Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(4): 9740–9760.
DOI: 10.3934/math.2023491
Received: 07 December 2022
Revised: 28 January 2023
Accepted: 28 January 2023
Published: 21 February 2023

Research article

On a general class of nth order sequential hybrid fractional differential equations with boundary conditions

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Abstract: This manuscript is related to consider a general class of nth order sequential hybrid fractional differential equations (S-HFDEs) with boundary conditions. With the help of the coincidence degree theory of topology, some appropriate results for the existence theory of the aforementioned class are developed. The mentioned degree theory is a powerful tool to investigate nonlinear problems for qualitative theory. A result related to Ulam-Hyers (U-H) stability is also developed for the considered problem. It should be kept in mind that the considered degree theory relaxes the strong compact condition by some weaker one. Hence, it is used as a sophisticated tool in the investigation of the existence theory of solutions to nonlinear problems. Also, an example is given.

Keywords: fractional Caputo derivative; sequential hybrid fractional differential equations; degree theory; Ulam-Hyers stability
Mathematics Subject Classification: 34A08, 34A12, 47H10

1. Introduction

Recently fractional calculus has gotten much attention from researchers like traditional calculus. The mentioned area has the ability to describe real-world phenomena in more excellent ways. Also, such derivatives have numerous applications in the description of those systems with memory effects. Due to various applications, the said calculus has been used to investigate various infectious disease models like in [1–3]. Also, in the field of physics, engineering, and cosmology, fractional calculus has very well been used. For instance, we refer [4–6]. It has been shown that in many applications, the use of fractional calculus provides more realistic models than those obtained via classical ordinary derivatives. Due to this reason, the study of fractional models has received great attention from many researchers in the last few years. As the fractional order derivatives have important characteristics known as the memory effect which ordinary derivatives do not involve. Also, fractional differential operators are nonlocal as compared to the local behavior of integer derivatives. Recently researchers have published some very important results in this regard like [7, 8]. Also, authors [9] have published some new results on the numerical scheme for fractional order SEIR epidemic of measles. Here it is remarked that some authors have discussed the geometry of fractional order derivatives. For instance, authors [10] have suggested a geometric interpretation of the fractional derivatives which is based on modern differential geometry and the geometry of jet bundles. In fact, the fractional differential operators are definite integrals that create the complete accumulation or spectrum of the function on whose these applications include the corresponding integer-order counterpart of a special case. In the same way, authors [11] have given the definition of the geometric interpretation of gradient of order $(0, 1]$. In this way, they have suggested some geometric interpretations of the differentiability of real order.

Different problems under the concept of fractional calculus have been studied for the existing theory and stability analysis. One of the important areas is devoted to investigating hybrid problems under the aforesaid concepts. In this regard, different problems under boundary and initial conditions have been studied by using nonlinear analysis and tools of advanced functional analysis. In the present time class of fractional differential equations devoted to the quadratic perturbation of nonlinear differential equations (called hybrid differential equations) has attracted much attention from researchers. This is due to the fact that they include several dynamical systems as special cases. For instance, Dhage and Lakshmikantham [12] discussed the existence theory of the following problem of hybrid differential equations

$$\frac{d}{dt}\Big[\frac{u(t)}{f(t, u(t))}\Big] = g(t, u(t)),\ a.e.t \in [t_0, t_0 + a],$$

where

$$f \in C([t_0, t_0 + a] \times \mathbb{R}, \mathbb{R} \setminus \{0\}),\ \text{and } g \in C([t_0, t_0 + a] \times \mathbb{R}, \mathbb{R}).$$

They used hybrid fixed point theory to establish the existence of solutions to the proposed problems. In the same way, Dhage and Jadhav [13] studied the existence and uniqueness results for the ordinary first-order hybrid differential equation with perturbation of second type given by

$$\begin{aligned} &\frac{d}{dt}\Big[u(t) - f(t, u(t))\Big] = g(t, u(t)),\ a.e.t \in [t_0, t_0 + a],\\ &u(t_0) = u_0, \end{aligned}$$

where $f, g \in C([t_0, t_0 + a] \times \mathbb{R}, \mathbb{R})$. Motivated from the mentioned mentioned work, Lu et al. [14] studied the following class of FHDEs by replacing the ordinary derivative by Caputo fractional type with $0 < \sigma \leq 1$ as

$$\begin{aligned} &{}^{c}\mathbf{D}^{\sigma}\Big[u(t) - f(t, u(t))\Big] = g(t, u(t)),\ a.e.t \in [t_0, t_0 + a],\\ &u(t_0) = u_0. \end{aligned}$$

They used the hybrid fixed point theory of Dhage to study the existence and uniqueness of the solution to the mentioned problem.

In the last few years, hybrid fixed point theory has been used to study different problems of hybrid nature for the existence theory in [15–18]. Also, using the aforementioned tools, authors have established various results devoted to existence theory for different boundary value problems of HFDEs in [19–23]. Also, some authors have studied integral-type FHDEs in [24, 25]. Authors [26] have studied a system of FHDEs of thermostats type by using a fixed point approach. Furthermore, authors [27] have investigated a class of FHDEs under mixed-type hybrid integral boundary conditions. In the same way authors in [28–32] have used the tools of nonlinear functional analysis for studying various problems and systems of HFDEs. Here, we remark that authors [33] established a review of the analytical solutions for some generalized classes. In the same line, a class of HFDEs has also been considered in [34]. Authors in [35–37] have studied different boundary value problems of fractional order using topological degree theory for the existence theory. Here it should be kept in mind that a hybrid system is a dynamic system that interacts with continuous and discrete dynamics. For, example, the novel multiplex engineering systems involve numerous kinds of process and abstract decision-making units that present the image of various systems simultaneously exhibiting continuous and discrete time dynamics (see details in [38]). Further, the applications of hybrid systems can be found in embedded control systems also (see [39]).

The existence theory of solutions to nonlinear problems is an important area of research in the current scenario. Because the said theory predicts the existence of a solution to a dynamic problem whether it has a solution or not. Usually, for the said theory fixed point theory has been used very well. But fixed point theory needs strong compact conditions which restricted its use in some situations. Also, to deal HFDEs, Dhage established some hybrid-type fixed point theorems to study the existence and uniqueness of the solution to the mentioned problems. However, it also needs the same strong compact conditions. To relax, the criteria and replace strong compact conditions with some weaker compactness, the degree theory has been introduced. It has a sophisticated tool to be used to investigate numerous nonlinear problems of integral, differential, and difference equations for their corresponding solution. The mentioned theory has been used very well for usual problems of fractional order equations. However, in the case of HFDEs, it has not been used properly. For some important contributions by using degree theory to study various problems in fractional calculus, we refer few papers as [40–43].

In this work, we study a more general class of nonlinear boundary value problems (BVPs) consisting of a more general class of nth order S-HFDEs) together subject to nonlinear boundary conditions. Also, we choose the general case in which the nonlinear functions involved depend on the non-integer order derivatives. Further, necessary conditions required for the uniqueness of a solution to the proposed problem, we implement Caratheódory conditions along with techniques of measure of non-compactness and degree theory. Some new and interesting results are developed. Also, a result devoted to U-H stability is derived for the considered problem. Our proposed problem is stated as

$$\begin{aligned} &{}^{c}\mathbf{D}^{\vartheta}\Big[\frac{{}^{c}\mathbf{D}^{\omega}u(t)-\sum_{1}^{m}\mathbf{I}^{\beta_i}h_i(t,u(t),\mathbf{D}^{\rho}u(t))}{f(t,u(t),\mathbf{D}^{\rho}u(t))}\Big]=g(t,u(t),\mathbf{I}^{\gamma}u(t)),\ t\in \mathrm{I}=[0,1]\\ &u(0)=\psi_1(u(\eta)),\ u'(0)=0,u''(0)=0,....,u^{(n-1)}(0)=0,\ u(1)=\psi_2(u(\eta)), \end{aligned} \tag{1.1}$$

where $0<\vartheta\leq 1$, $n-1<\omega,\beta_i,\ \gamma\leq n$, $n-2<\rho\leq n-1$ with $n\geq 2$, $0<\eta<1$, the functions $f:\mathrm{I}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}-\{0\}$, $h_i:\mathrm{I}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}\,(i=1,2,....,m)$ and $g:\mathrm{I}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfy the Caratheódory conditions. Moreover, $\psi_1,\ \psi_2:\mathbb{R}\to\mathbb{R}$ are nonlinear functions. Also, the notation ${}^{c}\mathbf{D}$

denotes the Caputo fractional order derivative and **I** represents fractional integral. Here, we use the tools mentioned in [44–46] to establish a detailed analysis of the considered problem. Also, stability is an important aspect of qualitative theory. In this regard, U-H stability analysis has also considered for some problems of HFDEs. For instance, see [47, 48].

The present article is organized as: Section 1 is devoted to the introduction. Section 2 is related to basic results from fractional calculus and degree theory. Section 3 is devoted to the first part of our main results. Section 4 is related to the second part of our main results. The section is consisted of applications to verify our results. Section 6 is devoted to a brief conclusion.

2. Fundamental results

Here it should be kept in mind that we have used the following basic definitions from [1, 2] of fractional order derivative and integration.

Definition 2.1. If $\vartheta > 0$, then the fractional order integration of a function $u \in L^1([0, 1])$ is given by

$$I_{0+}^{\vartheta} u(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} u(s) ds.$$

Definition 2.2. The fractional derivative in Caputo sense of a function u over the interval $[0, 1]$ is defined as

$${}^c\mathbf{D}^{\vartheta} u(t) = \frac{1}{\Gamma(m-\vartheta)} \int_0^t (t-s)^{n-\vartheta-1} \theta^{(n)}(s) ds,$$

where $n - 1 = [\vartheta]$.

Theorem 2.3. *The solution of*

$$\mathbf{I}^{\vartheta}[{}^c\mathbf{D}^{\vartheta} u(t)] = y(t),\ n-1 < \vartheta \leq n,$$

is derived as

$$\mathbf{I}^{\vartheta}[{}^c\mathbf{D}^{\vartheta} u(t)] = y(t) + \mathscr{C}_i t^{n-1},$$

for arbitrary $\mathscr{C}_i \in \mathbb{R}$, $\quad i = 0, 1, 2, \ldots, n-1 = [\vartheta]$.

Let $\mathbb{E} = \{u \in C(\mathrm{I}) :^c \mathbf{D}^{\omega-1} u \in C(\mathrm{I})\}$ is Banach space under the norm $\|u\|_{\rho} = \max_{0 \leq t \leq 1} |u(t)| + \max_{0 \leq t \leq 1} |{}^c\mathbf{D}^{\rho} u|$.

Let **P** represents family of all bounded sub sets of $\mathbb{E}$, then we define the following measure of non-compactness.

Definition 2.4. [44] The measure due to non-compactness $\mu : \mathbf{P} \to \mathcal{R}^+$ is Kuratowski measure which is defined as

$$\mu(\mathrm{S}) = \inf\{\varrho > 0 \text{ where } \mathrm{S} \in \mathbf{P} \text{ such that } diameter of \mathrm{S} \leq \varrho\}.$$

Definition 2.5. [25] If $\mathrm{T}_1, \mathrm{T}_2 : \mathbb{E} \to \mathbb{E}$ are μ-Lipschtiz with constants C and C' respectively, then $\mathrm{T}_1 + \mathrm{T}_2 : \mathbb{E} \to \mathbb{E}$ is μ-Lipschitz with constant $\mathrm{C} + \mathrm{C}'$.

Definition 2.6. [25] If $\mathrm{T}_1 : \mathrm{S} \to \mathbb{E}$ is compact, then T_1 is μ-Lipschitz with constant $\mathrm{C} = 0$.

Definition 2.7. [25] If $\mathrm{T}_1 : \mathrm{S} \to \mathbb{E}$ is Lipschitz with constant C, then T_1 is μ-Lipschitz with the same constant C.

We need the given theorem.

Theorem 2.8. *[25] Let* $\mathrm{T} : \mathbb{E} \to \mathbb{E}$ *be* μ*-condensing and*

$$\mathrm{S} = \{u \in \mathbb{E} : \textit{with} \quad \lambda \in [0,1] \quad \textit{as} \quad u = \lambda \mathrm{T} u\}.$$

If S *is a bounded set in* $\mathbb{E}$*, so we can find* $r > 0$*, such that* $\mathrm{S} \subset \mathrm{D}_r(0)$*, then the degree*

$$\deg(I - \lambda\mathrm{T}, \mathrm{D}_r(0), 0) = 1, \ \ \textit{for all} \ \ \lambda \in [0,1].$$

Thus T *has at least one fixed point and the set of the fixed points of* T *lies in* $\mathrm{D}_r(0)$*.*

3. Existence theory

Here, we derive first part of our main results.

Lemma 3.1. *The solution of* (1.1) *can be described as*

$$\begin{aligned} u(t) = {} & \mathbf{I}^{\omega}\Psi(t,u(t),{}^{c}\mathbf{D}^{\rho}u(t)) + \sum_{1}^{m}\mathbf{I}^{\omega-\beta_i}h_i(t,u(t),{}^{c}\mathbf{D}^{\rho}u(t)) + \psi_1(u(\eta)) + \\ & \frac{\mathbf{I}^{\omega}f(t,u(t),{}^{c}\mathbf{D}^{\rho-1}u(t))}{\mathbf{I}^{\omega}f(1,u(1),{}^{c}\mathbf{D}^{\rho}u(1))}(\psi_2(u(\eta)) - \psi_1(u(\eta)) - \mathbf{I}^{\omega}\Psi(1,u(1),{}^{c}\mathbf{D}^{\rho}u(1)) - \\ & \mathbf{I}^{\omega}f(1,u(1),{}^{c}\mathbf{D}^{\rho}u(1)) - \sum_{1}^{m}\mathbf{I}^{\omega+\beta_i}h_i(1,u(1),{}^{c}\mathbf{D}^{\rho}u(1)), \end{aligned} \tag{3.1}$$

such that

$$\Psi(t,u(t),{}^{c}\mathbf{D}^{\rho}u(t)) = f(t,u(t),{}^{c}\mathbf{D}^{\rho}u(t))\mathbf{I}^{\vartheta}g(t,u(t),\mathbf{I}^{\gamma}u(t)),$$

$$\mathbf{I}^{\omega}\psi(1,u(1),{}^{c}\mathbf{D}^{\rho}u(1)), \mathbf{I}^{\omega}f(1,u(1),{}^{c}\mathbf{D}^{\rho}u(1))$$

represent value of integral

$$\mathbf{I}^{\omega}\Psi(t,u(t),{}^{c}\mathbf{D}^{\rho}u(t)), \mathbf{I}^{\omega}f(1,u(1),{}^{c}\mathbf{D}^{\rho}u(1))$$

at $t = 1$ *and* $\mathbf{I}^{\omega+\beta_i}h_i(1,u(1),{}^{c}\mathbf{D}^{\rho}u(1))$ *denotes the value of the integral* $\mathbf{I}^{\omega+\beta_i}h_i(t,u(t),{}^{c}\mathbf{D}^{\rho}u(t))$ *at* $t = 1$*, for* $i = 1,2,3,...m$*.*

Proof. On using $\mathbf{I}^{\vartheta}$ at both sides of (1.1), one has

$$\begin{aligned} {}^{c}\mathbf{D}^{\omega}u(t) - \sum_{1}^{m}\mathbf{I}^{\beta_i}h_i(t,u(t),{}^{c}\mathbf{D}^{\rho}u(t)) &= f(t,u(t),{}^{c}\mathbf{D}^{\rho}u(t))\mathbf{I}^{\vartheta}g(t,u(t),\mathbf{I}^{\gamma}u(t)) \\ &+ \mathscr{C}_0 f(t,u(t),{}^{c}\mathbf{D}^{\rho}u(t)) = \Psi(t,u(t),{}^{c}\mathbf{D}^{\rho}u(t)) + C_0 f(t,u(t),{}^{c}\mathbf{D}^{\rho}u(t)). \end{aligned}$$

Hence, we obtain

$${}^{c}\mathbf{D}^{\omega}u(t) = \sum_{1}^{m}\mathbf{I}^{\beta_i}h_i(t,u(t),{}^{c}\mathbf{D}^{\rho}u(t)) + \Psi(t,u(t),{}^{c}\mathbf{D}^{\rho}u(t)) + C_0 f(t,u(t),{}^{c}\mathbf{D}^{\rho}u(t)).$$

Using $\mathbf{I}^{\omega}$ and the semi group property of integrals, one has

$$u(t) = \sum_{1}^{m} \mathbf{I}^{\omega+\beta_i} h_i(t, u(t), {}^c\mathbf{D}^{\rho} u(t)) + \mathbf{I}^{\omega}\Psi(t, u(t), {}^c\mathbf{D}^{\rho} u(t)) + \mathscr{C}_0 \mathbf{I}^{\omega} f(t, u(t), {}^c\mathbf{D}^{\rho} u(t))$$
$$+ \mathscr{C}_1 + \mathscr{C}_2 t + \cdots + \mathscr{C}_n t^{n-1}.$$

Taking jth order ordinary derivative, one has

$$u^j(t) = \sum \mathbf{I}^{\omega+\beta_i-j} h_i(t, u(t), {}^c\mathbf{D}^{\rho} u(t)) + \mathbf{I}^{\omega-j}\Psi(t, u(t), {}^c\mathbf{D}^{\rho} u(t)) +$$
$$\mathscr{C}_0 \mathbf{I}^{\omega-j} f(t, u(t), {}^c\mathbf{D}^{\rho} u(t)) + \sum_{1}^{n} \mathscr{C}_i \frac{i! t^{i-j}}{(i-j+1)!}.$$

Also, $u'(0) = 0,\ u''(0) = 0, ..., u^{n-1}(0) = 0$ yield $\mathscr{C}_2 = 0,\ \mathscr{C}_3 = 0, ..., \mathscr{C}_n = 0$. Thus

$$u(t) = \sum_{1}^{m} \mathbf{I}^{\omega+\beta_i} h_i(t, u(t), {}^c\mathbf{D}^{\rho} u(t)) + \mathbf{I}^{\omega}\Psi(t, u(t), {}^c\mathbf{D}^{\rho} u(t)) + \mathscr{C}_0 \mathbf{I}^{\omega} f(t, u(t), {}^c\mathbf{D}^{\rho} u(t)) + \mathscr{C}_1. \tag{3.2}$$

Further $u(0) = \psi_1(u(\eta))$ yields $\mathscr{C}_1 = \psi_1(u(\eta))$ and using $u(1) = \psi_2(u(\eta))$, one has

$$\psi_2(u(\eta)) = u(1) = \mathbf{I}^{\omega}\Psi(1, u(1), {}^c\mathbf{D}^{\rho} u(1)) + \sum_{1}^{m} \mathbf{I}^{\omega+\beta_i} h_i(1, u(1), {}^c\mathbf{D}^{\rho} u(1))$$
$$+ \mathscr{C}_0 \mathbf{I}^{\omega} f(1, u(1), {}^c\mathbf{D}^{\rho} u(1)) + \psi_1(u(\eta)),$$

which implies

$$[\psi_2(u(\eta)) - \psi_1(u(\eta)) - \mathbf{I}^{\omega}\Psi(1, u(1), {}^c\mathbf{D}^{\rho} u(1)) - \sum_{1}^{m} \mathbf{I}^{\omega+\beta_i} h_i(1, u(1), {}^c\mathbf{D}^{\rho} u(1))]$$
$$= \mathscr{C}_0 \mathbf{I}^{\omega} f(1, u(1), {}^c\mathbf{D}^{\rho} u(1)).$$

Hence, we get the given result

$$\mathscr{C}_0 = \frac{[\psi_2(u(\eta)) - \psi_1(u(\eta)) - \mathbf{I}^{\omega}\Psi(1, u(1), {}^c\mathbf{D}^{\rho} u(1)) - \sum_{1}^{m} \mathbf{I}^{\omega+\beta_i} h(1, u(1), {}^c\mathbf{D}^{\rho} u(1))]}{\mathbf{I}^{\omega} f(1, u(1), {}^c\mathbf{D}^{\rho} u(1))}.$$

Hence, (3.2) becomes

$$u(t) = \mathbf{I}^{\omega}\Psi(t, u(t), {}^c\mathbf{D}^{\rho} u(t)) + \sum_{1}^{m} \mathbf{I}^{\omega+\beta_i} h_i(t, u(t), {}^c\mathbf{D}^{\rho} u(t)) + \frac{\mathbf{I}^{\omega} f(t, u(t), {}^c\mathbf{D}^{\rho} u(t))}{\mathbf{I}^{\omega} f(1, u(1), {}^c\mathbf{D}^{\rho} u(1))} \times$$
$$(\psi_2(u(\eta)) - \mathbf{I}^{\omega}\Psi(1, u(1), {}^c\mathbf{D}^{\rho} u(1)) - \mathbf{I}^{\omega} f(1, u(1), {}^c\mathbf{D}^{\rho} u(1)) + \psi_1(u(\eta)) -$$
$$\sum_{1}^{m} \mathbf{I}^{\omega-\beta_i} h_i(1, u(1), {}^c\mathbf{D}^{\rho} u(1))) = \mathbf{I}^{\omega}\Psi(t, u(t), {}^c\mathbf{D}^{\rho} u(t)) + \sum_{1}^{m} \mathbf{I}^{\omega+\beta_i} h_i(t, u(t), {}^c\mathbf{D}^{\rho} u(t))$$
$$+ \psi_1(u(\eta)) + \frac{\mathbf{I}^{\omega} f(t, u(t), {}^c\mathbf{D}^{\rho} u(t))}{\mathbf{I}^{\omega} f(1, u(1), {}^c\mathbf{D}^{\rho} u(1))} (\psi_2(u(\eta)) - \psi_1(u(\eta)) - \mathbf{I}^{\omega}\Psi(1, u(1), {}^c\mathbf{D}^{\rho} u(1))$$
$$- \mathbf{I}^{\omega} f(1, u(1), {}^c\mathbf{D}^{\rho} u(1)) - \sum_{1}^{m} \mathbf{I}^{\omega+\beta_i} h_i(1, u(1), {}^c\mathbf{D}^{\rho} u(1))),$$

which can be rewritten as

$$
\begin{aligned}
&u(t) = \mathbf{I}^{\omega}\Psi(t, u(t),^{c} \mathbf{D}^{\rho}u(t)) + \sum_{1}^{m} \mathbf{I}^{\omega+\beta_i}h_i(t, u(t),^{c} \mathbf{D}^{\rho}u(t)) + \psi_1(u(\eta))+ \\
&\frac{\mathbf{I}^{\omega} f(t, u(t),^{c} \mathbf{D}^{\rho}u(t))}{\mathbf{I}^{\omega} f(1, u(1),^{c} \mathbf{D}^{\rho}u(1))}(\psi_2(u(\eta)) - \psi_1(u(\eta)) - \mathbf{I}^{\omega}\Psi(1, u(1),^{c} \mathbf{D}^{\rho}u(1))- \\
&\mathbf{I}^{\omega} f(1, u(1),^{c} \mathbf{D}^{\rho}u(1)) - \sum_{1}^{m} \mathbf{I}^{\omega+\beta_i}h_i(1, u(1),^{c} \mathbf{D}^{\rho}u(1))).
\end{aligned}
\tag{3.3}
$$

□

From (3.3), it follows that

$$
\begin{aligned}
{}^{c}\mathbf{D}^{\rho}u(t) = &\mathbf{I}^{\omega-\rho}\Psi(t, u(t),^{c} \mathbf{D}^{\rho}u(t)) + \sum_{1}^{m} \mathbf{I}^{\omega+\beta_i-\rho}h_i(t, u(t),^{c} \mathbf{D}^{\rho}u(t)) \\
&+ \frac{\mathbf{I}^{\omega-\rho} f(t, u(t),^{c} \mathbf{D}^{\rho}u(t))}{\mathbf{I}^{\omega} f(1, u(1),^{c} \mathbf{D}^{\rho}u(1))}(\psi_2(u(\eta)) - \psi_1(u(\eta)).
\end{aligned}
\tag{3.4}
$$

Let define $\mathbb{A}$, $\mathbb{B}, \mathbb{C} : \mathbb{E} \to \mathbb{E}$ by $\mathbb{A} = \bar{\mathbb{A}} + \psi_1(u(\eta))$, $\mathbb{C} = (\psi_2(u(\eta)) - \psi_1(u(\eta))) - \bar{\mathbb{C}}$, where

$$
\begin{aligned}
&(\bar{\mathbb{A}}u)(t) = \mathbf{I}^{\omega}\Psi(t, u(t),^{c} \mathbf{D}^{\rho}u(t)) + \sum_{1}^{m} \mathbf{I}^{\omega+\beta_i}h_i(t, u(t),^{c} \mathbf{D}^{\rho}u(t)), \\
&(\mathbb{B}u)(t) = \frac{\mathbf{I}^{\omega} f(t, u(t),^{c} \mathbf{D}^{\rho}u(t))}{\mathbf{I}^{\omega} f(1, u(1),^{c} \mathbf{D}^{\rho}u(1))}, \\
&(\bar{\mathbb{C}}u)(t) = \mathbf{I}^{\omega}\Psi(1, u(1),^{c} \mathbf{D}^{\rho}u(1)) + \mathbf{I}^{\omega} f(1, u(1),^{c} \mathbf{D}^{\rho}u(1))+ \\
&\sum_{1}^{m} \mathbf{I}^{\omega+\beta_i}h_i(1, u(1),^{c} \mathbf{D}^{\rho}u(1))),
\end{aligned}
\tag{3.5}
$$

then (3.3) takes the form of the operator equation

$$
u(t) = \mathbb{A}u(t) + \mathbb{B}u(t)\mathbb{C}u(t) = \mathbb{T}u(t), t \in \mathrm{I}, \tag{3.6}
$$

and fixed points of the operator Eq (3.6) are solutions of the BVP (1.1). Further, from (3.4), it follows that

$$
\begin{aligned}
&{}^{c}\mathbf{D}^{\rho}(\bar{\mathbb{A}}u)(t) = \mathbf{I}^{\omega-\rho}\Psi(t, u(t),^{c} \mathbf{D}^{\rho}u(t)) + \sum_{1}^{m} \mathbf{I}^{\omega+\beta_i-\rho}h_i(t, u(t),^{c} \mathbf{D}^{\rho}u(t)), \\
&{}^{c}\mathbf{D}^{\rho}(\mathbb{B}u)(t) = \frac{\mathbf{I}^{\omega-\rho} f(t, u(t),^{c} \mathbf{D}^{\rho}u(t))}{\mathbf{I}^{\omega} f(1, u(1),^{c} \mathbf{D}^{\rho}u(1))}, \\
&{}^{c}\mathbf{D}^{\rho}(\bar{\mathbb{C}}u)(t) = 0.
\end{aligned}
\tag{3.7}
$$

Using (3.5), and (3.7), we obtain

$$
\begin{aligned}
&|(\bar{\mathbb{A}}u)(t)| \leq |\mathbf{I}^{\omega}\Psi(t, u(t),^{c} \mathbf{D}^{\rho}u(t))| + \sum_{1}^{m} |\mathbf{I}^{\omega+\beta_i} h_i(t, u(t),^{c} \mathbf{D}^{\rho}u(t))|, \\
&|^{c}\mathbf{D}^{\rho}(\bar{\mathbb{A}}u)| \leq |\mathbf{I}^{\omega-\rho}\Psi(t, u(t),^{c} \mathbf{D}^{\rho}u(t))| + \sum_{1}^{m} |\mathbf{I}^{\omega+\beta_i-\rho} h_i(t, u(t),^{c} \mathbf{D}^{\rho}u(t))|, \\
&|(\mathbb{B}u)(t)| \leq \frac{|\mathbf{I}^{\omega} f(t, u(t),^{c} \mathbf{D}^{\rho}u(t))|}{|\mathbf{I}^{\omega} f(1, u(1),^{c} \mathbf{D}^{\rho}u(1))|}, \ |^{c}\mathbf{D}^{\rho}(\mathbb{B}u)(t)| \leq \frac{|\mathbf{I}^{\omega-\rho} f(t, u(t),^{c} \mathbf{D}^{\rho}u(t))|}{|\mathbf{I}^{\omega} f(1, u(1),^{c} \mathbf{D}^{\rho}u(1))|}, \\
&|(\bar{\mathbb{C}}u)(t)| = |\mathbf{I}^{\omega}\Psi(1, u(1),^{c} \mathbf{D}^{\rho}u(1))| + |\mathbf{I}^{\omega} f(1, u(1),^{c} \mathbf{D}^{\rho}u(1))| \\
&+ \sum_{1}^{m} |\mathbf{I}^{\omega+\beta_i} h_i(1, u(1),^{c} \mathbf{D}^{\omega-1}u(1)))|, \ |^{c}\mathbf{D}^{\rho}(\bar{\mathbb{C}}u)(t)| = 0.
\end{aligned}
\tag{3.8}
$$

The following data depended results need to be hold to establish our main results.

(H_1) f, h_i, g fulfill the criteria of Caratheódory conditions.
(H_2) For constants τ_1, τ_2, d_1, d_2, c_1, c_2, one has

$$
\begin{aligned}
&|\psi_i(u_1(t)) - \psi_i(u_2(t))| \leq \tau_i |u_1 - u_2|, \ i = 1, 2 \\
&|\psi_i(u)| \leq c_i |u| + d_i, \ i = 1, 2.
\end{aligned}
$$

(H_3) Let we have continuous mappings θ_i, μ, $\delta : \mathrm{I} \to \mathbb{R}$, with $0 < \xi$, λ, such that for $u \in \mathbb{E}$ that

$$
\begin{aligned}
&|h_i(t, u(t),^{c} \mathbf{D}^{\rho}u(t))| \leq |\theta_i(t)|(\|u\| + \|^{c}\mathbf{D}^{\rho}u\|) + \xi = |\theta_i(t)|\|u\|_{\rho} + \xi, \\
&|f(t, u(t),^{c} \mathbf{D}^{\rho}u(t))| \leq |\mu(t)|(\|u\| + \|^{c}\mathbf{D}^{\rho}u\|) + \lambda = |\mu(t)|\|u\|_{\rho} + \lambda, \\
&|g(t, u(t), \mathbf{I}^{\gamma}u(t))| \leq \delta(t).
\end{aligned}
$$

(H_4) There exists $\tau_i > 0$, such that for $u_1, u_2 \in \mathbb{E}$,

$$
\begin{aligned}
&|f(t, u_1(t),^{c} \mathbf{D}^{\rho}u_1(t)) - f(t, u_2(t),^{c} \mathbf{D}^{\rho}u_2(t))| \leq \mu_0 \|u_1 - u_2\|_{\rho}, \\
&|h_i(t, u_1(t),^{c} \mathbf{D}^{\rho}u_1(t)) - h_i(t, u_2(t),^{c} \mathbf{D}^{\rho}u_2(t))| \leq \theta_i \|u_1 - u_2\|_{\rho}, \\
&|g(t, u_1(t), (t)) - g(t, u_2(t), \mathbf{I}^{\gamma}u_2(t))| \leq \delta_0 \|u_1 - u_2\|, \\
&|\psi_i(u_1(t)) - \psi_i(u_2(t))| \leq \tau_i |u_1 - u_2|, \ i = 1, 2,
\end{aligned}
$$

where

$$
\mu_0 = \max_{t \in I} \mu(t), \ \delta_0 = \max_{t \in I} \delta(t), \ \theta_i = \max_{t \in I} |\theta_i(t)|, i = 1, 2, 3, \cdots, m.
$$

Under the hypothesis ($H3$), we have the following relation

$$
\begin{aligned}
&|\Psi(t,u(t),^c\mathbf{D}^{\rho}u(t))| \le \frac{\delta_0}{\Gamma(\vartheta+1)}(\mu_0\|u\|_\rho+\lambda),\\
&|\mathbf{I}^{\omega}\Psi(t,u(t),^c\mathbf{D}^{\rho}u(t))| \le \frac{\delta_0}{\Gamma(\omega+1)\Gamma(\vartheta+1)}(\mu_0\|u\|_\rho+\lambda),\\
&|\mathbf{I}^{\omega-\rho}\Psi(t,u(t),^c\mathbf{D}^{\rho}u(t))| \le \frac{\delta_0}{\Gamma(\omega-\rho+1)\Gamma(\vartheta+1)}(\mu_0\|u\|_\rho+\lambda),\\
&|\mathbf{I}^{\omega}f(t,u(t),^c\mathbf{D}^{\rho}u(t))| \le \frac{1}{\Gamma(\omega+1)}(\mu_0\|u\|_\rho+\lambda),\\
&|\mathbf{I}^{\omega+\beta_i}h_i(t,u(t),^c\mathbf{D}^{\rho}u(t))| \le \frac{1}{\Gamma(\omega+\beta_i+1)}(\theta_i\|u\|_\rho+\xi),\\
&|\mathbf{I}^{\omega+\beta_i-\rho}h_i(t,u(t),^c\mathbf{D}^{\rho}u(t))| \le \frac{1}{\Gamma(\omega+\beta_i-\rho+1)}(\theta_i\|u\|_\rho+\xi).
\end{aligned}
\tag{3.9}
$$

Using (3.8), and (3.9) together with the hypothesis H_2, H_3, we obtain the following relations

$$
\begin{aligned}
&|(\bar{\mathbb{A}}u)(t)| \le \frac{\delta_0(\mu_0\|u\|_\rho+\lambda)}{\Gamma(\vartheta+1)\Gamma(\omega+1)} + \sum_1^m \frac{(\theta_i\|u\|_\rho+\xi)}{\Gamma(\omega+\beta_i+1)},\\
&|^c\mathbf{D}^{\rho}(\bar{\mathbb{A}}u)| \le \frac{\delta_0(\mu_0\|u\|_\rho+\lambda)}{\Gamma(\vartheta+1)\Gamma(\omega-\rho+1)} + \sum_1^m \frac{(\theta_i\|u\|_\rho+\xi)}{\Gamma(\omega+\beta_i-\rho+1)},\\
&|(\mathbb{B}u)(t)| \le \frac{(\mu_0\|u\|_\rho+\lambda)}{\Lambda\Gamma(\omega+1)},\ |^c\mathbf{D}^{\rho}(\mathbb{B}u)(t)| \le \frac{(\mu_0\|u\|_\rho+\lambda)}{\Lambda\Gamma(\omega-\rho+1)},\\
&|(\bar{\mathbb{C}}u)(t) \le \frac{\delta_0(\mu_0\|u\|_\rho+\lambda)}{\Gamma(\omega+1)\Gamma(\vartheta+1)} + \sum_1^m \frac{(\theta_i\|u\|_\rho+\xi)}{\Gamma(\omega+\beta_i+1)},\\
&|^c\mathbf{D}^{\rho}(\bar{\mathbb{C}}u)(t)| = 0,
\end{aligned}
\tag{3.10}
$$

where $\Lambda = |\mathbf{I}^{\omega}\Psi(1,u(1),^c\mathbf{D}^{\rho}u(1))$. Thus, under the hypothesis H_4, we have the following relation

$$
\begin{aligned}
&|\mathbf{I}^{\vartheta}g(t,u_1(t),\mathbf{I}^{\gamma}u_1(t)) - \mathbf{I}^{\vartheta}g(t,u_2(t),\mathbf{I}^{\gamma}u_2(t))| \le \frac{\rho_0}{\Gamma(\vartheta+1)}\|u_1-u_2\|,\\
&|\Psi(t,u_1(t),^c\mathbf{D}^{\rho}u_1(t)) - \Psi(t,u_2(t),^c\mathbf{D}^{\rho}u_2(t))| \le \frac{(\delta\mu+\rho_0(\|u\|_\rho+\lambda))}{\Gamma(\vartheta+1)}\|u_1-u_2\|_\rho.
\end{aligned}
\tag{3.11}
$$

Further, we have

$$
\begin{aligned}
&\mathbf{I}^{\omega}\Psi(t_1,u(t_1),^c\mathbf{D}^{\rho}u(t_1)) - \mathbf{I}^{\omega}\Psi(t_2,u(t_2),^c\mathbf{D}^{\rho}u(t_2)) =\\
&\frac{1}{\Gamma(\omega)}[\int_0^{t_1}(t_1-s)^{\omega-1}\Psi(s,u(s),^c\mathbf{D}^{\rho}u(s))ds - \int_0^{t_2}(t_2-s)^{\omega-1}\Psi(s,u(s),^c\mathbf{D}^{\rho}u(s))ds]\\
&= \frac{1}{\Gamma(\omega)}[\int_0^{t_1}((t_1-s)^{\omega-1}-(t_2-s)^{\omega-1})\Psi(s,u(s),^c\mathbf{D}^{\rho}u(s)ds+\\
&\int_{t_1}^{t_2}((t_2-s)^{\omega-1}-(t_2-s)^{\omega-1})\Psi(s,u(s),^c\mathbf{D}^{\rho}u(s))ds].
\end{aligned}
$$

Thus, one has

$$|\mathbf{I}^{\omega}\Psi(_1t, u(t_1),^c \mathbf{D}^{\rho}u(t_1)) - \mathbf{I}^{\omega}\Psi(t_2, u(t_2),^c \mathbf{D}^{\rho}u(t_2))| \leq \frac{\|\Psi(s, u(s),^c \mathbf{D}^{\rho}u(s)\|}{\Gamma(\omega+1)}(2(t_2-t_1)^{\omega} + t_1^{\omega} - t_2^{\omega}),$$

which in view of (3.9) implies that

$$|\mathbf{I}^{\omega}\Psi(t_1, u(t_1),^c \mathbf{D}^{\rho}u(t_1)) - \mathbf{I}^{\omega}\Psi(t_2, u(t_2),^c \mathbf{D}^{\rho}u(t_2))| \leq \frac{\delta_0(\mu_0\|u\|_{\rho} + \lambda)}{\Gamma(\vartheta+1)\Gamma(\omega+1)}(2(t_2-t_1)^{\omega} + t_1^{\omega} - t_2^{\omega}). \tag{3.12}$$

Similarly, in view of (3.9), we obtain

$$|\mathbf{I}^{\omega-\rho}\Psi(t_1, u(t_1),^c \mathbf{D}^{\rho}u(t_1)) - \mathbf{I}^{\omega-\rho}\Psi(t_2, u(t_2),^c \mathbf{D}^{\rho}u(t_2))| \leq \frac{\delta_0(\mu_0\|u\|_{\rho} + \lambda)}{\Gamma(\vartheta+1)\Gamma(\omega-\rho)}(2(t_2-t_1)^{\omega-\rho} + t_1^{\omega-\rho} - t_2^{\omega-\rho}), \tag{3.13}$$

$$|\mathbf{I}^{\omega+\beta_i}h_i(t_1, u(t_1),^c \mathbf{D}^{\rho}u(t_1)) - \mathbf{I}^{\omega+\beta_i}h_i(t_2, u(t_2),^c \mathbf{D}^{\rho}u(t_2))| \leq \frac{(\theta_i\|u\|_{\rho} + \xi)}{\Gamma(\omega+\beta_i+1)}(2(t_2-t_1)^{\omega+\beta_i} + t_1^{\omega+\beta_i} - t_2^{\omega+\beta_i}), \tag{3.14}$$

$$|\mathbf{I}^{\omega+\beta_i-\rho}h_i(t_1, u(t_1),^c \mathbf{D}^{\rho}u(t_1)) - \mathbf{I}^{\omega+\beta_i-\rho}h_i(t_2, u(t_2),^c \mathbf{D}^{\rho}u(t_2))| \leq \frac{(\theta_i\|u\|_{\rho} + \xi)}{\Gamma(\omega+\beta_i-\rho+1)}(2(t_2-t_1)^{\omega+\beta_i-\rho} + t_1^{\omega+\beta_i-\rho} - t_2^{\omega+\beta_i-\rho}), \tag{3.15}$$

$$|\mathbf{I}^{\omega}f(t_1, u(t_1),^c \mathbf{D}^{\rho}u(t_1)) - \mathbf{I}^{\omega}f(t_2, u(t_2),^c \mathbf{D}^{\rho}u(t_2))| \leq \frac{((\mu_0\|u\|_{\rho} + \lambda)}{\Gamma(\omega+1)}(2(t_2-t_1)^{\omega} + t_1^{\omega} - t_2^{\omega}), \tag{3.16}$$

$$|\mathbf{I}^{\omega-\rho}f(t_1, u(t_1),^c \mathbf{D}^{\rho}u(t_1)) - \mathbf{I}^{\omega-\rho}f(t_2, u(t_2),^c \mathbf{D}^{\rho}u(t_2))| \leq \frac{((\mu_0\|u\|_{\rho} + \lambda)}{\Gamma(\omega-\rho+1)}(2(t_2-t_1)^{\omega-\rho} + t_1^{\omega-\rho} - t_2^{\omega-\rho}). \tag{3.17}$$

Hence, it follows that

$$|Au_1(t) - Au_2(t)| \leq \frac{1}{\Gamma(\omega+1)}|\Psi(t, u_1(t), D^{\rho}u_1(t)) - \Psi(t, u_2(t), D^{\rho}u_2(t))| + \sum_1^m \frac{|h_i(t, u_1(t), D^{\rho}u_1(t)) - h_i(t, u_2(t), D^{\rho}u_2(t))|}{\Gamma(\omega+\beta_i+1)} + |\psi_1(u_1)(\eta) - \psi_1(u_2)(\eta)|,$$

which in view (3.11), and H_4 implies that

$$\begin{aligned} |Au_1(t) - Au_2(t)| &\leq \frac{(\delta\mu + \rho_0(\|u\|_{\rho} + \lambda)}{\Gamma(\omega+1)\Gamma(\vartheta+1)}\|u_1 - u_2\|_{\rho} + \sum_1^m \frac{|\theta_i|\|u_1 - u_2\|_{\rho}}{\Gamma(\omega+\beta_i+1)} + \tau_1\|u_1 - u_2\| \\ &\leq (\frac{(\delta\mu + \rho_0(\|u\|_{\rho} + \lambda)}{\Gamma(\omega+1)\Gamma(\vartheta+1)} + \sum_1^m \frac{|\theta_i|}{\Gamma(\omega+\beta_i+1)} + \tau_1)\|u_1 - u_2\|_{\rho}. \end{aligned} \tag{3.18}$$

In addition, we have

$$|D^{\rho}Au_1(t)-D^{\rho}Au_2(t)|\leq \mathbf{I}^{\omega-\rho}|\Psi(t,u_1(t),D^{\rho}u_1(t))-\Psi(t,u_2(t),D^{\rho}u_2)(t)|+\\ \sum_1^m \mathbf{I}^{\omega+\beta_i-\rho}|(h_i(t,u_1(t),D^{\rho}u_1(t))-h_i(t,u_2(t),D^{\rho}u_2(t))|,$$

which in view (3.11), and H_4 implies that

$$|D^{\rho}Au_1(t)-D^{\rho}Au_2(t)|\leq \frac{(\delta\mu+\rho_0(\|u\|_{\rho}+\lambda)}{\Gamma(\omega-\rho+1)\Gamma(\vartheta+1)}\|u_1-u_2\|_{\rho}+\sum_1^m\frac{|\theta_i|\|u_1-u_2\|_{\rho}}{\Gamma(\omega+\beta_i-\rho+1)}\\ =(\frac{(\delta\mu+\rho_0(\|u\|_{\rho}+\lambda)}{\Gamma(\omega-\rho+1)\Gamma(\vartheta+1)}+\sum_1^m\frac{|\theta_i|}{\Gamma(\omega+\beta_i-\rho+1)})\|u_1-u_2\|_{\rho}. \tag{3.19}$$

From (3.23), and (3.19), it follows that

$$\|Au_1(t)-Au_2(t)\|_{\rho}\leq (\frac{(\delta\mu+\rho_0(\|u\|_{\rho}+\lambda)}{\Gamma(\vartheta+1)}(\frac{1}{\Gamma(\omega+1)}+\frac{1}{\Gamma(\omega-\rho+1)})\\ +\sum_1^m(\frac{|\theta_i|}{\Gamma(\omega+\beta_i+1)}+\frac{|\theta_i|}{\Gamma(\omega+\beta_i-\rho+1)})+\tau_1)\|u_1-u_2\|_{\rho}=\kappa_1\|u_1-u_2\|_{\rho}, \tag{3.20}$$

where $\kappa_1=\frac{(\delta\mu+\rho_0(\|u\|_{\rho}+\lambda)}{\Gamma(\vartheta+1)}(\frac{1}{\Gamma(\omega+1)}+\frac{1}{\Gamma(\omega-\rho+1)})+\sum_1^m(\frac{|\theta_i|}{\Gamma(\omega+\beta_i+1)}+\frac{|\theta_i|}{\Gamma(\omega+\beta_i-\rho+1)})+\tau_1$. Now

$$|\mathbb{B}u_1(t)-\mathbb{B}u_2(t)|\leq \frac{\mathbf{I}^{\omega}|(f(t,u_1(t),D^{\rho}u_1(t))-f(t,u_2(t),D^{\rho}u_2(t))}{\Lambda},\\ |D^{\rho}\mathbb{B}u_1(t)-D^{\rho}\mathbb{B}u_2(t)|\leq \frac{\mathbf{I}^{\omega-\rho}|(f(t,u_1(t),D^{\rho}u_1(t))-f(t,u_2(t),D^{\rho}u_2(t))|}{\Lambda},$$

using H_4 yields

$$|\mathbb{B}u_1(t)-\mathbb{B}u_2(t)|\leq \frac{\mu\|u_1-u_2\|}{\Lambda\Gamma(\omega+1)},\\ |D^{\rho}\mathbb{B}u_1(t)-D^{\rho}\mathbb{B}u_2(t)|\leq \frac{\mu\|u_1-u_2\|}{\Lambda\Gamma(\omega-\rho+1)}.$$

Hence, it follows that

$$\|\mathbb{B}u_1-\mathbb{B}u_2\|_{\rho}\leq \frac{\mu}{\Lambda}(\frac{1}{\Gamma(\omega+1)}+\frac{1}{\Gamma(\omega-\rho+1)})\|u_1-u_2\|_{\rho}=\kappa_2\|u_1-u_2\|_{\rho}, \tag{3.21}$$

where$\kappa_2=\frac{\mu}{\Lambda}(\frac{1}{\Gamma(\omega+1)}+\frac{1}{\Gamma(\omega-\rho+1)})$.

$$|\mathbb{C}u_1(t)-\mathbb{C}u_2(t)|\leq|\psi_2(u_1)-\psi_2(u_2)|+|\psi_1(u_1)-\psi_1(u_2)|+|\mathbf{I}^{\omega}\Psi(1,u_1(1),D^{\rho}u_1(1))\\ -\mathbf{I}^{\omega}\Psi(1,u_1(1),D^{\rho}u_1(1))|+|\mathbf{I}^{\omega}f(1,u_1(1),D^{\rho}u_1(1))-\mathbf{I}^{\omega}f(1,u_1(1),D^{\rho}u_1(1))|+\\ \sum_1^m|\mathbf{I}^{\omega+\beta_i}h_i(1,u_1(1),D^{\rho}u_1(1))-\mathbf{I}^{\omega+\beta_i}h_i(1,u_1(1),D^{\rho}u_1(1))|. \tag{3.22}$$

Using(3.11), and H_4, we obtain

$$|\mathbb{C}u_1(t)-\mathbb{C}u_2(t)|\le(\tau_1+\tau_2)|u_1-u_2|+\frac{(\delta\mu+\rho_0(\|u\|_\rho+\lambda)}{\Gamma(\omega+1)\Gamma(\vartheta+1)}\|u_1-u_2\|_\rho+ \frac{\mu_0}{\Gamma(w+1)}\|u_1-u_2\|_\rho+\sum_1^m\frac{\theta_i\|u_1-u_2\|_\rho}{\Gamma(\omega+\beta_i+1)}\le(\frac{(\delta\mu+\rho_0(\|u\|_\rho+\lambda)}{\Gamma(\omega+1)\Gamma(\vartheta+1)}+ \frac{\mu_0}{\Gamma(w+1)}+\sum_1^m\frac{|\theta_i|}{\Gamma(\omega+\beta_i+1)}+\tau_1+\tau_2)\|u_1-u_2\|_\rho=\kappa_3\|u_1-u_2\|_\rho, \tag{3.23}$$

where $\kappa_3=\frac{(\delta\mu+\rho_0(\|u\|_\rho+\lambda)}{\Gamma(\omega+1)\Gamma(\vartheta+1)}+\frac{\mu_0}{\Gamma(w+1)}+\sum_1^m\frac{|\theta_i|}{\Gamma(\omega+\beta_i+1)}+\tau_1+\tau_2$.

Theorem 3.2. *Under the hypothesis H_1–H_3, the operator $\bar{\mathbb{A}}$ is compact and satisfies the following growth condition $\|\bar{\mathbb{A}}u\|_\rho\le\Delta_1\|u\|_\rho+\Delta_2$, where*

$$\Delta_1=\frac{\delta_0\mu_0}{\Gamma(\vartheta+1)}(\frac{1}{\Gamma(\omega+1)}+\frac{1}{\Gamma(\omega-\rho+1)})+\sum_1^m(\frac{\theta_i}{\Gamma(\omega+\beta_i+1)}+\frac{\theta_i}{\Gamma(\omega+\beta_i-\rho+1)}), \tag{3.24}$$

and

$$\Delta_2=\frac{\delta_0\lambda}{\Gamma(\vartheta+1)}(\frac{1}{\Gamma(\omega+1)}+\frac{1}{\Gamma(\omega-\rho+1)})+\sum_1^m(\frac{\xi}{\Gamma(\omega+\beta_i+1)}+\frac{\xi}{\Gamma(\omega+\beta_1-\rho+1)}). \tag{3.25}$$

Proof. Here $(\bar{\mathbb{A}}u)(t)=\mathbf{I}^\omega\Psi(t,u(t),^c\mathbf{D}^\rho u(t))+\sum_1^m\mathbf{I}^{\omega+\beta_i}h_i(t,u(t),^c\mathbf{D}^\rho u(t))$, clearly, $\bar{\mathbb{A}}$ is continuous on $\mathbb{E}$. Now, for $u\in\mathbb{E}$, using (3.10), we have

$$\|\bar{\mathbb{A}}u\|_\rho=\|\bar{\mathbb{A}}u\|+\|^c\mathbf{D}^\rho\bar{\mathbb{A}}u\|\le\frac{\delta_0(\mu_0\|u\|_\rho+\lambda)}{\Gamma(\vartheta+1)}(\frac{1}{\Gamma(\omega+1)}+\frac{1}{\Gamma(\omega-\rho+1)})+ \sum_1^m(\theta_i\|u\|_\rho+\xi))(\frac{1}{\Gamma(\omega+\beta_i+1)}+\frac{1}{\Gamma(\omega+\beta_i-\rho+1)})= (\frac{\delta_0\mu_0}{\Gamma(\vartheta+1)}(\frac{1}{\Gamma(\omega+1)}+\frac{1}{\Gamma(\omega-\rho+1)})+\sum_1^m(\frac{\theta_i}{\Gamma(\omega+\beta_i+1)}+\frac{\theta_i}{\Gamma(\omega+\beta_i-\rho+1)}))\|u\|_\rho +\frac{\delta_0\lambda}{\Gamma(\vartheta+1)}(\frac{1}{\Gamma(\omega+1)}+\frac{1}{\Gamma(\omega-\rho+1)})+\sum_1^m(\frac{\xi}{\Gamma(\omega+\beta_i+1)}+\frac{\xi}{\Gamma(\omega+\beta_1-\rho+1)}).$$

Hence

$$\|Au\|_\rho\le\Delta_1\|u\|_\rho+\Delta_2. \tag{3.26}$$

(3.26) yields that $\bar{\mathbb{A}}$ is uniformly bounded for bounded set on $\mathbb{E}$. Let, $t_1<t_2\in I$, and consider

$$|\bar{\mathbb{A}}(u)t_2-\bar{\mathbb{A}}(u)t_1|+|(^c\mathbf{D}^\rho\bar{\mathbb{A}}u)t_2-(^c\mathbf{D}^\rho\bar{\mathbb{A}}u)t_1|\le |\mathbf{I}^\omega\Psi(t_1,u(t_1),^c\mathbf{D}^\rho u(t_1))-\mathbf{I}^\omega\Psi(t_2,u(t_2),^c\mathbf{D}^\rho u(t_2))| +\sum_1^m|\mathbf{I}^{\omega+\beta_i}h_i(t_1,u(t_1),^c\mathbf{D}^\rho u(t_1)-\mathbf{I}^{\omega+\beta_i}h_i(t_2,u(t_2),^c\mathbf{D}^\rho u(t_2)|.$$

In view of (3.12), and (3.14), we obtain

$$\begin{aligned}|\bar{\mathbb{A}}(u)t_2 - \bar{\mathbb{A}}(u)t_1| \leq \frac{\frac{\delta_0}{\Gamma(\vartheta+1)}(\mu_0\|u\|_\rho + \lambda)}{\Gamma(\omega+1)}(2(t_2-t_1)^{\omega} + t_1^{\omega} - t_2^{\omega}) + \\ \sum_1^m \frac{(\theta_i\|u\|_\rho + \xi)}{\Gamma(\omega+\beta_i+1)}(2(t_2-t_1)^{\omega+\beta_i} + t_1^{\omega+\beta_i} - t_2^{\omega+\beta_i}),,\end{aligned} \tag{3.27}$$

in view of (3.13), and (3.15), we obtain

$$\begin{aligned}&|{}^c\mathbf{D}^\rho\bar{\mathbb{A}}(u)t_2 -{}^c \mathbf{D}^\rho\bar{\mathbb{A}}(u)t_1| \leq |\mathbf{I}^{\omega-\rho}\Psi(t_1,u(t_1),{}^c\mathbf{D}^\rho u(t_1)) - \mathbf{I}^{\omega-\rho}\Psi(t_2,u(t_2),{}^c\mathbf{D}^\rho u(t_2))| + \\ &\sum_1^m |\mathbf{I}^{\omega+\beta_i-\rho}h_i(t_1,u(t_1),{}^c\mathbf{D}^\rho u(t_1)) - \mathbf{I}^{\omega+\beta_i-\rho}h_i(t_2,u(t_2),{}^c\mathbf{D}^\rho u(t_2))| \\ &\leq \frac{\frac{\delta_0}{\Gamma(\vartheta+1)}(\mu_0\|u\|_\rho + \lambda)}{\Gamma(\omega-\rho)}(2(t_2-t_1)^{\omega-\rho} + t_1^{\omega-\rho} - t_2^{\omega-\rho}) + \\ &\sum_1^m \frac{(\theta_i\|u\|_\rho + \xi)}{\Gamma(\omega+\beta_i-\rho+1)}(2(t_2-t_1)^{\omega+\beta_i-\rho} + t_1^{\omega+\beta_i-\rho} - t_2^{\omega+\beta_i-\rho}).\end{aligned} \tag{3.28}$$

From (3.27) and (3.28), it follows that

$$\|\bar{\mathbb{A}}(u)t_2 - \bar{\mathbb{A}}(u)t_1\|_\rho = \|\bar{\mathbb{A}}(u)t_2 - \bar{\mathbb{A}}(u)t_1\| + \|({}^c\mathbf{D}^\rho\bar{\mathbb{A}}u)t_2 - ({}^c\mathbf{D}^\rho\bar{\mathbb{A}}u)t_1\| \to 0 \text{ as } t_1 \to t_2. \tag{3.29}$$

Thus $\bar{\mathbb{A}}$ is equi continuous, and by Arzelá- Ascoli theorem $\bar{\mathbb{A}}$ is relatively compact. Hence, $\bar{\mathbb{A}}$ is μ-Lipschtiz with constant 0. □

Lemma 3.3. *Under the hypothesis H_1–H_3, the operator $\mathbb{A}$ is μ-Lipschtiz with constant τ_1 and satisfes the following growth condition*

$$\|\mathbb{A}u\|_\rho \leq (\Delta_1 + c_1)\|u\|_\rho + (\Delta_2 + d_1). \tag{3.30}$$

Proof. By H_2, the operator $\psi_1(\eta)$ is μ-Lipschtiz with constant τ_1 and by Lemma (3.2), the operator $\bar{\mathbb{A}}$ is μ-Lipschtiz with constant 0. Hence, the operator $\mathbb{A} = \bar{\mathbb{A}} + \psi_1(\eta)$ is μ-Lipschtiz with constant τ_1. Since, $\|\bar{\mathbb{A}}u\|_\rho \leq \Delta_1\|u\|_\rho + \Delta_2$ by Lemma (3.2), it follows that $\|\mathbb{A}u\|_\rho \leq (\Delta_1 + c_1)\|u\|_\rho + (\Delta_2 + d_1)$. □

Lemma 3.4. *Under the hypothesis H_1–H_3, the operator $\mathbb{B}$ is continuous and compact.*

Proof. Here, $(\mathbb{B}u)(t) = \frac{\mathbf{I}^\omega f(t,u(t),{}^c\mathbf{D}^\rho u(t))}{\mathbf{I}^\omega f(1,u(1),{}^c\mathbf{D}^\rho u(1))}$. Clearly, $\mathbb{B}$ is continuous on E and bounded as

$$|(\mathbb{B}u)(t)| = |\frac{|\mathbf{I}^\omega f(t,u(t),{}^c\mathbf{D}^\rho u(t))|}{\Lambda} \leq 1. \tag{3.31}$$

For equi-continuity, choose $t_1 < t_2 \in I$, and consider

$$\begin{gathered}|\mathbb{B}(u)t_2 - \mathbb{B}(u)t_1| + |({}^c\mathbf{D}^\rho\mathbb{B}u)t_2 - ({}^c\mathbf{D}^\rho\mathbb{B}u)t_1| \leq \\ \frac{|\mathbf{I}^\omega f(t_2,u(t_2),{}^c\mathbf{D}^\rho u(t_2)) - \mathbf{I}^\omega f(t_1,u(t_1),{}^c\mathbf{D}^\rho u(t_1))|}{\Lambda}.\end{gathered}$$

In view of (3.16) , we obtain

$$|\mathbb{B}(u)t_2 - \mathbb{B}(u)t_1| \leq \frac{((\mu_0\|u\|_\rho + \lambda)}{\Gamma(\omega+1)\Lambda}(2(t_2-t_1)^\omega + t_1^\omega - t_2^\omega), \tag{3.32}$$

in view of (3.17), we obtain

$$|{}^c\mathbf{D}^\rho\mathbb{B}(u)t_2 -{}^c \mathbf{D}^\rho\mathbb{B}(u)t_1| \leq \frac{((\mu_0\|u\|_\rho + \lambda)}{\Gamma(\omega-\rho+1)\Lambda}(2(t_2-t_1)^{\omega-\rho} + t_1^{\omega-\rho} - t_2^{\omega-\rho}). \tag{3.33}$$

From (3.32) and (3.33), it follows that

$$\|\mathbb{B}(u)t_2 - \mathbb{B}(u)t_1\|_\rho = \|\mathbb{B}(u)t_2 - \mathbb{B}(u)t_1\| + \|({}^c\mathbf{D}^\rho\mathbb{B}u)t_2 - ({}^c\mathbf{D}^\rho\mathbb{B}u)t_1\| \to 0 \text{ as } t_1 \to t_2. \tag{3.34}$$

Therefore, $\mathbb{B}$ is equi continuous. Therefore, using Arzelá- Ascoli theorem, $\mathbb{B}$ is compact. □

Lemma 3.5. *Under the hypothesis H_1–H_3, the operator $\bar{\mathbb{C}}$ is compact and satisfies the following growth condition*

$$\|\bar{\mathbb{C}}u\|_\rho \leq \Delta_3\|u\|_\rho + \Delta_4, \tag{3.35}$$

where

$$\Delta_3 = \frac{\delta_0\mu_0}{\Gamma(\omega+1)\Gamma(\vartheta+1)} + \frac{\mu_0}{\Gamma(\omega+1)} + \sum_1^m(\frac{\theta_i}{\Gamma\omega+\beta_i+1}),$$

and

$$\Delta_4 = \frac{\lambda\delta_0}{\Gamma(\omega+1)\Gamma(\vartheta+1)} + \frac{\lambda}{\Gamma(\omega+1)} + \sum_1^m \frac{\xi}{\Gamma(\omega+\beta_i+1)}.$$

Proof. The continuity of $\bar{\mathbb{C}}$ follows from the definition of $\bar{\mathbb{C}}$. In addition, we have

$$|\bar{\mathbb{C}}u(t)| \leq |\mathbf{I}^\omega\Psi(1,u(1),{}^c\mathbf{D}^\rho u(1))| + |\mathbf{I}^\omega f(1,u(1),{}^c\mathbf{D}^\rho u(1))| + \sum_1^m |\mathbf{I}^{\omega+\beta_i}h_i(1,u(1),{}^c\mathbf{D}^\rho u(1)))|$$

which is in view of (3.9) implies that

$$|\bar{\mathbb{C}}u(t)| \leq \frac{\delta_0(\mu_0\|u\|_\rho + \lambda)}{\Gamma(\omega+1)\Gamma(\vartheta+1)} + \frac{1}{\Gamma(\omega+1)}(\mu_0\|u\|_\rho + \lambda) + \sum_1^m \frac{(\theta_i\|u\|_\rho + \xi)}{\Gamma(\omega+\beta_i+1)}.$$

Hence, it follows that

$$\|\bar{\mathbb{C}}(u)\|_\rho \leq \frac{\delta_0(\mu_0\|u\|_\rho + \lambda)}{\Gamma(\omega+1)\Gamma(\vartheta+1)} + \frac{1}{\Gamma(\omega+1)}(\mu_0\|u\|_\rho + \lambda) + \sum_1^m \frac{(\theta_i\|u\|_\rho + \xi)}{\Gamma(\omega+\beta_i+1)}. \tag{3.36}$$

The equi-continuity of $\bar{\mathbb{C}}$ follows from the fact that

$$|\bar{\mathbb{C}}(u)t_2 - \bar{\mathbb{C}}(u)t_1| = 0, \text{ for all } t_1, t_2 \in I.$$

Hence, $\bar{\mathbb{C}}$ is compact and it follows that $\bar{\mathbb{C}}$ is μ-Lipschtiz with constant 0. □

Lemma 3.6. *Under the hypothesis H_1–H_3, the operator $\mathbb{C}$ is μ-Lipschtiz with constant τ and satisfies the following growth condition*

$$\|\mathbb{C}u\|_\rho \le (\Delta_3 + c_1)\|u\|_\rho + (\Delta_4 + d_1). \tag{3.37}$$

Proof. Define $F(u) = \psi_2(u(\eta)) - \psi_1(u(\eta))$, for $u_1,\ u_2 \in \mathbb{E}$, consider

$$|F(u_2)t - F(u_1)t| \le |\psi_1(u_2(\eta)) - \psi_1(u_1(\eta))| + |\psi_2(u_2(\eta)) - \psi_2(u_1(\eta))|,$$

which in view of H_2 implies that

$$|F(u_2)t - F(u_1)t| \le (\tau_1 + \tau_2)\|u_2 - u_1\| = \tau\|u_2 - u_1\|,$$

$\tau = \tau_1 + \tau_2$. Since $\mathbb{C} = (\psi_2(u(\eta)) - \psi_1(u(\eta))) - \bar{\mathbb{C}} = F(u) - \bar{\mathbb{C}}$ and by Lemma 3.5 $\bar{\mathbb{C}}$ is μ-Lipschtiz with constant 0. Hence, $\mathbb{C}$ is μ-Lipschtiz with constant τ. Further, $\|\bar{\mathbb{C}}u\|_\rho \le \Delta_3\|u\|_\rho + \Delta_4$ by Lemma 3.5, it follows that

$$\|\mathbb{C}u\|_\rho \le (\Delta_3 + c_1)\|u\|_\rho + (\Delta_4 + d_1). \tag{3.38}$$

□

Choose the parameters such that $\Delta_1 + \Delta_3 + 2c_1 + c_2 < 1$. Choose $R \ge \max\{\tau_1 + \tau, \frac{\Delta_2+\Delta_4+2d_1+d_2}{1-(\Delta_1+\Delta_3+2c_1+c_2)}\}$. Define $S = \{u \in \mathbb{E} : \|u\|_\rho \le R\}$, then S is closed, convex and bounded subset of $\mathbb{E}$.

Theorem 3.7. *Under the assumptions* (H_1)–(H_3)*, the system* (3.6) *has at least one solution $u \in \mathbb{E}$ provided* $\Delta_1 + \Delta_3 + 2c_1 + c_2 < 1$.

Proof. By Lemma 3.3, the operator $\mathbb{A}$ is μ-Lipschitz with constant τ_1. Using Lemma 3.6, the operator $\mathbb{C}$ is μ-Lipschitz with constant τ. By Lemma 3.4, the operator $\mathbb{B}$ is compact. Now for $v \in S$ and $u \in \mathbb{E}$, consider the equation $u = \mathbb{A}u + \mathbb{B}v\mathbb{C}u$, which implies that

$$\|u\|_\rho \le \|\mathbb{A}u\|_\rho + \|\mathbb{B}y\|_\rho\|\mathbb{C}u\|_\rho.$$

Using (3.30), (3.31) and (3.37), we obtain

$$\|u\|_\rho \le (\Delta_1 + c_1)\|u\|_\rho + (\Delta_2 + d_1) + (\Delta_3 + c_1 + c_2)\|u\|_\rho + (\Delta_4 + d_1 + d_2).$$

That implies

$$(1 - (\Delta_1 + \Delta_3 + 2c_1 + c_2))\|u\|_\rho \le (\Delta_2 + \Delta_4 + 2d_1 + d_2).$$

Hence, it follows that

$$\|u\|_\rho \le \frac{\Delta_2 + \Delta_4 + 2d_1 + d_2}{(1 - (\Delta_1 + \Delta_3 + 2c_1 + c_2)} \le R$$

which implies that $u \in S$. Further we have $M = \|\mathbb{B}u\|_\rho = 1$ and $R \ge \tau_1 + \tau$. Finally from above, we conclude that (3.6) has at least one solution $u \in \mathbb{E}$. □

Theorem 3.8. *Under the assumptions* (H_1)*–*(H_4)*, the system* (3.6) *has a unique solution in* S *provided that*

$$\kappa_1 + \kappa_2((\Delta_3 + c_1)R + (\Delta_4 + d_1)) + k_3 < 1. \tag{3.39}$$

Proof. For $u_1, u_2 \in$ S, Consider

$$\begin{aligned}&\|T(u_2) - T(u_1)\|_\rho = \|Au_2 + Bu_2Cu_2 - (Au_1 + Bu_1Cu_1)\|_\rho \leq \\ &\|Au_2 - Au_1\|_\rho + \|Cu_1\|_\rho\|Bu_2 - Bu_1\|_\rho + \|Bu_2\|_\rho\|Cu_2 - Cu_1\|_\rho.\end{aligned} \tag{3.40}$$

By (3.20), (3.21), and (3.23), we have

$$\begin{aligned}&\|\mathbb{A}u_2 - \mathbb{A}(u_1)\|_\rho \leq \kappa_1\|u_1 - u_2\|_\rho,\\ &\|\mathbb{B}u_2 - \mathbb{B}(u_1)\|_\rho \leq \kappa_2\|u_1 - u_2\|_\rho,\\ &\|\mathbb{C}u_2 - \mathbb{C}(u_1)\|_\rho \leq \kappa_3\|u_1 - u_2\|_\rho.\end{aligned} \tag{3.41}$$

Using (3.41) in (3.40), we obtain

$$\|Tu_2 - Tu_1\|_\rho \leq \kappa_1\|u_1 - u_2\|_\rho + \|Cu_1\|_\rho\kappa_2\|u_1 - u_2\|_\rho + \kappa_3\|u_1 - u_2\|_\rho\|Bu_2\|_\rho$$

which in view of (3.31), and (3.38) implies that

$$\begin{aligned}\|Tu_2 - Tu_1\|_\rho \leq &\kappa_1\|u_1 - u_2\|_\rho + ((\Delta_3 + c_1)\|u\|_\rho + (\Delta_4 + d_1))\kappa_2\|u_1 - u_2\|_\rho +\\ &\kappa_3\|u_1 - u_2\|_\rho.\end{aligned}$$

Further, the above relation implies that

$$\begin{aligned}\|\mathbb{T}u_1 - \mathbb{T}u_2\|_\rho &\leq (\kappa_1 + \kappa_2((\Delta_3 + c_1)\|u\|_\rho + (\Delta_4 + d_1)) + k_3)\|u_1 - u_2\|_\rho\\ &\leq (\kappa_1 + \kappa_2((\Delta_3 + c_1)R + (\Delta_4 + d_1)) + k_3)\|u_1 - u_2\|_\rho,\end{aligned} \tag{3.42}$$

and uniqueness follows by the Banach contraction principle. □

4. Ulam Hyers stability

U-H stability result is developed for (1.1). For detail introduction and results of U-H stability, we refer [47, 48].

Definition 4.1. The problem (3.6) is said to be U-H stable, if there exists a constant $\zeta > 0$, such that for a given $\varphi > 0$, and for each solution u of the inequality

$$\|u - (\mathbb{A}u + \mathbb{B}u\mathbb{C}u)\|_\rho < \varphi, \tag{4.1}$$

there exists a solution $\bar{u}(t)$ of (3.6). Then one has

$$\bar{u}(t) = \mathbb{A}\bar{u}(t) + \mathbb{B}\bar{u}(t)\mathbb{C}\bar{u}(t),$$

such that

$$\|u - \bar{u}\|_\rho < \varphi\zeta.$$

Theorem 4.2. *Under the assumptions* (H_2) *and* (H_4)*, the problem* (1.1) *is U-H stable provided* $k + k_1 < 1$.

Proof. Let $u \in \mathbb{E}$ satisfies the inequality (4.1), and $\bar{u} \in \mathbb{E}$ be a solution of (1.1) which satisfies the Eq (3.6). Consider

$$\begin{aligned} \|u - \bar{u}\|_\rho &= \|u - (\mathbb{A}\bar{u} + \mathbb{B}\bar{u}\mathbb{C}\bar{u})\|_\rho \leq \|u - (\mathbb{A}u + \mathbb{B}u\mathbb{C}u)\|_\rho \\ &+ \|(\mathbb{A}u + \mathbb{B}u\mathbb{C}u) - (\mathbb{A}\bar{u} + \mathbb{B}\bar{u}\mathbb{C}\bar{u})\|_\rho < \varphi + \|\mathbb{T}u - \mathbb{T}\bar{u}\|_\rho. \end{aligned} \tag{4.2}$$

Now using (3.42) and (4.1), we obtain

$$\begin{aligned} \|u - \bar{u}\|_\rho \leq& \varphi + (\kappa_1 + \kappa_2((\Delta_3 + c_1)R + (\Delta_4 + d_1)) + k_3)\|u_1 - u_2\|_\rho \\ &= \varphi + K\|u_1 - u_2\|_\rho, \end{aligned} \tag{4.3}$$

where $K = \kappa_1 + \kappa_2((\Delta_3 + c_1)R + (\Delta_4 + d_1)) + k_3$. Hence, it follows that

$$\|u - \bar{u}\|_\rho < \varphi\zeta, \text{ where } \zeta = \frac{1}{1 - K}.$$

□

5. Illustrative application

Here, we present an application to demonstrate our results.

Example 5.1. *Consider the following problem by taking $n = 2$ as*

$$\begin{aligned} &{}^c\mathbf{D}^{0.5}\Big[\frac{{}^c\mathbf{D}^{1.5}u(t) - \sum_1^m \mathbf{I}^{1.5}h_i(t, u(t), {}^c\mathbf{D}^{1.5}u(t))}{f(t, u(t), {}^c\mathbf{D}^{1.5}u(t))}\Big] = g(t, u(t), \mathbf{I}^{1.5}u(t)),\ t \in \mathrm{I} = [0, 1], \\ &u(0) = \psi_1(u(0.5)),\ u'(0) = 0,\ u(1) = \psi_2(u(0.5)). \end{aligned} \tag{5.1}$$

Consider

$$\begin{aligned} h_i(t, u(t), {}^c\mathbf{D}^{1.5}u(t)) &= \frac{\sin|u(t)| + \sin|{}^c\mathbf{D}^{1.5}u(t)|}{100 + t^2}, \\ f(t, u(t), {}^c\mathbf{D}^{1.5}u(t)) &= \frac{\sin|u(t)| + \sqrt{|{}^c\mathbf{D}^{1.5}u(t)|}}{50 + e^{-t^2}}, \\ g(t, u(t), \mathbf{I}^{1.5}u(t)) &= \frac{\sqrt{|u(t)|} + \mathbf{I}^{1.5}u(t)}{150 + t}, \\ \psi_1(u(0.5)) &= \frac{\sin|u(0.5)|}{50},\ \psi_2(u(0.5) = \frac{\sin|u(0.5)|}{50}. \end{aligned}$$

It is easy to show that the conditions of Theorem 3.2 and 3.8 are satisfied. Therefore, the given problem (5.1) *has at least one solution. Further, the solution uniqueness condition also holds. Also, one can obviously verified the condition of U-H stability given in Theorem 4.2.*

6. Conclusions

In this manuscript, a nonlinear problem of S-HFDEs has been investigated by using a sophisticated tool known as topological degree theory. We have used a degree of non-compactness along with

Carathеódory condition to establish appropriate results for the qualitative theory. Usually, fixed point theory involves strong compact conditions which require more restrictions on the nonlinear operators. Therefore, to replace the strong compact condition with some weaker compact condition, the proposed degree theory is a powerful tool. The concerned tool has the ability to relax the criteria and hence can be applied to large numbers of nonlinear problems of differential and integral equations. On the other hand, stability is an important consequence of nonlinear analysis. Therefore, a result based on U-H concepts for stability has been established. Finally, to verify our obtained results, we have given an illustrative problem. In the future, the degree theory will be applied in hybrid problems of fractal-fractional differential equations which have the ability to describe complex and irregular geometry in more diligent ways. Also, the mentioned degree theory has not yet been used in dealing with non-singular type hybrid fractional differential equations. Therefore, the aforesaid area will be our next target.

Acknowledgments

All the authors are thankful to Prince Sultan University for paying the APC and support through TAS research lab.

Conflict of interest

The authors declare no conflict of interest.

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