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Research article

Some new results in $\mathcal{F}$-metric spaces with applications

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Abstract: In this research article, we give the notion of graphic F-contraction in the setting of $\mathcal{F}$-metric space and establish some fixed point results. We also supply some examples to demonstrate the brilliance of the established results. We also establish some fixed point results for orbitally continuous and orbitally G-continuous graphic F-contractions as applications of our main results. Also, we discuss the existence and the uniqueness of solutions of functional equations involving in dynamic programming.

Keywords: fixed point; $\mathcal{F}$-complete metric space; generalized contraction; graphs; dynamic programming
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1. Introduction

Fixed point theory is one of the most celebrated and conventional theories in mathematics and has comprehensive applications in different fields. In this theory, the first and pioneer result is Banach contraction principle [1] in which the underlying space is the complete metric space. This principle has plenty of generalizations and extensions in different directions (see [2–5]). In 2006, Espinola et al. [6] combined fixed point theory and graph theory and published some useful results. In 2008, Jachymski [7] proposed considering partial order sets as graphs in metric spaces. He obtained novel contraction mappings using this concept, which generalized many of the prior contractions. Moreover, in a metric space endowed with a graph, some results of the fixed points under these contractions were successfully deduced. Several authors have used this contribution in various applications.

Wardowski [8] gave a contemporary kind of contraction by utilizing a certain function is said to be F-contraction and provided some examples to manifest the originality of such generalizations. Wardowski [8] established a fixed point theorem by using the notion of F-contraction and generalized well known Banach contraction principle. Later on, Batra et al. [9] proved fixed point results for

F-contractions on metric spaces endowed with graphs. For more details, we refer the readers to (see [10–23]).

On the other hand, the chief part in fixed point theory is the underlying space. The study of metric space was given by Maurice Fréchet in 1905 which plays an essential role in the fundamental result in fixed point theory. In previous fifty years, many authors have presented attractive generalizations and extensions of metric spaces. Generally, the generalizations of metric spaces are made by taking some alterations in the triangle inequality of the initial definition. Some of these familiar extensions are b-metric space given by Czerwik [24], generalized metric space given by Branciari [25] and JS-metric space given by Jleli et al. [26]. Among these extensions, Jleli et al. [27] introduced an absorbing generalization of a metric space that is named as $\mathcal{F}$-metric space. Fore more details in this direction, we refer the researchers to [26–33]. In this research article, we give the notion of graphic F-contraction in the framework of $\mathcal{F}$-metric space and establish some results. We also supply some examples to demonstrate the originality of the established results. As an application, we discuss the existence and the uniqueness of solutions of functional equations involving in dynamic programming.

2. Preliminaries

Banach contraction principle states that every self-mapping $\mathcal{V}$ defined on complete metric space $(\Re, \tau)$ satisfying

$$\tau(\mathcal{V}\hbar, \mathcal{V}\varsigma) \leq \eth\tau(\hbar, \varsigma)$$

for all $\hbar, \varsigma \in \mathfrak{R}$, where $\eth \in [0, 1)$ has a unique fixed point. Some concepts from graph theory given by Jachymski [7] will be presented here. Let $(\mathfrak{R}, \tau)$ be a metric space and let Δ denote the diagonal of $\mathfrak{R} \times \mathfrak{R}$. Let $G = (V(G), E(G))$ be a directed graph such that the set $V(G)$ of its vertices coincides with $\mathfrak{R}$ and the set $E(G)$ of its edges contains all loops, i.e., $\Delta \subseteq E(G)$. Also assume that the graph G has no parallel edges.

Jachymski [7] introduced the following definition of G-contraction:

Definition 1. *[7] Let $(\mathfrak{R}, \tau)$ be a metric space and $\mathcal{V} : \mathfrak{R} \to \mathfrak{R}$. Then $\mathcal{V}$ is said to be Banach graphic contraction if*

(a) for all $\hbar, \varsigma \in \mathfrak{R}$ with $(\hbar, \varsigma) \in E(G)$, we have $(\mathcal{V}(\hbar), \mathcal{V}(\varsigma)) \in E(G)$,

(b) there exists $\eth \in (0, 1)$ such that, for all $\hbar, \varsigma \in \mathfrak{R}$ with $(\hbar, \varsigma) \in E(G)$, we have

$$\tau(\mathcal{V}(\hbar), \mathcal{V}(\varsigma)) \leq \eth\tau(\hbar, \varsigma). \tag{2.1}$$

G^{-1} is the converse graph of G that is the edge of G^{-1} is established by reversing the direction of edges of G, that is

$$E\left(G^{-1}\right) = \{(\hbar, \varsigma) \in \mathfrak{R} \times \mathfrak{R} : (\varsigma, \hbar) \in E(G)\}.$$

If $\hbar$ and ς are vertices in a graph G, then a path in G from $\hbar$ to ς of length N ($N \in \mathbb{N}$) is a sequence $\{\hbar_i\}_{i=0}^{N}$ of $N + 1$ vertices such that $\hbar_0 = \hbar$, $\hbar_N = \varsigma$ and $(\hbar_{n-1}, \hbar_n) \in E(G)$ for each $i = 1, \cdots, N$. A graph G is connected if there exist a path between any two vertices and it is weakly connected if $\widetilde{G}$ is connected, where $\widetilde{G}$ represents the undirected graph obtained from G by ignoring the direction. Since it is more convenient to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric, under this convention, we have that

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

If G is such that $E(G)$ is symmetric, then for $\hbar \in E(G)$, the symbol $[\hbar]_G$ represents the equivalence class of the relation $\mathfrak{R}$ defined on $V(G)$ by the rule:

$$\varsigma \mathfrak{R} \varkappa \text{ if there is a path in } G \text{ from } \varsigma \text{ to } \varkappa.$$

Let $\Theta = \{G : G \text{ is a directed graph with } V(G) = \mathfrak{R} \text{ and } \Delta \subseteq E(G)\}$. If $\mathcal{V} : \mathfrak{R} \to \mathfrak{R}$, then we represent set of all fixed points of $\mathcal{V}$ by $\mathfrak{R}_{\mathcal{V}}$ and let $\mathfrak{R}_{\mathcal{V}} := \{\hbar \in \mathfrak{R} : (\hbar, \mathcal{V}(\hbar)) \in E(G)\}$.

In 2008, Jachymski [7] gave the following property which is also required in the proof of our result.

(P) for $\{\hbar_n\} \subseteq \mathfrak{R}$, if $\hbar_n \to \hbar$ as $n \to \infty$ and $(\hbar_n, \hbar_{n+1}) \in E(G)$, then there exists a subsequence $\{\hbar_{j_n}\}$ such that $(\hbar_{j_n}, \hbar) \in E(G)$, for all $n \in \mathbb{N}$.

Definition 2. *[34] Let $(\Re, \tau)$ be a metric space and $\mathcal{V} : \Re \to \Re$. Then $\mathcal{V}$ is called a Picard operator if $\mathcal{V}$ has a unique fixed point $\hbar^*$ and $\mathcal{V}^n \hbar \to \hbar^*$, as $n \to \infty$, for all $\hbar \in \mathfrak{R}$.*

Definition 3. *[7] Let $(\Re, \tau)$ be a metric space and $\mathcal{V} : \Re \to \Re$. Then $\mathcal{V}$ is called a weakly Picard operator if for any $\hbar \in \mathfrak{R}$, $\lim_{n\to\infty} \mathcal{V}^n \hbar$ exists and is a fixed point of $\mathcal{V}$.*

Wardowski [8] gave the notion of F-contraction in this way.

Definition 4. *Let $(\Re, \tau)$ be a metric space and $\mathcal{V} : \Re \to \Re$. Then $\mathcal{V}$ is called an F-contraction if there exists $\eth > 0$ such that*

$$\tau(\mathcal{V}\hbar, \mathcal{V}\varsigma) > 0 \Longrightarrow \eth + \Im(\tau(\mathcal{V}\hbar, \mathcal{V}\varsigma)) \leq \Im(\tau(\hbar, \varsigma))$$

for all $\hbar, \varsigma \in \Re$, where $\Im : (0, \infty) \to \mathbb{R}$ is a function satisfying ($\mathcal{F}_1$) $0 < \hbar_1 < \hbar_2 \Rightarrow \xi(\hbar_1) \leq \xi(\hbar_2)$.

($\mathcal{F}_2$) for all $\{\hbar_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \hbar_n = 0$ if and only if $\lim_{n\to\infty} \xi(\hbar_n) = -\infty$.

($\mathcal{F}_3$) there exists $0 < r < 1$ such that $\lim_{\hbar\to 0^+} \hbar^r \Im(\hbar) = 0$.

We represents Ψ, the family of the functions $\Im : \mathbb{R}^+ \to \mathbb{R}$ satisfying ($\mathcal{F}_1$)–($\mathcal{F}_3$).

Jleli et al. [27] gave an impressive extension of a metric space that is famous as $\mathcal{F}$-metric space by considering $\mathcal{F}$ as the set of functions $\xi : (0, +\infty) \to \mathbb{R}$ satisfying ($\mathcal{F}_1$)-($\mathcal{F}_2$).

Definition 5. *[27] Let $\mathfrak{R}$ be nonempty set, and let $\tau : \mathfrak{R} \times \mathfrak{R} \to [0, +\infty)$. Suppose that there exists $(\xi, \ell) \in \mathcal{F} \times [0, +\infty)$ such that*

(D_1) $(\hbar, \varsigma) \in \mathfrak{R} \times \mathfrak{R}$, $\tau(\hbar, \varsigma) = 0$ if and only if $\hbar = \varsigma$.

(D_2) $\tau(\hbar, \varsigma) = \tau(\varsigma, \hbar)$, for all $\hbar, \varsigma \in \mathfrak{R}$.

(D_3) for all $(\hbar, \varsigma) \in \mathfrak{R} \times \mathfrak{R}$, and $(\hbar_i)_{i=1}^N \subset \mathfrak{R}$, with $(\hbar_1, \hbar_N) = (\hbar, \varsigma)$, we have

$$\tau(\hbar, \varsigma) > 0 \Rightarrow \xi(\tau(\hbar, \varsigma)) \leq \xi\Big(\sum_{i=1}^{N-1} \tau(\hbar_i, \hbar_{i+1})\Big) + \ell$$

for all $N \geq 2$. Then $(\mathfrak{R}, \tau)$ is called an $\mathcal{F}$-metric space.

Example 1. *[27] Let $\mathfrak{R} = \mathbb{R}$. Then $\tau : \mathfrak{R} \times \mathfrak{R} \to [0, +\infty)$ defined by*

$$\tau(\hbar, \varsigma) = \begin{cases} (\hbar - \varsigma)^2 & \text{if } (\hbar, \varsigma) \in [0, 4] \times [0, 4], \\ |\hbar - \varsigma| & \text{if } (\hbar, \varsigma) \notin [0, 4] \times [0, 4] \end{cases}$$

with $\xi(\imath) = \ln(\imath)$ and $\ell = \ln(4)$ is an $\mathcal{F}$-metric.

Definition 6. *[27] Let $(\mathfrak{R}, \tau)$ be an $\mathcal{F}$-metric space.*

(i) Let $\{\hbar_n\} \subseteq \mathfrak{R}$. Then $\{\hbar_n\}$ is called an $\mathcal{F}$-convergent to $\hbar \in \mathfrak{R}$ if $\{\hbar_n\}$ is convergent to $\hbar$ in reference to an $\mathcal{F}$-metric τ.

(ii) A sequence $\{\hbar_n\}$ is $\mathcal{F}$-Cauchy, if

$$\lim_{n,m\to\infty} \tau(\hbar_n, \hbar_m) = 0.$$

Faraj et al. [35] defined the notion of F-G-contraction in this way.

Definition 7. *[35] Let $(\Re, \tau)$ be an $\mathcal{F}$-metric space equipped with a graph G. A mapping $\mathcal{V} : \Re \to \Re$ is said to be an F-G-contraction if for every $\hbar, \varsigma \in \Re$, the following two conditions*

(i)

$$(\hbar, \varsigma) \in E(G) \text{ implies } (\mathcal{V}\hbar, \mathcal{V}\varsigma) \in E(G); \tag{2.2}$$

(ii) there exists $\eth \in [0, 1)$ such that

$$(\hbar, \varsigma) \in E(G) \text{ implies } \tau(\mathcal{V}\hbar, \mathcal{V}\varsigma) \leq \eth\tau(\hbar, \varsigma) \tag{2.3}$$

are satisfied.

Theorem 1. *[35] Let $(\Re, \tau)$ be an $\mathcal{F}$-complete $\mathcal{F}$-metric space equipped with a graph G and let $\mathcal{V} : \Re \to \Re$ a self-mapping satisfy (2.2) and (2.3). Then $\mathcal{V}$ is a Picard operator.*

3. Main results

In the whole section, we suppose that $\Re$ is an $\mathcal{F}$-metric space with an $\mathcal{F}$-metric τ and $G = \{G : G$ is a directed graph with $V(G) = \Re$ and $\Delta \subseteq E(G)\}$. The set of all fixed points of $\mathcal{V} : \Re \to \Re$ will be denoted by $Fix(\mathcal{V})$. Now we introduce the notion of graphic F-contraction in this way.

Definition 8. *Let $(\Re, \tau)$ be an $\mathcal{F}$-metric space equipped with a graph G. A mapping $\mathcal{V} : \Re \to \Re$ is said to be graphic F-contraction if for every $\hbar, \varsigma \in \Re$, the following two conditions*

(i)

$$(\hbar, \varsigma) \in E(G) \text{ implies } (\mathcal{V}\hbar, \mathcal{V}\varsigma) \in E(G), \tag{3.1}$$

(ii) there exists $\eth > 0$ such that

$$(\hbar, \varsigma) \in E(G) \text{ implies } \eth + \Im\left(\tau(\mathcal{V}\hbar, \mathcal{V}\varsigma)\right) \leq \Im\left(\tau(\hbar, \varsigma)\right) \tag{3.2}$$

with $\tau(\mathcal{V}\hbar, \mathcal{V}\varsigma) \geq 0$, are satisfied.

Example 2. *Let $\Re \neq \emptyset$ and $(\Re, \tau)$ be an $\mathcal{F}$-metric space. Then for any $\Im \in \Psi$ and $G \in \Theta$, a constant function $\mathcal{V} : \Re \to \Re$ is graphic F-contraction because in this way $G = (\Re, E(G))$.*

Example 3. *Suppose $\Im \in \Psi$ be an arbitrary function. Then each F-contraction is a graphic F-contraction for the complete graph G_0 defined by $V(G_0) = \Re$ and $E(G_0) = \Re \times \Re$.*

Example 4. *Let $G \in \Theta$. Then each graphic contraction is graphic F-contraction for $\Im : \mathbb{R}^+ \to \mathbb{R}$ given by $\Im(\iota) = \ln \iota$, for $\iota > 0$.*

Example 5. *Define the sequence $\{\mu_n\}$ as follows:*

$\mu_1 = \ln(1)$
$\mu_2 = \ln(1+4)$
.
.
.
$\mu_n = \ln(1+4+7+...+(3n-2)) = \ln\left(\frac{n(3n-1)}{2}\right)$,
for all $n \in \mathbb{N}$. Let $\Re = \{\mu_n : n \in \mathbb{N}\}$ endowed with

$$\tau(\hbar, \varsigma) = \begin{cases} e^{|\hbar-\varsigma|}, \text{ if } \hbar \neq \varsigma, \\ 0, \text{ if } \hbar = \varsigma. \end{cases}$$

with $\xi(\iota) = \frac{-1}{\iota}$ and $a = 1$. Then, $(\Re, \tau)$ is an $\mathcal{F}$-complete $\mathcal{F}$-metric space equipped with the graph G given by $V(G) = \Re$ and

$$E(G) = \{(\mu_n, \mu_n) : n \in \mathbb{N}\} \cup \{(\mu_1, \mu_n) : n \in \mathbb{N}\}.$$

Define $\mathcal{V} : \Re \to \Re$ by

$$\mathcal{V}(\mu_n) = \begin{cases} \mu_1, & \text{if } n = 1, \\ \mu_{n-1}, & \text{if } n > 1. \end{cases}$$

Then it is simple to prove that $\mathcal{V}$ preserves edges. We prove that $\mathcal{V}$ satisfies Eq (3.2). Clearly $(\hbar, \varsigma) \in E(G)$ with $\mathcal{V}\hbar \neq \mathcal{V}\varsigma$ if and only if $\hbar = \mu_1$ and $\varsigma = \mu_n$ for some $n > 2$.

Let the mapping $\Im : (0, \infty) \to \mathbb{R}$ defined by

$$\Im(\iota) = \ln \iota + \iota, \ \iota > 0.$$

It is simple to show that $\Im \in \Psi$. Now for $n \in \mathbb{N}$, $n > 2$ and $\eth > 0$, we have

$$\begin{aligned} \tau(\mathcal{V}(\mu_1), \mathcal{V}(\mu_n)) \neq 0 &\Longrightarrow \frac{\tau(\mathcal{V}(\mu_1), \mathcal{V}(\mu_n))}{\tau(\mu_1, \mu_n)} e^{\tau(\mathcal{V}(\mu_1),\mathcal{V}(\mu_n))-\tau(\mu_1,\mu_n)} \leq e^{-\eth} \\ & \frac{\tau(\mu_1, \mu_{n-1})}{\tau(\mu_1, \mu_n)} e^{\tau(\mu_1,\mu_{n-1})-\tau(\mu_1,\mu_n)} \\ &= \frac{e^{\mu_{n-1}-\mu_1}}{e^{\mu_n-\mu_1}} e^{e^{\mu_{n-1}-\mu_1}-e^{\mu_n-\mu_1}} \\ &= \frac{(n-1)(3n-4)}{n(3n-1)} e^{-6n+4} < e^{-1}. \end{aligned}$$

Thus, the inequality (3.2) is satisfied with $\eth = 1 > 0$. Thus $\mathcal{V} : \Re \to \Re$ is a graphic F-contraction. Now as

$$\lim_{n \longrightarrow \infty} \frac{\tau(\mathcal{V}(\mu_1), \mathcal{V}(\mu_n))}{\tau(\mu_1, \mu_n)} = 1$$

so $\mathcal{V}$ is not a graphic contraction.

Definition 9. *Let $(\Re, \tau)$ be an $\mathcal{F}$-metric space. Then any two sequences $\{\hbar_n\}$ and $\{\varsigma_n\}$ are equivalent if $\tau(\hbar_n, \varsigma_n) \to 0$ as $n \to \infty$.*

Proposition 1. *Let $(\Re, \tau)$ be an $\mathcal{F}$-metric space and $\mathcal{V} : \Re \to \Re$ be a graphic F-contraction. Then $\mathcal{V} : \Re \to \Re$ is graphic F-contraction for both G^{-1} and $\widetilde{G}$, it means (3.1) and (3.2) holds for G^{-1} and $\widetilde{G}$.*

Proof. As $\mathcal{F}$-metric is symmetric, so $\mathcal{V} : \Re \to \Re$ is also graphic F-contraction for both G^{-1} and $\widetilde{G}$. □

Theorem 2. *Let $(\Re, \tau)$ be an $\mathcal{F}$-metric space. Then following conditions are equivalent:*
(i) G is weakly connected,
(ii) for any graphic F-contraction $\mathcal{V} : \Re \to \Re$ and $\hbar, \varsigma \in \Re$, $\{\hbar_n\}$ and $\{\varsigma_n\}$ are equivalent and Cauchy,
(iii) for any graphic F-contraction $\mathcal{V} : \Re \to \Re$, Card(Fix$\mathcal{V}$)$\leq 1$.

Proof. (i)$\Longrightarrow$(ii)
Suppose G is weakly connected $\mathcal{V} : \Re \to \Re$ is graphic F-contraction and $\hbar, \varsigma \in \Re$. Then $\Re = [\hbar]_{\widetilde{G}}$. Take $\varsigma = \mathcal{V}\hbar \in [\hbar]_{\widetilde{G}}$, so there exists a path $\{\hbar_i\}_{i=0}^{N}$ in $\widetilde{G}$ from $\hbar$ to ς with $\hbar_0 = \hbar$ and $\hbar_N = \varsigma$ and $(\hbar_{i-1}, \hbar_i) \in E(\widetilde{G})$ for all $i = 1, 2, ..., N$. If $\mathcal{V}^{j+1}\hbar = \mathcal{V}^j\hbar$, for some $j \in \mathbb{N}$, then the sequence $\{\mathcal{V}^n\hbar\}$ becomes constant sequence and hence it is Cauchy. So assume that $\tau_n = \tau\left(\mathcal{V}^n\hbar, \mathcal{V}^{n+1}\hbar\right) > 0$, for all $n \in \mathbb{N}$. From Proposition 1, $\mathcal{V} : \Re \to \Re$ is graphic F-contraction for $\widetilde{G}$, then we have

$$(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i) \in E(\widetilde{G})$$

consequently

$$\Im\left(\tau\left(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i\right)\right) \leq \Im\left(\tau\left(\mathcal{V}^{n-1}\hbar_{i-1}, \mathcal{V}^{n-1}\hbar_i\right)\right) - \eth$$

for all $n \in \mathbb{N}$ and $i = 1, 2, ..., N$. Thus continuing in this way, we get

$$\Im\left(\tau\left(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i\right)\right) \leq \Im\left(\tau\left(\hbar_{i-1}, \hbar_i\right)\right) - n\eth \tag{3.3}$$

for all $n \in \mathbb{N}$ and $i = 1, 2, ..., N$. Thus

$$\lim_{n\to\infty} \Im\left(\tau\left(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i\right)\right) = -\infty$$

which implies that

$$\lim_{n\to\infty} \tau\left(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i\right) = 0.$$

From the condition $(\mathcal{F}_3)$, there exists $0 < r_i < 1$ such that

$$\lim_{n\to\infty} [\tau\left(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i\right)]^{r_i} \Im\left(\tau\left(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i\right)\right) = 0. \tag{3.4}$$

From (3.3) and (3.4), we have

$$\begin{aligned}
& [\tau\left(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i\right)]^{r_i} \Im\left(\tau\left(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i\right)\right) - [\tau\left(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i\right)]^{r_i} \Im\left(\tau\left(\hbar_{i-1}, \hbar_i\right)\right) \\
\leq\ & [\tau\left(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i\right)]^{r_i} [\Im\left(\tau\left(\hbar_{i-1}, \hbar_i\right)\right) - n\eth] - [\tau\left(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i\right)]^{r_i} \Im\left(\tau\left(\hbar_{i-1}, \hbar_i\right)\right) \\
\leq\ & -n\tau[(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i)]^{r_i} \leq 0.
\end{aligned}$$

Taking $n \to \infty$, we have

$$\lim_{n\to\infty} n\tau\left(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i\right)^{r_i} = 0.$$

So there exists m_i (a positive integer) such that

$$n\tau\left(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i\right)^{r_i} < 1$$

for all $n \geq m_i$, or

$$\tau\left(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i\right) < \frac{1}{n^{\frac{1}{r_i}}}, \tag{3.5}$$

$$\sum_{i=1}^{m_i} \tau\left(\mathcal{V}^i\hbar_{i-1}, \mathcal{V}^i\hbar_i\right) \leq \sum_{i=1}^{m_i} \frac{1}{i^{\frac{1}{r_i}}}$$

for all $n \geq m_i$. Now let $(\xi, \ell) \in \mathcal{F} \times [0, +\infty)$ be such that (D_3) is satisfied and $\epsilon > 0$ be fixed. From $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < \imath < \delta \text{ implies } \xi(\imath) < \xi(\epsilon) - \ell. \tag{3.6}$$

Now since

$$0 < \sum_{i=1}^{m_i} \frac{1}{i^{\frac{1}{r_i}}} < \sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{r_i}}} < \delta,$$

for $n > m_i$. Hence, by (3.6) and $(\mathcal{F}_1)$, we get

$$\xi\left(\sum_{i=1}^{m_i} \tau\left(\mathcal{V}^i\hbar_{i-1}, \mathcal{V}^i\hbar_i\right)\right) \leq \xi\left(\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{r_i}}}\right) < \xi(\epsilon) - a \tag{3.7}$$

for $n > m_i$. Using (D_3) and (3.7), we get

$$\tau(\mathcal{V}^n\hbar, \mathcal{V}^n\varsigma) > 0,\ n > m_i \Longrightarrow \xi\left(\tau(\mathcal{V}^n\hbar, \mathcal{V}^n\varsigma)\right) \leq \xi\left(\sum_{i=1}^{m_i} \tau\left(\mathcal{V}^i\hbar_{i-1}, \mathcal{V}^i\hbar_i\right)\right) + a < \xi(\epsilon)$$

which, from $(\mathcal{F}_1)$, gives that

$$\tau(\mathcal{V}^n\hbar, \mathcal{V}^n\varsigma) < \epsilon$$

for $n > m_i$. Thus $\tau(\mathcal{V}^n\hbar, \mathcal{V}^n\varsigma) \to 0$ as $n \to \infty$ and hence $\{\mathcal{V}^n\hbar\}$ is Cauchy Sequence. □

Now, we show that (ii) $\Longrightarrow$ (iii). Let $\mathcal{V} : \Re \to \Re$ be a graphic F-contraction and $\hbar, \varsigma \in Fix\mathcal{V}$. By (ii), $\{\hbar_n\}$ and $\{\varsigma_n\}$ are equivalent. Then we obtain

$$\tau(\hbar, \varsigma) = \tau(\mathcal{V}^n\hbar, \mathcal{V}^n\varsigma) \to 0$$

as $n \to \infty$, i.e, $\hbar = \varsigma$.

In the end, we show that (iii) $\Longrightarrow$ (i). We suppose on the contrary that G is not weakly connected, i.e, $\widetilde{G}$ is disconnected. Assume that there exists $\hbar_0 \in \Re$ such that both sets $[\hbar_0]_{\widetilde{G}} \neq \emptyset$ and $\Re - [\hbar_0]_{\widetilde{G}} \neq \emptyset$. Suppose $\varsigma_0 \in \Re - [\hbar_0]_{\widetilde{G}}$ and define

$$\mathcal{V}\hbar = \hbar_0 \text{ if } \hbar \in [\hbar_0]_{\widetilde{G}},\ \mathcal{V}\hbar = \varsigma_0 \text{ if } \hbar \in \Re - [\hbar_0]_{\widetilde{G}}.$$

Therefore, $Fix(\mathcal{V}) = \{\hbar_0, \varsigma_0\}$. Now, we prove that $\mathcal{V}$ is graphic F-contraction. Assume $(\hbar, \varsigma) \in E(G)$, so $[\hbar]_{\widetilde{G}} = [\varsigma]_{\widetilde{G}}$, i.e., $\hbar, \varsigma \in [\hbar_0]_{\widetilde{G}}$ or $\hbar, \varsigma \in \Re - [\hbar_0]_{\widetilde{G}}$. Then, we have $\mathcal{V}\hbar = \mathcal{V}\varsigma$, so $(\mathcal{V}\hbar, \mathcal{V}\varsigma) \in E(G)$ as $\triangle \subset E(G)$. Thus Eq (3.1) holds. Also as there is no $(\hbar, \varsigma) \in E(G)$ with $\mathcal{V}\hbar \neq \mathcal{V}\varsigma$, therefore, inequality (3.2) is satisfied. Thus $\mathcal{V}$ is graphic F-contraction having two fixed points that contravenes (iii). Thus G is necessarily weakly connected.

Corollary 1. *Let $(\Re,\tau)$ be an $\mathcal{F}$-complete $\mathcal{F}$-metric space equipped with a weakly connected graph G. Then for any graphic F-contraction $\mathcal{V}:\Re\to\Re$, there exists $\hbar^*\in\Re$ such that* $\lim_{n\to\infty}\mathcal{V}^n\hbar=\hbar^*$ *for all* $\hbar\in\Re$.

Proof. Let $\mathcal{V}:\Re\to\Re$ is a graphic F-contraction and and fix any point $\hbar\in\Re$. Let $m,n\in\mathbb{N}$ with $m>n\geq 0$. As G is a weakly connected, so by Theorem 2, $\{\mathcal{V}^n\hbar\}$ and $\{\mathcal{V}^n\mathcal{V}^{m-n}\hbar\}$ are equvailent. Then

$$\lim_{m,n\to\infty}\tau(\mathcal{V}^n\hbar,\mathcal{V}^m\hbar)=0$$

that is $\{\mathcal{V}^n\hbar\}$ is $\mathcal{F}$-Cauchy sequence in $\Re$. Thus there exists $\hbar^*\in\Re$ such that $\lim_{n\to\infty}\mathcal{V}^n\hbar=\hbar^*$ as $n\to\infty$. Assume that $\varsigma\in\Re$, then by Theorem 2, the sequences $\{\mathcal{V}^n\hbar\}$ and $\{\mathcal{V}^n\varsigma\}$ are equivalent. Now by (D_3), we obtain

$$\xi\left(\tau(\mathcal{V}^n\varsigma,\hbar^*)\right)\leq\xi\left(\tau\left(\mathcal{V}^n\hbar,\mathcal{V}^n\varsigma\right)+\tau\left(\mathcal{V}^n\hbar,\hbar^*\right)\right)+a$$

for all $n\in\mathbb{N}$. As $\tau\left(\mathcal{V}^n\hbar,\mathcal{V}^n\varsigma\right)+\tau\left(\mathcal{V}^n\hbar,\hbar^*\right)\to 0$ as $n\to\infty$, so

$$\lim_{n\to\infty}\xi\left(\tau\left(\mathcal{V}^n\hbar,\mathcal{V}^n\varsigma\right)+\tau\left(\mathcal{V}^n\hbar,\hbar^*\right)\right)+a=-\infty.$$

Then $\tau(\mathcal{V}^n\varsigma,\hbar^*)\to 0$ as $n\to\infty$. □

Theorem 3. *Let $(\Re,\tau)$ be an $\mathcal{F}$-metric space equipped with a graph G and $\mathcal{V}:\Re\to\Re$ be a graphic F-contraction. Then $[\hbar_0]_{\widetilde{G}}$ is $\mathcal{V}$-invariant and $\mathcal{V}|_{[\hbar_0]_{\widetilde{G}}}$ is graphic F-contraction for $\widetilde{G}_{\hbar_0}$ where $\hbar_0\in\Re$ and $\mathcal{V}(\hbar,)\in[\hbar_0]_{\widetilde{G}}$. Furthermore, if $\hbar,\varsigma\in[\hbar_0]_{\widetilde{G}}$, then $\{\mathcal{V}^n\hbar\}$ and $\{\mathcal{V}^n\varsigma\}$ are equivalent and Cauchy.*

Proof. Let $\hbar\in[\hbar_0]_{\widetilde{G}}$, so there exists a path $\{\upsilon_i\}_{i=0}^N$ in $\widetilde{G}$ from $\hbar$ to $\hbar_0$ with $\hbar=\upsilon_0$ and $\hbar_0=\upsilon_N$ and $(\upsilon_{i-1},\upsilon_i)\in E(\widetilde{G})$. Since $\mathcal{V}:\Re\to\Re$ is graphic F-contraction for graph G, so for all $i\in\mathbb{N}$, we get $(\mathcal{V}\upsilon_{i-1},\mathcal{V}\upsilon_i)\in E(G)$. Then $\mathcal{V}\hbar\in[\mathcal{V}\hbar_0]_{\widetilde{G}}=[\hbar_0]_{\widetilde{G}}$, i.e. $[\hbar_0]_{\widetilde{G}}$ is $\mathcal{V}$-invariant. Now, let $(\hbar,\varsigma)\in E(\widetilde{G}_{\hbar_0})$. As $\mathcal{V}:\Re\to\Re$ is graphic F-contraction, so $(\mathcal{V}\hbar,\mathcal{V}\varsigma)\in E(G)$. Since, $[\hbar_0]_{\widetilde{G}}$ is $\mathcal{V}$-invariant, then $(\mathcal{V}\hbar,\mathcal{V}\varsigma)\in E(\widetilde{G}_{\hbar_0})$. As $\widetilde{G}_{\hbar_0}$ is a subgraph of G, we get $\mathcal{V}|_{[\hbar_0]_{\widetilde{G}}}$ is graphic F-contraction for $\widetilde{G}_{\hbar_0}$. Eventually, from the connectedness of $\widetilde{G}_{\hbar_0}$ and Theorem 2, $\{\mathcal{V}^n\hbar\}$ and $\{\mathcal{V}^n\varsigma\}$ are equivalent and Cauchy. □

Theorem 4. *Let $(\Re,\tau)$ be an $\mathcal{F}$-complete $\mathcal{F}$-metric space equipped with a graph G satisfying the property (P), $\mathcal{V}:\Re\to\Re$ is a graphic F-contraction and the set*

$$\Re_{\mathcal{V}}=\{\hbar\in\Re:(\hbar,\mathcal{V}\hbar)\in E(G)\}$$

is nonempty. Then

(i) Card(Fix$\mathcal{V}$)=Card$\left\{[\hbar]_{\widetilde{G}}:\hbar\in\Re_{\mathcal{V}}\right\}$,
(ii) Fix$\mathcal{V}\neq\emptyset$ if and only if $\Re_{\mathcal{V}}\neq\emptyset$,
(iii) $\mathcal{V}$ possess a unique fixed point if and only if there exists $\hbar_0\in\Re_{\mathcal{V}}$ such that $\Re_{\mathcal{V}}\subseteq[\hbar_0]_{\widetilde{G}}$,
(iv) $\mathcal{V}|_{[\hbar]_{\widetilde{G}}}$ is Picard Operator, for any $\hbar\in\Re_{\mathcal{V}}$,
(v) if G is weakly connected and $\Re_{\mathcal{V}}\neq\emptyset$, then $\mathcal{V}$ is Picard Operator,
(vi) if $\Re'=\cup\left\{[\hbar]_{\widetilde{G}}:\hbar\in\Re_{\mathcal{V}}\right\}$, then $\mathcal{V}|_{\Re'}$ is weakly Picard Operator,
(vii) if $\mathcal{V}\subseteq E(G)$, then $\mathcal{V}$ is weakly Picard Operator.

Proof. First we prove (iv) that is $\mathcal{V}|_{[\hbar_0]_{\widetilde{G}}}$ is Picard Operator, for any $\hbar \in \Re_{\mathcal{V}}$. Let $\hbar \in \Re_{\mathcal{V}}$, then $(\hbar, \mathcal{V}\hbar) \in E(G)$ which implies that $\mathcal{V}\hbar \in [\hbar]_{\widetilde{G}}$. Then the sequence $\{\mathcal{V}^n\hbar\}$ and $\{\mathcal{V}^n\varsigma\}$, for $\varsigma \in \Re$ are equivalent and Cauchy by Theorem 3. Now as $(\Re, \tau)$ be an $\mathcal{F}$-complete, so there exists $\hbar^* \in \Re$ such that $\mathcal{V}^n\hbar \to \hbar^* \longleftarrow \mathcal{V}^n\varsigma$ as $n \to \infty$. As $(\hbar, \mathcal{V}\hbar) \in E(G)$, so from (3.1), we have

$$(\mathcal{V}^n\hbar, \mathcal{V}^{n+1}\hbar) \in E(G) \tag{3.8}$$

for all $n \in \mathbb{N}$. As G satisfies the property (P), so there exists a subsequence $\{\mathcal{V}^{j_n}\hbar\}$ of $\{\mathcal{V}^n\hbar\}$ such that

$$(\mathcal{V}^{j_n}\hbar, \hbar^*) \in E(G)$$

for all $n \in \mathbb{N}$. Now from (3.8), there exists a path $\left(\hbar, \mathcal{V}\hbar, \mathcal{V}^2\hbar, ..., \mathcal{V}^{j_1}\hbar, \hbar^*\right)$ in G from $\hbar$ to $\hbar^*$. So $\hbar^* \in [\hbar]_{\widetilde{G}}$. Now

$$\tau\left(\mathcal{V}^{j_n+1}\hbar, \mathcal{V}\hbar^*\right) \leq \tau\left(\mathcal{V}^{j_n}\hbar, \hbar^*\right)$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$, we have

$$\tau(\hbar^*, \mathcal{V}\hbar^*) = 0$$

which implies $\hbar^* = \mathcal{V}\hbar^*$. Thus $\mathcal{V}|_{[\hbar]_{\widetilde{G}}}$ is Picard Operator. □

Now we prove the condition (v). Let $\Re_{\mathcal{V}} \neq \emptyset$ and $\hbar \in \Re_{\mathcal{V}}$. Also suppose that G is weakly connected. Then $\Re = [\hbar]_{\widetilde{G}}$ and $\mathcal{V}$ is Picard Operator. Condition (vi) is direct consequence of (iv).

Now we prove the condition (vii). Let $\mathcal{V} \subseteq E(G)$. This implies $\Re = \Re_{\mathcal{V}}$ which gives $\Re' = \Re$. Thus $\mathcal{V}$ is weakly Picard Operator directly from the condition (iv).

Now we prove the condition (i). For this, we define $\wp : Fix\mathcal{V} \to Ⓢ$ by $\wp(\hbar) = [\hbar]_{\widetilde{G}}$, for all $\hbar \in Fix\mathcal{V}$, where

$$Ⓢ = \left\{[\hbar]_{\widetilde{G}} : \hbar \in \Re_{\mathcal{V}}\right\}.$$

Then we just have to prove that $\wp$ is bijective mapping. Now let $\hbar \in \Re_{\mathcal{V}}$. Then from (iv), $\mathcal{V}|_{[\hbar]_{\widetilde{G}}}$ is Picard Operator. Now let

$$\hbar^* = \lim_{n\to\infty} \mathcal{V}^n\hbar.$$

Then

$$\hbar^* \in Fix\mathcal{V} \cap [\hbar]_{\widetilde{G}}$$

and $\wp(\hbar^*) = [\hbar^*]_{\widetilde{G}} = [\hbar]_{\widetilde{G}}$. Hence the function $\wp : Fix\mathcal{V} \to Ⓢ$ is onto. Also suppose $\hbar_1, \hbar_2 \in Fix\mathcal{V}$ with $[\hbar_1]_{\widetilde{G}} = [\hbar_2]_{\widetilde{G}}$. Then $\hbar_2 \in [\hbar_1]_{\widetilde{G}}$. Now from (iv),

$$\lim_{n\to\infty} \mathcal{V}^n\hbar_2 \in Fix\mathcal{V} \cap [\hbar_1]_{\widetilde{G}} = \{\hbar_1\}.$$

But $\mathcal{V}^n\hbar_2 = \hbar_2$, for all $n \in \mathbb{N}$. Hence we have $\hbar_1 = \hbar_2$, which shows that the function $\wp : Fix\mathcal{V} \to Ⓢ$ is one-to-one. Thus the $\wp : Fix\mathcal{V} \to Ⓢ$ is bijective function. Lastly, (i) directly follows from (ii) and (iii).

Corollary 2. *Let $(\Re, \tau)$ be an $\mathcal{F}$-complete $\mathcal{F}$-metric space equipped with a graph G satisfying the property (P). Then these conditions are equivalent*

(i) G is weakly connected,

(ii) $\mathcal{V}$ is Picard Operator for every graphic F-contraction $\mathcal{V} : \Re \to \Re$ such that $(\hbar_0, \mathcal{V}\hbar_0) \in E(G)$ for some $\hbar_0 \in \Re$

(iii) for any graphic F-contraction $\mathcal{V} : \Re \to \Re$, Card($Fix\mathcal{V}$)$\leq 1$.

Proof. From condition (v) of Theorem (4), (i)$\Longrightarrow$ (ii). Now we prove (ii)$\Longrightarrow$ (iii). Suppose $\mathcal{V}:\Re\to\Re$ is graphic F-contraction. If $\Re_{\mathcal{V}}=\emptyset$, then it is obvious that Card($Fix\mathcal{V}$)≤ 1. Now if $\Re_{\mathcal{V}}\neq\emptyset$, then by (ii) $Fix\mathcal{V}$ is singleton. Thus Card($Fix\mathcal{V}$)≤ 1. Lastly, (iii)$\Longrightarrow$(i) follows directly from Theorem 4. □

Definition 10. *[35] Let $(\Re,\tau)$ be an $\mathcal{F}$-metric space equipped with a graph G and $\mathcal{V}:\Re\to\Re$. Then $\mathcal{V}$ is said to be orbitally continuous if $\forall\hbar,\varsigma\in\Re$ and any sequence $\{j_n\}$ of positive numbers, then $\mathcal{V}^{j_n}\hbar\to\varsigma$ implies $\mathcal{V}(\mathcal{V}^{j_n}\hbar)\to\mathcal{V}\varsigma$ as $n\to\infty$.*

Definition 11. *[35] Let $(\Re,\tau)$ be an $\mathcal{F}$-metric space equipped with a graph G and $\mathcal{V}:\Re\to\Re$. Then $\mathcal{V}$ is said to be G-continuous if for $\hbar\in\Re$ and a sequence $\{\hbar_n\}$ with $\hbar_n\to\hbar$ as $n\to\infty$ and $(\hbar_n,\hbar_{n+1})\in E(G)$, then $\mathcal{V}\hbar_n\to\mathcal{V}\hbar$ as $n\to\infty$.*

Definition 12. *[35] Let $(\Re,\tau)$ be an $\mathcal{F}$-metric space equipped with a graph G and $\mathcal{V}:\Re\to\Re$. Then $\mathcal{V}$ is said to be orbitally G- continuous if for all $\hbar,\varsigma\in\Re$ and any sequence $\{j_n\}$ of positive numbers $\mathcal{V}^{j_n}\hbar\to\varsigma$ and $(\mathcal{V}^{j_n}\hbar,\mathcal{V}^{j_n+1}\hbar)\in E(G)$ implies $\mathcal{V}(\mathcal{V}^{j_n}\hbar)\to\mathcal{V}\varsigma$ as $n\to\infty$.*

Remark 1. *[35] Let $(\Re,\tau)$ be an $\mathcal{F}$-metric space endowed with a graph G and $\mathcal{V}:\Re\to\Re$. Clearly, we have the following relations.*

Continuity $\Longrightarrow$ G-continuity $\Longrightarrow$ orbital G-continuity,
Continuity $\Longrightarrow$ orbital continuity $\Longrightarrow$ orbital G-Continuity.

Theorem 5. *Let $(\Re,\tau)$ be an $\mathcal{F}$-complete $\mathcal{F}$-metric space equipped with a graph G and $\mathcal{V}:\Re\to\Re$ is orbitally G-continuous graphic F-contraction. Then these conditions hold:*

(i) $\Re_{\mathcal{V}}\neq\emptyset\Longleftrightarrow Fix(\mathcal{V})\neq\emptyset$,

(ii) for any $\hbar\in\Re_{\mathcal{V}}$ and $\varsigma\in[\hbar]_{\widetilde{G}}$, the sequence $\{\mathcal{V}^n\varsigma\}$ converges to the fixed point of $\mathcal{V}$ and $\lim_{n\to\infty}\mathcal{V}^n\varsigma$ does not depend on ς,

(iii) if G is weakly connected and $\Re_{\mathcal{V}}\neq\emptyset$, then $\mathcal{V}$ is Picard Operator,

(iv) if $\mathcal{V}\subseteq E(G)$, then $\mathcal{V}$ is weakly Picard Operator.

Proof. We prove (i)$\Longrightarrow$ (ii). Let $\hbar\in\Re$ with $(\hbar,\mathcal{V}\hbar)\in E(G)$ and $\varsigma\in[\hbar]_{\widetilde{G}}$. From Theorem 3, $\{\mathcal{V}^n\hbar\}$ and $\{\mathcal{V}^n\varsigma\}$ converges to a point $\hbar^*$. Also $\left(\mathcal{V}^n\hbar,\mathcal{V}^{n+1}\hbar\right)\in E(G)$, $\forall n\in\mathbb{N}$. Now using the orbitally G-continuity of $\mathcal{V}$ we have

$$\mathcal{V}(\mathcal{V}^n\hbar)\to\mathcal{V}(\hbar^*).$$

Also since $\mathcal{V}(\mathcal{V}^n\hbar)=\mathcal{V}^{n+1}\hbar\to\hbar^*$, thus we obtain $\mathcal{V}(\hbar^*)=\hbar^*$. Thus (i) implies (ii). Now since $\Delta\subseteq E(G)$, so (ii) implies (i). Also since $\mathcal{V}\subseteq E(G)$ so $\Re_{\mathcal{V}}=\Re$, so (iv) follows directly from (ii). Now if $\Re_{\mathcal{V}}\neq\emptyset$ and $\hbar_0\in\Re_{\mathcal{V}}$, then $[\hbar_0]_{\widetilde{G}}=\Re$, so $\mathcal{V}$ is a Picard Operator by (ii) which proved (iii). □

The following result can be a generalization of above result in the sense of above remark.

Theorem 6. *Let $(\Re,\tau)$ be an $\mathcal{F}$-complete $\mathcal{F}$-metric space equipped with a graph G and $\mathcal{V}:\Re\to\Re$ is orbitally continuous graphic F-contraction. Then these conditions hold:*

(i) $Fix(\mathcal{V})\neq\emptyset$ if and only if there exists $\hbar_0\in\Re$ with $\mathcal{V}\hbar_0\in[\hbar_0]_{\widetilde{G}}$,

(ii) for any $\hbar\in\Re_{\mathcal{V}}$ and $\varsigma\in[\hbar]_{\widetilde{G}}$, the sequence $\{\mathcal{V}^n\varsigma\}$ converges to the fixed point of $\mathcal{V}$ and $\lim_{n\to\infty}\mathcal{V}^n\varsigma$ does not depend on ς,

(iii) if G is weakly connected and $\Re_{\mathcal{V}}\neq\emptyset$, then $\mathcal{V}$ is Picard Operator,

(iv) $\mathcal{V}$ is weakly Picard Operator if $\mathcal{V}(\hbar)\in[\hbar]_{\widetilde{G}}$ for any $\hbar\in\Re$.

Proof. We prove (i)$\Longrightarrow$ (ii). Let $\hbar \in \Re$ such that $\mathcal{V}(\hbar) \in [\hbar]_{\widetilde{G}}$ and $\varsigma \in [\hbar]_{\widetilde{G}}$. From Theorem 3, $\{\mathcal{V}^n\hbar\}$ and $\{\mathcal{V}^n\varsigma\}$ converges to a point $\hbar^*$. Now using the orbitally continuity of $\mathcal{V}$ we have

$$\mathcal{V}(\mathcal{V}^n\hbar) \to \mathcal{V}(\hbar^*).$$

Also since $\mathcal{V}(\mathcal{V}^n\hbar) = \mathcal{V}^{n+1}\hbar \to \hbar^*$, thus we obtain $\mathcal{V}(\hbar^*) = \hbar^*$. Thus (i) implies (ii). Now we prove (ii) implies (i). Since $\hbar \in [\hbar]_{\widetilde{G}}$, for any $\hbar \in \Re$, so (ii) implies (i). Now if G is weakly connected then, $[\hbar]_{\widetilde{G}} = \Re$ for any $\hbar \in \Re$ so $\mathcal{V}$ is a Picard Operator. Thus (iii) hold. Also (iv) follows directly from (ii). $\square$

Corollary 3. *Let $(\Re, \tau)$ be an $\mathcal{F}$-complete $\mathcal{F}$-metric space equipped with a graph G and $\mathcal{V} : \Re \to \Re$. Then these conditions are equivalent.*
(i) G is weakly connected,
(ii) every orbitally continuous graphic F-contraction $\mathcal{V} : \Re \to \Re$ is Picard Operator,
(iii) for every orbitally continuous graphic F-contraction $\mathcal{V} : \Re \to \Re$,Card(Fix$\mathcal{V}$)$\leq 1$.

Remark 2. *(i) If we take $E(G) = \Re \times \Re$ in Theorem 4, then we get main result of Asif et al. [21].*
(ii) If we take $\Im(\iota) = \ln \iota$, for $\iota > 0$ in Theorem 4, then we get a result (Corollary 2.11) of Faraj et al. [35].
(iii) If we take $\xi(\iota) = \ln \iota$, for $\iota > 0$ and $\ell = 0$ in Definition 5, then the notion of $\mathcal{F}$-metric space reduces to metric space and from our Theorem 4, we get the main result of Batra et al. [9].
(iv) If we take $\xi(\iota) = \ln \iota$, for $\iota > 0$ and $\ell \geq 1$ in Definition 5, then the notion of $\mathcal{F}$-metric space reduces to b-metric space. Now with $E(G) = \Re \times \Re$, Theorem 4 reduces to main result of Cosentino et al. [14].

4. Application

In the present section, we discuss the solution of functional equations

$$\mu(\omega) = \sup_{\varpi \in D} \{f(\omega, \varpi) + K(\omega, \varpi, \mu(g(\omega, \varpi)))\}, \ \omega \in \Phi_1 \tag{4.1}$$

given in dynamic programming connected to multistage process [36] to apply the Theorem 2. In this equation, $f : \Phi_1 \times \Phi_2 \to \mathbb{R}$, $K : \Phi_1 \times \Phi_2 \times \mathbb{R} \to \mathbb{R}$ and $g : \Phi_1 \times \Phi_2 \to \Phi_1$, Φ_1 and Φ_2 are Banach spaces. Also Φ_1 and Φ_2 represent a state space and decision space respectively.

Let $B(\Phi_1)$ represents the family of all bounded real valued functions defined on Φ_1. For $h \in B(\Phi_1)$, consider

$$\|h\| = \sup_{\omega \in \Phi_1} |h(\omega)|.$$

Clearly, $(B(\Phi_1), \|h\|)$ is a Banach space. We equipe $B(\Phi_1)$ with the $\mathcal{F}$-metric (with $f(\iota) = \ln \iota$, for $\iota > 0$ and $\ell = 1$) defined by

$$d(h, k) = \sup_{\iota \in \Phi_1} |h(\iota) - k(\iota)|.$$

Evidently, $(B(\Phi_1), d)$ is $\mathcal{F}$-complete $\mathcal{F}$-metric space.

Theorem 7. *Assume that that these assertions are satisfied:*
(i) f and K are bounded,

(ii) for $\omega \in \Phi_1$ and $h \in B(\Phi_1)$, define $\mathcal{V} : B(\Phi_1) \to B(\Phi_1)$ by

$$\mathcal{V}(h)(\omega) = \sup_{\varpi \in \Phi_2} \{f(\omega, \varpi) + K(\omega, \varpi, \omega(g(\omega, \varpi)))\} \tag{4.2}$$

$\forall\ h \in B(\Phi_1)$. *Evidently, $\mathcal{V}$ is well-defined as f and K are bounded,*

(iii) there exists $\Game > 1$ such that

$$|K(\omega, \varpi, h(\omega)) - K(\omega, \varpi, k(\omega))| \leq e^{-\Game}(|h(\omega) - k(\omega)|),$$

where $h, k \in B(\Phi_1)$, $\omega \in \Phi_1$ and $\varpi \in \Phi_2$.

Then, the functional Eq (4.1) has a unique and bounded solution.

Proof. Note that $(B(\Phi_1), d)$ is a $\mathcal{F}$-complete $\mathcal{F}$-metric space. Assume that $\varepsilon > 0$ and $h_1, h_2 \in B(\Phi_1)$, then there exist $\varpi_1, \varpi_2 \in \Phi_2$ such that

$$\mathcal{V}(h_1)(\omega) < f(\omega, \varpi_1) + K(\omega, \varpi_1, h_1(g(\omega, \varpi_1))) + \varepsilon, \tag{4.3}$$

$$\mathcal{V}(h_2)(\omega) < f(\omega, \varpi_2) + K(\omega, \varpi_2, h_2(g(\omega, \varpi_2))) + \varepsilon, \tag{4.4}$$

$$\mathcal{V}(h_2)(\omega) \geq f(\omega, \varpi_2) + K(\omega, \varpi_2, h_2(g(\omega, \varpi_2))) \tag{4.5}$$

$$\mathcal{V}(h_1)(\omega) \geq f(\omega, \varpi_1) + K(\omega, \varpi_1, h_1(g(\omega, \varpi_1))). \tag{4.6}$$

Now, from (4.3) and (4.6), it follows easily that

$$\begin{aligned}
\mathcal{V}(h_1)(\omega) - \mathcal{V}(h_2)(\omega) &< K(\omega, \varpi_1, h_1(g(\omega, \varpi_1))) - K(\omega, \varpi_1, h_2(g(\omega, \varpi_1))) + \varepsilon \\
&\leq |K(\omega, \varpi_1, h_1(g(\omega, \varpi_1))) - K(\omega, \varpi_1, h_2(g(\omega, \varpi_1)))| + \varepsilon \\
&\leq e^{-\Game}(|h_1(\omega) - h_2(\omega)|) + \varepsilon.
\end{aligned}$$

Hence we get

$$\mathcal{V}(h_1)(\omega) - \mathcal{V}(h_2)(\omega) < e^{-\Game}(|h_1(\omega) - h_2(\omega)|) + \varepsilon. \tag{4.7}$$

Similarly, from (4.4) and (4.5) we obtain

$$\mathcal{V}(h_2)(\omega) - \mathcal{V}(h_1)(\omega) < e^{-\Game}(|h_1(\omega) - h_2(\omega)|) + \varepsilon. \tag{4.8}$$

Therefore, from (4.7) and (4.8) we have

$$|\mathcal{V}(h_1)(\omega) - \mathcal{V}(h_2)(\omega)| < e^{-\Game}(|h_1(\omega) - h_2(\omega)|) + \varepsilon. \tag{4.9}$$

for all $\varepsilon > 0$. Hence

$$d(\mathcal{V}h_1(\omega), \mathcal{V}h_2(\omega)) \leq e^{-\Game} d(h_1(\omega), h_2(\omega))$$

that is,

$$d(\mathcal{V}h_1, \mathcal{V}h_2) \leq e^{-\Game} d(h_1, h_2).$$

Taking natural logarithm on both side, we have

$$\ln\left(d(\mathcal{V}h_1, \mathcal{V}h_2)\right) \leq \ln\left(e^{-\Game} d(h_1, h_2)\right).$$

Consequently if we define $\Im : \mathbb{R}^+ \to \mathbb{R}$ define by $\Im(\iota) = \ln \iota, \iota > 0$, we have

$$\Game + \Im\left(d(\mathcal{V}h_1, \mathcal{V}h_2)\right) \leq \Im\left(d(h_1, h_2)\right)$$

Hence, all the hypotheses of Theorem 2 are satisfied. Thus, there exists a point h, such that $\mathcal{V}(h) = h$, that is the bounded solution of the functional Eq (4.1). □

5. Conclusions

In this article, we introduced the notion of graphic F-contraction in the context of $\mathcal{F}$-metric space and established some new fixed point theorems in this newly introduced space. We have given an example to manifest the authenticity of the obtained results. We also have presented some results for orbitally continuous and orbitally G-continuous graphic F-contractions. For future work, one can establish fixed point results for multivalued graphic F-contraction in $\mathcal{F}$-metric space.

Conflict of interest

The author declares that he had no conflict of interest.

References

1. S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, *Fund. Math.*, **3** (1922), 133–181.
2. S. Nadler, Multi-valued contraction mappings, *Pacidic Journal of Mathematics*, **30** (1969), 475–488. http://dx.doi.org/10.2140/pjm.1969.30.475
3. M. Kikkawa, T. Suzuki, Three fixed point theorems for generalized contractions with constants in complete metric spaces, *Nonlinear Anal.-Theor.*, **69** (2008), 2942–2949. http://dx.doi.org/10.1016/j.na.2007.08.064
4. J. Ahmad, A. Al-Mazrooei, Y. Cho, Y. Yang, Fixed point results for generalized Θ-contractions, *J. Nonlinear Sci. Appl.*, **10** (2017), 2350–2358. http://dx.doi.org/10.22436/jnsa.010.05.07
5. P. Proinov, Fixed point theorems for generalized contractive mappings in metric spaces, *J. Fixed Point Theory Appl.*, **22** (2020), 21. http://dx.doi.org/10.1007/s11784-020-0756-1
6. R. Espinola, W. Kirk, Fixed point theorems in R-trees with applications to graph theory, *Topol. Appl.*, **153** (2006), 1046–1055. http://dx.doi.org/10.1016/j.topol.2005.03.001
7. J. Jachymski, The contraction principle for mappings on a metric space with a graph, *Proc. Am. Math. Soc.*, **136** (2008), 1359–1373.
8. D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2012** (2012), 94. http://dx.doi.org/10.1186/1687-1812-2012-94
9. R. Batra, S. Vashistha, Fixed points of an F-contraction on metric spaces with a graph, *Int. J. Comput. Math.*, **91**, (2014), 2483–2490. http://dx.doi.org/10.1080/00207160.2014.887700
10. N. Secelean, Iterated function system consisting of F-contarctions, *Fixed Point Theory Appl.*, **2013** (2013), 277. http://dx.doi.org/10.1186/1687-1812-2013-277
11. H. Piri, P. Kumam, Some fixed point theorems concerning F-contraction in complete metic spaces, *Fixed Point Theory Appl.*, **2014** (2014), 210. http://dx.doi.org/10.1186/1687-1812-2014-210
12. O. Popescu, G. Stan, Two new fixed point theorems concerning F-contractions in complete metric spaces, *Symmetry*, **12** (2020), 58. http://dx.doi.org/10.3390/sym12010058

13. M. Abbas, B. Ali, S. Romaguera, Coincidence points of generalized multivalued (f, L)-almost F-contraction with applications, *J. Nonlinear Sci. Appl.*, **8** (2015), 919–934. http://dx.doi.org/10.22436/jnsa.008.06.03
14. M. Cosentino, M. Jleli, B. Samet, C. Vetro, Solvability of integrodifferential problems via fixed point theory in b-metric spaces, *Fixed Point Theory Appl.*, **2015** (2015), 70. http://dx.doi.org/10.1186/s13663-015-0317-2
15. J. Ahmad, A. Al-Rawashdeh, A. Azam, New fixed point theorems for generalized F-contractions in complete metric spaces, *Fixed Point Theory Appl.*, **2015** (2015), 80. http://dx.doi.org/10.1186/s13663-015-0333-2
16. N. Hussain, J. Ahmad, A. Azam, On Suzuki-Wardowski type fixed point theorems, *J. Nonlinear Sci. Appl.*, **8** (2015), 1095–1111. http://dx.doi.org/10.22436/jnsa.008.06.19
17. N. Hussain, P. Salimi, Suzuki-Wardowski type fixed point theorems for α-GF-contractions, *Taiwan. J. Math.*, **18** (2014), 1879–1895. http://dx.doi.org/10.11650/tjm.18.2014.4462
18. H. Nashine, D. Gopal, D. Jain, A. Al-Rawashdeh, Solutions of initial and boundary value problems via F-contraction mappings in metric-like space, *Int. J. Nonlinear Anal. Appl.*, **9** (2018), 129–145. http://dx.doi.org/10.22075/IJNAA.2017.1725.1452
19. H. Nashine, Z. Kadelburg, R. Agarwal, Existence of solutions of integral and fractional differential equations using α-type rational F-contractions in metric-like spaces, *Kyungpook Math. J.*, **58** (2018), 651–675. http://dx.doi.org/10.5666/KMJ.2018.58.4.651
20. M. Nazam, M. Arshad, M. Abbas, Existence of common fixed points of improved F-contraction on partial metric spaces, *Appl. Gen. Topol.*, **18** (2017), 277–287. http://dx.doi.org/10.4995/agt.2017.6776
21. A. Asif, M. Nazam, M. Arshad, S. Kim, $\mathcal{F}$-metric, F-contraction and common fixed point theorems with applications, *Mathematics*, **7** (2019), 586. http://dx.doi.org/10.3390/math7070586
22. G. Mani, A. Gnanaprakasam, N. Kausar, M. Munir, Salahuddin, Orthogonal F-contraction mapping on 0-complete metric space with applications, *Int. J. Fuzzy Log. Inte.*, **21** (2021), 243–250. http://dx.doi.org/10.5391/IJFIS.2021.21.3.243
23. M. Maurice Fréchet, Sur quelques points du calcul fonctionnel, *Rend. Circ. Matem. Palermo*, **22** (1906), 1–72. http://dx.doi.org/10.1007/BF03018603
24. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Mathematica et Informatica Universitatis Ostraviensis*, **1** (1993), 5–11.
25. A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalizedmetric spaces, *Publ. Math. Debrecen*, **57** (2000), 31–37. http://dx.doi.org/10.5486/pmd.2000.2133
26. M. Jleli, B. Samet, A generalized metric space and related fixed point theorems, *Fixed Point Theory Appl.*, **2015** (2015), 61. http://dx.doi.org/10.1186/s13663-015-0312-7
27. M. Jleli, B. Samet, A new generalization of the Banach contraction principle, *J. Inequal. Appl.*, **2014** (2014), 38. http://dx.doi.org/10.1186/1029-242X-2014-38
28. L. Alnaser, D. Lateef, H. Fouad, J. Ahmad, Relation theoretic contraction results in $\mathcal{F}$-metric spaces, *J. Nonlinear Sci. Appl.*, **12** (2019), 337–344.

29. S. Al-Mezel, J. Ahmad, G. Marino, Fixed point theorems for generalized ($\alpha\beta$-ψ)-contractions in $\mathcal{F}$-metric spaces with applications, *Mathematics*, **8** (2020), 584. http://dx.doi.org/10.3390/math8040584

30. M. Alansari, S. Mohammed, A. Azam, Fuzzy fixed point results in $\mathcal{F}$-metric spaces with applications, *J. Funct. Space.*, **2020** (2020), 5142815. http://dx.doi.org/10.1155/2020/5142815

31. D. Lateef, J. Ahmad, Dass and Gupta's Fixed point theorem in $\mathcal{F}$-metric spaces, *J. Nonlinear Sci. Appl.*, **12** (2019), 405–411. http://dx.doi.org/10.22436/jnsa.012.06.06

32. A. Hussain, T. Kanwal, Existence and uniqueness for a neutral differential problem with unbounded delay via fixed point results, *Trans. A. Razmadze Math. In.*, **172** (2018), 481–490. http://dx.doi.org/10.1016/j.trmi.2018.08.006

33. T. Kanwal, A. Hussain, H. Baghani, M. De La Sen, New fixed point theorems in orthogonal $\mathcal{F}$-metric spaces with application to fractional differential equation, *Symmetry*, **12** (2020), 832. http://dx.doi.org/10.3390/sym12050832

34. A. Petrusel, I. Rus, Fixed point theorems in ordered L-spaces, *Proc. Amer. Math. Soc.*, **134** (2006), 411–418.

35. H. Faraji, S. Radenović, Some fixed point results for F-G-contraction in $\mathcal{F}$-metric spaces endowed with a graph, *J. Math. Ext.*, **16** (2022), 1–13.

36. F. Vetro, F-contractions of Hardy-Rogers type and application to multistage decision processes, *Nonlinear Anal.-Mdel.*, **21** (2016), 531–546. http://dx.doi.org/10.15388/NA.2016.4.7

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