 - s) - \mathcal{V}_\xi(\ell_1 - s)\| \\
&+ 2j_3 \int_{\ell_1-\epsilon}^{\ell_1} \ell_1^{(1+\xi\mathfrak{a})(1-\nu)}(\ell_1 - s)^{-1-\xi\mathfrak{a}}\delta(s)ds.
\end{aligned}$$

It follows from the uniformly continuity of the $\mathcal{V}_\xi(\ell)$ and $\lim_{\ell_2\to\ell_1} \mathfrak{b}_2 = 0$, that $\mathfrak{b}_4 \to 0$, as $\ell_2 \to \ell_1$. And this means the independence of $\mathfrak{m} \in \mathcal{B}_r(\mathcal{L})$. Therefore, $\|\eth\mathfrak{m}(\ell_2) - \eth\mathfrak{m}(\ell_1)\| \to 0$, as $\ell_2 \to \ell_1$. Thus, $\{\eth\mathfrak{m} : \mathfrak{m} \in \mathcal{B}_r(\mathcal{L})\}$ is equicontinuous.

Theorem 3.2. *Suppose that the conditions ($\mathscr{B}_1 - \mathscr{B}_4$) are hold true and $\mathcal{A} \in \Psi_\kappa^{\mathfrak{a}}$. Then the element of $\{\eth\mathfrak{m} : \mathfrak{m} \in \mathcal{B}_r(\mathcal{L})\}$, are continuous, bounded and $\wp_0 \in \mathcal{D}(\mathcal{A}^\theta)$, such that $\theta > \mathfrak{a} + 1$.*

Proof. We shall show that $\eth$ is a self-mapping on $\mathcal{B}_r(\mathcal{L})$. To achieve this, we choose $\mathfrak{m} \in \mathcal{B}_r(\mathcal{L})$, and put $\wp(\ell) = \ell^{-(1+\xi\mathfrak{a})(1-\nu)}\mathfrak{m}(\ell)$, $\Xi = -(1 + \xi\mathfrak{a})(1 - \nu)$, then we have $\wp \in \mathcal{B}_r(\mathcal{L})$. Now, assume that $\ell \in [0, L]$

$$\|\eth\| \leq \|\ell^{(1+\xi\mathfrak{a})(1-\nu)}\mathcal{U}_{\xi,\nu}(\ell)\wp_0\| + \ell^{(1+\xi\mathfrak{a})(1-\nu)}\left\| \int_0^\ell (\ell - s)^{\xi-1}\mathcal{V}_\xi(\ell - s)\mathscr{F}(s, \wp(s), \wp'(s), \hbar(s))ds \right\|.$$

From $\mathscr{B}_2$ and $\mathscr{B}_3$, we get

$$\begin{aligned}
\|\eth\| &\leq \ell^{(1+\xi\mathfrak{a})(1-\nu)}\|\mathcal{U}_{\xi,\nu}(\ell)\wp_0\| + \ell^{(1+\xi\mathfrak{a})(1-\nu)} \int_0^\ell (\ell - s)^{-\xi\mathfrak{a}-1}\delta(\ell)ds \\
&\leq \sup_{[0,L]} \ell^{(1+\xi\mathfrak{a})(1-\nu)} \int_0^\ell (\ell - s)^{-\xi\mathfrak{a}-1}\delta(\ell)ds \leq \eta.
\end{aligned}$$

Therefore, for each $\mathfrak{m} \in \mathcal{B}_r(\mathcal{L})$, we have $\|\eth\mathfrak{m}\| \leq \eta$. Now, at this step, we examine the continuity of $\eth$ in $\mathcal{B}_r(\mathcal{L})$. For achieve this, get $\mathfrak{m}_k, \mathfrak{m} \in \mathcal{B}_r(\mathcal{L})$ such that $\lim_{k\to\infty} \mathfrak{m}_k = \mathfrak{m}$, namely, $\lim_{k\to\infty} \ell^{-(1+\xi\mathfrak{a})(1-\nu)}\mathfrak{m}_k = \ell^{-(1+\xi\mathfrak{a})(1-\nu)}\mathfrak{m}$. In view of $\mathscr{B}_1$, we have

$$\mathscr{F}(s, \wp_k(s), \wp'_k(s), \hbar_k(s)) = \mathscr{F}(s, s^\Xi\mathfrak{m}_k, \Xi s^{\Xi-1}\mathfrak{m}_k + s^\Xi\mathfrak{m}'_k, s^\Xi\hbar_{\mathfrak{m}_k})$$

$$\to \mathscr{F}(s, s^{\Xi}\mathfrak{m}, \Xi s^{\Xi-1}\mathfrak{m} + s^{\Xi}\mathfrak{m}', s^{\Xi}\hbar_{\mathfrak{m}}),$$

as $k \to \infty$. By using $\mathscr{B}_2$, we deduce that

$$(\ell_1 - s)^{-1-\xi\mathfrak{a}}|\mathscr{F}(s, \wp_k(s), \wp_k'(s), \hbar_k(s))| \leq 2(\ell_1 - s)^{-\xi\mathfrak{a}(1-\nu)}\delta(s),$$

which yields that

$$\int_0^{\ell} (\ell - s)^{-\xi\mathfrak{a}-1}\left\|\mathscr{F}(s, \wp_k(s), \wp_k'(s), \hbar_k(s)) - \mathscr{F}(s, \wp(s), \wp'(s), \hbar(s))\right\|ds \to 0, \text{ as } k \to \infty.$$

Hence, we get

$$\|\eth\mathfrak{m}_k - \eth\mathfrak{m}\| \leq \ell^{(1+\xi\mathfrak{a})(1-\nu)}\left\|\int_0^{\ell}(\ell - s)^{\xi-1}\mathcal{V}_{\xi}(\ell - s)\mathscr{F}(s, \wp_k(s), \wp_k'(s), \hbar_k(s)) - \mathscr{F}(s, \wp(s), \wp'(s), \hbar(s))ds\right\|.$$

According to Remark 2.1, as $k \to \infty$, we obtain

$$\|\eth\mathfrak{m}_k - \eth\mathfrak{m}\| \leq j_3\ell^{(1+\xi\mathfrak{a})(1-\nu)}\int_0^{\ell}(\ell - s)^{-\xi\mathfrak{a}-1}\left\|\mathscr{F}(s, \wp_k(s), \wp_k'(s), \hbar_k(s)) - \mathscr{F}(s, \wp(s), \wp'(s), \hbar(s))\right\|ds \to 0.$$

Thus, $\eth\mathfrak{m}_k \to \eth\mathfrak{m}$, pointwise on $\mathcal{L}$. Furthermore, it follows from Theorem 3.1 that $\eth\mathfrak{m}_k \to \eth\mathfrak{m}$ uniformly on $\mathcal{L}$, which $k \to \infty$. Hence $\eth$ is continuous.

Theorem 3.3. *Suppose that the conditions* ($\mathscr{B}_1 - \mathscr{B}_3$) *are hold true,* $\mathcal{A} \in \Psi_{\kappa}^{\mathfrak{a}}$ *and the semigroup* $\{\mathcal{T}(\ell)\}_{\ell\geq 0}$ *be compact. Then* $\forall \wp_0 \in \mathcal{D}(\mathcal{A}^{\theta})$ *there exists a mild solution of* (1.1)*, in* $\mathcal{B}_r(\mathcal{L})$ *such that* $\theta > \mathfrak{a} + 1$.

Proof. The equicontinuity of semigroup $\mathcal{T}(\ell)$ obtains from the assumption of its compactness. Furthermore, continuity and boundedness of $\eth : \mathcal{B}_r(\mathcal{L}) \to \mathcal{B}_r(\mathcal{L})$ follows from Theorems 3.1 and 3.2. Thus, ${}_*\eth : \mathcal{B}_r(\mathcal{L}) \to \mathcal{B}_r(\mathcal{L})$ is bounded, continuous and the operators $\{{}_*\eth : \mathfrak{m} \in \mathcal{B}_r(\mathcal{L})\}$ are equicontinuous. Define ${}_*\eth : \mathcal{B}_r(\mathcal{L}) \to \mathcal{B}_r(\mathcal{L})$ via

$${}_*\eth\mathfrak{m}(\ell) =_* \eth^1\mathfrak{m}(\ell) +_* \eth^2\mathfrak{m}(\ell),$$

such that

$$\begin{aligned}
{}_*\eth^1\mathfrak{m}(\ell) &= \ell^{(1+\xi\mathfrak{a})(1-\nu)}\mathcal{U}_{\xi,\nu}(\ell)\wp_0 = \ell^{(1+\xi\mathfrak{a})(1-\nu)}\mathcal{I}^{\nu(1-\xi)}\ell^{\xi-1}\mathcal{V}_{\xi}(\ell)\wp_0 \\
&= \frac{\ell^{(1+\xi\mathfrak{a})(1-\nu)}}{\Gamma(\nu(1-\xi))}\int_0^{\ell}(\ell - s)^{\nu(1-\xi)-1}s^{\xi-1}\int_0^{\infty}\xi z\mathfrak{R}_{\xi}(z)\mathcal{T}(\ell^{\xi}z)dzd\ell \\
&= \frac{\xi\ell^{(1+\xi\mathfrak{a})(1-\nu)}}{\Gamma(\nu(1-\xi))}\int_0^{\ell}\int_0^{\infty}(\ell - s)^{\nu(1-\xi)-1}s^{\xi-1}z\mathfrak{R}_{\xi}(z)\mathcal{T}(\ell^{\xi}z)dzd\ell,
\end{aligned}$$

and

$${}_*\eth^2\mathfrak{m}(\ell) = \ell^{(1+\xi\mathfrak{a})(1-\nu)}\int_0^{\ell}(\ell - s)^{\xi-1}\mathcal{V}_{\xi}(\ell - s)\mathscr{F}(s, \wp_k(s), \wp_k'(s), \hbar_k(s)).$$

Now, we define an operator ${}_{*}\eth^{1}_{z,\Lambda}$ on $\mathcal{B}_r(\mathcal{L})$, such that $\Lambda > 0$ and $0 < z < \ell$

$$\begin{aligned}{}_{*}\eth^{1}_{z,\Lambda}\mathfrak{m}(\ell) &= \frac{\ell^{(1+\xi\mathfrak{a})(1-\nu)}}{\Gamma(\nu(1-\xi))}\int_{z}^{\ell}\int_{\Lambda}^{\infty}(\ell-s)^{(1-\xi)\nu-1}s^{\xi-1}z\mathfrak{R}_{\xi}(z)\mathcal{T}(\ell^{\xi}z)\wp_0 dzd\ell\\ &= \frac{\xi\ell^{(1+\xi\mathfrak{a})(1-\nu)}}{\Gamma(\nu(1-\xi))}\mathcal{T}(z^{\xi}\Lambda)\int_{z}^{\ell}\int_{\Lambda}^{\infty}(\ell-s)^{(1-\xi)\nu-1}s^{\xi-1}z\mathfrak{R}_{\xi}(z)\mathcal{T}(\ell^{\xi}z-z^{\xi}\Lambda)\wp_0 dzd\ell.\end{aligned}$$

Since $\mathcal{T}(z^{\xi}\Lambda)$ is compact, the set $\{{}_{*}\eth^{1}_{z,\Lambda}\mathfrak{m} : \mathfrak{m}\in\mathcal{B}_r(\mathcal{L})\}$ is relatively compact. Furthermore, $\forall\mathfrak{m}\in\mathcal{B}_r(\mathcal{L})$, we can write

$$\begin{aligned}\|{}_{*}\eth^{1}\mathfrak{m}(\ell)-{}_{*}\eth^{1}_{z,\Lambda}\mathfrak{m}(\ell)\| &\leq j_4\left\|\ell^{(1+\xi\mathfrak{a})(1-\nu)}\int_{0}^{\ell}\int_{0}^{\Lambda}(\ell-s)^{(1-\xi)\nu-1}s^{\xi-1}z\mathfrak{R}_{\xi}(z)\mathcal{T}(\ell^{\xi}z)\wp_0 dzd\ell\right\|\\ &+ j_4\left\|\ell^{(1+\xi\mathfrak{a})(1-\nu)}\int_{0}^{z}\int_{\Lambda}^{\infty}(\ell-s)^{(1-\xi)\nu-1}s^{\xi-1}z\mathfrak{R}_{\xi}(z)\mathcal{T}(\ell^{\xi}z)\wp_0 dzd\ell\right\|\\ &\leq j_4\ell^{(1+\xi\mathfrak{a})(1-\nu)}\int_{0}^{\ell}\int_{0}^{\Lambda}(\ell-s)^{(1-\xi)\nu-1}s^{\xi-1}z\mathfrak{R}_{\xi}(z)s^{-\xi\nu-\xi}\|\wp_0\|z^{-\mathfrak{a}-1}dzd\ell\\ &+ j_4\ell^{(1+\xi\mathfrak{a})(1-\nu)}\int_{0}^{z}\int_{\Lambda}^{\infty}(\ell-s)^{(1-\xi)\nu-1}s^{\xi-1}z\mathfrak{R}_{\xi}(z)s^{-\xi\mathfrak{a}-\xi}z^{-\mathfrak{a}-1}\|\wp_0\|dzd\ell\\ &= j_4\ell^{(1+\xi\mathfrak{a})(1-\nu)}\int_{0}^{\ell}(\ell-s)^{(1-\xi)\nu-1}s^{-\xi\mathfrak{a}-1}\|\wp_0\|d\ell\int_{0}^{\Lambda}z^{-\mathfrak{a}}\mathfrak{R}_{\xi}(z)dz\\ &+ j_4\ell^{(1+\xi\mathfrak{a})(1-\nu)}\int_{0}^{z}(\ell-s)^{(1-\xi)\nu-1}s^{-\xi\mathfrak{a}-1}\|\wp_0\|d\ell\int_{\Lambda}^{\infty}z^{-\mathfrak{a}}\mathfrak{R}_{\xi}(z)dz\\ &\leq j_4\ell^{-\xi\nu(1+\mathfrak{a})}\|\wp_0\|\int_{0}^{\Lambda}z^{-\mathfrak{a}}\mathfrak{R}_{\xi}(z)dz\\ &+ j_4\ell^{-\xi\nu(1+\mathfrak{a})}\|\wp_0\|\int_{0}^{z}(1-s)^{(1-\xi)\nu-1}s^{-\xi\mathfrak{a}-1}d\ell\int_{\Lambda}^{\infty}z^{-\mathfrak{a}}\mathfrak{R}_{\xi}(z)dz\\ &\to 0, \quad \text{as } z\to 0 \text{ and } \Lambda\to 0,\end{aligned}$$

which $j_4 = \dfrac{\xi}{\Gamma(\nu(1-\xi))}$. Hence, the set $\{{}_{*}\eth^{1}_{z,\Lambda}\mathfrak{m} : \mathfrak{m}\in\mathcal{B}_r(\mathcal{L})\}$ is arbitrarily closed to the set $\{{}_{*}\eth^{1}\mathfrak{m}(\ell) : \mathfrak{m}\in\mathcal{B}_r(\mathcal{L})\}$. Thereby, the set $\{{}_{*}\eth^{1}\mathfrak{m}(\ell) : \mathfrak{m}\in\mathcal{B}_r(\mathcal{L})\}$ is relatively compact in $\mathscr{H}$. Now, similar to the process above, we define the operator ${}_{*}\eth^{2}_{z,\Lambda}\mathfrak{m}$, as follows

$$\begin{aligned}&{}_{*}\eth^{2}_{z,\Lambda}\mathfrak{m}(\ell) = \xi\ell^{(1+\xi\mathfrak{a})(1-\nu)}\\ &\int_{0}^{\ell-z}\int_{\Lambda}^{\infty}z\mathfrak{R}_{\xi}(z)(\ell-s)^{\xi-1}\mathcal{T}((\ell-s)^{\xi-1}z)\mathscr{F}(s,\wp_k(s),\wp_k'(s),\hbar_k(s))dzd\ell\\ &= \xi\ell^{(1+\xi\mathfrak{a})(1-\nu)}\mathcal{T}(z^{\xi}\Lambda)\\ &\int_{0}^{\ell-z}\int_{\Lambda}^{\infty}z\mathfrak{R}_{\xi}(z)(\ell-s)^{\xi-1}\mathcal{T}((\ell-s)^{\xi-1}z-z^{\xi}\Lambda)\mathscr{F}(s,\wp_k(s),\wp_k'(s),\hbar_k(s))dzd\ell,\end{aligned}$$

and, $\forall\mathfrak{m}\in\mathcal{B}_r(\mathcal{L})$, we find

$$\|{}_{*}\eth^{2}\mathfrak{m}(\ell)-{}_{*}\eth^{2}_{z,\Lambda}\mathfrak{m}(\ell)\|$$

$$\leq \left\|\xi\ell^{(1+\xi\mathfrak{a})(1-\nu)}\int_0^\ell\int_0^\Lambda z\mathfrak{R}_\xi(z)(\ell-s)^{\xi-1}\mathcal{T}((\ell-s)^{\xi-1}z)\mathscr{F}(s,\wp_k(s),\wp_k'(s),\hbar_k(s))dzd\ell\right\|$$
$$+\left\|\xi\ell^{(1+\xi\mathfrak{a})(1-\nu)}\int_{\ell-z}^\ell\int_\Lambda^\infty z\mathfrak{R}_\xi(z)(\ell-s)^{\xi-1}\mathcal{T}((\ell-s)^{\xi-1}z)\mathscr{F}(s,\wp_k(s),\wp_k'(s),\hbar_k(s))dzd\ell\right\|$$
$$\leq j_0\xi\ell^{(1+\xi\mathfrak{a})(1-\nu)}\Big(\int_0^\ell(\ell-s)^{-\xi\mathfrak{a}-1}\delta(\ell)d\ell\int_0^\Lambda z^{-\mathfrak{a}}\mathfrak{R}_\xi(z)dz\Big)$$
$$+\ j_0\xi\ell^{(1+\xi\mathfrak{a})(1-\nu)}\Big(\int_{\ell-z}^\ell(\ell-s)^{-\xi\mathfrak{a}-1}\delta(\ell)d\ell\int_0^\infty z^{-\mathfrak{a}}\mathfrak{R}_\xi(z)dz\Big)$$
$$\leq j_0\xi\ell^{(1+\xi\mathfrak{a})(1-\nu)}\Big(\int_0^\ell(\ell-s)^{-\xi\mathfrak{a}-1}\delta(\ell)d\ell\int_0^\Lambda z^{-\mathfrak{a}}\mathfrak{R}_\xi(z)dz\Big)$$
$$+\ \frac{\Gamma(1-\mathfrak{a})}{\Gamma(1-\xi\mathfrak{a})}j_0\xi\ell^{(1+\xi\mathfrak{a})(1-\nu)}\Big(\int_{\ell-z}^\ell(\ell-s)^{-\xi\mathfrak{a}-1}\delta(\ell)d\ell\Big)\to 0,\ \text{ as }\ \Lambda\to 0.$$

Hence, the set $\{{}_*\eth^2_{z,\Lambda}\mathfrak{m}:\mathfrak{m}\in\mathcal{B}_r(\mathcal{L})\}$ is arbitrarily closed to the set $\{{}_*\eth^2\mathfrak{m}(\ell):\mathfrak{m}\in\mathcal{B}_r(\mathcal{L})\}$. Thereby, the set $\{{}_*\eth^2\mathfrak{m}(\ell):\mathfrak{m}\in\mathcal{B}_r(\mathcal{L})\}$ is relatively compact in $\mathscr{H}$. As a result of the Arzela-Ascoli theorem, $\{\eth\mathfrak{m}(\ell):\mathfrak{m}\in\mathcal{B}_r(\mathcal{L})\}$ is relatively compact. Moreover $\eth$ is a completely continuous operator. According to the Schauder fixed point theorem, this operator has at least a fixed point $\mathfrak{m}_*\in\mathcal{B}_r(\mathcal{L})$. Put $\wp_*(\ell)=\ell^{(1+\xi\mathfrak{a})(1-\nu)}\mathfrak{m}_*$. And this means $\wp_*$ is a mild solution to the problem mentioned in (1.1).

To continue the work, we need the following hypothesis in the case that the semigroup $\mathcal{T}(\ell)$ is noncompact.

($\mathscr{B}_5$) For each bounded $\mathscr{G}_1,\mathscr{G}_2,\mathscr{G}_3\subset\mathscr{H}$ there exist a constant $\mathcal{N}$ such that

$$\boldsymbol{\mu}^*(\mathscr{F}(s,\mathscr{G}_1,\mathscr{G}_2,\mathscr{G}_3))\leq\mathcal{N}\boldsymbol{\mu}^*(\mathscr{G}_1,\mathscr{G}_2,\mathscr{G}_3),$$

which $\boldsymbol{\mu}^*$ is the same as mentioned in Definition 2.4.

Theorem 3.4. *Suppose that the conditions ($\mathscr{B}_1-\mathscr{B}_5$) are hold true, $\mathcal{A}\in\Psi^{\mathfrak{a}}_{\boldsymbol{\kappa}}$ and the semigroup $\{\mathcal{T}(\ell)\}_{\ell\geq 0}$ is noncompact. Then $\forall\wp_0\in\mathcal{D}(\mathcal{A}^\theta)$ there exists a mild solution of* (1.1)*, in $\mathcal{B}_r(\mathcal{L})$ such that $\theta>\mathfrak{a}+1$.*

Proof. In Theorems 3.1 and 3.2, we proved that $\eth:\mathcal{B}_r(\mathcal{L})\to\mathcal{B}_r(\mathcal{L})$ is bounded, continuous and $\{\eth\mathfrak{m}:\mathfrak{m}\in\mathcal{B}_r(\mathcal{L})\}$ is equicontinuous. Moreover, we showed $\exists C\subset\mathcal{B}_r(\mathcal{L})$, such that $\eth$ is compact in C. For each bounded $C\subset\mathcal{B}_r(\mathcal{L})$, put

$$\eth^{(1)}(C)=\eth(C),\ldots,\eth^{(n)}(C)=\eth(\overline{co}(\eth^{(n-1)}(C))),\quad n=2,3,\ldots$$

In view of properties of a measure of noncompactness (Lemmas 2.3 and 2.4), we can find a subsequence $\{\mathfrak{m}_n^{(1)}\}_{n=1}^\infty\subset C$, such that

$$\boldsymbol{\mu}^*(\eth^{(1)}(C(\ell)))$$
$$\leq 2\boldsymbol{\mu}^*\Big(\ell^{(1+\xi\mathfrak{a})(1-\nu)}\int_0^\ell(\ell-s)^{\xi-1}\mathcal{V}_\xi(\ell-s)\mathscr{F}(s,\{s^\Xi\mathfrak{m}_n^{(1)}(s),\Xi s^{\Xi-1}\mathfrak{m}_n^{(1)}(s)+s^\Xi\mathfrak{m}'^{(1)}_n(s),\Im_{\mathfrak{m}_n^{(1)}(s)}\}_{n=1}^\infty)ds\Big)$$
$$\leq 4j_3\ell^{(1+\xi\mathfrak{a})(1-\nu)}\Big(\int_0^\ell(\ell-s)^{-\xi\mathfrak{a}-1}\boldsymbol{\mu}^*(\mathscr{F}(s,\{s^\Xi\mathfrak{m}_n^{(1)}(s),\Xi s^{\Xi-1}\mathfrak{m}_n^{(1)}(s)+s^\Xi\mathfrak{m}'^{(1)}_n(s),\Im_{\mathfrak{m}_n^{(1)}(s)}\}_{n=1}^\infty))ds\Big)$$

$$\leq 4j_3\mathcal{N}\ell^{(1+\xi\mathfrak{a})(1-\nu)}\boldsymbol{\mu}^*(C)\Big(\int_0^\ell (\ell-s)^{-\xi\mathfrak{a}-1}s^{-3(1+\xi\mathfrak{a})(1-\nu)}ds\Big)$$
$$\leq 4j_3\mathcal{N}\ell^{-\xi\mathfrak{a}}\boldsymbol{\mu}^*(C)\Big(\frac{\Gamma(-\xi\mathfrak{a})\Gamma(-3\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)}{\Gamma(-4\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)}\Big).$$

It follows from the fact that $\Game$ is arbitrary

$$\boldsymbol{\mu}^*(\Game^{(1)}(C(\ell))) \leq 4j_3\mathcal{N}\ell^{-\xi\mathfrak{a}}\boldsymbol{\mu}^*(C)\Big(\frac{\Gamma(-\xi\mathfrak{a})\Gamma(-3\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)}{\Gamma(-4\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)}\Big),$$

which implies that

$$\boldsymbol{\mu}^*(\Game^{(n)}(C(\ell))) \leq \frac{(4j_3\mathcal{N})^n\ell^{-n\xi\mathfrak{a}}\Gamma^n(-\xi\mathfrak{a})\Gamma(-3\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)}{\Gamma(-(n+3)\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)}. \tag{3.1}$$

Put $\mathbb{A} = 4j_3\mathcal{N}\ell^{-\xi\mathfrak{a}}\Gamma(-\xi\mathfrak{a})$, Then by rewriting Eq (3.1), we find

$$\frac{(4j_3\mathcal{N})^n\ell^{-n\xi\mathfrak{a}}\Gamma^n(-\xi\mathfrak{a})\Gamma(-3\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)}{\Gamma(-(n+3)\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)} = \frac{\mathbb{A}^n\Gamma(-3\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)}{\Gamma(-(n+3)\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)}.$$

We can choose $x, y \in \mathbb{N}$, so large where $\frac{1}{x} < \xi\mathfrak{a} < \frac{1}{x-1}$, and $\frac{n+3}{x+1} > 2$ for

$$\Gamma(\frac{n+3}{x+1}) < y\Gamma(-(n+3)\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu) < n,$$

then, we obtain

$$\frac{\mathbb{A}^n\Gamma(-3\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)}{\Gamma(-(n+3)\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)} \leq \frac{\mathbb{A}^n\Gamma(-3\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)}{\Gamma(\frac{n+3}{x+1})}. \tag{3.2}$$

By substitution $n+3 \to (t+1)(x+1)$, the Eq (3.2) becomes

$$\frac{\mathbb{A}^{(t+1)(x+1)-3}\Gamma(-3\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)}{\Gamma(t+1)} = \frac{[\mathbb{A}^{(x+1)}]^{t+1}\mathbb{A}^{-3}\Gamma(-3\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)}{t!} \to 0, \quad \text{as} \quad t \to \infty.$$

So, $\exists n_* \in \mathbb{N}$, which

$$\frac{\mathbb{A}^n\Gamma(-3\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)}{\Gamma(-(n+3)\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)} \leq \frac{\mathbb{A}^{n_*}\Gamma(-3\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)}{\Gamma(-(n_*+3)\xi\mathfrak{a}+(1+\xi\mathfrak{a})\nu)} = \pounds < 1,$$

thus, we deduced that

$$\boldsymbol{\mu}^*(\Game^{(n_*)}(C(\ell))) \leq \pounds\boldsymbol{\mu}^*(C).$$

$\Game^{(n_*)}(C(\ell))$ is equicontinuous and bounded, therefore by Lemma 2.3, we get

$$\boldsymbol{\mu}^*(\Game^{(n_*)}(C)) = \max_{\ell\in\mathcal{L}}\boldsymbol{\mu}^*(\Game^{(n_*)}(C(\ell))).$$

Hence,

$$\boldsymbol{\mu}^*(\Game^{n_*}(C)) \leq \pounds\boldsymbol{\mu}^*(C),$$

such that $\pounds < 1$. Similar to what was done in the previous Theorem 3.3, we get $\mathcal{G}$ in $\mathcal{B}_r(\mathcal{L})$, which $\Game(\mathcal{G}) \subset \mathcal{G}$, and $\Game(\mathcal{G})$ is compact. Thanks to Schauder fixed point theorem, we find a fixed point $\mathfrak{m}_* \in \mathcal{B}_r(\mathcal{L})$ for the operator $\Game$. Put $\wp_*(\ell) = \ell^{(1+\xi\mathfrak{a})(1-\nu)}\mathfrak{m}_*$, and this means $\wp_*$ is a mild solution to the problem mentioned in (1.1).

4. Examples

Example 4.1. *Consider the following problem*

$$\begin{cases} \mathfrak{D}^{\frac{7}{13},\frac{1}{2}}\wp(\ell) + \mathcal{A}\wp(\ell) = \ell^{\frac{-1}{3}}cos^2(\wp(\ell)) + \dfrac{\wp'(\ell)}{1+e^{\wp'(\ell)}}, \\ I^{(1-\frac{7}{13})(1-\frac{1}{2})}\wp(0) = \wp_0. \end{cases} \tag{4.1}$$

In this case put $\xi = \frac{7}{13}$, $\nu = \frac{1}{2}$, $\ell \in [0,1] = \mathcal{L}$, *and* $\mathscr{F}(\ell, \wp(\ell), \wp'(\ell), \hbar(\ell)) = \ell^{\frac{-1}{3}}cos^2(\wp(\ell)) + \dfrac{\wp'(\ell)}{1+e^{\wp'(\ell)}}$. *Set* $\mathscr{H} = C^{\lambda}$, *which* $\lambda \in (0,1)$ *and* $\mathcal{D}(\mathcal{A}) = \{\wp \in C^{2+\lambda}(\mathcal{L}) : \wp(0) = 0\}$, *then from [53], we deduce that* $\exists w, y > 0$, *such that* $\mathcal{A} + w \in \Psi^{\frac{\lambda}{2}-1}_{\frac{\pi}{2}-y}(C^{\lambda}(\mathcal{L}))$. *We choose* $\delta(\ell) = \ell^{\frac{-1}{3}}$ *and*

$$\eta = \sup_{[0,1]}\left(\ell^{(1+\frac{7}{13}\alpha)(1-\frac{1}{2})}\|\mathcal{U}_{\frac{7}{13},\frac{1}{2}}(\ell)\|\right) + \frac{\Gamma(-\frac{7\alpha}{13})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{7\alpha}{13})}. \tag{4.2}$$

Thus, all the desired assumptions ($\mathscr{B}_1 - \mathscr{B}_4$) *are correctly established, then according to Theorem 3.3, problem* (4.1) *has a mild solution. To better understand this example, we present some graphs for system* (4.1) *in Figures 1–3 and numerical result in Table 1.*

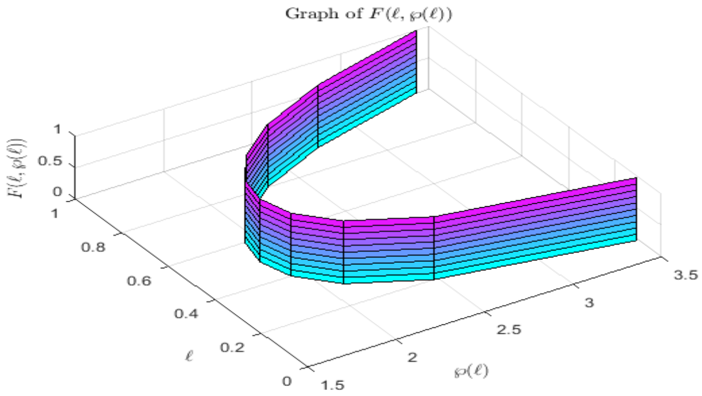

Figure 1. The graph of $\ell^{\frac{-1}{3}}cos^2(\wp(\ell))$ in Example 4.1.

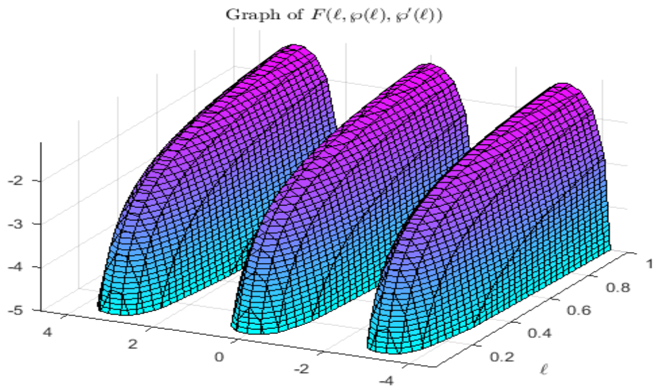

Figure 2. The graph of $\ell^{\frac{-1}{3}}cos^2(\wp(\ell)) + \dfrac{\wp'(\ell)}{1+e^{\wp'(\ell)}}$ in Example 4.1.

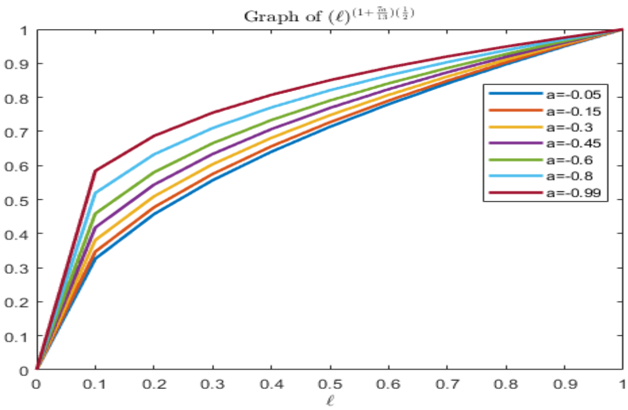

Figure 3. The graph of $\ell^{(1+\frac{7}{13}\mathfrak{a})(1-\frac{1}{2})}$ for different values of $\mathfrak{a}$ in Eq (4.2).

Table 1. Numerical results for some functions in Example 4.1.

		$\mathfrak{a}$=-0.3	$\mathfrak{a}$=-0.8
ℓ	$\delta(\ell)$	η	η
0	0	3.8674	0.2617
0.1	2.1544	$0.3808\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.1)\|+3.8674$	$0.5192\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.1)\|+0.2617$
0.2	1.7099	$0.5093\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.2)\|+3.8674$	$0.6325\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.2)\|+0.2617$
0.3	1.4938	$0.6036\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.3)\|+3.8674$	$0.7098\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.3)\|+0.2617$
0.4	1.3572	$0.6810\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.4)\|+3.8674$	$0.7704\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.4)\|+0.2617$
0.5	1.2599	$0.7478\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.5)\|+3.8674$	$0.8209\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.5)\|+0.2617$
0.6	1.1856	$0.8072\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.6)\|+3.8674$	$0.8646\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.6)\|+0.2617$
0.7	1.1262	$0.8611\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.7)\|+3.8674$	$0.9034\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.7)\|+0.2617$
0.8	1.0772	$0.9106\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.8)\|+3.8674$	$0.9384\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.8)\|+0.2617$
0.9	1.0357	$0.9567\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.9)\|+3.8674$	$0.9704\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.9)\|+0.2617$
1	1	$\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(1)\|+3.8674$	$\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(1)\|+0.2617$

Example 4.2. *Consider the following problem*

$$\begin{cases} \mathfrak{D}^{\frac{2}{7},\frac{1}{2}}\wp(\ell) + \mathcal{A}\wp(\ell) = \ell^{\frac{-1}{7}} sin(\wp(\ell)) + tan(\wp'(\ell)), \\ I^{(1-\frac{2}{7})(1-\frac{1}{2})}\wp(0) = \wp_0. \end{cases} \tag{4.3}$$

In this case put $\xi = \frac{2}{7}$, $\nu = \frac{1}{2}$, $\ell \in [0,1] = \mathcal{L}$, *and* $\mathscr{F}(\ell, \wp(\ell), \wp'(\ell), \hbar(\ell)) = \ell^{\frac{-1}{7}} sin(\wp(\ell)) + tan(\wp'(\ell))$. *Set* $\mathscr{H} = C^{\lambda}$, *which* $\lambda \in (0,1)$ *and* $\mathcal{D}(\mathcal{A}) = \{\wp \in C^{2+\lambda}(\mathcal{L}) : \wp(0) = 0\}$, *then from [53], we deduce that* $\exists w, y > 0$, *such that* $\mathcal{A} + w \in \Psi^{\frac{\lambda}{2}-1}_{\frac{\pi}{2}-y}(C^{\lambda}(\mathcal{L}))$. *We choose* $\delta(\ell) = \ell^{\frac{-1}{7}}$ *and*

$$\eta = \sup_{[0,1]} \left(\ell^{(1+\frac{2}{7}\mathfrak{a})(1-\frac{1}{2})}\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(\ell)\|\right) + \frac{\Gamma(-\frac{2\mathfrak{a}}{7})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} - \frac{2\mathfrak{a}}{7})}. \tag{4.4}$$

Now, the conditions ($\mathscr{B}_1 - \mathscr{B}_4$) *are satisfied and by Theorem 3.3, problem* (4.3) *has a mild solution. To better understand this example,we present some graphs for system* (4.3) *in Figures 4–6 and numerical result in Table 2.*

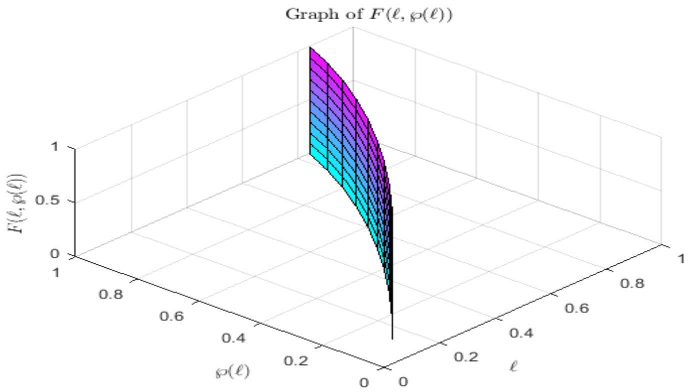

Figure 4. The graph of $\ell^{\frac{-1}{7}} sin(\wp(\ell))$ in Example 4.2.

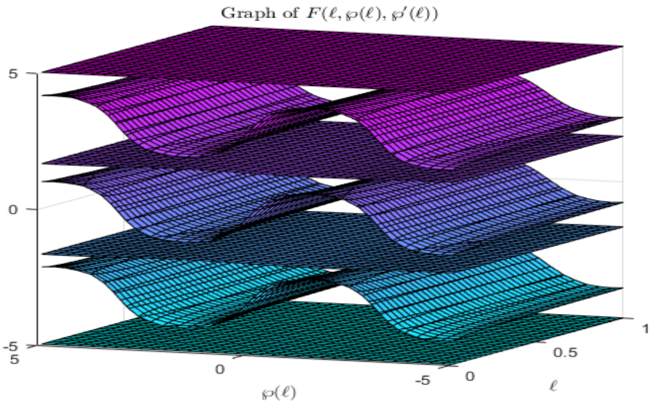

Figure 5. The graph of $\ell^{\frac{-1}{7}} sin(\wp(\ell)) + tan(\wp'(\ell))$ in Example 4.2.

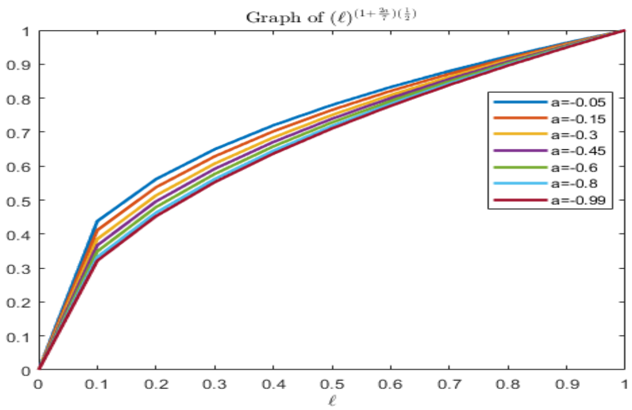

Figure 6. The graph of $\ell^{(1+\frac{2}{7}\mathfrak{a})(1-\frac{1}{2})}$ for different values of $\mathfrak{a}$ in Eq (4.4).

Table 2. Numerical results for some functions in Example 4.2.

		α=-0.15	α=-0.45
ℓ	$\delta(\ell)$	η	η
0	0	20.8574	5.4155
0.1	1.3894	$0.3010\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.1)\|$+20.8574	$0.2727\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.1)\|$+5.4155
0.2	1.2584	$0.4320\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.2)\|$+20.8574	$0.4033\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.2)\|$+5.4155
0.3	1.1876	$0.5337\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.3)\|$+20.8574	$0.5069\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.3)\|$+5.4155
0.4	1.1398	$0.6201\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.4)\|$+20.8574	$0.5963\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.4)\|$+5.4155
0.5	1.1040	$0.6966\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.5)\|$+20.8574	$0.6763\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.5)\|$+5.4155
0.6	1.0757	$0.7661\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.6)\|$+20.8574	$0.7496\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.6)\|$+5.4155
0.7	1.0522	$0.8302\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.7)\|$+20.8574	$0.8177\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.7)\|$+5.4155
0.8	1.0323	$0.8901\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.8)\|$+20.8574	$0.8817\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.8)\|$+5.4155
0.9	1.0151	$0.9465\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.9)\|$+20.8574	$0.9422\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(0.9)\|$+5.4155
1	1	$\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(1)\|$+20.8574	$\|\mathcal{U}_{\frac{2}{7},\frac{1}{2}}(1)\|$+5.4155

5. Conclusions

In this paper, we showed that for the existence of a mild solution to the desired problem, namely system (1.1), which involves Hilfer fractional derivative and almost sectorial operator (ASO), the semigroup $\{\mathcal{T}(\ell)\}$ need not be compact. We have guaranteed this issue in Theorem 3.4. To perform this feature, we introduced special conditions ($\mathscr{B}_1 - \mathscr{B}_5$). Krasnoselskii's fixed point theorem and Arzela-Ascoli's theorem were central to our proofs. Although various works have been done with the almost sectorial operator (ASO), the novelty of our work is in using it in fractional integro-differential equations of Hilfer type. We provided two examples to illustrate our result. Other researchers can test our results with other fractional operators and pave the way.

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Conflict of interest

The authors declare no conflict of interest.

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