AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(4): 9624–9646.
DOI: 10.3934/math.2023486
Received: 07 November 2022
Revised: 05 February 2023
Accepted: 06 February 2023
Published: 20 February 2023

Research article

Extinction and permanence of a general non-autonomous discrete-time SIRS epidemic model

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Abstract: We investigate a non-autonomous discrete-time SIRS epidemic model with nonlinear incidence rate and distributed delays combined with a nonlinear recovery rate taken into account the impact of health care resources. Two threshold parameters $\mathcal{R}_0, \mathcal{R}_\infty$ are obtained so that the disease dies out when $\mathcal{R}_0 < 1$; and the infective persists indefinitely when $\mathcal{R}_\infty > 1$.

Keywords: non-autonomous SIRS model; threshold parameters; extinction; permanence; number of hospital beds
Mathematics Subject Classification: 34D05, 34D23, 92D30

1. Introduction

In recent years, mathematical epidemic models have been extensively investigated in order to understand the dynamics and control the spreading of various diseases, see for instance [4–7, 11, 13–15, 19, 20, 23–27] and the references therein. Related to this work are the following studies. Zhang et al. [26] investigated a non-autonomous SIRS epidemic model with a standard incidence rate and distributed delays of the form

$$\begin{cases} \dot{S}(t) = \Lambda(t) - \beta(t) \displaystyle\int_0^\tau p(\xi) I(t-\xi) S(t)\, d\xi - \mu_1(t) S(t) + \gamma(t) R(t), \\ \dot{I}(t) = \beta(t) \displaystyle\int_0^\tau p(\xi) I(t-\xi) S(t)\, d\xi - (\mu_2(t) + k(t)) I(t), \\ \dot{R}(t) = k(t) I(t) - (\mu_3(t) + \gamma(t)) R(t), \end{cases} \tag{1.1}$$

where $S(t)$, $I(t)$, and $R(t)$ are the numbers of susceptible, infectious, and recovered individuals at time t, respectively. The distributed delays are used to model the infection mechanisms of some diseases, where infected individuals may not be infectious until some time after becoming infected, and that the infectivity is a function of the duration since infection, up to some maximum duration. Various (time-dependent) parameters in (1.1) are defined as follows: Λ is the growth (or recruitment) rate of the population, β is the daily contact rate, that is the average number of contacts per day, μ_i, for $i = 1, 2, 3$, are the instantaneous per capita mortality rates of S-, I-, and R-classes, respectively, γ is the rate that the recovered individual loses immunity and returns to be susceptible, ξ is a time taken for an infected individual to become infectious and $p(\xi)$ is the distributed proportion of the population taking time ξ after being infected to become infectious, τ is the infected period, and k is the recovery rate that can incorporate basic medical treatment and prevention of the disease. The main result in [26] is that they obtain two threshold values $R_\star$ and $R^\star$, which depend on the parameters of the model, so that the disease is permanent if $R_\star > 1$ while if $R^\star < 1$ then the disease extincts. Global behavior of the model is also studied using the Lyapunov functional method.

Enatsu et al. [3] investigated an autonomous version of (1.1), i.e. $\Lambda, \beta, \mu_1, \gamma, \mu_2, k, \mu_3$ are constants, but with a general nonlinear incidence rate and distributed delays of the form

$$\beta \int_0^\tau p(\xi) g(I(t-\xi)) S(t)\, d\xi, \tag{1.2}$$

where the nonlinear function $g(I)$ satisfies

(A1) $g(I)$ is continuous and monotone increasing on $[0, \infty)$ with $g(0) = 0$, and

(A2) $g(I)/I$ is monotone decreasing on $(0, \infty)$ with $\lim_{I \to 0^+} g(I)/I = 1$.

Using the Lyapunov functional method, the authors derived the basic reproduction number R_0 and established sufficient conditions of the rate of immunity loss for the global asymptotic stability of an endemic equilibrium for the model.

To mitigate the impact of the disease, some control and prevention should be included into modeling the disease. [16, 22] investigated SIR and SIRS epidemic models where the impact of health care resources especially hospital beds is included, so that the recovery rate is a nonlinear function of I of the form

$$k(I, t) = k_0 + (k_1 - k_0) \frac{b(t)}{I + b(t)} \quad (0 < k_0 \le k_1, b(t) > 0). \tag{1.3}$$

Here, according to [22], k_1 is the maximum per capita recovery rate when health care resources are adequate and the number of infected people is low, k_0 is the minimum per capita recovery rate due to the lack of basic clinical resources, and b is a parameter measuring the availability of hospital beds. Complex dynamics of the models are then derived in those papers.

Based on the above discussions, it is interesting to consider the following general non-autonomous SIRS epidemic model

$$\begin{cases} \dot{S}(t) = \Lambda(t) - \beta(t) \displaystyle\int_0^\tau p(\xi) g(I(t-\xi)) S(t)\, d\xi - \mu_1(t) S(t) + \gamma(t) R(t), \\ \dot{I}(t) = \beta(t) \displaystyle\int_0^\tau p(\xi) g(I(t-\xi)) S(t)\, d\xi - (\mu_2(t) + k(I(t), t)) I(t), \\ \dot{R}(t) = k(I(t), t) I(t) - (\mu_3(t) + \gamma(t)) R(t), \end{cases} \tag{1.4}$$

where $S(t)$, $I(t)$, and $R(t)$ and $\Lambda(t), \beta(t), \mu_1(t), \gamma(t), \mu_2(t), \mu_3(t), p(\xi)$ are the same as explained in (1.1) and (1.2). In the model (1.4), however, we relax the assumptions (A1), (A2) and also we do not require the nonlinear recovery rate $k(I, t)$ to have the form (1.3). Namely, for $k(I, t)$, we assume that there are positive constants k_0, k_1 such that

$$k_0 \leq k(I, t) \leq k_1 \tag{1.5}$$

for all $I \geq 0$ and $t \geq 0$. For $g(I)$, it is assumed to be continuous, $g(I) \geq 0$ on $[0, \infty)$, $g(0) = 0$ and $g(I) > 0$ for $I > 0$, and also that $\lim_{x \to 0^+}(g(x)/x)$ exists and is positive. Now we define the function

$$f(I) = \begin{cases} g(I)/I & \text{if } I > 0, \\ \lim_{x \to 0^+}(g(x)/x) & \text{if } I = 0. \end{cases} \tag{1.6}$$

In epidemiology, f is the transmission function of the disease. Observe that f is continuous and $f(I) > 0$ on $[0, \infty)$. Clearly, $g(I) = I f(I)$, so the incidence rate appearing in (1.4) can be expressed as

$$\beta(t) \int_0^\tau p(\xi) f(I(t - \xi)) I(t - \xi) S(t)\, d\xi. \tag{1.7}$$

Let us mention some specific examples of incidence rates. If $f(I) = 1$ is constant, then $g(I)S = IS$ is simply the standard (or bilinear) incidence rate; if $f(I) = \alpha_3/(1 + \alpha_1 I)$, then $g(I)S = \alpha_3 IS/(1 + \alpha_1 I)$ is the saturated incidence rate [2]; if $f(I) = \alpha_3/(1 + \alpha_2 I + \alpha_1 I^2)$, then $g(I)S = \alpha_3 IS/(1 + \alpha_2 I + \alpha_1 I^2)$ is the non-monotone incidence rate [8, 19, 20].

An interesting and important question is whether the results obtained for the continuous-time SIRS model (1.4) can be extended to the corresponding discrete-time epidemic model. In this work, we investigate the following discrete-time non-autonomous SIRS epidemic model: For a step size $h > 0$ and the maximum infectious period $m \in \mathbb{N}$, consider

$$\begin{cases} S_{n+1} = h\Big(\Lambda(n) - \beta(n) \sum_{j=0}^{m} p_j f(I_{n-j}) I_{n-j} S_{n+1} - \mu_1(n) S_{n+1} + \gamma(n) R_{n+1}\Big) + S_n, \\ I_{n+1} = h\Big(\beta(n) \sum_{j=0}^{m} p_j f(I_{n-j}) I_{n-j} S_{n+1} - (\mu_2(n) + k(I_n, n)) I_{n+1}\Big) + I_n, \\ R_{n+1} = h\left(k(I_n, n) I_{n+1} - (\mu_3(n) + \gamma(n)) R_{n+1}\right) + R_n, \\ S_j \geq 0, \quad I_j \geq 0, \quad R_j \geq 0 \quad (-m \leq j \leq 0), \quad S_0 > 0, \quad I_0 > 0, \end{cases} \tag{1.8}$$

where S_n, I_n, and R_n are the numbers of susceptible, infectious, and recovered individuals at time n, respectively, and $p_j \geq 0\, (j = 0, 1, \ldots, m)$ is the distributed proportion of the population taking time j to become infectious, and $\Lambda(n)$, $\beta(n)$, $\mu_1(n)$, $\gamma(n)$, $\mu_2(n)$, $k(I_n, n)$, $\mu_3(n)$ are the discretized values of the corresponding continuous-time functions explained in (1.1), (1.2), and (1.5). We have obtained (1.8) from (1.4) by applying Mickens's nonstandard finite difference discretization (see [10, 17, 27]; see also Appendix for a brief introduction). Let us list some known results for (1.8). A global attractivity for the autonomous version of (1.8) with $k = k(n)$ and a bit more general f is investigated in [17]. In [27], the threshold conditions are obtained for (1.8) with the non-delay standard incidence rate.

This paper is organized as follows. In Section 2 we introduce notations and basic results used in this work. Then we derive the positivity and boundedness of solutions (S_n, I_n, R_n) of (1.8) in Section 3. In Section 4, a threshold condition is obtained so that the permanence of disease for the model (1.8) is guaranteed. Then, we derive in Section 5 a threshold condition so that the extinction of the disease for (1.8) holds. In Section 6, some numerical simulations are presented. Finally, a discussion is given in Section 7.

2. Notations and preliminaries

Notation. In this work, we denote

- $G_n = \beta(n) \sum_{j=0}^{m} p_j f(I_{n-j}) I_{n-j}$ and $K_n = k(I_n, n)$.
- $\Pi_1(n) = 1 + h\mu_1(n)$, $\Pi_2(n) = 1 + h\mu_2(n)$, and $\Pi_3(n) = 1 + h(\mu_3(n) + \gamma(n))$.
- For a sequence $\{a(n)\}$, denote $a^u = \limsup_{n\to\infty} a(n)$, $a^l = \liminf_{n\to\infty} a(n)$, $a^M = \sup_{n\in\mathbb{N}_0} a(n)$, $a^L = \inf_{n\in\mathbb{N}_0} a(n)$. Same notations are applied to functions as well, e.g. $f^M = \sup_{I\geq 0} f(I)$ and $G^M = \sup_{n\in\mathbb{N}_0} G_n$.

The system (1.8) is said to be *permanent* if there exist constants $l_i, L_i > 0$ $(i = 1, 2, 3)$ such that any solution (S_n, I_n, R_n) satisfies

$$\begin{cases} l_1 \leq \liminf_{n\to\infty} S_n \leq \limsup_{n\to\infty} S_n \leq L_1, \\ l_2 \leq \liminf_{n\to\infty} I_n \leq \limsup_{n\to\infty} I_n \leq L_2, \\ l_3 \leq \liminf_{n\to\infty} R_n \leq \limsup_{n\to\infty} R_n \leq L_3. \end{cases} \tag{2.1}$$

On the other hand, (1.8) is said to exhibit *extinction* provided $\lim_{n\to\infty} I_n = 0$ for any solution.

We assume the following conditions for (1.8). First of all, $\Lambda(n), \beta(n), \mu_1(n), \mu_2(n), \mu_3(n), \gamma(n)$ are bounded and positive for all $n \in \mathbb{N}_0$, $\sum_{j=0}^{m} p_j = 1$, and there are constants $k_0 > 0, k_1 > 0$ such that $k_0 \leq K(I, n) \leq k_1$ for all $I \geq 0$ and $n \in \mathbb{N}_0$. We list all other assumptions that will be used in this work.

(H1) $f \geq 0$ is a bounded continuous function on $[0, \infty)$.

(H2) $\mu_1(n) \leq \min\{\mu_2(n), \mu_3(n)\}$ for all $n \in \mathbb{N}_0$.

(H3) There exist $n_0 \in \mathbb{N}, \chi \geq 1$, and $0 < q < 1$ such that

$$\prod_{n=n_1}^{N} \frac{1}{1 + h\mu_1(n)} \leq \chi q^{N-n_1+1} \quad \forall N \geq n_1 \geq n_0.$$

(H4) There exist $\lambda > 0$ and $r \in \mathbb{N} \cup \{0\}$ such that

$$\sum_{n=n_1}^{n_1+r} \Lambda(n) \geq \lambda \quad \forall n_1 \geq 0.$$

The following key result will be used in the study of disease-free states of (1.8) and also employed repeatedly in the investigation of the permanence and extinction phenomena. The proof is inspired by Lemma 2 in [21] and Lemma 2.2 in [24].

Lemma 1. *Let* $\{a(n)\}, \{b(n)\}$ *be sequences such that* $a(n) > 0$ *and* $0 < b(n) < 1$ *for all* $n \in \mathbb{N}_0$. *Assume that* $a^u < \infty$, $b^l > 0$, *and that there exist* n_0, $\chi > 0$, *and* $q \in (0, 1)$ *such that*

$$b(n_1)b(n_1 + 1)\cdots b(n_2) \le \chi q^{n_2 - n_1 + 1} \quad \forall\, n_2 \ge n_1 \ge n_0. \tag{2.2}$$

Let x_n *be a positive solution to the equation*

$$x_{n+1} = a(n)b(n) + b(n)x_n \quad (n \ge 0) \tag{2.3}$$

with initial value $x_0 \ge 0$. *Then the following results hold:*

(1) There exist constants $x^\star \ge 0, x_\star \ge 0$ *depending only upon* $\{a(n)\}$ *and* $\{b(n)\}$ *such that*

$$\limsup_{n\to\infty} x_n = x^\star \quad \text{and} \quad \liminf_{n\to\infty} x_n = x_\star,$$

independent of the initial condition x_0. *In fact, we have*

$$x^\star = \limsup_{N\to\infty} \sum_{n=n_1}^{N} a(n)b(n)b(n+1)\cdots b(N), \tag{2.4}$$

$$x_\star = \liminf_{N\to\infty} \sum_{n=n_1}^{N} a(n)b(n)b(n+1)\cdots b(N), \tag{2.5}$$

for any $n_1 \ge n_0$, *and the following estimate holds*

$$x^\star \le a^u \frac{\chi q}{1-q}. \tag{2.6}$$

Moreover, each fixed solution of (2.3) is globally uniformly attractive, i.e. if x'_n *is also a solution of (2.3) with initial value* $x'_0 \ge 0$, *then*

$$\lim_{n\to\infty}(x_n - x'_n) = 0.$$

(2) Suppose there exist constants $a_0 > 0$ *and* $r \in \mathbb{N} \cup \{0\}$ *such that*

$$\sum_{n=n_1}^{n_1+r} a(n) \ge a_0 \quad \forall\, n_1 \ge 0. \tag{2.7}$$

Then $x^\star > 0$ *and* $x_\star > 0$.

(3) Suppose y_n *satisfies*

$$y_{n+1} \le a(n)b(n) + b(n)y_n \quad \forall\, n \ge 0.$$

Then

$$\limsup_{n\to\infty} y_n \le x^\star.$$

If in addition $y_0 \le x_0$ *then* $y_n \le x_n$ *for all n.*

(4) *Suppose z_n satisfies*

$$z_{n+1} \geq a(n)b(n) + b(n)z_n \quad \forall\, n \geq 0.$$

Then

$$\liminf_{n\to\infty} z_n \geq x_\star.$$

If in addition $z_0 \geq x_0$ then $z_n \geq x_n$ for all n.

(5) *Let $\{a(n)\}, \{b(n)\}$ be as above and $\{\tilde{b}(n)\}$ be another sequence satisfying $\tilde{b}(n) \in (0,1)$ for all $n \in \mathbb{N}_0$, $\tilde{b}^l > 0$, and (2.2) holds with $b(n), \chi, q, n_0$ replaced by $\tilde{b}(n), \chi, q, \tilde{n}_0$, respectively. Assume also that there exist constants $\varepsilon > 0$ and $N_0 \in \mathbb{N}$ such that*

$$|b(n) - \tilde{b}(n)| < \varepsilon \quad \forall\, n \geq N_0. \tag{2.8}$$

Let $\tilde{x}_n$ be a positive solution of the equation

$$\tilde{x}_{n+1} = a(n)\tilde{b}(n) + \tilde{b}(n)\tilde{x}_n \quad (n \geq 0).$$

Then there is a constant $C = C(a^u, x^\star, \chi, q) > 0$ such that

$$|\tilde{x}^\star - x^\star| < C\varepsilon, \quad |\tilde{x}_\star - x_\star| < K\varepsilon,$$

where according to (1), $\tilde{x}^\star = \limsup_{n\to\infty} \tilde{x}_n$ and $\tilde{x}_\star = \liminf_{n\to\infty} \tilde{x}_n$.

(6) *For each $\varepsilon > 0$ small, there exist positive integers N_ε and D_ε such that if $n_1 \geq N_\varepsilon$ and $n_2 - n_1 \geq D_\varepsilon$, then*

$$\sum_{n=n_1}^{n_2} a(n)b(n)\cdots b(n_2) > x_\star - \varepsilon.$$

Proof. (1) For $N > n_1$, it is directly to show that

$$\begin{aligned} x_N &= A(N,n_1) + B(N,n_1)x_{n_1}, \\ y_N &\leq A(N,n_1) + B(N,n_1)y_{n_1}, \\ z_N &\geq A(N,n_1) + B(N,n_1)z_{n_1}, \end{aligned}$$

where $A(N,n_1) = \sum_{n=n_1}^{N-1} a(n)b(n)b(n+1)\cdots b(N-1)$ and $B(N,n_1) = b(n_1)b(n_1+1)\cdots b(N-1)$. For fixed $n_1 \geq n_0$, we have by assumption (2.2) that

$$B(N,n_1)x_{n_1} = b(n_1)b(n_1+1)\cdots b(N-1)x_{n_1} \leq \chi q^{N-n_1} x_{n_1} \to 0 \quad \text{as } N \to \infty,$$

and

$$A(N,n_1) \leq (\sup_{n\geq n_1} a(n)) \sum_{n=n_1}^{N-1} b(n)b(n+1)\cdots b(N-1) \leq (\sup_{n\geq n_1} a(n)) \sum_{n=n_1}^{N-1} \chi q^{N-n} \leq (\sup_{n\geq n_1} a(n)) \frac{\chi q}{1-q},$$

so $\{A(N,n_1)\}_{N=n_1+1}^{\infty}$ is bounded. Now $\lim_{N\to\infty}(x_N - A(N,n_1)) = \lim_{N\to\infty} B(N,n_1)x_{n_1} = 0$, it follows by the boundedness of $\{A(N,n_1)\}_{N=n_1+1}^{\infty}$ that

$$\limsup_{N\to\infty} x_N = \limsup_{N\to\infty} A(N,n_1) = x^\star, \quad \liminf_{N\to\infty} x_N = \liminf_{N\to\infty} A(N,n_1) = x_\star$$

and the two limits do not depend on n_1. This proves the first part and (2.4), (2.5).

For each $\varepsilon > 0$, by taking n_1 large enough, we get $\sup_{n\geq n_1} a(n) \leq a^u + \varepsilon$, hence

$$x^\star \leq a^u \frac{\chi q}{1-q},$$

proving (2.6).

The last part of (1) is true because

$$\lim_{N\to\infty}(x_N - x'_N) = \lim_{N\to\infty}(x_N - A(N,n_1)) - \lim_{N\to\infty}(x'_N - A(N,n_1)) = 0.$$

(3) and (4) immediately follow from the proof of (1). We prove (2). By assumption $b^l > 0$, so there exist $b_0 \in (0,1)$ and $N_0 \in \mathbb{N}$ such that $b(n) \geq b_0$ for all $n \geq N_0$. Fix $n_1 \geq \max\{n_0, N_0\}$. Then we have for all $N \geq n_1 + r + 1$ that

$$A(N,n_1) \geq \sum_{n=N-1-r}^{N-1} a(n)b(n)b(n+1)\cdots b(N-1) \geq \sum_{n=N-1-r}^{N-1} a(n)b_0^{r+1} \geq a_0 b_0^{r+1}.$$

Since $a_0 b_0^{r+1} > 0$, we have $x^\star \geq x_\star = \liminf_{N\to\infty} A(N,n_1) > 0$.

(5) Recall the well-known fact* that, for bounded sequences,

$$\max\{|\liminf_{n\to\infty} \tilde{x}_n - \liminf_{n\to\infty} x_n|, |\limsup_{n\to\infty} \tilde{x}_n - \limsup_{n\to\infty} x_n|\} \leq \limsup_{n\to\infty} |\tilde{x}_n - x_n|.$$

So it suffices to show that $\limsup_{n\to\infty} |\tilde{x}_n - x_n| \leq C\varepsilon$. For each n, we have

$$\begin{aligned}\tilde{x}_{n+1} - x_{n+1} &= (a(n)\tilde{b}(n) + \tilde{b}(n)\tilde{x}_n) - (a(n)b(n) + b(n)x_n)\\ &= a(n)(\tilde{b}(n) - b(n)) + \tilde{b}(n)\tilde{x}_n - b(n)x_n + \tilde{b}(n)x_n - \tilde{b}(n)x_n\\ &= a(n)(\tilde{b}(n) - b(n)) + (\tilde{b}(n) - b(n))x_n + \tilde{b}(n)(\tilde{x}_n - x_n).\end{aligned}$$

By the assumption (2.8), it follows that

$$|\tilde{x}_{n+1} - x_{n+1}| \leq \varepsilon(a(n) + x_n) + \tilde{b}(n)|\tilde{x}_n - x_n|.$$

Note that $\limsup_{n\to\infty} \varepsilon(a(n) + x_n) \leq \varepsilon(a^u + x^\star)$. Using part (3) and (2.6), we have

$$\limsup_{n\to\infty} |\tilde{x}_n - x_n| \leq (a^u + x^\star)\frac{\chi q}{1-q}\varepsilon.$$

so $\limsup_{n\to\infty} |\tilde{x}_n - x_n| \leq C\epsilon$, where $C = (a^u + x^\star)\chi q/(1-q) > 0$.

(6) We have $x_n \leq x^M := \max x_n$ for all n. For $\varepsilon > 0$, there exists $N_\varepsilon \geq n_0$ such that if $n \geq N_\varepsilon$ then $x_n \geq x_\star - \frac{\varepsilon}{2}$. By (2.2), there exists D_ε such that if $N - n_1 \geq D_\varepsilon$ then $b(n_1)b(n_1+1)\cdots b(N)x^M \leq \frac{\varepsilon}{2}$. Now let $n_1 \geq N_\varepsilon$ and $N \geq n_1 + D_\varepsilon$. Then we have

$$\sum_{n=n_1}^{N-1} a(n)b(n)b(n+1)\cdots b(N-1) = x_N - b(n_1)b(n_1+1)\cdots b(N-1)x_{n_1} \geq x_\star - \varepsilon,$$

as desired. □

*Let $c = \limsup |\tilde{x}_n - x_n|$, $a = \limsup \tilde{x}_n$, $b = \limsup x_n$. By the triangle inequality $\tilde{x}_n \leq |\tilde{x}_n - x_n| + x_n$, we get $a \leq c + b$, similarly, we also have $b \leq c + a$. Thus $|a - b| \leq c$. For the other fact, we simply use that $\liminf x_n = -\limsup(-x_n)$.

The following elementary result is employed throughout this work.

Lemma 2. *The system (1.8) can be expressed as*

$$\begin{cases} S_{n+1} = \dfrac{h\Lambda(n) + S_n + h\gamma(n)R_{n+1}}{\Pi_1(n) + hG_n}, \\ I_{n+1} = \dfrac{I_n + hG_nS_{n+1}}{\Pi_2(n) + hK_n}, \\ R_{n+1} = \dfrac{R_n + hK_nI_{n+1}}{\Pi_3(n)}. \end{cases} \tag{2.9}$$

where $\Pi_1(n) = 1 + h\mu_1(n), \Pi_2(n) = 1 + h\mu_2(n)$*,* $K_n = k(I_n, n)$*, and* $\Pi_3(n) = 1 + h(\mu_3(n) + \gamma(n))$*.*

Proof. By (1.8), we have $S_{n+1} = h\Lambda(n) - hG_nS_{n+1} - h\mu_1(n)S_{n+1} + h\gamma(n)R_{n+1} + S_n$, which directly leads to the first expression. Similarly, we have $I_{n+1} = hG_nS_{n+1} - h(\mu_2(n) + k(I_n, n))I_{n+1} + I_n$ and $R_{n+1} = hk(I_n, n)I_{n+1} - h(\mu_3(n) + \gamma(n))R_{n+1} + R_n$, so the other two expressions follow. □

3. Positivity and boundedness

We prove the positivity of solutions of (1.8).

Proposition 3. *Assume* $f(I) \geq 0$ *for all* $I \geq 0$*. Then every solution of the problem (1.8) is positive, that is* $S_n > 0, I_n > 0, R_n > 0$ *for all* $n > 0$*.*

Proof. For simplicity, we omit the dependence of $\Pi_1, \Pi_2, \Pi_3, \gamma$ on n in the following calculations. By (2.9), one can directly manipulate the identities to obtain

$$\begin{aligned} (\Pi_2 + K_n)\Pi_3(\Pi_1 + hG_n)S_{n+1} &= (\Pi_2 + K_n)\Pi_3(h\Lambda + S_n) + h\gamma(\Pi_2 + K_n)\Pi_3R_{n+1} \\ &= (\Pi_2 + K_n)\Pi_3(h\Lambda + S_n) + h\gamma(\Pi_2 + K_n)(R_n + K_nI_{n+1}) \\ &= (\Pi_2 + K_n)\Pi_3(h\Lambda + S_n) + h\gamma(\Pi_2 + K_n)R_n + h\gamma K_n(I_n + hG_nS_{n+1}) \\ &= (\Pi_2 + K_n)\Pi_3(h\Lambda + S_n) + h\gamma(\Pi_2 + K_n)R_n + h\gamma K_nI_n + h^2\gamma K_nG_nS_{n+1} \end{aligned}$$

thus

$$S_{n+1} = \frac{(\Pi_2 + K_n)\Pi_3(h\Lambda + S_n) + h\gamma(\Pi_2 + K_n)R_n + h\gamma K_nI_n}{\Pi_1(\Pi_2 + K_n)\Pi_3 + ((\Pi_2 + K_n)\Pi_3 - h\gamma K_n)hG_n}.$$

From the initial condition in (1.8), it is easily seen that $G_0 \geq 0$ so $S_1 > 0$. Observe that $\Pi_i > 1$ for $i = 1, 2, 3$ and $(\Pi_2 + K_n)\Pi_3 - h\gamma K_n > 0$. Using (2.9), we also have $I_1 > 0$ and $R_1 > 0$. Applying the same argument, we obtain $G_n, K_n \geq 0$ and so $S_{n+1}, I_{n+1}, R_{n+1} > 0$ for all n. □

In this work, we are interested in the extinction and permanence of disease of the model (1.8), so the asymptotic behaviors of disease-free states of the system is crucial. By definition, a disease-free state of (1.8) is a solution (S_n, I_n, R_n) where $I_n = 0$ for all $n \geq 0$. From Lemma 2, the system is reduced in this case to

$$S_{n+1} = h\Lambda(n)\frac{1}{1 + h\mu_1(n)} + \frac{1}{1 + h\mu_1(n)}S_n + \frac{h\gamma(n)R_{n+1}}{1 + h\mu_1(n)}, \quad R_{n+1} = \frac{R_n}{1 + h(\mu_3(n) + \gamma(n))}.$$

Let $N > n_1 \geq n_0$. By (H2), (H3), and Proposition 3, we have that

$$\begin{aligned} h\gamma(N)R_{N+1} &= h\gamma(N)\left\{\prod_{n=n_1}^{N}\frac{1}{1+h(\mu_3(n)+\gamma(n))}\right\}R_{n_1} \\ &\leq \left\{\prod_{n=n_1}^{N-1}\frac{1}{1+h\mu_1(n)}\right\}R_{n_1} \leq \chi q^{N-n_1}R_{n_1} \to 0, \end{aligned}$$

as $N \to \infty$. Similarly, $R_N \to 0$. Thus the asymptotic behaviors of the disease-free states of (1.8) are closely related to the properties of solutions S_n^0 of the following equation

$$S_{n+1}^0 = h\Lambda(n)\frac{1}{1+h\mu_1(n)} + \frac{1}{1+h\mu_1(n)}S_n^0. \tag{3.1}$$

The following proposition gives the uniform upper and lower bounds for any positive solution of (3.1).

Proposition 4. *Suppose that (H3) and (H4) hold. Then there exist constants* $S^{0,\star}, S_\star^0 > 0$ *such that*

$$S_\star^0 = \liminf_{n\to\infty} S_n^0 \leq \limsup_{n\to\infty} S_n^0 = S^{0,\star},$$

for any positive solution S_n^0 *of (3.1) regardless of the initial condition. In fact, we have*

$$S^{0,\star} = \limsup_{N\to\infty}\sum_{n=n_1}^{N}\frac{h\Lambda(n)}{(1+h\mu_1(n))\cdots(1+h\mu_1(N))}, \tag{3.2}$$

$$S_\star^0 = \liminf_{N\to\infty}\sum_{n=n_1}^{N}\frac{h\Lambda(n)}{(1+h\mu_1(n))\cdots(1+h\mu_1(N))}, \tag{3.3}$$

for all $n_1 \geq n_0$

Proof. Set $a(n) = h\Lambda(n)$ and $b(n) = \frac{1}{1+\mu_1(n)}$. It is obvious that $a^u < \infty$ and $b^l > 0$. Also, the hypotheses (H3) and (H4) are equivalent to (2.2) and (2.7) in Lemma 1, respectively. Hence, the proof immediately follows from Lemmas 1(1) and (2). □

The following proposition is a generalized version of the preceding one. It will be used in the proof of the permanence of disease (Theorem 8).

Proposition 5. *Suppose that (H3) and (H4) hold. Let* $\nu \geq 0$ *and define* μ_1^ν *by*

$$\mu_1^\nu(n) := \mu_1(n) + \beta^M f^M \nu \quad \textit{for all } n.$$

Then there exist constants $S^{\nu,\star}, S_\star^\nu > 0$ *such that if* S_n^ν *is a positive solution of the equation*

$$S_{n+1}^\nu = h\Lambda(n)\frac{1}{1+h\mu_1^\nu(n)} + \frac{1}{1+h\mu_1^\nu(n)}S_n^\nu, \tag{3.4}$$

then

$$S_\star^\nu = \liminf_{n\to\infty} S_n^\nu \leq \limsup_{n\to\infty} S_n^\nu = S^{\nu,\star}.$$

Proof. As in the proof of Proposition 4, choosing $a(n) = h\Lambda(n)$ and $b(n) = \frac{1}{1+\mu_1^\nu(n)}$, and applying (H3) and (H4), we get the conditions (2.2) and (2.7). By Lemmas 1(1) and (2), the result is true and the quantities $S^{\nu,\star}, S^\nu_\star$ are given by replacing μ_1 in the formulas (3.2) and (3.3) with μ_1^ν. ◻

Notice that

$$S^{\nu,\star} = \limsup_{N\to\infty} \sum_{n=n_1}^{N} \frac{h\Lambda(n)}{(1 + h\mu_1^\nu(n)) \cdots (1 + h\mu_1^\nu(N))}, \tag{3.5}$$

$$S^\nu_\star = \liminf_{N\to\infty} \sum_{n=n_1}^{N} \frac{h\Lambda(n)}{(1 + h\mu_1^\nu(n)) \cdots (1 + h\mu_1^\nu(N))}. \tag{3.6}$$

The following result shows that the "total population" of the model (1.8) is bounded above.

Proposition 6. *Assume $f(I) \geq 0$ for all $I \geq 0$ and that (H2), (H3) hold. Then the total population $T_n := S_n + I_n + R_n$ for (1.8) satisfies*

$$\limsup_{n\to\infty} T_n \leq S^{0,\star},$$

where $S^{0,\star}$ is given by (3.2).

Proof. Adding the equations in (1.8), applying the hypothesis (H2), and Proposition 3, we get

$$\begin{aligned} T_{n+1} &= T_n + h\left(\Lambda(n) - \mu_1(n)S_{n+1} - \mu_2(n)I_{n+1} - \mu_3(n)R_{n+1}\right) \\ &\leq T_n + h(\Lambda(n) - \mu_1(n)T_{n+1}), \end{aligned}$$

which implies

$$T_{n+1} \leq h\Lambda(n)\frac{1}{1 + h\mu_1(n)} + \frac{1}{1 + h\mu_1(n)}T_n.$$

The desired conclusion now follows from Lemma 1(3) and (H3). ◻

As a corollary, we obtain the boundedness of solutions (S_n, I_n, R_n) of (1.8).

Corollary 7. *Assume $f(I) \geq 0$ for all $I \geq 0$ and that (H2), (H3) hold. Then*

$$\limsup_{n\to\infty} S_n \leq S^{0,\star}, \quad \limsup_{n\to\infty} I_n \leq S^{0,\star}, \quad \limsup_{n\to\infty} R_n \leq S^{0,\star}.$$

Proof. Since $f \geq 0$, we have by Proposition 3 that $S_n > 0, I_n > 0, R_n > 0$ for all $n > 0$. Then, we have $S_n \leq T_n, I_n \leq T_n$, and $R_n \leq T_n$, hence the desired conclusion follows from Proposition 6. ◻

4. Permanence of disease

Now we prove our first main result.

Theorem 8. *Suppose that (H1)-(H4) hold, and that*

$$\mathcal{R}_\infty := \frac{\beta^l f(0) S^0_\star}{\mu_2^u + k_1} > 1, \tag{4.1}$$

where $S^0_\star$ *is given by (3.3),* $\beta^l = \liminf_{n\to\infty} \beta(n)$*, and* $\mu_2^u = \limsup_{n\to\infty} \mu_2(n)$*. Then the model (1.8) is permanent. Moreover, we will show that if*

$$\frac{\beta^l f(0) S^0_\star}{\mu_2^u + K^u} > 1, \tag{4.2}$$

where $K^u = \limsup_{n\to\infty} K_n \le k_1$*, then (1.8) is permanent.*

Proof. It suffices to prove the second statement because $\frac{\beta^l f(0) S^0_\star}{\mu_2^u + K^u} \ge \mathcal{R}_\infty$. Our proof is inspired by [18] and [14]. By the positivity (Proposition 3) and boundedness of solutions (Corollary 7), it suffices to prove uniform lower bounds (2.1) for S_n, I_n, and R_n of (1.8).

By the assumption (4.2) and (H1), there exist $\theta_0 > 0$ and $\varepsilon_0 > 0$ sufficiently small such that

$$\xi := \frac{(\beta^l - \theta_0)(\inf_{I\in[0,\varepsilon_0]} f(I))(S^0_\star - \theta_0)}{\mu_2^u + K^u + \theta_0} > 1. \tag{4.3}$$

This also implies $\beta^l - \theta_0 > 0$ and $S^0_\star - \theta_0 > 0$. We also assume $K^l - \theta_0 > 0$.

Choose $Q_0 \in \mathbb{N}$ independent of solutions to (1.8) such that

$$\text{if } n \ge n_0 + Q_0 \quad \text{then} \quad \beta^l - \theta_0 \le \beta(n) \le \beta^u + \theta_0 \text{ and } \mu_2(n) + K_n \le \mu_2^u + K^u + \theta_0. \tag{4.4}$$

Estimate for S_n. There is a constant $l_S > 0$ such that $\liminf_{n\to\infty} S_n \ge l_S$.

Proof of Estimate for S_n. By the second estimate in Corollary 7, we can take $P_0 \in \mathbb{N}$ which depend on solution (S_n, I_n, R_n) so that

$$\text{if } n \ge n_0 + P_0 \quad \text{then} \quad \max_{0\le j\le m}\{S(n-j), I(n-j), R(n-j)\} \le S^{0,\star} + \theta_0.$$

Let $n > n_1 \ge n_0 + \max\{P_0, Q_0\}$. By Proposition 3, we have $R_{n+1} \ge 0$. Also, $\beta(n)\sum_{j=0}^m p_j f(I_{n-j}) I_{n-j} \le (\beta^u + \theta_0) f^M (S^{0,\star} + \theta_0) < \infty$. Using the first equation in (1.8), we get

$$S_{n+1} \ge h(\Lambda(n) - (\beta^u + \theta_0) f^M (S^{0,\star} + \theta_0) S_{n+1} - \mu_1(n) S_{n+1}) + S_n,$$

that is

$$S_{n+1} \ge h\Lambda(n) \frac{1}{1 + h\tilde{\mu}_1(n)} + \frac{1}{1 + h\tilde{\mu}_1(n)} S_n,$$

where $\tilde{\mu}_1(n) = \mu_1(n) + (\beta^u + \theta_0) f^M (S^{0,\star} + \theta_0)$. Applying Lemma 1(4) and Proposition 5, we find that $\liminf_{n\to\infty} S_n \ge \tilde{S}^0_\star > 0$ where

$$\tilde{S}^0_\star = \liminf_{N\to\infty} \sum_{n=n_1}^{N} \frac{h\Lambda(n)}{(1 + h\tilde{\mu}_1(n)) \cdots (1 + h\tilde{\mu}_1(N))}. \tag{4.5}$$

Estimate for I_n. There is a constant $l_I > 0$ such that $\liminf_{n\to\infty} I_n \ge l_I$.

Proof of Estimate for I_n. We begin by proving a continuity result for the sum in (3.5) as $\nu \to 0$.

Claim 1. Let $\theta_0 > 0$ be as specified in (4.3). Then there exist ν_0, n_0', ρ, depending only on parameters in (1.8) and θ_0 and satisfying

$$0 < \nu_0 \le \varepsilon_0, \quad n_0' \ge n_0 + \max\{m, Q_0\}, \quad \rho \in \mathbb{N},$$

such that the following statement holds:

$$\text{if } n_1 \ge n_0',\ N \ge n_1 + \rho m \quad \text{then} \quad \sum_{n=n_1}^{N} \frac{h\Lambda(n)}{(1 + h\mu_1^{\nu_0}(n)) \cdots (1 + h\mu_1^{\nu_0}(N))} > S_\star^0 - \theta_0,$$

where $\mu_1^{\nu_0} := \mu_1 + \beta^M f^M \nu_0$ and $S_\star^0$ is given by (3.3).

Proof of Claim 1. First we specify ν_0. By (3.6), we have

$$S_\star^\nu = \liminf_{N\to\infty} \sum_{n=n_1}^{N} \frac{h\Lambda(n)}{(1 + h\mu_1^\nu(n)) \cdots (1 + h\mu_1^\nu(N))},$$

and the limit does not depend on n_1. We apply Lemma 1(5) with $a(n) = h\Lambda(n), b(n) = 1/(1 + h\mu_1(n))$, and $\tilde{b}(n) = 1/(1 + h\mu_1^\nu(n))$ and $\nu > 0$. Note that $\tilde{b}(n) \ge \frac{1}{1+h(\mu_1^M+\beta^M f^M \nu)} > 0$, where $\mu_1^M = \max_n \mu_1(n)$, and $\tilde{b}(n) \le b(n)$ for all n, so $\tilde{b}^l > 0$ and $\tilde{b}$ satisfies (2.2). Also, $|b(n) - \tilde{b}(n)| \le h\beta^M f^M \nu$, so applying Lemma 1(5), there is a constant $C > 0$ independent of ν such that

$$|S_\star^\nu - S_\star^0| \le Ch\beta^M f^M \nu.$$

Now we choose ν_0 small enough so that $\nu_0 \le \varepsilon_0$ and

$$Ch\beta^M f^M \nu_0 < \frac{\theta_0}{2}. \tag{4.6}$$

Then we have

$$S_\star^{\nu_0} > S_\star^0 - \frac{\theta_0}{2}.$$

Now we choose n_0', ρ. By Lemma 1(6), there exist n_0', ρ depending only on the Λ, μ_1, f, m, and θ_0, so that

$$\text{if } n_1 \ge n_0',\ N \ge n_1 + \rho m \quad \text{then} \quad \sum_{n=n_1}^{N} \frac{h\Lambda(n)}{(1 + h\mu_1^{\nu_0}(n)) \cdots (1 + h\mu_1^{\nu_0}(n_2))} > S_\star^{\nu_0} - \frac{\theta_0}{2}.$$

Combining the above two estimates, we obtain the desired result. □

Fix $\theta_0 > 0$ and ν_0, n_0', ρ as above. Notice that $n_0' \ge n_0 + Q_0$, so according to (4.4) we get

$$\forall\, n \ge n_0' : \quad \beta^l - \theta_0 \le \beta(n) \le \beta^u + \theta_0, \quad \mu_2(n) + K_n \le \mu_2^u + K^u + \theta_0. \tag{4.7}$$

Next, we prove that if $I_n \le \nu_0$ on an interval of length at least $\rho(m + 1)$, then $S_{N+1} > S_\star^0 - \theta_0$, where N is the right endpoint.

Claim 2. Let $n_1 \geq n'_0$, $N \geq n_1 + \rho m$, and (S_n, I_n, R_n) be a solution of (1.8). Then the following statement holds. If $0 \leq I_n \leq \nu_0$ for all $n \in [n_1 - m, N]$, then

$$S_{N+1} > S^0_\star - \theta_0. \tag{4.8}$$

Proof of Claim 2. Since $I_p \leq \nu_0$ on $[n_1 - m, N]$, we have for all $n \in [n_1, N]$ that

$$G_n = \beta(n) \sum_{j=0}^{m} p_j f(I_{n-j}) I_{n-j} \leq \beta^M \sum_{j=0}^{m} p_j f^M \nu_0 = \beta^M f^M \nu_0.$$

This gives using (1.8) that $S_{n+1} \geq h(\Lambda(n) - \beta^M f^M \nu_0 S_{n+1} - \mu_1(n) S_{n+1}) + S_n$, i.e.

$$S_{n+1} \geq h\Lambda(n) \frac{1}{1 + h\mu_1^{\nu_0}(n)} + \frac{1}{1 + h\mu_1^{\nu_0}(n)} S_n \quad \text{for } n = n_1, n_1 + 1, \ldots, N.$$

By induction, it is directly to see that

$$\begin{aligned} S_{N+1} &\geq \sum_{n=n_1}^{N} \frac{h\Lambda(n)}{(1 + h\mu_1^{\nu_0}(n)) \cdots (1 + h\mu_1^{\nu_0}(N))} + \frac{1}{(1 + h\mu_1^{\nu_0}(n_1)) \cdots (1 + h\mu_1^{\nu_0}(N))} S_{n_1} \\ &\geq \sum_{n=n_1}^{N} \frac{h\Lambda(n)}{(1 + h\mu_1^{\nu_0}(n)) \cdots (1 + h\mu_1^{\nu_0}(N))}, \end{aligned}$$

by the positivity of S_{n_1}. Applying Claim 1, the last term on the right hand side is greater than $S^0_\star - \theta_0$, which implies $S_{N+1} > S^0_\star - \theta_0$. So the claim is proved. □

In addition to the previous estimate for S_{N+1}, at a right endpoint N, we also have the following boosting estimate for I_{N+1}.

Claim 3. Let $n_1 \geq n'_0$, $N \geq n_1 + \rho m$, and (S_n, I_n, R_n) be a solution of (1.8). Then the following statement holds. If $0 \leq I_n \leq \nu_0$ for all $n \in [n_1 - m, N]$, then

$$I_{N+1} \geq \kappa \mathcal{I}_N, \quad \text{where } \mathcal{I}_N := \min_{p \in [N-m, N]} I_p, \tag{4.9}$$

and $\kappa > 1$ is the constant given by

$$\kappa := \frac{1 + h\xi(\mu_2^u + K^u + \theta)}{1 + h(\mu_2^u + K^u + \theta)}. \tag{4.10}$$

Proof of Claim 3. Observe that $\kappa > 1$ because $\xi > 1$. We can apply Claim 2 to get $S_{N+1} > S^0_\star - \theta_0$ and by (4.7) with that $N \geq n'_0$, we also have $\beta(N) \geq \beta^l - \theta_0$. Using (4.3) and that $\nu_0 \leq \varepsilon_0$, we obtain

$$\beta(N) f(I_{N-j}) S_{N+1} \geq (\beta^l - \theta_0) \left(\inf_{I \in [0, \varepsilon_0]} f(I) \right) (S^0_\star - \theta_0) = \xi(\mu_2^u + K^u + \theta_0)$$

for $j = 0, 1, \ldots, m$. Also, noting that $\mu_2(N) + K_N \leq \mu_2^u + K^u + \theta_0$ by (4.7). Applying Lemma 2, we have

$$I_{N+1} = \frac{I_N + h\beta(N) \sum_{j=0}^{m} p_j f(I_{N-j}) I_{N-j} S_{N+1}}{1 + h(\mu_2(N) + K_N)}$$

$$\geq \frac{\mathcal{I}_N + h\beta(N)\sum_{j=0}^{m} p_j f(I_{N-j})\mathcal{I}_N S_{N+1}}{1 + h(\mu_2(N) + K_N)}$$
$$\geq \frac{1 + h\sum_{j=0}^{m} p_j \xi(\mu_2^u + K^u + \theta_0)}{1 + h(\mu_2^u + K^u + \theta_0)}\mathcal{I}_N$$
$$\geq \kappa \mathcal{I}_N,$$

where we have used that $\sum_{j=0}^{m} p_j = 1$. □

Claim 4. It is impossible that $I_n \leq \nu_0$ for all sufficiently large n.

Proof of Claim 4. Suppose on the contrary that there is $N_0 \geq n_0'$ such that $I_n \leq \nu_0$ for all $n \geq N_0$. Denote $N_1 = N_0 + (\rho + 1)m$, and define $\mathcal{I}_n$ as in (4.9) above. Since $I_n \leq \nu_0$ for all $n \in [N_0, N_1]$, we have by Claim 3 that

$$I_{N_1+1} \geq \kappa \mathcal{I}_{N_1}.$$

This implies in particular that $\mathcal{I}_{N_1+1} = \min_{n\in[N_1+1-m,N_1+1]} I_n \geq \mathcal{I}_{N_1}$ because $\kappa > 1$. Repeating the preceding argument with that $\mathcal{I}_{N_1+1} \geq \mathcal{I}_{N_1}$, we get $I_{N_1+2} \geq \kappa\mathcal{I}_{N_1+1} \geq \kappa\mathcal{I}_{N_1}$. Continuing the argument, we finally obtain

$$I_n \geq \kappa \mathcal{I}_{N_1} \quad \text{for all } n \geq N_1 + 1.$$

This implies

$$\forall\, n \geq N_1 + 1 + m : \quad \mathcal{I}_n = \min\{I_{n-m}, \ldots, I_n\} \geq \kappa \mathcal{I}_{N_1}.$$

Let $N_2 = N_1 + 1 + \rho m$. Since $I_n \leq \nu_0$ for all $n \in [N_1 + 1 - m, N_2]$ and observe that $\mathcal{I}_{N_2} \geq \kappa\mathcal{I}_{N_1}$, it follows by the same argument as above that $I_n \geq \kappa\mathcal{I}_{N_2} \geq \kappa^2\mathcal{I}_{N_1}$ for all $n \geq N_2 + 1$. Repeating the process, we obtain, for $N_l := N_{l-1} + 1 + \rho m$, that

$$I_n \geq \kappa^l \mathcal{I}_{N_1} \quad \text{for all } n \geq N_l + 1.$$

Recalling $\kappa > 1$, so $\kappa^l \to \infty$ as $l \to \infty$. Now by selecting l large enough, we get from the above that there is $n > N_0$ such that $I_n \geq \kappa^l \mathcal{I}_{N_1} > \nu_0$. This contradicts that $I_n \leq \nu_0$ for all $n \geq N_0$. Therefore the claim is proved. □

After the above preparations, now we can prove the lower estimate for I_n. According to Claim 4, there are two possibilities: either

(i) $I_n > \nu_0$ for all n sufficiently large, or

(ii) I_n oscillates about ν_0 for large n.

Obviously, if (i) occurs then the desired estimate follows, namely

$$\liminf_{n\to\infty} I_n \geq \nu_0.$$

So assume (ii).

It suffices to prove the following statement. Suppose n_1, N are such that $N > n_1 \geq n_0'$ and

$$I_{n_1} \geq \nu_0, \quad I_N \geq \nu_0, \quad \text{and} \quad I_n < \nu_0 \quad \text{for all } n_1 < n < N.$$

Then

$$\forall\, n \in (n_1, N): \quad I_n \geq l_I := \frac{\nu_0}{(1 + h(\mu_2^u + K^u + \theta_0))^{\Delta}}, \quad \Delta := 1 + (\rho + 1)m. \tag{4.11}$$

Denote

$$n_2 = n_1 + 1 + \Delta, \quad n_3 = n_2 + 1 + \Delta, \quad n_4 = n_3 + 1 + \Delta, \quad \ldots$$

Let $n > n_1$, so $n - 1 \geq n_1 \geq n_0'$. By (4.7), we have $\mu_2(n-1) + K_{n-1} \leq \mu_2^u + K^u + \theta_0$. Employing the second identity in Lemma 2 and the positivity of G_{n-1}, S_n, we get

$$I_n \geq \frac{I_{n-1}}{1 + h(\mu_2(n-1) + K_{n-1})} \geq \frac{I_{n-1}}{1 + h(\mu_2^u + K^u + \theta_0)}.$$

By induction, we have for all $p \in \mathbb{N}$ satisfying $n - p \geq n_1$ that

$$I_n \geq \frac{I_{n-p}}{(1 + h(\mu_2^u + K^u + \theta_0))^p}.$$

Case $N \leq n_2$. For any $n \in (n_1, N)$, it follows by taking $p = n - n_1$, which satisfies $p \leq \Delta$, that

$$I_n \geq \frac{I_{n_1}}{(1 + h(\mu_2^u + K^u + \theta_0))^{n-n_1}} \geq \frac{\nu_0}{(1 + h(\mu_2^u + K^u + \theta_0))^{\Delta}} = l_I,$$

where we have employed the fact that $I_{n_1} \geq \nu_0$. Thus in this case (4.11) is true.

Case $n_2 < N \leq n_3$. We use the preceding case on (n_1, n_2) to conclude that $I_n \geq l_I$ for all $n \in (n_1, n_2]$. This implies $\mathcal{I}_{n_2} \geq l_I$. We show that $I_n \geq l_I$ for $n \in (n_2, N]$ as well. Since $I_n \leq \nu_0$ on $[n_1 + 1, n_2]$ and $n_2 \geq (n_1 + 1) + (\rho + 1)m$, it follows by Claim 3 that

$$I_{n_2+1} \geq \kappa \mathcal{I}_{n_2} \geq \kappa l_I > l_I \quad (\because\ \kappa > 1).$$

If $n_2 + 1 = N$ then we are done. Otherwise, we employ that $I_n \leq \nu_0$ on $[n_1 + 2, n_2 + 1]$ to continue the preceding argument and get

$$I_{n_2+2} \geq \kappa \min_{p \in [n_2+1-m, n_2+1]} I_p \geq \kappa l_I > l_I.$$

By induction, we finally conclude that $I_n \geq l_I$ for all $n \in (n_2, N]$ as desired.

Cases that $n_3 < N \leq n_4$ and so on can be proved by the same argument and is omitted.

Now (4.11) is proved hence establishing the desired estimate for I_n.

Estimate for R_n. There exists a positive constant l_R such that $\liminf_{n\to\infty} R_n \geq l_R$.

Proof of Estimate for R_n. First observe that, from the estimate for I_n, we get that

$$\forall\, n \geq n_0': \quad I_n \geq l_I,$$

where l_I is given by (4.11). Choose $Q_0' > 0$ so that

$$\text{if } n \geq n_0' + Q_0' \quad \text{then} \quad \mu_3(n) + \gamma(n) \leq \mu_3^u + \gamma^u + \theta_0,\ K_n \geq K^l - \theta_0.$$

Note that $K^l - \theta_0 > 0$ according to (4.3). Let $n \geq n_0' + Q_0'$. Using the third equation of Lemma 2, we get

$$R_{n+1} = \frac{R_n + hK_nI_{n+1}}{\Pi_3(n)} \geq hl_I(K^l - \theta_0)\frac{1}{1 + h(\mu_3^u + \gamma^u + \theta_0)} + \frac{1}{1 + h(\mu_3^u + \gamma^u + \theta_0)}R_n. \tag{4.12}$$

We can apply parts (2) and (4) of Lemma 1 to find that there is a constant $l_R > 0$, independent of solutions to (1.8) such that

$$\liminf_{n\to\infty} R_n \geq l_R. \tag{4.13}$$

This establish the estimate for R_n, therefore it completes the proof of Theorem 8. □

5. Extinction of disease

We prove that under a certain condition the disease of the model (1.8) always extincts.

Theorem 9. *Suppose that (H1)-(H4) hold and that*

$$\mathcal{R}_0 := \frac{\beta^u(\sup_{I\in[0,S^{0,\star}]} f(I))S^{0,\star}}{\mu_2^l + k_0} < 1, \tag{5.1}$$

where $S^{0,\star}$ is given by (3.2), $\beta^u = \limsup_{n\to\infty} \beta(n)$, and $\mu_2^l = \liminf_{n\to\infty} \mu_2(n)$. Then (1.8) exhibits the extinction of the disease. Moreover, we will show that if

$$\frac{\beta^u(\sup_{I\in[0,S^{0,\star}]} f(I))S^{0,\star}}{\mu_2^l + K^l} < 1, \tag{5.2}$$

where $K^l := \liminf_{n\to\infty} K_n \geq k_0$, then (1.8) exhibits the extinction of the disease.

Proof of Theorem 9. It suffices to prove the second statement because $\frac{\beta^u(\sup_{I\in[0,S^{0,\star}]} f(I))S^{0,\star}}{\mu_2^l+K^l} \leq \mathcal{R}_0$. We shall repeatedly employ the results on positivity (Proposition 3) and boundedness of solutions (Proposition 6 and Corollary 7).

By the assumption (5.2) and the continuity of f (H1), we choose θ_0 and $\varepsilon_0 > 0$ such that

$$\xi := \frac{(\beta^u + \theta_0)(\sup_{I\in[0,S^{0,\star}+\varepsilon_0]} f(I))(S^{0,\star} + \theta_0)}{\mu_2^l + K^l - \theta_0} < 1. \tag{5.3}$$

Warning! Here and below, for simplicity, the parameters $\theta_0, \varepsilon_0, \xi, n_0', \kappa, \ldots$ have been reused and are in no connection to corresponding ones appeared in the proof of Theorem 8.

Let (S_n, I_n, R_n) be a solution of (1.8). We also choose $n_0' > 0$ such that if $n \geq n_0'$ then

$$\beta(n) \leq \beta^u + \theta_0, \quad \mu_2(n) + K_n \geq \mu_2^l + K^l - \theta_0$$

and

$$I_n \leq S^{0,\star} + \varepsilon_0, \quad S_n \leq S^{0,\star} + \theta_0.$$

Now let $n \geq n_0' + m$. We observe that

$$f(I_{n-j}) \leq \sup_{I\in[0,S^{0,\star}+\varepsilon_0]} f(I), \quad j = 0, 1, \ldots, m.$$

So $G_n = \beta(n) \sum_{j=0}^{m} p_j f(I_{n-j}) I_{n-j} \leq (\beta^u + \theta_0)(\sup_{I\in[0,S^{0,\star}+\varepsilon_0]} f(I))\tilde{\mathcal{I}}_n$, where

$$\tilde{\mathcal{I}}_n := \max_{p\in[n-m,n]} I_p. \tag{5.4}$$

Applying the second equation in (2.9) and that $S_{n+1} \leq S^{0,\star} + \theta_0$, we get

$$\begin{aligned} I_{n+1} &= \frac{I_n + hG_n S_{n+1}}{\Pi_2(n) + hK_n} \leq \frac{\tilde{\mathcal{I}}_n + h(\beta^u + \theta_0)(\sup_{I\in[0,S^{0,\star}+\varepsilon_0]} f(I))\tilde{\mathcal{I}}_n(S^{0,\star} + \theta_0)}{1 + h(\mu_2(n) + K_n)} \\ &= \frac{1 + h\xi(\mu_2^l + K^l - \theta_0)}{1 + h(\mu_2(n) + K_n)}\tilde{\mathcal{I}}_n \leq \frac{1 + h\xi(\mu_2^l + K^l - \theta_0)}{1 + h(\mu_2^l + K^l - \theta_0)}\tilde{\mathcal{I}}_n. \end{aligned}$$

In other words, we now obtain

$$I_{n+1} \leq \kappa \tilde{\mathcal{I}}_n, \tag{5.5}$$

where $0 < \kappa < 1$ is the constant given by

$$\kappa = \frac{1 + h\xi(\mu_2^l + K^l - \theta_0)}{1 + h(\mu_2^l + K^l - \theta_0)}. \tag{5.6}$$

Employing (5.5), we are going to show that

$$\tilde{\mathcal{I}}_{n+m+1} \leq \kappa \tilde{\mathcal{I}}_n \tag{5.7}$$

for all $n \geq n_0' + m$.

By (5.5) we have $I_{n+1} \leq \kappa\tilde{\mathcal{I}}_n$, so $\tilde{\mathcal{I}}_{n+1} = \max_{p\in[n+1-m,n+1]} I_p \leq \tilde{\mathcal{I}}_n$. Then by (5.5) again $I_{n+2} \leq \kappa\tilde{\mathcal{I}}_{n+1}$, hence $I_{n+2} \leq \kappa\tilde{\mathcal{I}}_n$. Similarly, (5.5) gives $I_{n+3} \leq \kappa\tilde{\mathcal{I}}_{n+2}$ and $\tilde{\mathcal{I}}_{n+2} = \max_{[n+2-m,n+2]} I_p \leq \tilde{\mathcal{I}}_n$, so $I_{n+3} \leq \kappa\tilde{\mathcal{I}}_n$. By induction, we get $I_{n+p} \leq \kappa\tilde{\mathcal{I}}_n$ for all $p = 1, 2, \ldots$. Now

$$\tilde{\mathcal{I}}_{n+m+1} = \max\{I_{n+1}, \ldots, I_{n+m+1}\} \leq \kappa\tilde{\mathcal{I}}_n,$$

proving (5.7).

We apply the conclusion from (5.7). Setting

$$N_0 = n_0' + m, \quad N_{j+1} = N_j + m + 1, \quad j = 0, 1, 2, \ldots,$$

we directly get

$$\tilde{\mathcal{I}}_{N_{j+1}} \leq \kappa\tilde{\mathcal{I}}_{N_j}.$$

Since $\kappa < 1$, the sequence $\{\tilde{\mathcal{I}}_{N_j}\}$ is monotonically decreasing. It is easy to see that $\tilde{\mathcal{I}}_{N_j} \leq \kappa^j\tilde{\mathcal{I}}_{N_0}$, hence

$$\lim_{j\to\infty} \tilde{\mathcal{I}}_{N_j} = 0. \tag{5.8}$$

Now we prove the extinction of the disease, i.e. $\lim_{n\to\infty} I_n = 0$.

Let $\varepsilon > 0$. By (5.8) we can find j such that $\tilde{\mathcal{I}}_{N_j} \leq \varepsilon$. Now let $n \geq N_j$. We have $n \in [N_{r-1}, N_r)$ for some $r \geq j + 1$. Then $I_n \leq \tilde{\mathcal{I}}_{N_r} \leq \tilde{\mathcal{I}}_{N_j} \leq \varepsilon$. Therefore $I_n \to 0$ as desired. □

6. Numerical simulations

In this section, the model (1.8) is considered with non-monotone incidence rate

$$g(I) = \frac{\alpha_3 I}{1 + \alpha_2 I + \alpha_1 I^2}.$$

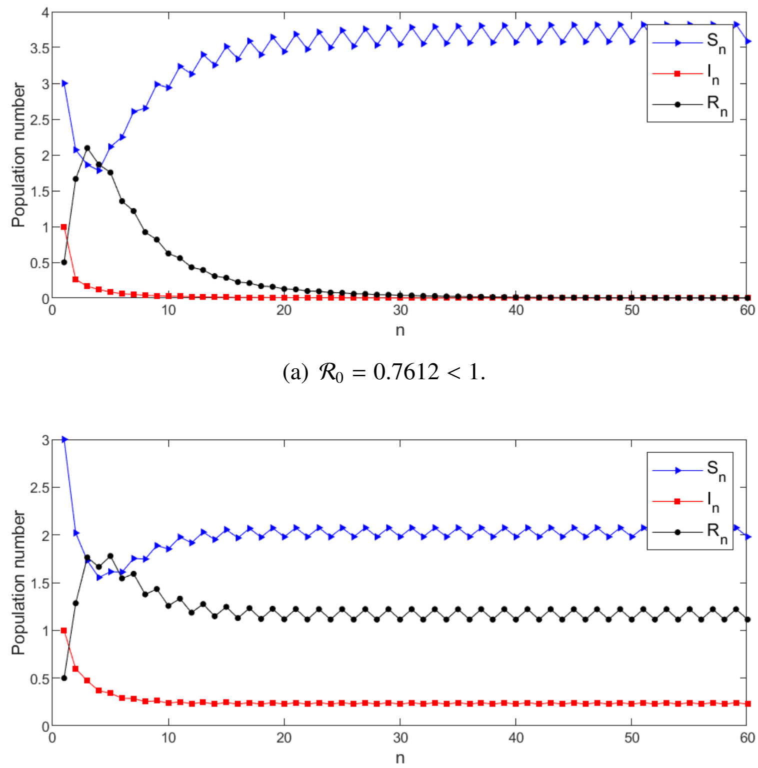
For simplicity, some parameters are fixed as follows: $h = 1$, $m = 2$, $p_0 = 0.2$, $p_1 = 0.3$, $p_2 = 0.5$, $\alpha_3 = 2$, $\alpha_1 = 1$, $\{\Lambda(n)\} = (1, 1.2, 1, 1.2, \dots)$, $\{\gamma(n)\} = (0.3, 0.3, 0.3, \dots)$, $\{\mu_2(n)\} = (0.5, 0.8, 0.5, 0.8, \dots)$, $\{\mu_1(n)\} = \{\mu_3(n)\} = (0.2, 0.4, 0.2, 0.4, \dots)$, $\{\beta(n)\} = (1, 1, 1, \dots)$. We also impose the following initial conditions: $S_j = 3$, $I_j = 1$, $R_j = 0.5$ ($j = 0, -1, -2$). From the formulas (3.2) and (3.3), we have $S^{0,\star} = S^0_\star = 3.2353$.

Now, we present the examples and numerical simulations for different α_2, k_0, k_1.

Example 1. *Choose $\alpha_2 = 0.3$. We have $f(I) = \frac{g(I)}{I} = \frac{2}{1+0.3I+I^2}$. Since f is a decreasing function, we have $f(0) = \sup_{I\in[0,S^{0,\star}]} f(I) = 2$. Also $\beta^u = \beta^l = 1$. We use $K_n = k_0 + (k_1 - k_0)\frac{10}{I_n+10}$.*

(a) If $k_0 = 8$ and $k_1 = 9$, then we get $K^u = K^l = 9$ and $\frac{\beta^u(\sup_{I\in[0,S^{0,\star}]} f(I))S^{0,\star}}{\mu_2^l+K^l} = 0.6811$, $\mathcal{R}_0 = 0.7612 < 1$. Figure 1(a) indicates that the disease exhibits extinction.

(b) If $k_0 = 1$ and $k_1 = 3$, we get $K^u = 2.9556$, $K^l = 2.9525$, and $\frac{\beta^l f(0) S^0_\star}{\mu_2^u+K^u} = 1.7229$, and $\mathcal{R}_\infty =$ 1.7029 > 1. *Figure 1(b) indicates that the disease is permanent.*



(a) $\mathcal{R}_0 = 0.7612 < 1$.

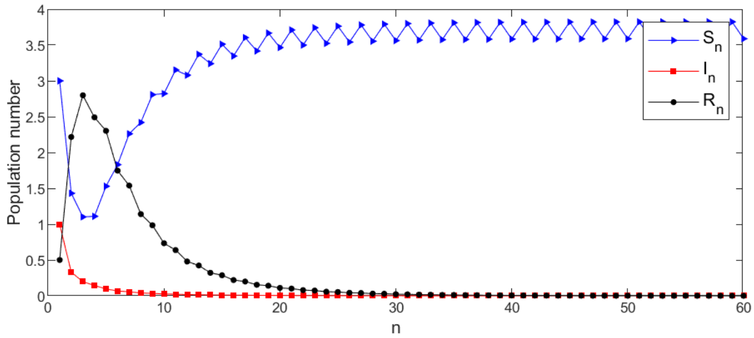
(b) $\mathcal{R}_\infty = 1.7029 > 1$.

Figure 1. Numerical solution (S_n, I_n, R_n) of model (1.8) with $g(I) = \frac{2I}{1+0.3I+I^2}$.

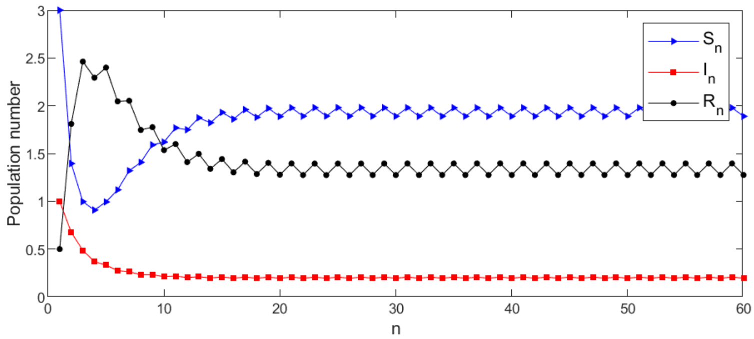
Example 2. *Choose* $\alpha_2 = -1$. *We have* $f(I) = \frac{g(I)}{I} = \frac{2}{1-I+I^2}$. *It is directly to calculate that* $f(0) = 2$ *and* $\sup_{I\in[0,S^{0,\star}]} f(I) = 2.6667$. *Again* $\beta^u = \beta^l = 1$. *We use* $K_n = k_0 + (k_1 - k_0)\frac{10}{I_n+10}$.

(a) If $k_0 = 9$ *and* $k_1 = 10$, *then we get* $K^u = K^l = 10$, $\frac{\beta^u(\sup_{I\in[0,S^{0,\star}]} f(I))S^{0,\star}}{\mu_2^l+K^l} = 0.8217$, *and* $\mathcal{R}_0 = 0.9082 < 1$. *Figure 2(a) indicates that the disease exhibits extinction.*

(b) If $k_0 = 2$ *and* $k_1 = 4$, *we get* $K^u = 3.9618$, $K^l = 3.9595$, $\frac{\beta^l f(0)S^0_\star}{\mu_2^u+K^u} = 1.3589$, *and* $\mathcal{R}_\infty = 1.3480 > 1$. *Figure 2(b) indicates that the disease is permanent.*



(a) $\mathcal{R}_0 = 0.9082 < 1$.



(b) $\mathcal{R}_\infty = 1.3480 > 1$.

Figure 2. Numerical solution (S_n, I_n, R_n) of model (1.8) with $g(I) = \frac{2I}{1-I+I^2}$.

7. Conclusions

In this paper, we investigate the discrete-time non-autonomous SIRS epidemic model (1.8), which is a discretization by the nonstandard finite difference method of the continuous-time model (1.4). In the model, a general nonlinear incidence rate with distributed delays is included together with a nonlinear recovery rate which takes into account the effect of health care resources such as the hospital beds (1.3). Two threshold parameters $\mathcal{R}_0$ and $\mathcal{R}_\infty$ are obtained so that if $\mathcal{R}_0 < 1$ then the disease dies out while if $\mathcal{R}_\infty > 1$ then the disease is permanent.

In the special case of autonomous (1.8), f is a decreasing function, and k a constant, our results imply that $\mathcal{R}_0 = \mathcal{R}_\infty = \frac{\beta f(0)\Lambda}{\mu_1(\mu_2+k)}$, which gives the basic reproduction number of the model. The same number is reported in Corollary 5.4 of [27] when $f = 1$. This result is also in line with the known results for the continuous-time models [12] when $f = 1$, and [3] when f is decreasing and $f(0) = 1$.

For the non-autonomous model (1.8), we obtain the following corollary. Suppose that, as $n \to \infty$, we have $\beta(n) \to \beta'$, $\Lambda(n) \to \Lambda'$, $\mu_1(n) \to \mu_1'$, $\mu_2(n) \to \mu_2'$, $k(n) \to k'$ (k does not depend on I_n), and $f = 1$. Then we obtain the reproduction number of the model to be $\mathcal{R}_0 = \mathcal{R}_\infty = \frac{\beta' S'}{\mu_2' + k'}$, which is the same number implied by Theorem 4.1 and Theorem 5.1 in [27]. Comparing this result to that from the continuous-time model in [25, 26], we find an improvement because the threshold parameters obtained in those two papers do not lead to the reproduction number of the model.

A novelty of this paper is that our results are mathematically more tractable (compared e.g. to [27]) so they can be effective tools in practice to help the policy-makers fight the spread of the disease. To employ the condition (5.2), for example, one can first set the "ultimate" contact rate β^∞ and health care resources K^∞ so that the condition is met with β^u, K^l replaced by β^∞, K^∞ (assuming all other parameters are available and fixed). Then set a certain starting time N_0 and control the transmission rate and the hospital beds to satisfy $\beta(n) \le \beta^\infty$ and $K_n \ge K^\infty$ for all $n \ge N_0$ onward. It then follows from Theorem 9 that the spreading of the disease will be suppressed eventually.

For further investigations, it is interesting to extend the results of this paper to the model (1.8) where the transmission function f depends not only on I but also on S, such as the Beddington-DeAngelis function $f(S, I) = S/(1 + m_1 S + m_2 I)$ and the saturated incidence $f(S, I) = S/(1 + m_1 S)(1 + m_2 I)$. It is also interesting to apply the technique in this paper to explore the threshold dynamics for other non-autonomous models such as a model with vaccination, a model with age structures, multi-strain diseases, etc.

Another interesting question is the chaotic dynamics of epidemic models with seasonality in the transmission rate [1, 9]. It was shown in [1] for the classical SIR model that the disease dies out when $R_0 < 1$, where R_0 is the reproduction number, while if $R_0 > 1$ the model admits periodic and aperiodic patterns together with sensitive dependence on the initial conditions of the solution. The SIR model with logistic growth rate was explored in [9] and it was shown that the condition $R_0 < 1$ is not sufficient to guarantee the extinction of the disease due to backward bifurcation and the model exhibits persistent strange attractors. For our model (1.8), the condition $\mathcal{R}_0 < 1$ always implies the elimination of the disease regardless of the initial infectives. Moreover, for the periodic forced model with $f = 1$, it was shown in Corollary 5.3 [27] that if $R_0 < 1$ the disease dies out while it is permanent when $R_0 > 1$, where R_0 is given explicitly by $R_0 = \prod_{k=0}^{\omega-1}\left(\frac{1+\beta(k) z_k^*}{1+\mu(k)+\gamma(k)+\alpha(k)}\right)^{1/\omega}$ and z_k^* is a unique ω-periodic solution of (3.1). In our work, f is a nonlinear function, so it is an interesting open problem to see whether a chaotic behavior can happen when $\mathcal{R}_0 > 1$ due to the seasonally forced contact rate β.

Appendix

We briefly discuss the basic idea of the nonstandard finite difference (NSFD) method that enables to get the discrete model (1.8) from the continuous model (1.4). NSFD is a discretization technique consisting of some rules with the aim to preserve significant properties of the related continuous problem (such as positivity, boundedness, stability, etc.) and avoid numerical instabilities. The following rules were proposed by Mickens [10] for constructing a NSFD scheme from a continuous problem:

Rule 1. The orders of the discrete derivatives of the scheme should be equal to the orders of the corresponding derivatives of the differential equation.

Rule 2. The denominator function for each discrete derivative should, in general, be expressed as a function of step-size which is more complicated than the conventional one. This rule is not strict and in this work, to get (1.8), we choose $\phi(h) = h$.

Example. Consider a first-order differential equation of the form

$$\frac{dx}{dt} = f(t, x). \tag{7.1}$$

Conventionally, the discrete derivative for dx/dt in (7.1) is given by

$$\frac{x_{k+1} - x_k}{\Delta t},$$

but, in the NSFD scheme, the denominator Δt is replaced by a denominator function $\phi(h)$ with the property

$$\phi(h) = h + O(h^2) \quad \text{as } h \to 0.$$

Rule 3. Nonlinear terms should, in general, be replaced by nonlocal discrete representations using more than one mesh point. For example, the nonlinear term x^2 can be replaced by a nonlocal representation evaluated at two mesh points such as $x_{k+1}x_k$ or $2x_k^2 - x_{k+1}x_k$.

To get (1.8), the incidence rate (1.7) is discretized by $\beta(n) \sum_{j=0}^{m} p_j f(I_{n-j})I_{n-j}S_{n+1}$.

Rule 4. Special conditions that hold for the solutions of the differential equations should also hold for the solutions of the finite difference scheme. An important example is the positivity of solutions when the solutions represent some physical positive quantities. If the discrete equations allow their solutions to become negative, then numerical instabilities will occur.

We have shown the basic properties including Propositions 3 and 6.

Rule 5. The finite difference scheme should not introduce extraneous or spurious solutions that do not correspond to any solution of the corresponding differential equations.

Conflict of interest

The authors declare that there are no conflicts of interest.

Acknowledgments

The authors would like to thank the reviewers for valuable comments and suggestions which are tremendously helpful to improve the manuscript. We are thankful for financial support by Science Achievement Scholarship of Thailand (SAST). We are also supported by the 90th Anniversary of Chulalongkorn University Fund, Thailand (Ratchadaphiseksomphot Endowment Fund).

References

1. P. G. Barrientos, J. A. Rodríguez, A. Ruiz-Herrera, Chaotic dynamics in the seasonally forced SIR epidemic model, *J. Math. Biol.*, **75** (2017), 1655–1668. https://doi.org/10.1007/s00285-017-1130-9
2. V. Capasso, G. Serio, A generalization of the Kermack-Mckendrick deterministic epidemic model, *Math. Biosci.*, **42** (1978), 43–61. https://doi.org/10.1016/0025-5564(78)90006-8

3. T. Enatsu, Y. Nakata, Y. Muroya, Lyapunov functional techniques for the global stability analysis of a delayed SIRS epidemic model, *Nonlinear Anal.: Real World Appl.*, **13** (2012), 2120–2133. https://doi.org/10.1016/j.nonrwa.2012.01.007
4. Y. Gu, S. Ullah, M. A. Khan, M. Y. Alshahrani, M. Abohassan, M.B. Riaz, Mathematical modeling and stability analysis of the COVID-19 with quarantine and isolation, *Results Phys.,* **34** (2022), 105284. https://doi.org/10.1016/j.rinp.2022.105284
5. H. F. Huo, Z. P. Ma, Dynamics of a delayed epidemic model with non-monotonic incidence rate, *Commun. Nonlinear Sci. Numer. Simul.*, **15** (2010), 459–468. https://doi.org/10.1016/j.cnsns.2009.04.018
6. Z. Jiang, W. Ma, Permanence of a delayed SIR epidemic model with general nonlinear incidence rate, *Math. Meth. Appl. Sci.*, **38** (2015), 505–516. https://doi.org/10.1002/mma.3083
7. M. A. Khan, A. Atangana, Mathematical modeling and analysis of COVID-19: A study of new variant Omicron, *Phys. A: Stat. Mech. Appl.*, **599** (2022), 127452. https://doi.org/10.1016/j.physa.2022.127452
8. M. Lu, J. Huang, S. Ruan, P. Yu, Bifurcation analysis of an SIRS epidemic model with a generalized nonmonotone and saturated incidence rate, *J. Differ. Equ.,* **267** (2019), 1859–1898. https://doi.org/10.1016/j.jde.2019.03.005
9. J. P. S. Mauŕicio de Carvalho, A. A. P. Rodrigues, Strange attractors in a dynamical system inspired by a seasonally forced SIR model, *Phys. D: Nonlinear Phenom.*, **434** (2022), 133268. https://doi.org/10.1016/j.physd.2022.133268
10. R. E. Mickens, *Nonstandard finite difference models of differential equations*, Singapore: World Scientific, 1994. https://doi.org/10.1142/2081
11. Y. Muroya, Y. Enatsu, Y. Nakata, Monotone iterative techniques to SIRS epidemic models with nonlinear incidence rate and distributed delays, *Nonlinear Anal.: Real World Appl.,* **12** (2011), 1897–1910. https://doi.org/10.1016/j.nonrwa.2010.12.002
12. Y. Nakata, Y. Enatsu, Y. Muroya, On the global stability of an SIRS epidemic model with distributed delays, *Conference Publications*, 2011(Special), 1119–1128. https://doi.org/10.3934/proc.2011.2011.1119
13. S. Ottaviano, M. Sensi, S. Sottile, Global stability of SAIRS epidemic models, *Nonlinear Anal.: Real World Appl.*, **65** (2022), 103501. https://doi.org/10.1016/j.nonrwa.2021.103501
14. M. Sekiguchi, Permanence of a discrete SIRS epidemic model with time delays, *Appl. Math. Lett.,* **23** (2010), 1280–1285. https://doi.org/10.1016/j.aml.2010.06.013
15. M. Sekiguchi, E. Ishiwata, Global dynamics of a discretized SIRS epidemic model with time delay, *J. Math. Anal. Appl.,* **371** (2010), 195–202. https://doi.org/10.1016/j.jmaa.2010.05.007
16. C. Shan, H. Zhu, Bifurcations and complex dynamics of an SIR model with the impact of the number of hospital beds, *J. Differ. Equ.*, **257** (2014), 1662–1688. https://doi.org/10.1016/j.jde.2014.05.030
17. Z. Teng, L. Wang, L. Nie, Global attractivity for a class of delayed discrete SIRS epidemic models with general nonlinear incidence rate, *Math. Meth. Appl. Sci.*, **38** (2015), 4741–4759. https://doi.org/10.1002/mma.3389

18. W. Wang, Global behavior of an SEIRS epidemic model with time delays, *Appl. Math. Lett.,* **15** (2002), 423–428. https://doi.org/10.1016/S0893-9659(01)00153-7
19. D. Xiao, S. Ruan, Global analysis of an epidemic model with nonmonotone incidence rate, *Math. Biosci.,* **208** (2007), 419–429. https://doi.org/10.1016/j.mbs.2006.09.025
20. D. Xiao, Y. Zhou, Qualitative analysis of an epidemic model, *Can. Appl. Math. Q.,* **14** (2006), 469–492.
21. J. Xu, Z. Teng, S. Gao, Almost sufficient and necessary conditions for permanence and extinction of nonautonomous discrete logistitc systems with time-varying delays and feedback control, *Appl. Math-Czech,* **56** (2011), 207–225. https://doi.org/10.1007/s10492-011-0003-6
22. Y. Xu, L. Wei, X. Jiang, Complex dynamics of a SIRS epidemic model with the influence of hospital bed number, *Discrete Contin. Dynam. Syst. Ser. B*, **56** (2021), 6229–6252. http://doi.org/10.3934/dcdsb.2021016
23. L. Zhang, X. Fan, Z. Teng, Global dynamics of a nonautonomous SEIRS epidemic model with vaccination and nonlinear incidence, *Math. Meth. Appl. Sci.*, **44** (2021), 9315–9333. https://doi.org/10.1002/mma.7359
24. T. Zhang, Permanence and extinction in a nonautonomous discrete SIRVS epidemic model with vaccination, *Appl. Math. Comput.*, **271** (2015), 716–729. https://doi.org/10.1016/j.amc.2015.09.071
25. T. Zhang, Z. Teng, Permanence and extinction for a nonautonomous SIRS epidemic model with time delay, *Appl. Math. Model.*, **33** (2009), 1058–1071. https://doi.org/10.1016/j.apm.2007.12.020
26. T. Zhang, J. Liu, Z. Teng, Dynamic behavior for a nonautonomous SIRS epidemic model with distributed delays, *Appl. Math. Comput.*, **214** (2009), 624–631. https://doi.org/10.1016/j.amc.2009.04.029
27. T. Zhang, J. Liu, Z. Teng, Threshold conditions for a discrete nonautonomous SIRS model, *Math. Meth. Appl. Sci.*, **38** (2015), 1781–1794. https://doi.org/10.1002/mma.3186

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