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Research article

Bifurcation and one-sign solutions for semilinear elliptic problems in $\mathbb{R}^N$

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Abstract: In this work, we study the existence of one-sign solutions without signum condition for the following problem:

$$\begin{cases} -\Delta u = \lambda a(x) f(u), & x \in \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to +\infty, \end{cases}$$

where $N \geq 3$, λ is a real parameter and $a \in C^{\alpha}_{loc}(\mathbb{R}^N, \mathbb{R})$ for some $\alpha \in (0, 1)$ is a weighted function, $f \in C^{\alpha}(\mathbb{R}, \mathbb{R})$, and there exist two constants $s_2 < 0 < s_1$, such that $f(s_1) = f(s_2) = f(0) = 0$ and $sf(s) > 0$ for $s \in \mathbb{R}\backslash\{s_1, 0, s_2\}$. Furthermore, we consider the exact multiplicity of one-sign solutions for above problem under more strict hypotheses. We use bifurcation techniques and the approximation of connected components to prove our main results.

Keywords: unilateral global bifurcation; one-sign solutions; semilinear elliptic problems
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1. Introduction

Consider the following semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda a(x) f(u), & x \in \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to +\infty, \end{cases} \tag{1.1}$$

where λ is a real parameter, $N \geq 3$, and $a \in C^{\alpha}_{loc}(\mathbb{R}^N, \mathbb{R})$ for some $\alpha \in (0, 1)$ is a weighted function which can be sign-changing and $f \in C^{\alpha}(\mathbb{R}, \mathbb{R})$, and $f(s)s > 0$ for any $s \neq 0$. Edelson and Rumbos [1] have shown that problem (1.1) with $f(u) \equiv u$ has a positive, simple and principal eigenvalue λ_1 and the positive principle eigenfunction ϕ satisfies the asymptotic decay law $\lim_{|x|\to+\infty} |x|^{N-2}\phi_1(x) = c$ for some constant c(where a satisfied the following condition (A1)). Edelson and Rumbos [1,2] have also studied the existence of positive solution and the existence of global branches of minimal solutions of the problem (1.1) by the Schauder-Tychonoff fixed point theorem and the Dancer global bifurcation theorems [3]. By using the Rabinowitz global bifurcation method [4], Edelson and Furi [5] have shown the existence of positive minimal solution of the problem (1.1). In 2017, Dai [6] have established a global bifurcation result for the problem (1.1).

By [6], set

$$M(\Omega) := \{a \in C^{\alpha}_{loc}(\Omega, \mathbb{R}) : \{x \in \Omega : a(x) > 0\} \neq \emptyset\}.$$

For any $u \in C^{\infty}_c(\Omega)$ with $\Omega \subseteq \mathbb{R}^N$, we define

$$||u||_1 = (\int_{\Omega} |\nabla u|^2 dx)^{1/2}.$$

Denote by $\mathcal{D}^{1,2}(\Omega)$ the completion of $C^{\infty}_c(\Omega)$ with respect to the norm $||u||_1$. Denote by $S(\mathbb{R}^N)$ the set of all measurable real functions defined on $\mathbb{R}^N$. Two functions in $S(\mathbb{R}^N)$ are considered as the same element of $S(\mathbb{R}^N)$ when they are equal almost everywhere. Let $L^2(\mathbb{R}^N; |a|) =: \{u \in S(\mathbb{R}^N) : \int_{\mathbb{R}^N} |a|u^2 dx < +\infty\}$. For $u \in \mathcal{D}^{1,2}(\mathbb{R}^N), u \neq 0$, define the Rayleigh quotient

$$R(u) = \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\int_{\mathbb{R}^N} au^2 dx}.$$

Dai et al. [6] also assumed that a satisfied the following condition:

(A1) Let $a \in M(\mathbb{R}^N)$. Assume that $p, P \in C(\mathbb{R}^N, \mathbb{R})$ are positive, radially symmetric and satisfies

$$0 < p(|x|) \leq a(x) \leq P(|x|), \forall x \in \mathbb{R}^N$$

and

$$\int_{\mathbb{R}^N} |x|^{2-N} P(|x|) dx < +\infty.$$

Furthermore, if P satisfies the following more strong condition (with $r = |x|$)

$$\int_0^{\infty} r^{N-1} P(r) dr < +\infty. \tag{1.2}$$

Dai [5] established the following spectrum structure:

Lemma 1.1 (see [6,Theorem 1.1]). Let (A1) hold. Then there exists an orthonormal basis $\{\varphi_k\}_1^{+\infty}$ of $L^2(\mathbb{R}^N; |a|)$ and a sequence of positive real numbers $\{\lambda_k\}_1^{+\infty}$with $\lambda_k \to +\infty$ as $k \to +\infty$, such that

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots,$$

$$\begin{cases} -\Delta \varphi_k = \lambda a(x) \varphi_k, & x \in \mathbb{R}^N, \\ \varphi_k \in D^{1,2}(\mathbb{R}^N) \cap C^{2,\alpha}_{loc}(\mathbb{R}^N), & \\ \varphi_k(x) \to 0, & \text{as} |x| \to +\infty. \end{cases} \tag{1.3}$$

Moreover, one has that

$$\lambda_1 = \min_{u \neq 0, u \in D^{1,2}(\mathbb{R}^N)} R(u) = R(\varphi_1)$$

and

$$\lambda_k = R(\varphi_k) = \max_{u\neq 0, u\in span\{\varphi_1,\cdots,\varphi_{k-1}\}} R(u) = \min_{u\neq 0, u\perp\{\varphi_1,\cdots,\varphi_{k-1}\}} R(u) = \min_{\dim W=k, W\subset D^{1,2}(\mathbb{R}^N)} \max_{u\in W} R(u).$$

For $k \geq 2$. λ_1 is simple and principal eigenvalue. Furthermore, if P satisfies (1.2), λ_1 is the unique positive principal eigenvalue and

$$\lim_{|x|\to+\infty} |x|^{N-2}\varphi_1(x) = c$$

for some constant c.

By the Rabinowitz global bifurcation theorem [2, Theorem 1.3] and the Dancer unilateral global bifurcation theorem [7, Theorem 2], Dai [6] obtained [6, Theorem 1.3]. The signum condition $f(s)s > 0$ for $s \neq 0$ plays an important role in the [6, Theorem 1.3].

Of course, the natural question is that of what would happen without signum condition for the problem (1.1). Recently, Dai [8,9] studied the global behavior of the components of positive solutions for the Schrödinger equation and one-sign solutions for the p-Laplacian without the signum condition, respectively.

Motivated by the above interesting and important studies, we shall show that the branches bifurcating from infinity and the trivial solution line for the problem (1.1) are disjoint and the existence results of radial nodal solutions to problem (1.1) without signum condition.

We now present the following assumptions on f :

(A2) $f \in C(\mathbb{R}, \mathbb{R})$, and there exist two constants $s_2 < 0 < s_1$, such that $f(s_1) = f(s_2) = f(0) = 0$ and $sf(s) > 0$ for $s \in \mathbb{R}\backslash\{s_1, 0, s_2\}$.

(A3) There exist two constants $\gamma_1 > 0$ and $\gamma_2 < 0$ such that

$$\lim_{s\to s_1^-} \frac{f(s)}{s_1 - s} = \gamma_1, \lim_{s\to s_2^+} \frac{f(s)}{s - s_2} = \gamma_2.$$

Let

$$f_0 = \lim_{|s|\to 0} \frac{f(s)}{s}, f_\infty = \lim_{|s|\to+\infty} \frac{f(s)}{s}.$$

The main results of this section are the following interesting results:

Theorem 1.1. Let (A1)–(A3) hold.

(a) If $f_0, f_\infty \in (0, \infty)$, then the problem (1.1) has at least two solutions u_∞^+ and u_∞^- for $\lambda \in (\min\{\frac{\lambda_1}{f_0}, \frac{\lambda_1}{f_\infty}\}, \max\{\frac{\lambda_1}{f_0}, \frac{\lambda_1}{f_\infty}\}]$ such that u_∞^+ is positive in $\mathbb{R}^N$ and u_∞^- is negative in $\mathbb{R}^N$; the problem (1.1) has at least four solutions u_∞^+ and u_∞^-, u_0^+ and u_0^- for $\lambda \in (\max\{\frac{\lambda_1}{f_0}, \frac{\lambda_1}{f_\infty}\}, \infty)$ such that u_∞^+, u_0^+ are positive in $\mathbb{R}^N$ and u_∞^-, u_0^- are negative in $\mathbb{R}^N$. Moreover, if P satisfies (1.2), we have that

$$\lim_{|x|\to+\infty} |x|^{N-2}u_i^+(x) = c_1^i$$

for all $x \in \mathbb{R}^N$ and some constants $c_1^i \neq 0$, where $i = 0, \infty$. Do the same for u_i^-.

(b) If $f_0 \in (0, \infty)$ and $f_\infty = \infty$, then the problem (1.1) has at least two solutions u_∞^+ and u_∞^- for $\lambda \in (0, \frac{\lambda_1}{f_0}]$ such that u_∞^+ is positive in $\mathbb{R}^N$ and u_∞^- is negative in $\mathbb{R}^N$; the problem (1.1) has at least four solutions u_∞^+ and u_∞^-, u_0^+ and u_0^- for $\lambda \in (\frac{\lambda_1}{f_0}, \infty)$ such that u_∞^+, u_0^+ are positive in $\mathbb{R}^N$ and u_∞^-, u_0^- are negative in $\mathbb{R}^N$. Moreover, if P satisfies (1.2), we have that

$$\lim_{|x|\to+\infty} |x|^{N-2}u_i^+(x) = c_1^i$$

for all $x \in \mathbb{R}^N$ and some constants $c_1^i \neq 0$, where $i = 0, \infty$. Do the same for u_i^-.

(c) If $f_0 = \infty$ and $f_\infty \in (0, \infty)$, then the problem (1.1) has at least two solutions u_∞^+ and u_∞^- for $\lambda \in (0, \frac{\lambda_1}{f_\infty}]$ such that u_∞^+ is positive in $\mathbb{R}^N$ and u_∞^- is negative in $\mathbb{R}^N$; the problem (1.1) has at least four solutions u_∞^+ and u_∞^-, u_0^+ and u_0^- for $\lambda \in (\frac{\lambda_1}{f_\infty}, +\infty)$ such that u_∞^+, u_0^+ are positive in $\mathbb{R}^N$ and u_∞^-, u_0^- are negative in $\mathbb{R}^N$. Moreover, if P satisfies (1.2), we have

$$\lim_{|x| \to +\infty} |x|^{N-2} u_i^+(x) = c_1^i$$

for all $x \in \mathbb{R}^N$ and some constants $c_1^i \neq 0$, where $i = 0, \infty$. Do the same for u_i^-.

(d) If $f_0 = \infty$ and $f_\infty = \infty$, then the problem (1.1) has at least four solutions u_∞^+ and u_∞^-, u_0^+ and u_0^- for $\lambda \in (0, \infty)$ such that u_∞^+, u_0^+ are positive in $\mathbb{R}^N$ and u_∞^-, u_0^- are negative in $\mathbb{R}^N$. Moreover, if P satisfies (1.2), we have that

$$\lim_{|x| \to +\infty} |x|^{N-2} u_i^+(x) = c_1^i$$

for all $x \in \mathbb{R}^N$ and some constants $c_1^i \neq 0$, where $i = 0, \infty$. Do the same for u_i^-.

Furthermore, we can get the exact multiplicity of one-sign solutions for problem (1.1) under more strict hypotheses.

(A4) $f(s) \equiv 0$ for any $s \notin [s_2, s_1]$, $f(s)$ is C^1 with respect to $s \in [s_2, s_1]$, and such that $f(s)/s$ is decreasing in $[0, s_1]$ and is increasing in $[s_2, 0]$.

The following are the main results of this section.

Theorem 1.2. Let (A1), (A2) and (A4) hold. Assume that $f_0 \in (0, \infty)$. Then,

(i) the problem (1.1) has exactly two solutions $u^+(\lambda, \cdot)$ and $u^-(\lambda, \cdot)$ for $\lambda \in (\frac{\lambda_1}{f_0}, +\infty)$, such that $0 < u^+(\lambda, \cdot) \leq s_1$ and $s_2 \leq u^-(\lambda, \cdot) < 0$ in $\mathbb{R}^N$;

(ii) all one-sign solutions of problem (1.1) lie on two smooth curves

$$\mathscr{C}^\nu = \{(\lambda, u^\pm(\lambda, \cdot)) : \lambda \in (\frac{\lambda_1}{f_0}, +\infty)\},$$

$\mathscr{C}^+$ and $\mathscr{C}^-$ join at $(\lambda_1/f_0, 0)$;

(iii) $u^+(\lambda, \cdot)(u^-(\lambda, \cdot))$ is increasing (decreasing) with respect to λ.

(iv) If P satisfies (1.2), we have that

$$\lim_{|x| \to +\infty} |x|^{N-2} u_1^+(x) = c_1 \text{ and } \lim_{|x| \to +\infty} |x|^{N-2} u_1^-(x) = c_2$$

for all $x \in \mathbb{R}^N$ and some constants $c_1, c_2 \neq 0$.

Theorem 1.3. Let (A1), (A2) and (A4) hold. Assume that $f_0 = \infty$. Then,

(i) the problem (1.1) has exactly two solutions $u^+(\lambda, \cdot)$ and $u^-(\lambda, \cdot)$ for $\lambda \in (0, +\infty)$, such that $0 < u^+(\lambda, \cdot) \leq s_1$ and $s_2 \leq u^-(\lambda, \cdot) < 0$ in $\mathbb{R}^N$;

(ii) all one-sign solutions of problem (1.1) lie on two smooth curves

$$\mathscr{C}^\nu = \{(\lambda, u^\pm(\lambda, \cdot)) : \lambda \in (\frac{\lambda_1}{f_0}, +\infty)\},$$

$\mathscr{C}^+$ and $\mathscr{C}^-$ join at $(\lambda_1/f_0, 0)$;

(iii) $u^+(\lambda, \cdot)(u^-(\lambda, \cdot))$ is increasing (decreasing) with respect to λ.

(iv) If P satisfies (1.2), we have that

$$\lim_{|x|\to+\infty}|x|^{N-2}u_1^+(x)=c_1 \text{ and } \lim_{|x|\to+\infty}|x|^{N-2}u_1^-(x)=c_2$$

for all $x\in\mathbb{R}^N$ and some constants $c_1, c_2\neq 0$.

The rest of this paper is arranged as follows: Section 2, provides some preliminaries. In Section 3, we prove Theorem 1.1, which considers the existence of one-sign solutions for the problem (1.1) without signum condition. In Section 4, we consider exact multiplicity of one-sign solutions for the problem (1.1) and give the proof of Theorems 1.2 and 1.3.

2. Preliminaries

Let

$$E=\{u\in C(\mathbb{R}^N,\mathbb{R}):\sup_{x\in\mathbb{R}^N}|u(x)|<+\infty\}$$

with the norm

$$\|u\|=\sup_{x\in\mathbb{R}^N}|u(x)|, \text{ for all } u\in E.$$

Clearly, E is a Banach space. Let $P^+=\{u\in E|u>0, \text{ for all } x\in\mathbb{R}^N\}$ and set $P^-=-P^+$ and $P=P^+\cup P^-$.

By [6], we can show that u is a one-sign $C^{2+\alpha}$ solution of problem (1.1) if and only if u is a solution of the operator equation

$$u=L(f(u))=\lambda\int_{\mathbb{R}^N}\Gamma_N(x-y)a(y)f(u(y))dy, \tag{2.1}$$

where $\Gamma_N(x-y)=\frac{1}{N(N-2)\omega_N}|x-y|^{2-N}$, ω_N being the volume of the unit ball in $\mathbb{R}^N$. Dai [6] also can show that $L:E\to E$ is completely continuous.

Consider the following problem

$$\begin{cases} -\Delta u=\lambda a(x)u(x)+g(\lambda,x,u), & x\in\mathbb{R}^N,\\ u(x)\to 0, & \text{as } |x|\to+\infty.\end{cases} \tag{2.2}$$

Suppose $g\in C(\mathbb{R}^N\times E\times\mathbb{R},E)$ satisfies

$$\lim_{|s|\to 0}\frac{g(x,s,\lambda)}{s}=0 \tag{2.3}$$

uniformly for $x\in\mathbb{R}^N$ and λ on bounded sets.

Similar the proof of [6, Theorem 1.3], we can obtain that the following result:

Theorem 2.1. Assume that (A1) and (2.3) hold. The pair $(\lambda_1,0)$ is a bifurcation point of the problem (2.2) and there are two distinct unbounded continuum $\mathscr{C}^+$ and $\mathscr{C}^-$ in $\mathbb{R}\times E$ of solutions of the problem (2.2) emanating from $(\lambda_1,0)$. Moreover, we have

$$\mathscr{C}^\nu\subset((\mathbb{R}\times P^\nu)\cup\{(\lambda_1,0)\}),$$

where $\nu\in\{+,-\}$.

By (2.1), Eq (2.2) is equivalent to

$$u = \lambda Lu + H(\lambda, u) = G(\lambda, u),$$

where $H(\lambda, u) = o(\|u\|)$ at $u = 0$ uniformly on bounded λ intervals, $H(\lambda, u) = L[g(x, u, \lambda)]$. $G(\lambda, u) : \mathbb{R} \times E \to E$ is completely continuous.

In order to prove Theorem 1.1, we also need to establish the unilateral global bifurcation result of the problem (2.2) from infinity under assumption

$$\lim_{s \to +\infty} \frac{g(x, s, \lambda)}{s} = 0 \tag{2.4}$$

uniformly for $x \in \mathbb{R}^N$ and λ on bounded sets.

By Rabinowitz [10], we have the following theorem.

Theorem 2.2. Let (A1) and (2.4) hold. There exists a connected component $\mathscr{D}^\nu$ of of solutions of the problem (2.2), containing $\lambda_1 \times \{\infty\}$. Moreover, if $\Lambda \subset \mathbb{R}$ is an interval such that $\Lambda \backslash \{\lambda_1\}$ doesn't contain any other eigenvalue of problem (1.3), and $\mathscr{M}$ is a neighborhood of $\lambda_1 \times \{\infty\}$ whose projection on $\mathbb{R}$ lies in Λ and whose projection on E is bounded away from 0, then either

1^o $\mathscr{D}^\nu - \mathscr{M}$ is bounded in $\mathbb{R} \times E$ in which case $\mathscr{D}^\nu - \mathscr{M}$ meets $\mathscr{R} = \{(\lambda, 0) | \lambda \in \mathbb{R}\}$ or

2^o $\mathscr{D}^\nu - \mathscr{M}$ is unbounded.

If 2^o occurs and $\mathscr{D}^\nu - \mathscr{M}$ has a bounded projection on $\mathbb{R}$, then $\mathscr{D}^\nu - \mathscr{M}$ meets $\lambda_j \times \{\infty\}$ for some $j \neq 1$, and $\nu \in \{+, -\}$.

To prove our main results, we need the following results:

Lemma 2.1. (see [9]) Let X be a normal space and let $\{C_n | n = 1, 2, ...\}$ be a sequence of unbounded connected subsets of X. Assume that:

(i) there exists $z^* \in \liminf_{n \to +\infty} C_n$ with $\|z^*\| < +\infty$;

(ii) for every $R > 0$, $\left(\cup_{n=1}^{+\infty} C_n\right) \cap \overline{B}_R$ is a relatively compact set of X, where

$$B_R = \{x \in X | \|x\| \leq R\}.$$

Then, $\mathbb{D} := \limsup_{n \to \infty} C_n$ is unbounded, closed and connected.

In order to treat the problems with non-asymptotic nonlinearity at ∞, we shall need the following lemmas.

Lemma 2.2. (see [9]) Let (X, ρ) be a metric space. If $\{C_i\}_{i \in \mathbb{N}}$ is a sequence of sets whose limit superior is L and there exists a homeomorphism $T : X \to X$ such that for every $R > 0$, $\left(\cup_{i=1}^{+\infty} T(C_i)\right) \cap \overline{B}_R$ is a relatively compact set, then for each $\epsilon > 0$ there exists an m such that for every $n > m, C_n \subset V_\epsilon(L)$, where $V_\epsilon(L)$ denotes the set of all points p with $\rho(p, x) < \epsilon$ for any $x \in L$.

Now, in order to study the exact multiplicity of one-sign solutions for (1.1), let $\mathbb{E} = \mathbb{R} \times E$, $\Phi(\lambda, u) = u - G(\lambda, u)$ and

$$\mathcal{S} = \overline{\{(\lambda, u) \in \mathbb{E} : \Phi(\lambda, u) = 0, u \neq 0\}}^{\mathbb{R} \times E}.$$

For $\lambda \in \mathbb{R}$ and $0 < s < +\infty$, define an open neighborhood of $(\lambda_1, 0)$ in $\mathbb{E}$ as follows:

$$\mathbb{B}_s(\lambda_1, 0) = \{(\lambda, u) \in \mathbb{E} : \|u\| + |\lambda - \lambda_1| < s\}.$$

Let E_0 be a closed subset of E satisfying $E = \mathrm{span}\{\psi_1\} \oplus E_0$, where ψ_1 is an eigenfunction corresponding to λ_1 with $\|\psi_1\| = 1$. According to the Hahn-Banach theorem, we have $l \in E^*$ satisfying

$$l(\psi_1) = 1 \text{ and } E_0 = \{u \in E : l(u) = 0\},$$

where E^* denotes the dual space of E. For any $0 < \varepsilon < +\infty$ and $0 < \eta < 1$, define

$$K_{\varepsilon,\eta} = \{(\lambda, u) \in \mathbb{E} : |\lambda - \lambda_1| < \varepsilon, |l(u)| > \eta\|u\|\}.$$

Obviously, $K_{\varepsilon,\eta}$ is an open subset of E, $K_{\varepsilon,\eta} = K^+_{\varepsilon,\eta} \cup K^-_{\varepsilon,\eta}$, with $K^+_{\varepsilon,\eta} = \{(\lambda, u) \in \mathbb{E} : |\lambda - \lambda_1| < \varepsilon, l(u) > \eta\|u\|\}$, $K^-_{\varepsilon,\eta} = -K^+_{\varepsilon,\eta}$, which are disjoint and open in E.

Similar to that of [11, Lemma 6.4.1], we can show the following lemma.

Lemma 2.3. Let $\eta \in (0, 1)$, there is $\delta_0 > 0$ such that for each $\delta : 0 < \delta < \delta_0$, it holds that

$$((\mathcal{S}\backslash\{(\lambda_1, 0)\}) \cap \mathbb{B}_\delta(\lambda_1, 0)) \subseteq K_{\varepsilon,\eta}.$$

And there exist $s \in \mathbb{R}$ and a unique $y \in E_0$ such that

$$v = s\psi_1 + y \text{ and } |s| > \eta\|v\|$$

for each $(\lambda, v) \in ((\mathcal{S}\backslash\{(\lambda_1, 0)\}) \cap \mathbb{B}_\delta(\lambda_1, 0))$. Further, $\lambda = \lambda_1 + o(1)$ and $y = o(s)$ as $s \to 0$ for these solutions (λ, v).

Remark 2.1. From Lemma 2.3, we can see that $\mathscr{D} = \mathscr{D}^+ \cup \mathscr{D}^-$ near $(\lambda_1, 0)$ is given by a curve $(\lambda(s), u(s)) = (\lambda_1 + o(1), s\psi_1 + o(s))$ for s near 0. Moreover, we can distinguish between two portions of this curve by $s \geq 0$ and $s \leq 0$.

3. One-sign solutions without the signum condition

When $f_0 \in (0, \infty)$, let $\zeta(u) \in C(\mathbb{R}, \mathbb{R})$ be such that

$$f(u) = f_0 u + \zeta(u)$$

with

$$\lim_{|u|\to 0} \frac{\zeta(u)}{u} = 0.$$

Let us consider

$$\begin{cases} -\Delta u = \lambda a(x) f_0 u(x) + \lambda a(x)\zeta(u), & x \in \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to +\infty \end{cases} \tag{3.1}$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

Applying Theorem 2.1 to (3.1), we have the following result.

Remark 3.1. There is an unbounded continuum $\mathscr{C}^\nu$ of solutions of the problem (1.1) emanating from $(\frac{\lambda_1}{f_0}, 0)$, such that $\mathscr{C}^\nu \subset ((\mathbb{R} \times P^\nu) \cup \{(\frac{\lambda_1}{f_0}, 0)\})$, where $\nu \in \{+, -\}$.

We now analyze the global behavior of $\mathscr{C}^+$+ and $\mathscr{C}^-$.

Lemma 3.1. Let (A1)–(A3) hold. Then

(i) for $(\lambda, u) \in (\mathscr{C}^+ \cup (\mathscr{C}^-)$, we have that $s_2 < u(x) < s_1$ for all $x \in \mathbb{R}^N$;

(ii) for $(\lambda, u) \in (\mathscr{D}^+ \cup \mathscr{D}^-)$, we have that either $\sup_{x\in\mathbb{R}^N} u(x) > s_1$ or $\inf_{x\in\mathbb{R}^N} u(x) < s_2$.

Proof. Suppose on the contrary that there exists $(\lambda, u) \in (\mathscr{C}^+ \cup \mathscr{C}^- \cup \mathscr{D}^+ \cup \mathscr{D}^-)$ such that either $\sup_{x\in\mathbb{R}^N} u(x) = s_1$ or $\inf_{x\in\mathbb{R}^N} u(x) = s_2$.

We only treat the case of $\sup_{x\in\mathbb{R}^N} u(x) = s_1$ because the proof for the case of $\inf_{x\in\mathbb{R}^N} u(x) = s_2$ can be given similarly.

We claim that there exists $0 < m < +\infty$ such that $f(s) \leq m(s_1 - s)$ for any $s \in [0, s_1]$. Clearly, the claim is true for the case of $s = 0$ or $s = s_1$ by virtue of (A2).

For any $\epsilon \in (0, \gamma_1)$, it follows from (A3) that there exists $\delta > 0$, such that

$$f(s) < (\gamma_1 + \epsilon)(s_1 - s)$$

for any $s \in (s_1 - \delta, s_1)$. From (A2), it arrives

$$\max_{s\in[0,s_1-\delta]} \frac{f(s)}{s_1 - s} = \rho > 0.$$

So, the claim is verified by choosing $m = \max\{\rho, \gamma_1 + \epsilon\}$.

Now, let us consider the following problem

$$\begin{cases} -\Delta(s_1 - u) + \lambda a(x)m(s_1 - u) = \lambda a(x)[m(s_1 - u) - f(u)], & x \in \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to +\infty. \end{cases}$$

It is obvious that $f(s) \leq m(s_1 - s)$ for any $s \in [0, s_1]$ implies

$$\begin{cases} -\Delta(s_1 - u) + \lambda a(x)m(s_1 - u) \geq 0, & x \in \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to +\infty. \end{cases}$$

The strong maximum principle of [12] implies that $s_1 > u$ in $\mathbb{R}^N$. This is a contradiction.

Lemma 3.2. Let (A1)–(A3) hold. Then

$$(\frac{\lambda_1}{f_0}, \infty) \subseteq \mathtt{Proj}_{\mathbb{R}}(\mathscr{C}^+), (\frac{\lambda_1}{f_0}, \infty) \subseteq \mathtt{Proj}_{\mathbb{R}}(\mathscr{C}^-).$$

Proof. We show that the projection of $\mathscr{C}^+$ on $\mathbb{R}$ is unbounded. It is sufficient to show that the set $\{(\lambda, u) \in \mathscr{C}^+ | \lambda \in [0, d]\}$ is bounded for any fixed $d \in (0, +\infty)$. Suppose on the contrary that there exists $(\lambda_n, u_n) \in \mathscr{C}^+, n \in \mathbb{N}$, such that $\lambda_n \to \mu \leq d$, $||u_n|| \to +\infty$ as $n \to +\infty$. Let $v_n = u_n/||u_n||$. Then, v_n should be the solutions of problem

$$\begin{cases} -\Delta v_n = \lambda a(x)\frac{f(u_n)}{||u_n||}, & x \in \mathbb{R}^N, \\ v_n(x) \to 0, & \text{as } |x| \to +\infty. \end{cases} \tag{3.2}$$

By Lemma 3.1 (i), we have that

$$f(u_n) \leq \max_{s\in[0,s_1]} |f(s)|.$$

By (A2), one can obtain that $f(s) \in C([0, s_1])$.

Thus, we can show

$$\lim_{n\to+\infty} \frac{f(u_n)}{||u_n||} = 0. \tag{3.3}$$

By the compactness of $L(.)$, it follows from (3.2) that $v_n \to v_0 \equiv 0$ as $n \to +\infty$. This contradicts $\|v_0\| = 1$.

This together with the fact that $\mathscr{C}^+$ joins $(\frac{\lambda_1}{f_0}, 0)$ to infinity yields that

$$(\frac{\lambda_1}{f_0}, \infty) \subseteq \mathtt{Proj}_{\mathbb{R}}(\mathscr{C}^+).$$

Similarly, we can show that

$$(\frac{\lambda_1}{f_0}, \infty) \subseteq \mathtt{Proj}_{\mathbb{R}}(\mathscr{C}^-).$$

In the following we will investigate the other one-sign solutions of problem (1.1).

When $f_\infty \in (0, \infty)$, let $\xi(u) \in C(\mathbb{R}, \mathbb{R})$ be such that

$$f(u) = f_\infty u + \xi(u)$$

with

$$\lim_{|u|\to\infty} \frac{\xi(u)}{u} = 0.$$

Let us consider

$$\begin{cases} -\Delta u = \lambda a(x) f_\infty u(x) + \lambda a(x)\xi(u), & x \in \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to +\infty \end{cases} \tag{3.4}$$

as a bifurcation problem from infinity. We add the points $\{(\lambda, \infty) | \lambda \in \mathbb{R}\}$ to space $\mathbb{R} \times E$.

Applying Theorem 2.2 to (3.4), we have the following result.

Remark 3.2. There exists an unbounded continua $\mathscr{D}^\nu$ of solutions of (1.1), emanating from $(\frac{\lambda_1}{f_\infty}, \infty)$, such that $\mathscr{D}^\nu \subset ((\mathbb{R} \times P^\nu) \cup \{(\frac{\lambda_1}{f_\infty}, \infty)\})$, where $\nu \in \{+, -\}$.

We now analyze the global behavior of $\mathscr{D}^+$+ and $\mathscr{D}^-$.

Lemma 3.3. Let (A1)–(A3) hold. Then

$$(\frac{\lambda_1}{f_\infty}, \infty) \subseteq \mathtt{Proj}_{\mathbb{R}}(\mathscr{D}^+), (\frac{\lambda_1}{f_\infty}, \infty) \subseteq \mathtt{Proj}_{\mathbb{R}}(\mathscr{D}^-).$$

Proof. We show that $\mathscr{D}^\nu - \mathscr{M}$ has an unbounded projection on $\mathbb{R}$.

Applying Theorem 2.2 to (3.4), one can obtain that (1^0) of Theorem 2.2 does not occur by Lemma 3.1 (ii). So $\mathscr{D}^\nu - \mathscr{M}$ is unbounded.

Now, we show that the case of $\mathscr{D}^\nu - \mathscr{M}$ meeting $\lambda_j \times \{\infty\}$ for some $j > 1$ is impossible, where λ_j denotes the jth eigenvalue of the problem (1.3). Assume on the contrary that $\mathscr{D}^\nu - \mathscr{M}$ meets $\lambda_j \times \{\infty\}$ for some $j > 1$. So there exists a neighborhood $\widetilde{\mathscr{N}} \subset \widetilde{\mathscr{M}}$ of $\lambda_j \times \{\infty\}$ such that u must change sign for any $(\lambda, u) \in (\mathscr{D}^\nu - \mathscr{M}) \cap (\widetilde{\mathscr{N}} \setminus (\lambda_j \times \{\infty\}))$, where $\widetilde{\mathscr{M}}$ is a neighborhood of $\lambda_j \times \{\infty\}$ which satisfies the assumptions of Theorem 2.2, which contradicts $\mathscr{D}^\nu \subset ((\mathbb{R} \times P^\nu) \cup \{(\frac{\lambda_1}{f_\infty}, \infty)\})$.

Thus,

$$(\frac{\lambda_1}{f_\infty}, \infty) \subseteq \mathtt{Proj}_{\mathbb{R}}(\mathscr{D}^+).$$

Similarly, we have

$$(\frac{\lambda_1}{f_\infty}, \infty) \subseteq \mathtt{Proj}_{\mathbb{R}}(\mathscr{D}^-).$$

Now, we give Proof of Theorem 1.1

Proof of Theorem 1.1.

(a) Since problem (1.1) has a unique solution $u \equiv 0$ for $\lambda \equiv 0$, we get

$$(\mathscr{C}^+ \cup \mathscr{C}^- \cup \mathscr{D}^+ \cup \mathscr{D}^-) \subset \{(\mu, z) \in \mathbb{R} \times E | \mu \geq 0\}.$$

By Lemmas 3.1–3.3, we conclude the desired results.

We only derive the rate of decay of u_∞^+ since the proof for the other case is completely analogous.

By (2.1), we have

$$u_\infty^+ = \lambda \int_{\mathbb{R}^N} \Gamma_n(x-y) a(y) f(u_\infty^+(y)) dy,$$

where $\Gamma_N(x-y) = \frac{1}{N(N-2)\omega_N} |x-y|^{2-N}$, ω_N being the volume of the unit ball in $\mathbb{R}^N$.

By $f_0, f_\infty \in (0, \infty)$, there exist some constant $\varrho > 0$ such that $|f(s)| \leq \varrho |s|$ for any $s \in \mathbb{R}$. Then, we have that

$$u_\infty^+ = \lambda \int_{\mathbb{R}^N} \Gamma_n(x-y) a(y) f(u_\infty^+(y)) dy \leq \varrho\lambda \int_{\mathbb{R}^N} \Gamma_n(x-y) a(y) u_\infty^+(y) dy.$$

Since u_∞^+ is bounded, one can get $u_\infty^+ \leq c_3$ for some constants $c_3 > 0$. By condition (1.2), it follows that

$$\int_{\mathbb{R}^N} a(y) u_\infty^+(y) dy \leq c_3 \int_{\mathbb{R}^N} P(y) dy \leq c_3 \int_0^{+\infty} r^{N-1} P(r) dr < +\infty. \tag{3.5}$$

By (3.5), for any $\varepsilon > 0$, there exists a $R > 0$ such that for all $x \in \mathbb{R}^N$

$$|x|^{N-2} \int_{\Omega_R} \Gamma_N(x-y) a(y) u_\infty^+(y) dy < \frac{\varepsilon}{4}, \int_{\Omega_R} a(y) u_\infty^+(y) dy < \frac{\varepsilon}{4c_N} \tag{3.6}$$

and

$$\lim_{|x| \to +\infty} |x|^{N-2} \int_{B_R} \Gamma_N(x-y) a(y) u_\infty^+(y) dy = c_N \int_{B_R} a(y) u_\infty^+(y) dy, \tag{3.7}$$

where $\Omega_R = \{y \in \mathbb{R}^N : |y| > R\}$, $B_R = \{y \in \mathbb{R}^N : |y| < R\}$.

Furthermore, by (3.6), (3.7), and proof of [6, Theorem 1.1: p. 5948–5949], one can obtain

$$\lim_{|x| \to +\infty} |x|^{N-2} \int_{\mathbb{R}^N} \Gamma_N(x-y) a(y) u_\infty^+(y) dy = c_N \int_{\mathbb{R}^N} a(y) u_\infty^+(y) dy.$$

where $c_N = \frac{1}{N(N-2)\omega_N}$.

By (3.5), it follows that

$$\begin{aligned}
\lim_{|x| \to +\infty} |x|^{N-2} u_\infty^+(x) &= \lambda \cdot \lim_{|x| \to +\infty} |x|^{N-2} \int_{\mathbb{R}^N} \Gamma_n(x-y) a(y) f(u_\infty^+(y)) dy \\
&\leq \lambda\varrho \cdot \lim_{|x| \to +\infty} |x|^{N-2} \int_{\mathbb{R}^N} \Gamma_n(x-y) a(y) u_\infty^+(y) dy \\
&\leq c_N \lambda\varrho \int_{\mathbb{R}^N} a(y) u_\infty^+(y) dy < \infty.
\end{aligned}$$

Therefore, we have that

$$\lim_{|x| \to +\infty} |x|^{N-2} u_i^+(x) = c_1^i$$

for all $x \in \mathbb{R}^N$ and some constants $c_1^i \neq 0$, where $i = 0, \infty$. Do the same for u_i^-.

(b) Inspired by the idea of [13], we define the cut-off function of f as the following

$$f^{[n]}(s) := \begin{cases} f(s), & s \in [-n, n], \\ \frac{2n^2 - f(n)}{n}(s-n) + f(n), & s \in (n, 2n), \\ \frac{2n^2 + f(-n)}{n}(s+n) + f(-n), & s \in (-2n, -n), \\ ns, & s \in (-\infty, -2n] \cup [2n, +\infty). \end{cases}$$

We consider the following problem

$$\begin{cases} -\Delta u = \lambda a(x) f^{[n]}(u), & x \in \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to +\infty. \end{cases} \tag{3.8}$$

Clearly, we can see that $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = f_0$ and $(f^{[n]})_\infty = n$.

By Remarks 3.1 and 3.2, there are two distinct unbounded continuum $\mathscr{C}^\nu$ and $\mathscr{D}^{\nu[n]}$ of solutions of the problem (3.8) emanating from $(\frac{\lambda_1}{f_0}, 0)$, and $(\frac{\lambda_1}{n}, \infty))$ respectively, such that they are disjoint, unbounded in the direction of λ and

$$\mathscr{C}^\nu \subset (\mathbb{R} \times P^\nu) \cup \{(\frac{\lambda_1}{f_0}, 0)\}, \mathscr{D}^{\nu[n]} \subset (\mathbb{R} \times P^\nu) \cup \{(\frac{\lambda_1}{n}, \infty)\},$$

where $\nu = +, -$.

By Lemma 2.2, one derives that for each $\epsilon > 0$ there exists an N, such that for $n > N$, $\mathscr{D}^{\nu[n]} \subset V_\epsilon(\mathscr{D}^\nu)$, where $\mathscr{D}^\nu = \limsup_{n\to+\infty} \mathscr{D}^{\nu[n]}$. So it achieves

$$(\frac{\lambda_1}{n}, +\infty) \subseteq \mathtt{Proj}_{\mathbb{R}}(\mathscr{D}^{\nu[n]}) \subseteq \mathtt{Proj}_{\mathbb{R}} V_\epsilon(\mathscr{D}^\nu).$$

It follows $(\frac{\lambda_1}{n} + \epsilon, +\infty) \subseteq \mathtt{Proj}_{\mathbb{R}}(\mathscr{D}^\nu)$. The arbitrariness of $\epsilon > 0$ and n imply

$$(0, \infty) \subseteq \mathtt{Proj}_{\mathbb{R}}(\mathscr{D}^\nu).$$

Similar the proof of Lemma 3.2, we have that

$$(\frac{\lambda_1}{f_0}, \infty) \subseteq \mathtt{Proj}_{\mathbb{R}}(\mathscr{C}^\nu).$$

Similar the proof of (a), we have that

$$\lim_{|x|\to+\infty} |x|^{N-2} u_i^+(x) = c_1^i$$

for all $x \in \mathbb{R}^N$ and some constants $c_1^i \neq 0$, where $i = 0, \infty$. Do the same for u_i^-.

(c) Define

$$f^{[n]}(s) := \begin{cases} ns, & s \in [-\frac{1}{n}, \frac{1}{n}], \\ \left[f(\frac{2}{n}) - 1\right](ns - 2) + f(\frac{2}{n}), & s \in (\frac{1}{n}, \frac{2}{n}), \\ -\left[f(-\frac{2}{n}) + 1\right](ns + 2) + f(-\frac{2}{n}), & s \in (-\frac{2}{n}, -\frac{1}{n}), \\ f(s), & s \in (-\infty, -\frac{2}{n}] \cup [\frac{2}{n}, +\infty). \end{cases}$$

We consider the following problem

$$\begin{cases} -\Delta u = \lambda a(x) f^{[n]}(u), & x \in \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to +\infty. \end{cases} \tag{3.9}$$

Clearly, we can see that $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = n$ and $(f^{[n]})_\infty = f_\infty$.

By Remarks 3.1 and 3.2, there are two distinct unbounded continuum $\mathscr{C}^\nu$ and $\mathscr{D}^{\nu[n]}$ of solutions of the problem (3.9) emanating from $(\frac{\lambda_1}{n}, 0)$, and $(\frac{\lambda_k}{f_\infty}, \infty))$ respectively, such that they are disjoint, unbounded in the direction of λ and

$$\mathscr{C}^{\nu[n]} \subset (\mathbb{R} \times P^\nu) \cup \{(\frac{\lambda_1}{n}, 0)\}, \mathscr{D}^\nu \subset (\mathbb{R} \times P^\nu) \cup \{(\frac{\lambda_1}{f_\infty}, \infty)\},$$

where $\nu = +, -$.

Taking $z^* = (0, 0)$, and applying Lemma 2.1 again, one derives that $\mathscr{C}^\nu = \limsup_{n\to+\infty} \mathscr{C}^{\nu[n]}$ is unbounded and connected, moreover, $z^* \in \mathscr{C}^\nu$.

We claim that $\mathscr{C}^\nu \cap (\mathbb{R} \times P^\nu) = \{(0, 0)\})$.

Suppose on the contrary that there exists a sequence $(\lambda_n, u_n) \in \mathscr{C}^\nu \setminus \{(0, 0)\} = \limsup_{n\to\infty} \mathscr{C}^{\nu[n]} \setminus \{(0, 0)\}$ such that $\lim_{n\to\infty} \lambda_n = \mu \neq 0$ and $\lim_{n\to\infty} \|u_n\| = 0$. Hence, for any $N_0 \in \mathbb{N}$, there exists $n_0 \geq N_0$ such that $(\lambda_n, u_n) \in \mathscr{C}^{\nu[n_0]}$. By (3.9), it follows that $\lambda_{n_0} = \frac{\lambda_1}{n}$ for $n_0 \geq N_0$. From the arbitrary of N_0, it implies that $n_0 \to \infty$, i.e., $\mu = 0$, which contradicts the assumption of $\mu \neq 0$.

Lemma 3.1(i) implies that the projection of $\mathscr{C}^+_{k,0}$ on $\mathbb{R}$ is unbounded.

Furthermore, we have

$$(0, \infty) \subseteq \mathtt{Proj}_{\mathbb{R}}(\mathscr{C}^\nu_{k,0}).$$

Similar the proof of Lemma 3.3, we have that

$$(\frac{\lambda_1}{f_\infty}, \infty) \subseteq \mathtt{Proj}_{\mathbb{R}}(\mathscr{D}^\nu).$$

(d) Similar the proof of (b) and (c), respectively, we have that

$$(0, \infty) \subseteq \mathtt{Proj}_{\mathbb{R}}(\mathscr{D}^\nu)$$

and

$$(0, \infty) \subseteq \mathtt{Proj}_{\mathbb{R}}(\mathscr{C}^\nu).$$

4. Exact multiplicity of one-sign solutions

To prove Theorems 1.2 and 1.3, by Dai and Han [14], Afrouzi and Rasouli [15], we first give the definition of linearly stable solution for the problem (1.1). For any $\phi \in E$ and positive solution u of problem (1.1), we can calculate that the linearized eigenvalue problem of (1.1) at the direction ϕ is

$$\begin{cases} -\Delta\phi - \lambda a(x) f'(u)\phi = \mu\phi, & x \in \mathbb{R}^N, \\ \phi(x) \to 0, & \text{as } |x| \to +\infty, \end{cases} \tag{4.1}$$

where $f'(u)\phi$ denotes the Fr´echet derivative of f about u at the direction ϕ. A solution u of problem (1.1) is stable if all eigenvalues of problem (4.1) are positive, otherwise it is unstable. We define the Morse index $M(u)$ of u to problem (1.1) to be the number of negative eigenvalues of problem (4.1). A solution u of problem (1.1) is degenerate if 0 is an eigenvalue of problem (4.1), otherwise it is non-degenerate. The following lemma is our main stability result for the one-sign solution.

Lemma 4.1. Under the assumptions of Theorem 1.2 (a), then any positive or negative solution u of problem (1.1) is stable and non-degenerate, and Morse index $M(u) = 0$.

Proof. Without loss of generality, let u be a positive solution of problem (1.1), and let (μ_1, ϕ_1) be the corresponding principal eigenpair of problem (4.1) with $\phi_1 > 0$ in $\mathbb{R}^N$. Notice that u and ϕ_1 satisfy

$$\begin{cases} -\Delta u = \lambda a(x) f(u), & x \in \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to +\infty \end{cases} \tag{4.2}$$

and

$$\begin{cases} -\Delta \phi_1 - \lambda a(x) f'(u)\phi_1 = \mu\phi_1, x \in \mathbb{R}^N, \\ \phi_1(x) \to 0, & \text{as } |x| \to +\infty. \end{cases} \tag{4.3}$$

Multiplying the first equation of problem (4.3) by u and the first equation of problem (4.2) by ϕ_1, subtracting and integrating, we obtain

$$\mu_1 \int_{\mathbb{R}^N} \phi_1 u dx = \lambda \int_{\mathbb{R}^N} a(x)\phi_1 (f(u) - f'(u)u) dx.$$

By some simple computations, we can show that it follows from (A4) that $f(s) - f'(s)s \geq 0$ for any $s \geq 0$. Since $u > 0$ and $\phi_1 > 0$ in $\mathbb{R}^N$, we have $\mu_1 > 0$ and the positive solution u must be stable.

Proof of Theorem 1.2

(a) By Theorem 2.1, we can obtain the problem (1.1) possesses at least two one-sign solutions u^+ and u^- such that $u^+ > 0$ and $u^- < 0$ in $\mathbb{R}^N$. In order to prove exact multiplicity of one-sign solutions for (1.1). Define $F : \mathbb{R} \times E \to \mathbb{R}$ by

$$F(\lambda, u) = -\Delta u - \lambda a(x) f(u).$$

From Lemma 4.1, we know that any one-sign solution (λ, u) of problem (1.1) is stable. Therefore, at any one-sign solution (λ^*, u^*) for the problem (1.1), we can apply Implicit Function Theorem to $F(\lambda, u) = 0$, and all the solutions of $F(\lambda, u) = 0$ near (λ^*, u^*) are on a curve $(\lambda, u(\lambda))$ with $|\lambda - \lambda^*| \leq \varepsilon$ for some small $\varepsilon > 0$. Furthermore, by virtue of Remark 2.2, the unbounded continua $\mathscr{D}^+$ and $\mathscr{D}^-$ are all curves.

To complete the proof, it suffices to show that $u^+(\lambda, \cdot)(u^-(\lambda, \cdot))$ is increasing (decreasing) with respect to λ. We only prove the case of $u^+(\lambda, \cdot)$. The proof of $u^-(\lambda, \cdot)$ can be given similarly. Since $u^+(\lambda, \cdot)$ is differentiable with respect to λ (as a consequence of Implicit Function Theorem), taking the derivative of the first equation of problem (4.2) by λ, one can obtain that

$$-\Delta(\frac{du^+}{d\lambda}) = \lambda a(x) f'(u^+) \frac{du^+}{d\lambda} + a(x) f'(u^+). \tag{4.4}$$

Multiplying the first equation of problem (4.4) by u and the first equation of problem (4.2) by $\frac{du^+}{d\lambda}$, subtracting and integrating, we obtain

$$\int_{\mathbb{R}^N} \left[\lambda a(x)(f'(u^+)u^+ - f(u^+))\frac{du^+}{d\lambda} + f(u^+)u^+ \right] dx = 0.$$

(A2) implies $f(s)s \geq 0$ for any $s \in \mathbb{R}$. So we get $(f'(u^+)u^+ - f(u^+))\frac{du^+}{d\lambda} \leq 0$ by (A1). While (A4) shows that $f'(u^+)u^+ - f(u^+) \leq 0$. Therefore, we have $\frac{du^+}{d\lambda} \geq 0$.

Next, we only prove the case of the uniqueness of positive solution of problem (1.1) since the proof of the uniqueness of negative solution of problem (1.1) is similar. Suppose on the contrary that there exist two solutions v_1^+ and v_2^+ corresponding to λ with $v_1^+ \in \mathscr{D}^+$ of the problem (1.1) for $\lambda \in (\lambda_1/f_0, +\infty)$. For $\varepsilon > 0$, take $(\lambda - \varepsilon, v_{\lambda-\varepsilon}^+), (\lambda + \varepsilon, v_{\lambda+\varepsilon}^+) \in \mathscr{D}^+$, then $v_{\lambda\pm\varepsilon} \to v_1^+$ as $\varepsilon \to 0$. By the monotonicity of v_2^+ with respect to λ, we get $v_{\lambda-\varepsilon}^+ \leq v_2^+ \leq v_{\lambda+\varepsilon}^+$. Then $v_2^+ = v_1^+$.

Similar the proof of (a) of Theorem 1.1, we have that

$$\lim_{|x|\to+\infty} |x|^{N-2}u_1^+(x) = c_1 \text{ and } \lim_{|x|\to+\infty} |x|^{N-2}u_1^-(x) = c_2$$

for all $x \in \mathbb{R}^N$ and some constants $c_1, c_2 \neq 0$.

Proof of Theorem 1.3. By Theorem 4.1 and Lemma 2.1, there is a distinct unbounded continuum $\mathscr{D}^\nu(\nu \in \{+, -\})$ of solutions of the problem (1.2) emanating from $(0, 0)$. Furthermore, we have that the problem (1.1) possesses at least two one-sign solutions u^+ and u^- such that $u^+ > 0$ and $u^- < 0$ in $(0, +\infty)$. In view of the argument of Theorem 1.2, the desired conclusion can be obtained immediately.

5. Conclusions

In this study, we have proved the existence of one-sign solutions without signum condition for the semilinear elliptic problems (1.1) by the bifurcation techniques and the approximation of connected components. We also obtained the exact multiplicity of one-sign solutions for the problem (1.1) under more strict hypotheses. Our fifindings can be applied to further other differential equations with various other boundary conditions, high dimensional case, and so on as future work.

Conflict of interest

The author declares no conflicts of interest in this paper.

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