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Research article

The Hermite-Hadamard type inequalities for quasi p-convex functions

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Abstract: In this paper, the Hermite-Hadamard inequality and its generalization for quasi p-convex functions are provided. Also several new inequalities are established for the functions whose first derivative in absolute value is quasi p-convex, which states some bounds for sides of the Hermite-Hadamard inequalities. In the context of the applications of results, we presented some relations involving special means and some inequalities for special functions including digamma function and Fresnel integral for sinus. In addidition, an upper bound for error in numerical integration of quasi p-convex functions via composite trapezoid rule is given.

Keywords: Hermite-Hadamard inequality; p-convex function; quasi p-convex function
Mathematics Subject Classification: 26A51, 26B25

1. Introduction

The concept of quasi convexity is notable one among different types of convexity such as pseudoconvexity, B-convexity, B^{-1}-convexity, s-convexity, m-convexity, p-convexity etc. [2, 3, 10, 13, 15, 17]. A real valued function f defined over a convex subset A of vector space X is called quasi convex function if

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$$

for all $x \in A$ and $\lambda \in [0,1]$. As seen in its definition, while classical convex functions are based on the sum of function values, it stands on the comparison of function values, thus making it handy tool, for example, to represent customer preferences in economical applications. Overtime, different approaches and needs result in different kinds of quasi convexity e.q., harmonically quasi convexity, φ-quasi convexity, γ-quasi convexity, ω-quasi convex functions [5, 11, 20]. A novel one is quasi p-convexity, which associates quasi convexity and p-convexity .

For a fixed $p \in (0,1]$, p-convex functions are defined on a p-convex set, which is the set that contains a special type curve $x(t)$ joining any different two points a and b of it, namely, $x_{a,b}(t) = at^{\frac{1}{p}} + b(1-t)^{\frac{1}{p}}$, $t \in [0,1]$. Similarly, the definiton of p-convex function is based on the curves joining these points and their images, i.e., a function f defined on a p-convex set A is called p-convex function if

$$f(x_{a,b}(t)) \leq x_{f(a),f(b)}(t)$$

for $a, b \in A$ and $t \in [0, 1]$.

On the other hand, as in convex functions, the quasi convex funtions are commonly characterized or given by an inequality. For quasi convex functions, the restatement of some preeminent inequalities that is given for classical convex functions such as Jensen, Hermite-Hadamard are of interest among researchers (see [6,7,12] and the references therein). Especially in recent years, the studies on Hermite Hadamard inequality take vast place in literature [1,8,9,16,18,22,23], which is stated as follows:

Let f be real valued function defined on a real interval $[a, b]$. If f is convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

This inequality indicates that the average value of a convex function on an interval interpolates between the image of the arithmetic mean of the points and the arithmetic mean of images of the points.

In this paper, we state the Hermite-Hadamard type inequality and related extensions for quasi p-convex functions and some applications are given. Results are presented in two subsections. In the first, the Hermite-Hadamard inequalities for quasi p-convex functions defined on nonnegative numbers, then, real numbers are stated and a generalization of the inequality is given. In the second, some upper bounds for left and right sides of the Hermite-Hadamard inequality is obtained for the functions whose derivative in absolute value is quasi p-convex function. In applications, some inequalities involving special means and special functions such as digamma and Fresnel integral functions are presented. Moreover, an upper bound for error in numerical integration of quasi p-convex functions via composite trapezoid rule is given.

Let us give some essential notations and formal definitions of p-convex set and quasi p-convex functions. Throughout the paper, $\mathbb{R}, \mathbb{R}_+, \mathbb{R}^n$ denote the set of real numbers, the set of nonnegative real numbers, and n-dimensional Euclidean space, respectively.

Definition 1. [17] Let U be a subset of $\mathbb{R}^n$ and $0 < p \leq 1$. U is called p-convex set if

$$\lambda x + \mu y \in U$$

for all $x, y \in U$ and $\lambda, \mu \in [0, 1]$ such that $\lambda^p + \mu^p = 1$.

It is known that any interval of real numbers including zero or accepting zero as a boundary point is a p-convex set [18].

Definition 2. [21] Let $0 < p \leq 1$ and $U \subseteq \mathbb{R}^n$ be a p-convex set. A function $f : U \to \mathbb{R}$ is called quasi p-convex function if f provides

$$f(\lambda x + \mu y) \leq \max\{f(x), f(y)\}$$

for each $x, y \in U$; $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$.

Let $U \subseteq \mathbb{R}^n$ be a p-convex set. If we define the function f such that

$$f : U \to \mathbb{R}\ ,\ \ f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} |kx_i|,$$

for $k \in \mathbb{R}$, then f is a quasi p-convex function.

A fundamental characterization of quasi p-convex functions can be given as in the following theorem:

Theorem 1.1. *[21] Let $0 < p < 1$. A function $f : \mathbb{R}_+ \to \mathbb{R}$ is a quasi p-convex function if and only if f is an increasing function.*

We present Jensen inequality for the quasi p-convex functions, which is needed next.

Theorem 1.2. *[21] Let $U \subseteq \mathbb{R}^n$ and $f : U \to \mathbb{R}_+$ be a quasi p-convex function. Let $x_1, \ldots, x_m \in U$ and $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1^p + \ldots + \lambda_m^p = 1$. Then*

$$f\left(\lambda_1 x_1 + \ldots + \lambda_m x_m\right) \leq \max\{f(x_1), f(x_2), \ldots, f(x_m)\}.$$

2. Main results

2.1. The Hermite-Hadamard inequalities for quasi p-convex functions

In this subsection, the Hermite-Hadamard type inequality for quasi p-convex functions on nonnegative real numbers is stated. Then its version for the quasi-p-convex fuctions whose domain is extended to real numbers is obtained. Finally, a generalization of the Hermite-Hadamard inequality is given such that it is expressed in terms of internal points in the interval.

The following theorem states Hermite-Hadamard type inequality with nonnegative integral bounds.

Theorem 2.1. *Let $0 < p < 1$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an integrable quasi p-convex function. For $a, b \in \mathbb{R}_+$ with $a < b$, the following inequality holds*

$$2^{-1} f(\frac{a+b}{2^{\frac{1}{p}}})(b-a) \leq \int_a^b f(x)dx \leq (b-a)f(b). \tag{2.1}$$

Proof. It is clear from Theorem 1.1 that $f(a) < f(b)$ for $a < b$. First we show the right part of the inequality. By changing variable $x = t^{\frac{1}{p}} b + (1-t)^{\frac{1}{p}} a$, we have

$$\int_a^b f(x)dx = \frac{1}{p} \int_0^1 f(t^{\frac{1}{p}} b + (1-t)^{\frac{1}{p}} a)(bt^{\frac{1}{p}-1} - a\,(1-t)^{\frac{1}{p}-1})dt. \tag{2.2}$$

From the quasi p-convexity of f and the triangle inequality, we can write

$$\int_a^b f(x)dx \leq \frac{f(b)}{p} \int_0^1 \left(bt^{\frac{1}{p}-1} - a\,(1-t)^{\frac{1}{p}-1}\right) dt \leq f(b)(b-a).$$

For the first part of the inequality, from the quasi p-convexity of f, for all $x, w > 0$, we have

$$f(\frac{x+w}{2^{\frac{1}{p}}}) \leq \max\{f(x), f(w)\}.$$

Let $x = ta + (1-t)b$ and $w = tb + (1-t)a$ for $t \in [0,1]$. Then

$$f(\frac{a+b}{2^{\frac{1}{p}}}) \leq \max\{f(ta + (1-t)b), f((1-t)a + tb)\}.$$

Since $(1-t)a + tb < ta + (1-t)b$ for $t \in [0, \frac{1}{2})$ and $ta + (1-t)b < (1-t)a + tb$ for $t \in (\frac{1}{2}, 1]$,

$$\max\{f(ta + (1-t)b), f((1-t)a + tb)\} = \begin{cases} f(ta + (1-t)b), & if \;\; t \in [0, \frac{1}{2}) \\ \\ f(tb + (1-t)a), & if \;\; t \in (\frac{1}{2}, 1]. \end{cases}$$

Integrating both side and using the fact that

$$\int_0^{\frac{1}{2}} f\left(ta + (1-t)\, b\right) dt = \int_{\frac{1}{2}}^{1} f\left(tb + (1-t)\, a\right) dt = \frac{1}{b-a} \int_{\frac{a+b}{2}}^{b} f(x)dx$$

one can have

$$f(\frac{a+b}{2^{\frac{1}{p}}}) \leq \frac{2}{b-a} \int_{\frac{a+b}{2}}^{b} f(x)dx \leq \frac{2}{b-a} \int_a^b f(x)dx.$$

□

Remark 2.2. Similar to the proof of Theorem 2.1, the inequality (2.1) for $p = 1$ gives the inequality

$$2^{-1} f(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \max\{f(a), f(b)\}$$

in [6].

Next theorem generalizes the result in Theorem 2.1 by extending integral bounds to all real numbers.

Theorem 2.2. *Let $0 < p < 1$ and $f : \mathbb{R} \to \mathbb{R}_+$ be an integrable quasi p-convex function. For $a, b \in \mathbb{R}$ with $a < b$, the following inequality holds*

$$2^{-1} f(\frac{a+b}{2^{\frac{1}{p}}})(b-a) \leq \int_a^b f(x)dx$$

$$
\leq \max\{f(a), f(b)\} \times \begin{cases} 2a\,(1-t_*)^{\frac{1}{p}} + 2bt_*^{\frac{1}{p}} - (a+b), & if \quad a<b<0 \\ \frac{1}{p}\max\{-a, b, bt^{\star\,\frac{1}{p}-1} - a\,(1-t^\star)^{\frac{1}{p}-1}\}, & if \quad a<0<b \\ (a+b) - 2bt_*^{\frac{1}{p}} - 2a\,(1-t_*)^{\frac{1}{p}}, & if \quad 0<a<b \\ b & if \quad a=0 \\ -a & if \quad b=0 \end{cases}
$$

where $t^\star = \left(1 + (\frac{-b}{a})^{\frac{p}{1-2p}}\right)^{-1}$ *and* $t_* = \left(1 + (\frac{b}{a})^{\frac{p}{1-p}}\right)^{-1}$.

Proof. The proof of the left part of the inequality is the same as in proof of Theorem 2.1. To show the right part of the inequality, let us examine the behavior of the function $g(t) = (bt^{\frac{1}{p}-1} - a\,(1-t)^{\frac{1}{p}-1})$ with respect to signs of a and b in (2.2). There exist three cases:

i) If $a < b < 0$, $g(t)$ is decreasing function such that $g : [0,1] \to [b, -a]$ and $g(t_*) = 0$ where $t_* = \left(1 + (\frac{b}{a})^{\frac{p}{1-p}}\right)^{-1}$.

ii) If $a < 0 < b$, it is seen from the derivative of g that g is unimodal and positive function such that it takes maximum value at $t = 0$ or $t = 1$ or $t^\star = \left(1 + (\frac{-b}{a})^{\frac{p}{1-2p}}\right)^{-1}$. In case of $p = \frac{1}{2}$ the maximum value of $g(t)$ is $\max\{-a, b\}$.

iii) If $0 < a < b$, g is a increasing function such that $g : [0,1] \to [-a, b]$ and $g(t_*) = 0$ where $t_* = \left(1 + (\frac{b}{a})^{\frac{p}{1-p}}\right)^{-1}$.

Case i) Let $a < b < 0$ and $f(a) \leq f(b)$. Then

$$
\begin{aligned}
\int_a^b f(x)dx &= \frac{1}{p}\int_0^1 f(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)(bt^{\frac{1}{p}-1} - a\,(1-t)^{\frac{1}{p}-1})dt \\
&\leq \frac{1}{p}\int_0^1 f(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)\left|bt^{\frac{1}{p}-1} - a\,(1-t)^{\frac{1}{p}-1}\right| dt \\
&= \frac{1}{p}\int_0^{t_*} f(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)(bt^{\frac{1}{p}-1} - a\,(1-t)^{\frac{1}{p}-1})dt \\
&+ \frac{1}{p}\int_{t_*}^1 f(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)(a\,(1-t)^{\frac{1}{p}-1} - bt^{\frac{1}{p}-1})dt \\
&= I_1 + I_2.
\end{aligned}
$$

Let us study on I_1 and I_2.

$$
I_1 = \frac{1}{p}\int_0^{t_*} f(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)(bt^{\frac{1}{p}-1} - a\,(1-t)^{\frac{1}{p}-1})dt
$$

$$\leq \frac{1}{p}\int_{0}^{t_*} f(b)(bt^{\frac{1}{p}-1} - a\,(1-t)^{\frac{1}{p}-1})dt$$

$$= f(b)(bt_*^{\frac{1}{p}} + a\,(1-t_*)^{\frac{1}{p}} - a). \tag{2.3}$$

$$I_2 = \frac{1}{p}\int_{t_*}^{1} f(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)(a\,(1-t)^{\frac{1}{p}-1} - bt^{\frac{1}{p}-1})dt$$

$$\leq \frac{1}{p}\int_{t_*}^{1} f(b)(a\,(1-t)^{\frac{1}{p}-1} - bt^{\frac{1}{p}-1})dt$$

$$= f(b)(bt_*^{\frac{1}{p}} + a\,(1-t_*)^{\frac{1}{p}} - b). \tag{2.4}$$

Consequently, combining (2.3) and (2.4), we have

$$\int_a^b f(x)dx \leq f(b)(2bt_*^{\frac{1}{p}} + 2a\,(1-t_*)^{\frac{1}{p}} - (a+b)).$$

Since the same idea works for the case $f(b) < f(a)$, we have the following result

$$\int_a^b f(x)dx \leq f(a)(2bt_*^{\frac{1}{p}} + 2a\,(1-t_*)^{\frac{1}{p}} - (a+b)).$$

Eventually, we have

$$\int_a^b f(x)dx \leq \max\{f(a), f(b)\}(a(2\,(1-t_*)^{\frac{1}{p}} - 1) + b(2t_*^{\frac{1}{p}} - 1)).$$

Case ii) Let $a < 0 < b$ and $f(a) \leq f(b)$. Then,

$$\int_a^b f(x)dx = \frac{1}{p}\int_0^1 f(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)(bt^{\frac{1}{p}-1} - a\,(1-t)^{\frac{1}{p}-1})dt \leq \frac{1}{p}f(b)\max\{-a, b, g(t^\star)\}.$$

where $t^\star = \left(1 + (\frac{-b}{a})^{\frac{p}{1-2p}}\right)^{-1}$. Similarly, the following is obtained for the case $f(b) < f(a)$

$$\int_a^b f(x)dx \leq \frac{1}{p}f(a)\max\{-a, b, g(t^\star)\}.$$

Combining these, we have

$$\int_a^b f(x)dx \leq \frac{1}{p}\max\{f(a), f(b)\}\max\{-a, b, g(t^\star)\}.$$

Case iii) Assume $0 < a < b$ and $f(a) < f(b)$. Then

$$\begin{aligned}
\int_a^b f(x)dx &= \frac{1}{p}\int_0^1 f(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)(bt^{\frac{1}{p}-1} - a\,(1-t)^{\frac{1}{p}-1})dt \\
&\leq \frac{1}{p}\int_0^1 f(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)\left|bt^{\frac{1}{p}-1} - a\,(1-t)^{\frac{1}{p}-1}\right|dt \\
&= \frac{1}{p}\int_0^{t_*} f(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)(bt^{\frac{1}{p}-1} - a\,(1-t)^{\frac{1}{p}-1})dt \\
&\quad + \frac{1}{p}\int_{t_*}^1 f(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)(a\,(1-t)^{\frac{1}{p}-1} - bt^{\frac{1}{p}-1})dt \\
&= -I_1 - I_2.
\end{aligned}$$

Making the similar calculations to obtain (2.3) and (2.4), we have

$$\begin{aligned}
-I_1 &= -\frac{1}{p}\int_0^{t_*} f(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)(bt^{\frac{1}{p}-1} - a\,(1-t)^{\frac{1}{p}-1})dt \\
&\leq \frac{1}{p}\int_0^{t_*} f(b)(-bt^{\frac{1}{p}-1} + a\,(1-t)^{\frac{1}{p}-1})dt \\
&= f(b)(a - bt_*^{\frac{1}{p}} - a\,(1-t_*)^{\frac{1}{p}}). \qquad (2.5)
\end{aligned}$$

$$\begin{aligned}
-I_2 &= \frac{1}{p}\int_{t_*}^1 f(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)(bt^{\frac{1}{p}-1} - a\,(1-t)^{\frac{1}{p}-1})dt \\
&\leq \frac{1}{p}\int_{t_*}^1 f(b)(bt^{\frac{1}{p}-1} - a\,(1-t)^{\frac{1}{p}-1})dt \\
&= f(b)(b - bt_*^{\frac{1}{p}} - a\,(1-t_*)^{\frac{1}{p}}). \qquad (2.6)
\end{aligned}$$

Consequently, combining (2.5) and (2.6), we have

$$\int_a^b f(x)dx \leq f(b)((a+b) - 2bt_*^{\frac{1}{p}} - 2a\,(1-t_*)^{\frac{1}{p}}).$$

Similarly, the following is obtained for the case $f(b) < f(a)$

$$\int_a^b f(x)dx \leq f(a)((a+b) - 2bt_*^{\frac{1}{p}} - 2a\,(1-t_*)^{\frac{1}{p}}).$$

So, we have

$$\int_a^b f(x)dx \le \max\{f(a), f(b)\}((a+b) - 2bt_*^{\frac{1}{p}} - 2a(1-t_*)^{\frac{1}{p}}).$$

The results for the cases $a = 0$ and $b = 0$ can be easily obtained from (2.2). □

Although Hermite-Hadamard inequality is originally stated for the end points of the interval in Theorem 2.1, we try to generalize it also in terms of internal points of the interval as follows.

Theorem 2.3. *Let $0 < p < 1$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ be quasi p-convex function and $[a,b] \subseteq \mathbb{R}_+$. Suppose that f integrable on $[a,b]$, $x_i \in [a,b]$ such that $x_1 < x_2 < \cdots < x_{n+1}$ for $i \in [1, n+1]$, then the following inequality holds*

$$2^{-1} f\left(\frac{x_1 + 2(x_2 + \cdots + x_n) + x_{n+1}}{(2n)^{\frac{1}{p}}}\right) \le \sum_{k=1}^{n} \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} f(t)dt \le \sum_{i=1}^{n} f(x_{i+1}).$$

Proof. Letting $a = x_i$, $b = x_{i+1}$ in Theorem 2.1, one gets

$$2^{-1} f\left(\frac{x_i + x_{i+1}}{2^{\frac{1}{p}}}\right) \le \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(t)dt \le f(x_{i+1})$$

for $i \in [1, n]$. Summing up these inequalities side by side, we have

$$2^{-1} \sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1}}{2^{\frac{1}{p}}}\right) \le \sum_{i=1}^{n} \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(t)dt \le \sum_{i=1}^{n} f(x_{i+1}). \tag{2.7}$$

From Theorem 1.2

$$f\left(\frac{2^{-\frac{1}{p}}(x_1 + x_2) + \cdots + 2^{-\frac{1}{p}}(x_n + x_{n+1})}{n^{\frac{1}{p}}}\right) \le \max\left\{f\left(\frac{x_1 + x_2}{2^{\frac{1}{p}}}\right), \cdots, f\left(\frac{x_n + x_{n+1}}{2^{\frac{1}{p}}}\right)\right\},$$

so

$$f\left(\frac{x_1 + 2(x_2 + \cdots + x_n) + x_{n+1}}{(2n)^{\frac{1}{p}}}\right) \le \sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1}}{2^{\frac{1}{p}}}\right). \tag{2.8}$$

Combining (2.7) and (2.8), we get the desired inequality. □

Corollary 2.4. *Let $[a,b] \subseteq \mathbb{R}_+$ such that $b - a = n$ where n is a natural number, $0 < p < 1$ and let $f : [a,b] \to \mathbb{R}_+$ be quasi p-convex function and integrable on $[a,b]$. Suppose $x_i = a + (i-1)\frac{(b-a)}{n}$ for $i \in [1, n+1]$, then the following inequality holds*

$$2^{-1} f\left(\frac{n}{(2n)^{\frac{1}{p}}}(a+b)\right) \le \frac{n}{b-a} \int_a^b f(t)dt \le \sum_{i=1}^{n} f(x_{i+1}).$$

2.2. Some bounds related to Hermite-Hadamard inequalities for the functions whose first derivative in absolute value is quasi p-convex functions

In this subsection, using two integral identities expressed as lemmas, we determine an upper bound for the right and left sides of the Hermite-Hadamard inequality for the functions whose derivative in absolute value is quasi p-convex function. Also we present sharper versions of this bounds.

By using the following lemma, we determine an upper bound for the right side of Hermite-Hadamard type inequality for the mentioned functions:

Lemma 2.5. *[18] Let $a, b \in \mathbb{R}$ with $a < b$, $0 < p \leq 1$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function. If $f' \in L[a, b]$, then the following equality holds:*

$$
\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \\
&= \frac{1}{2p(a-b)}\int_0^1 \left[a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right] f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\left[t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right]dt.
\end{aligned}
$$

Theorem 2.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable function such that $|f'|$ is integrable on $[a, b]$ and quasi p-convex function on $\mathbb{R}$. Then the following inequality holds:*

$$
\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{3}{2p(b-a)}(|a|+|b|)^2 \max\{|f'(a)|, |f'(b)|\}. \tag{2.9}
$$

Proof. From Lemma 2.5, triangle inequality and the quasi p-convexity of $|f'|$,

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{1}{2p(b-a)}\int_0^1 \left|a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|\left|t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right|\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|dt \\
&\leq \frac{1}{2p(b-a)}\int_0^1 \left|a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|\left|t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right|\max\{|f'(a)|, |f'(b)|\}dt \\
&\leq \frac{3}{2p(b-a)}(|a|+|b|)(|a|+|b|)\max\{|f'(a)|, |f'(b)|\}.
\end{aligned}
$$

□

Surely, some sharper versions of inequality (2.9) can be obtained. In the following two theorems, we present only two versions of them as examples.

Theorem 2.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable function such that $|f'|$ is integrable on $[a, b]$ and quasi p-convex function on $\mathbb{R}$. Then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{1}{2p(b-a)}\max\{|g(a)|, |g(b)|, |g(t_1)|\}\max\{|h(a)|, |h(b)|, |h(t_2)|\}\max\{|f'(a)|, |f'(b)|\}
\end{aligned}
$$

where

$$g(t)=t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a \ \ and \ \ h(t)=a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)$$

and for $a\neq 0$,

$$t_1=\left(1+\left(\left|\frac{b}{a}\right|\right)^{\frac{p}{1-2p}}\right)^{-1} \ \ and \ \ t_2=\left(1+\left(\left|\frac{b}{a}\right|\right)^{\frac{p}{1-p}}\right)^{-1}$$

for $a=0$, t_1,t_2 *equal to* 0 *or* 1.

Proof. From Lemma 2.5, as in the proof of Theorem 2.6, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2p(b-a)}\int_0^1|h(t)g(t)|\max\{|f'(a)|,|f'(b)|\}dt$$

where $g(t)=t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a$ and $h(t)=a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)$.

Let $a\neq 0$. From the first derivatives of $g(t)$ and $h(t)$, it is seen that according to the values of a,b,p, these functions are either monotonic functions or unimodal functions, i.e., the functions which has only one extremum point . So $g(t)$ and $h(t)$ take extremum values at the points $t=0$ or $t=1$ in common or $t_1=\left(1+\left(\frac{-b}{a}\right)^{\frac{p}{1-2p}}\right)^{-1}$ and $t_2=\left(1+\left(\frac{b}{a}\right)^{\frac{p}{1-p}}\right)^{-1}$ for proper values of a,b, respectively. Thus

$$|g(t)|\leq\max\{|g(a)|,|g(b)|,|g(t_1)|\} \text{ and } |h(t)|\leq\max\{|h(a)|,|h(b)|,|h(t_2)|\}$$

is derived. For the case $a=0$, extremum values are obtained for $t=0$, $t=1$, which is included in the inequality above. In a similar way in the proof of Theorem 2.6, by using the quasi p-convexity of $|f'|$, we get the desired result. □

Theorem 2.8. *Let* $f:\mathbb{R}\longrightarrow\mathbb{R}$ *be differentiable function such that* $|f'|$ *is integrable on* $[a,b]$ *and quasi* p*-convex function on* $\mathbb{R}$*. Then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{\max\{|f'(a)|,|f'(b)|\}}{b-a}\left\{a^2+|ab|+b^2+\frac{1}{p}B(\frac{1}{p},\frac{1}{p})\,|ab|\right\}$$

where $B(x,y)$ *is beta function.*

Proof. Let us suppose $g(t)$ and $h(t)$ as in the proof of Theorem 2.7. We have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2p(b-a)}\int_0^1|h(t)g(t)|\max\{|f'(a)|,|f'(b)|\}dt.$$

Using triangle inequality, we have

$$\begin{aligned}|h(t)g(t)|&=\Big|(ab+b^2)t^{\frac{1}{p}-1}-(a^2+ab)(1-t)^{\frac{1}{p}-1}-2b^2t^{\frac{2}{p}-1}\\&\quad+2ab(t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1}-t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}})+2a^2(1-t)^{\frac{2}{p}-1}\Big|\\&\leq\left(|ab|+b^2\right)t^{\frac{1}{p}-1}+\left(a^2+|ab|\right)(1-t)^{\frac{1}{p}-1}+2b^2t^{\frac{2}{p}-1}\\&\quad+2\,|ab|\,(t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1}+t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}})+2a^2(1-t)^{\frac{2}{p}-1}.\end{aligned}\tag{2.10}$$

If we multiply (2.10) with $\max\{|f'(a)|, |f'(b)|\}$ then expand and integrate on $[0,1]$ with respect to t, we get

$$\int_0^1 |h(t)g(t)| \max\{|f'(a)|, |f'(b)|\}dt \leq \frac{\max\{|f'(a)|, |f'(b)|\}}{b-a}\left\{a^2 + |ab| + b^2 + \frac{1}{p}B(\frac{1}{p}, \frac{1}{p})|ab|\right\}.$$

□

By means of the following lemma we will determine an upper bound for the left Hermite-Hadamard type inequality for the functions whose derivative's absolute value is quasi p-convex function.

Lemma 2.9. *[18] Let $a, b \in \mathbb{R}$ with $a < b$, $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $f' \in L[a,b]$, then the following equality holds:*

$$\begin{aligned}
&f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx \\
&= \frac{1}{p(b-a)}\int_0^1 \left[t^{\frac{1}{p}}\frac{a+b}{2} + ((1-t)^{\frac{1}{p}} - 1)a\right] f'(t^{\frac{1}{p}}\frac{a+b}{2} + (1-t)^{\frac{1}{p}}a)\left[t^{\frac{1}{p}-1}\frac{a+b}{2} - (1-t)^{\frac{1}{p}-1}a\right]dt \\
&+ \frac{1}{p(b-a)}\int_0^1 \left[b(t^{\frac{1}{p}} - 1) + (1-t)^{\frac{1}{p}}\frac{a+b}{2}\right] f'(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}\frac{a+b}{2})\left[t^{\frac{1}{p}-1}b - (1-t)^{\frac{1}{p}-1}\frac{a+b}{2}\right]dt.
\end{aligned}$$

Theorem 2.10. *Let a, b be real numbers with $a < b$ and $f : \mathbb{R} \to \mathbb{R}$ be differentiable function such that $|f'|$ is integrable on $[a,b]$ and quasi p-convex function on $\mathbb{R}$. Then the following inequality holds:*

$$\begin{aligned}
&\left|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{1}{p(b-a)}\left[\left(\frac{3|a| + |b|}{2}\right)^2 \max\left\{|f'(a)|, \left|f'(\frac{a+b}{2})\right|\right\}\right. \\
&\left.+\left(\frac{|a| + 3|b|}{2}\right)^2 \max\left\{|f'(b)|, \left|f'(\frac{a+b}{2})\right|\right\}\right].
\end{aligned}$$

Proof. From Lemma 2.9, triangle inequality and the quasi p-convexity of $|f'|$,

$$\begin{aligned}
&\left|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{1}{p(b-a)}\int_0^1 \left|t^{\frac{1}{p}}\frac{a+b}{2} + ((1-t)^{\frac{1}{p}} - 1)a\right|\left|t^{\frac{1}{p}-1}\frac{a+b}{2} - (1-t)^{\frac{1}{p}-1}a\right|\left|f'(t^{\frac{1}{p}}\frac{a+b}{2} + (1-t)^{\frac{1}{p}}a)\right|dt \\
&+ \frac{1}{p(b-a)}\int_0^1 \left|b(t^{\frac{1}{p}} - 1) + (1-t)^{\frac{1}{p}}\frac{a+b}{2}\right|\left|t^{\frac{1}{p}-1}b - (1-t)^{\frac{1}{p}-1}\frac{a+b}{2}\right|\left|f'(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}\frac{a+b}{2})\right|dt \\
&\leq \frac{1}{p(b-a)}\int_0^1 \left(\frac{|a| + |b|}{2} + |a|\right)^2 \max\left\{|f'(a)|, \left|f'(\frac{a+b}{2})\right|\right\}dt \\
&+ \frac{1}{p(b-a)}\int_0^1 \left(\frac{|a| + |b|}{2} + |b|\right)^2 \max\left\{|f'(b)|, \left|f'(\frac{a+b}{2})\right|\right\}dt \\
&\leq \frac{1}{p(b-a)}\left[\left(\frac{3|a| + |b|}{2}\right)^2 \max\left\{|f'(a)|, \left|f'(\frac{a+b}{2})\right|\right\} + \left(\frac{|a| + 3|b|}{2}\right)^2 \max\left\{|f'(b)|, \left|f'(\frac{a+b}{2})\right|\right\}\right].
\end{aligned}$$

□

Corollary 2.11. *In the theorem above, if the domain of f is restricted to positive real numbers, then the following inequality is obtained*

$$\left|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{p(b-a)}\left[\left(\frac{3a+b}{2}\right)^2\left|f'(\frac{a+b}{2})\right|+\left(\frac{a+3b}{2}\right)^2|f'(b)|\right].$$

Sharper version of Theorem 2.10 can be obtained by finding maximum values of some expressions existing inside the right hand side integral in Lemma 2.9.

Theorem 2.12. *Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable function such that $|f'|$ is integrable on $[a,b]$ and quasi p-convex function on $\mathbb{R}$. Let*

$$g_1(t)=t^{\frac{1}{p}}\frac{a+b}{2}+((1-t)^{\frac{1}{p}}-1)a \ \ and \ \ g_2(t)=b(t^{\frac{1}{p}}-1)+(1-t)^{\frac{1}{p}}\frac{a+b}{2}$$
$$h_1(t)=t^{\frac{1}{p}-1}\frac{a+b}{2}-(1-t)^{\frac{1}{p}-1}a \ \ and \ \ h_2(t)=t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}\frac{a+b}{2}.$$

Then

$$\left|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{p(b-a)}\left(w_1(t)\max\left\{|f'(a)|,\left|f'(\frac{a+b}{2})\right|\right\}\right.$$
$$\left.+w_2(t)\max\left\{|f'(b)|,\left|f'(\frac{a+b}{2})\right|\right\}\right)$$

where

$$\Omega_1(t)=\max\{|g_1(a)|,|g_1(b)|,|g_1(t_1)|\}\max\{|h_1(a)|,|h_1(b)|,|h_1(s_1)|\},$$
$$\Omega_2(t)=\max\{|g_2(a)|,|g_2(b)|,|g_2(t_2)|\}\max\{|h_2(a)|,|h_2(b)|,|h_2(s_2)|\}$$

and for a,b which makes t_1,t_2,s_1,s_2 defined,

$$t_1=\left(1+\left(\frac{a+b}{2a}\right)^{\frac{p}{p-1}}\right)^{-1}, \ \ t_2=\left(1+\left(\frac{a+b}{2b}\right)^{\frac{p}{p-1}}\right)^{-1},$$
$$s_1=\left(1+\left(\frac{a+b}{2a}\right)^{\frac{p}{2p-1}}\right)^{-1}, \ \ s_2=\left(1+\left(\frac{a+b}{2a}\right)^{\frac{p}{2p-1}}\right)^{-1}$$

for a,b which makes any of t_1,t_2,s_1,s_2 undefined, that one will be zero or one.

Proof. When their first derivatives of these functions are investigated, it is seen that $g_1(t)$, $g_2(t)$, $h_1(t)$, $h_2(t)$ with respect to values of a,b are either monotonic functions or unimodal functions on $[0,1]$, the maximum values of $|g_1(t)|$, $|g_2(t)|$, $|h_1(t)|$, $|h_2(t)|$ are attained at either boundary points of $[0,1]$ or extremum points. The extremum points for these functions with respect to values of a,b making the following values defined are

$$t_1=\left(1+\left(\frac{a+b}{2a}\right)^{\frac{p}{p-1}}\right)^{-1}, \ \ t_2=\left(1+\left(\frac{a+b}{2b}\right)^{\frac{p}{p-1}}\right)^{-1},$$

$$s_1 = \left(1 + \left(\frac{a+b}{2a}\right)^{\frac{p}{2p-1}}\right)^{-1} \quad , \quad s_2 = \left(1 + \left(\frac{a+b}{2a}\right)^{\frac{p}{2p-1}}\right)^{-1},$$

respectively.

For the values of a and b that make $\frac{a+b}{2b}$ or $\frac{a+b}{2b}$ negative, these functions will be monotone function. Therefore for $i = 1, 2$

$$|g_i(t)| \leq \max\{|g_i(a)|, |g_i(b)|, |g_i(t_i)|\} \text{ and } |h_i(t)| \leq \max\{|h_i(a)|, |h_i(b)|, |h_i(s_i)|\}.$$

From Lemma 2.9, we have

$$\begin{aligned}
&\left| f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\
&\leq \frac{1}{p(b-a)}\int_0^1 |g_1(t)|\,|h_1(t)| \left| f'(t^{\frac{1}{p}}\frac{a+b}{2} + (1-t)^{\frac{1}{p}}a) \right| dt \\
&+ \frac{1}{p(b-a)}\int_0^1 |g_2(t)|\,|h_2(t)| \left| f'(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}\frac{a+b}{2}) \right| dt \\
&\leq \frac{1}{p(b-a)}\int_0^1 \max\{|g_1(a)|, |g_1(b)|, |g_1(t_1)|\} \max\{|h_1(a)|, |h_1(b)|, |h_1(s_1)|\} \\
&\quad \times \max\left\{|f'(a)|, \left|f'(\frac{a+b}{2})\right|\right\} dt \\
&+ \frac{1}{p(b-a)}\int_0^1 \max\{|g_2(a)|, |g_2(b)|, |g_2(t_2)|\} \max\{|h_2(a)|, |h_2(b)|, |h_2(s_2)|\} \\
&\quad \times \max\left\{|f'(b)|, \left|f'(\frac{a+b}{2})\right|\right\} dt \\
&\leq \frac{1}{p(b-a)}\left(\Omega_1(t)\max\left\{|f'(a)|, \left|f'(\frac{a+b}{2})\right|\right\} + \Omega_2(t)\max\left\{|f'(b)|, \left|f'(\frac{a+b}{2})\right|\right\}\right).
\end{aligned}$$

□

3. Applications

In this section, performing the results, we obtained some inequalities involving special means, digamma function, Fresnel integral for sinus. In addition, numerical integration of a function whose derivative's absolute value is quasi p-convex via composite trapezoid rule is majorized with respect to chosen points in the interval.

Proposition 3.1. *Let $a, b > 0$ where $a < b$. Then the following inequality holds:*

$$\left|[M_{n+1}(a,b)]^{n+1} - [L_{n+2}(a,b)]^{n+1}\right| \leq \frac{3(n+1)}{2(b-a)}(a+b)^2 b^n$$

where the following means are defined in [4, 19], respectively

$$M_s(a,b) = \left(\frac{a^s + b^s}{2}\right)^{\frac{1}{s}} \quad \text{and} \quad L_s(a,b) = \begin{cases} a, & a = b \\ \left(\frac{a^s - b^s}{s(a-b)}\right)^{1/(s-1)}, & a \neq b;\, s \neq 0, 1. \end{cases}$$

Proof. Using Theorem 2.6 for $f(x) = \frac{x^{n+1}}{n+1}$ on $[a, b] \subseteq \mathbb{R}_+$ that is $f'(x) = x^n$ quasi p-convex from (1.1), we have

$$\left| \frac{a^{n+1} + b^{n+1}}{2} - \frac{b^{n+2} - a^{n+2}}{(n+2)(b-a)} \right| \leq \frac{3(n+1)}{2(b-a)} (a+b)^2 b^n.$$

Since $0 < p \leq 1$, the desired result is obtained. □

Proposition 3.2. *Let $a, b > 0$ where $a < b$. Then,*

$$\left| \frac{[M_{n+1}(a,b)]^{n+1} - [L_{n+2}(a,b)]^{n+1}}{3H(a^2, b^2) + \frac{1}{p} B(\frac{1}{p}, \frac{1}{p}) G(a^2, b^2)} \right| \leq \frac{(n+1)b^n}{(b-a)}$$

where $H(x,y) = \frac{x + \sqrt{xy} + y}{3}$ and $G(x,y) = \sqrt{xy}$.

Proof. Using Theorem 2.8 for $f(x) = \frac{x^{n+1}}{n+1}$ on $[a, b] \subset [0, \infty)$, we have the desired result. □

Using some of the results, many inequalities related to special functions can be obtained. We will express only three examples of them in the following propositions.

Proposition 3.3. *Let $x \geq 2$. Then*

$$2^{\frac{-x+2}{p} - 1}(2^{x-1} - 1) - \gamma \leq \Psi(x)$$

where $\Psi(x)$ is digamma function, i.e.,

$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

for $x > 0$ and γ is Euler-Mascheroni constant, that is, $\gamma \approx 0.5772156649...$

Proof. Let $x \geq 2$. Then, $g(x) = x^n$ $(n \in \mathbb{R}_+)$ is quasi p-convex from Theorem 1.1. Let us consider $g(x)$ and $[a, b] = [t, 1]$ where $t \in [0, 1]$ in Theorem 2.1. Then, the following inequality is obtained

$$2^{-\frac{n}{p} - 1}(1+t)^n (1-t) \leq \frac{1 - t^{n+1}}{n+1}.$$

Dividing both side by $1 - t$ and integrating both side on $[0, 1]$ with respect to t, we have

$$2^{-\frac{n}{p} - 1}(2^{n+1} - 1) \leq \int_0^1 \frac{1 - t^{n+1}}{1 - t} dt.$$

By means of the integral representation of digamma function [14], i.e.,

$$\int_0^1 \frac{1 - t^{x-1}}{1 - t} dt - \gamma = \Psi(x)$$

for $x > 0$, one can write

$$2^{-\frac{n}{p} - 1}(2^{n+1} - 1) \leq \Psi(n+2) + \gamma.$$

The substitution $x = n + 2$ yields to desired inequality. □

Proposition 3.4. *Let* $x \in [0, \sqrt{\frac{\pi}{2}}]$. *Then,*

$$2^{-1}\sin(\frac{x^2}{2^{\frac{2}{p}}})x \leq S(x)$$

where $S(x)$ *is the Fresnel integral for sinus, i.e.,*

$$S(x) = \int_0^x \sin(t^2)dt.$$

Proof. Since $f(x) = \sin(x^2)$ is quasi p-convex in $[0, \sqrt{\frac{\pi}{2}}]$ according to Theorem 1.1, taking $f(x) = \sin(x^2)$ in Theorem 2.1, we have the desired result. □

Proposition 3.5. *Let* $0 < x < \frac{1}{2}$. *Then the following inequality holds:*

$$\left|\frac{x\sin(x^2)}{2} - S(x)\right| \leq 3x^3\cos(x^2).$$

Proof. Let $f(x) = \sin(x^2)$ in Theorem 2.6. Then $f'(x) = 2x\cos(x^2)$ and it is quasi p-convex function on $[0, \frac{1}{2}]$ from Theorem 1.1. Let us consider $f(x)$ and $[a, b] = [t, 1]$ where $t \in [0, 1]$ in Theorem 2.1. Then, we get the desired inequality. □

Finally using Theorem 2.6, we can find an upper bound for the error in numerical integration for the functions whose absolute value of first derivatives are quasi p-convex via composite trapezoid rule.

Let f be an integrable function on $[a, b]$ and P be a partition of the interval $[a, b]$, i.e., $P : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and $\Delta x_{i+1} = x_{i+1} - x_i$. Then

$$\int_a^b f(x)dx = \sum_{k=0}^{n-1} \frac{f(x_k) + f(x_{k+1})}{2}\Delta x_{k+1} + E(f, P) \tag{3.1}$$

where $E(f, P)$ is called the error of integral with respect to P. There are some ways to estimate an upper bound for $E(f, P)$. For quasi p-convex functions we suggest the following proposition:

Proposition 3.6. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be differentiable function such that* $|f'|$ *be integrable on* $[a, b]$ *and quasi* p*-convex function on* $\mathbb{R}$. *Suppose that* P *is a partition of* $[a, b]$. *Then the following inequality holds*

$$|E(f, P)| \leq \frac{3}{2p}\sum_{k=0}^{n-1}(|x_k| + |x_{k+1}|)^2 \max\{|f'(x_k)|, |f'(x_{k+1})|\}.$$

Proof. Applying Theorem 2.6 on $[x_k, x_{k+1}]$, we have

$$\left|\frac{f(x_k) + f(x_{k+1})}{2} - \frac{1}{x_{k+1} - x_k}\int_{x_k}^{x_{k+1}} f(x)dx\right| \leq \frac{3}{2p(x_{k+1} - x_k)}(|x_k| + |x_{k+1}|)^2 \max\{|f'(x_k)|, |f'(x_{k+1})|\}. \tag{3.2}$$

Then using (3.1) and (3.2), we get the desired result as follows:

$$
\begin{aligned}
|E(f,P)| &= \left| \sum_{k=0}^{n-1} \frac{f(x_k)+f(x_{k+1})}{2}\Delta x_{k+1} - \int_a^b f(x)dx \right| \\
&= \left| \sum_{k=0}^{n-1} \left(\frac{f(x_k)+f(x_{k+1})}{2}\Delta x_{k+1} - \int_{x_k}^{x_{k+1}} f(x)dx \right) \right| \\
&\le \sum_{k=0}^{n-1} \left| \frac{f(x_k)+f(x_{k+1})}{2}\Delta x_{k+1} - \int_{x_k}^{x_{k+1}} f(x)dx \right| \\
&= \sum_{k=0}^{n-1} \Delta x_{k+1} \left| \frac{f(x_k)+f(x_{k+1})}{2} - \frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}} f(x)dx \right|.
\end{aligned}
$$

□

Corollary 3.7. *If f is restricted to $\mathbb{R}_+$ in proposition above, then the following inequality holds:*

$$|E(f,P)| \le \frac{3}{2p}\sum_{k=0}^{n-1} (x_k + x_{k+1})^2 \left| f'(x_{k+1}) \right|.$$

4. Conclusions

In this article, the Hermite-Hadamard inequality and its generalization for quasi p-convex functions are obtained. In addition, several new inequalities are established for the functions whose first derivative in absolute value is quasi p-convex, which states some bounds for sides of the Hermite-Hadamard inequalities. The applications related to some relations involving special means and some inequalities for special functions including digamma function and Fresnel integral for sinus are presented. In addiditon, an upper bound for error in numerical integration of quasi p-convex functions via composite trapezoid rule is given. In the future, more interesting inequalities regarding special functions can be obtained through different examples of quasi p-convex functions. The introduction of quasi p-convex functions and their properties for n dimensional case are given in [21]. By making use of that study, the existence of similar results can be investigated for multiple integrals.

Conflict of interest

We declare that all the authors have no any conflicts of interest about this submission and publication of this article.

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