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Research article

An efficient Fourier spectral method and error analysis for the fourth order problem with periodic boundary conditions and variable coefficients

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Abstract: We propose in this paper an efficient algorithm based on the Fourier spectral-Galerkin approximation for the fourth-order elliptic equation with periodic boundary conditions and variable coefficients. First, by using the Lax-Milgram theorem, we prove the existence and uniqueness of weak solution and its approximate solution. Then we define a high-dimensional L^2 projection operator and prove its approximation properties. Combined with Céa lemma, we further prove the error estimate of the approximate solution. In addition, from the Fourier basis function expansion and the properties of the tensor, we establish the equivalent matrix form based on tensor product for the discrete scheme. Finally, some numerical experiments are carried out to demonstrate the efficiency of the algorithm and correctness of the theoretical analysis.

Keywords: fourth-order problems; periodic boundary conditions; variable coefficients; Fourier spectral method; error estimation
Mathematics Subject Classification: 65N15,65N35

1. Introduction

Though periodic boundary conditions do not belong to three traditional boundary conditions (Dirichlet, Neumann, and Robin boundary conditions), which are commonly used in mathematical physics, they still can be found in the research of scientific and engineering problems, such as the interaction between solutions of the nonlinear Schrödinger equation [1] or KdV equation [2], isotropic uniform turbulence problem [3], and so on. In addition, under the polar, cylindrical, and spherical coordinates, we note that they are also periodic [4, 5] in the θ direction. Thus, it's obvious that physical models with periodic boundary conditions also have significant application. As a model, we first consider the following two dimensional fourth-order problem with periodic boundary conditions and variable coefficients:

$$\Delta^2\psi - \nabla(\alpha\nabla\psi) + \beta\psi = f,\ \mathbf{x} \in \Omega, \tag{1.1}$$

$$\psi(\mathbf{x}) = \psi(x_1 + L_{x_1}, x_2),\ \frac{\partial \psi(\mathbf{x})}{\partial x_1} = \frac{\partial \psi(x_1 + L_{x_1}, x_2)}{\partial x_1}, \tag{1.2}$$

$$\psi(\mathbf{x}) = \psi(x_1, x_2 + L_{x_2}),\ \frac{\partial \psi(\mathbf{x})}{\partial x_2} = \frac{\partial \psi(x_1, x_2 + L_{x_2})}{\partial x_2}, \tag{1.3}$$

where α is a nonnegative bounded periodic function, β is a positive bounded function, $\mathbf{x} = (x_1, x_2)$, $L_{x_1} = x_{1R} - x_{1L}$, $L_{x_2} = x_{2R} - x_{2L}$, $\Omega = (x_{1L}, x_{1R}) \times (x_{2L}, x_{2R})$.

The fourth-order problems can be found in the applications to thin beams and plates [6, 7]. Besides, many complex nonlinear problems also need to solve a fourth order problem repeatedly [8–13]. In the past decades, there have been many existing results for the theoretical analysis and numerical research of the fourth-order problems, mainly including various finite element methods [14–17], spectral methods and some high-order numerical methods [18–27]. However, to the best of our knowledge, there are few report on the fourth-order problems with periodic boundary conditions and variable coefficients [28]. As aforementioned, periodic boundary conditions have significant applications in some science and engineering [29, 30]. Thus, it is meaningful to construct an efficient and high-order numerical scheme for the fourth-order problems with periodic boundary conditions and variable coefficients.

The aim of this paper is to propose an efficient algorithm based on the Fourier spectral-Galerkin approximation for the fourth-order elliptic equation with periodic boundary conditions and variable coefficients. First, by using the Lax-Milgram theorem, we prove the existence and uniqueness of weak solution and its approximate solution. Then we define a high-dimensional L^2 projection operator and prove its approximation properties. Combined with Céa lemma, we further prove the error estimate of the approximate solution. In addition, from the Fourier basis function expansion and the properties of the tensor, we establish the equivalent matrix form based on tensor product for the discrete scheme. Finally, some numerical experiments are carried out to demonstrate the efficiency of the algorithm and correctness of the theoretical analysis.

The rest of this paper is organized as follows. In next section, we derive the weak form and associated discrete scheme. We give the error estimation of approximate solutions in section 3. In section 4, we present an efficient implementation of the algorithm. In section 5, we extend the algorithm to a three-dimensional case. In section 6, we carry out some numerical experiments. Finally, we make some concluding remarks in section 7.

2. Weak form and discrete scheme

We shall derive the weak form and discrete scheme associated with problem (1.1)–(1.3). Denote by $H^m(\Omega)$ the usual m-order Sobolev space, $\|\cdot\|_m$ and $|\cdot|_m$ denote the norm and semi-norm in $H^m(\Omega)$, respectively. In particular, we have

$$H^0(\Omega) = L^2(\Omega) = \left\{\psi : \int_\Omega |\psi|^2 d\mathbf{x} < \infty\right\}$$

with the following inner product and norm

$$(\psi, \varphi) = \int_\Omega \psi\bar{\varphi} d\mathbf{x},\ \|\psi\| = \Big(\int_\Omega |\psi|^2 d\mathbf{x}\Big)^{\frac{1}{2}},$$

where $\bar{\varphi}$ is the complex conjugate of φ. Define

$$H_p^2(\Omega) = \Big\{\psi \in H^2(\Omega) : \psi \text{ satisfies the periodic boundary conditions (1.2) } \textit{and} \text{ (1.3)}\Big\}$$

with the following inner product, norm and semi-norm:

$$(\psi, \varphi)_{2,\Omega} = \sum_{|\boldsymbol{\alpha}|=0}^{2} \int_\Omega D^{\boldsymbol{\alpha}}\psi D^{\boldsymbol{\alpha}}\bar{\varphi} d\mathbf{x},$$

$$\|\psi\|_{2,\Omega} = \Big(\sum_{|\boldsymbol{\alpha}|=0}^{2} \|D^{\boldsymbol{\alpha}}\psi\|^2\Big)^{\frac{1}{2}},$$

$$|\psi|_{2,\Omega} = \Big(\sum_{|\boldsymbol{\alpha}|=2} \|D^{\boldsymbol{\alpha}}\psi\|^2\Big)^{\frac{1}{2}},$$

where $D^{\boldsymbol{\alpha}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial {x_1}^{\alpha_1} \partial {x_2}^{\alpha_2}}$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2$. We further denote by $H_p^m(\Omega)$ the subspace of $H^m(\Omega)$, which consists of functions with derivatives of order up to $m-1$ being 2π-periodic.

Then a weak form of problem (1.1)–(1.3) is: Find $\psi \in H_p^2(\Omega)$, such that

$$a(\psi, \varphi) = F(\varphi), \quad \forall \varphi \in H_p^2(\Omega), \tag{2.1}$$

where

$$a(\psi, \varphi) = \int_\Omega \Delta\psi \Delta\bar{\varphi} d\mathbf{x} + \int_\Omega \alpha \nabla\psi \nabla\bar{\varphi} d\mathbf{x} + \int_\Omega \beta\psi\bar{\varphi} d\mathbf{x},$$
$$F(\varphi) = \int_\Omega f\bar{\varphi} d\mathbf{x}.$$

Define an approximation space of $H_p^2(\Omega)$ as follows:

$$X_M(\Omega) = \text{span}\{e^{i2\pi t \frac{x_1 - x_{1L}}{L_{x_1}}} e^{i2\pi q \frac{x_2 - x_{2L}}{L_{x_2}}} : |t| = 0, 1, \cdots, M, \ |q| = 0, 1, \cdots, M\}.$$

Then the corresponding discrete scheme of the weak form (2.1) is: Find $\psi_M \in X_M(\Omega)$, such that

$$a(\psi_M, \varphi_M) = F(\varphi_M), \quad \forall \varphi_M \in X_M(\Omega). \tag{2.2}$$

3. Error estimation of the approximation solution

In this section, we shall first prove the existence and uniqueness of weak solution and its approximate solution, and then further prove the error estimate between them.

3.1. Existence and uniqueness

For the sake of brevity, we denote by $a \lesssim b$ that $a \le cb$, where c is a positive constant. Without loss of generality, we shall confine our discussion to the following assumptions:

$$\alpha_* := \inf_{\mathbf{x}\in\Omega} \alpha(\mathbf{x}) \ge 0, \ \alpha^* := \sup_{\mathbf{x}\in\Omega} \alpha(\mathbf{x}) < \infty, \tag{3.1}$$

$$\beta_* := \inf_{\mathbf{x}\in\Omega} \beta(\mathbf{x}) > 0, \ \beta^* := \sup_{\mathbf{x}\in\Omega} \beta(\mathbf{x}) < \infty, \tag{3.2}$$

$$\Omega = (0, 2\pi) \times (0, 2\pi). \tag{3.3}$$

Lemma 1. *For any $\psi, \varphi \in H_p^2(\Omega)$, the following equalities hold:*

$$\int_\Omega \frac{\partial^2\psi}{\partial x_1\partial x_2}\frac{\partial^2\bar{\varphi}}{\partial x_1\partial x_2}d\mathbf{x} = \int_\Omega \frac{\partial^2\psi}{\partial x_1^2}\frac{\partial^2\bar{\varphi}}{\partial x_2^2}d\mathbf{x} = \int_\Omega \frac{\partial^2\psi}{\partial x_2^2}\frac{\partial^2\bar{\varphi}}{\partial x_1^2}d\mathbf{x},$$
$$\int_\Omega \frac{\partial^2\psi}{\partial x_1\partial x_2}\frac{\partial^2\bar{\psi}}{\partial x_1\partial x_2}d\mathbf{x} = \int_\Omega \frac{\partial^2\psi}{\partial x_1^2}\frac{\partial^2\bar{\psi}}{\partial x_2^2}d\mathbf{x} = \int_\Omega \frac{\partial^2\bar{\psi}}{\partial x_1^2}\frac{\partial^2\psi}{\partial x_2^2}d\mathbf{x}.$$

Proof. Using the integration by parts, we have

$$\begin{aligned}\int_\Omega \frac{\partial^2\psi}{\partial x_1\partial x_2}\frac{\partial^2\bar{\varphi}}{\partial x_1\partial x_2}d\mathbf{x} &= \int_0^{2\pi} \frac{\partial^2\psi}{\partial x_1\partial x_2}\frac{\partial\bar{\varphi}}{\partial x_2}\Big|_{x_1=0}^{x_1=2\pi}dx_2 - \int_\Omega \frac{\partial^3\psi}{\partial x_1^2\partial x_2}\frac{\partial\bar{\varphi}}{\partial x_2}d\mathbf{x}\\ &= \int_0^{2\pi} \frac{\partial^2\psi}{\partial x_1\partial x_2}\frac{\partial\bar{\varphi}}{\partial x_2}\Big|_{x_1=0}^{x_1=2\pi}dx_2 - \int_0^{2\pi} \frac{\partial^2\psi}{\partial x_1^2}\frac{\partial\bar{\varphi}}{\partial x_2}\Big|_{x_2=0}^{x_2=2\pi}dx_1\\ &+ \int_\Omega \frac{\partial^2\psi}{\partial x_1^2}\frac{\partial^2\bar{\varphi}}{\partial x_2^2}d\mathbf{x}.\end{aligned}$$

From (1.2) and (1.3), we derive that

$$\frac{\partial^2\psi(\mathbf{x})}{\partial x_1\partial x_2} = \frac{\partial^2\psi(x_1, x_2 + 2\pi)}{\partial x_1\partial x_2},\ \frac{\partial\bar{\varphi}}{\partial x_2}(\mathbf{x}) = \frac{\partial\bar{\varphi}(x_1 + 2\pi, x_2)}{\partial x_2},$$
$$\frac{\partial^2\psi(\mathbf{x})}{\partial x_1\partial x_2} = \frac{\partial^2\psi((x_1 + 2\pi, x_2)}{\partial x_1\partial x_2},\ \frac{\partial\bar{\varphi}(\mathbf{x})}{\partial x_2} = \frac{\partial\bar{\varphi}(x_1, x_2 + 2\pi)}{\partial x_2}.$$

Then we have

$$\int_0^{2\pi} \frac{\partial^2\psi}{\partial x_1\partial x_2}\frac{\partial\bar{\varphi}}{\partial x_2}\Big|_{x_1=0}^{x_1=2\pi}dx_2 = \int_0^{2\pi} \frac{\partial^2\psi}{\partial x_1^2}\frac{\partial\bar{\varphi}}{\partial x_2}\Big|_{x_2=0}^{x_2=2\pi}dx_1 = 0.$$

It follows from the above equality that

$$\int_\Omega \frac{\partial^2\psi}{\partial x_1\partial x_2}\frac{\partial^2\bar{\varphi}}{\partial x_1\partial x_2}d\mathbf{x} = \int_\Omega \frac{\partial^2\psi}{\partial x_1^2}\frac{\partial^2\bar{\varphi}}{\partial x_2^2}d\mathbf{x}.$$

We can obtain the following equalities in the same way

$$\int_\Omega \frac{\partial^2\psi}{\partial x_1\partial x_2}\frac{\partial^2\bar{\varphi}}{\partial x_1\partial x_2}d\mathbf{x} = \int_\Omega \frac{\partial^2\psi}{\partial x_1^2}\frac{\partial^2\bar{\varphi}}{\partial x_2^2}d\mathbf{x} = \int_\Omega \frac{\partial^2\psi}{\partial x_2^2}\frac{\partial^2\bar{\varphi}}{\partial x_1^2}d\mathbf{x},$$
$$\int_\Omega \frac{\partial^2\psi}{\partial x_1\partial x_2}\frac{\partial^2\bar{\psi}}{\partial x_1\partial x_2}d\mathbf{x} = \int_\Omega \frac{\partial^2\psi}{\partial x_1^2}\frac{\partial^2\bar{\psi}}{\partial x_2^2}d\mathbf{x} = \int_\Omega \frac{\partial^2\bar{\psi}}{\partial x_1^2}\frac{\partial^2\psi}{\partial x_2^2}d\mathbf{x}.$$

□

Lemma 2. *For any $\psi \in H_p^2(\Omega)$, the following inequalities hold:*

$$\int_\Omega |\frac{\partial\psi}{\partial x_1}|^2 d\mathbf{x} \le \frac{1}{2}\|\psi\|^2 + \frac{1}{2}\int_\Omega |\frac{\partial^2\psi}{\partial x_1^2}|^2 d\mathbf{x},$$
$$\int_\Omega |\frac{\partial\psi}{\partial x_2}|^2 d\mathbf{x} \le \frac{1}{2}\|\psi\|^2 + \frac{1}{2}\int_\Omega |\frac{\partial^2\psi}{\partial x_2^2}|^2 d\mathbf{x}.$$

Proof. We derive from integration by parts that

$$\begin{aligned}\int_\Omega |\frac{\partial\psi}{\partial x_1}|^2 d\mathbf{x} &= \int_0^{2\pi}\int_0^{2\pi} \frac{\partial\psi}{\partial x_1}\frac{\partial\bar{\psi}}{\partial x_1} d\mathbf{x}\\ &= \int_0^{2\pi} \psi\frac{\partial\bar{\psi}}{\partial x_1}\Big|_{x_1=0}^{x_1=2\pi} dx_2 - \int_0^{2\pi}\int_0^{2\pi} \psi\frac{\partial^2\bar{\psi}}{\partial x_1^2} d\mathbf{x}\\ &= -\int_0^{2\pi}\int_0^{2\pi} \psi\frac{\partial^2\bar{\psi}}{\partial x_1^2} d\mathbf{x} \le \Big(\int_\Omega |\psi|^2 d\mathbf{x}\Big)^{\frac{1}{2}}\Big(\int_\Omega |\frac{\partial^2\psi}{\partial x_1^2}|^2 d\mathbf{x}\Big)^{\frac{1}{2}}\\ &\le \frac{1}{2}||\psi||^2 + \frac{1}{2}\int_\Omega |\frac{\partial^2\psi}{\partial x_1^2}|^2 d\mathbf{x}.\end{aligned}$$

Similarly, we can obtain

$$\int_\Omega |\frac{\partial\psi}{\partial x_2}|^2 d\mathbf{x} \le \frac{1}{2}||\psi||^2 + \frac{1}{2}\int_\Omega |\frac{\partial^2\psi}{\partial x_2^2}|^2 d\mathbf{x}.$$

This finishes our proof. □

Lemma 3. *Let $\alpha(\mathbf{x}), \beta(\mathbf{x}) \in L^\infty(\Omega)$ satisfy the assumptions* (3.1) *and* (3.2)*. Then $a(\psi,\varphi)$ is a continuous and coercive bilinear form in $H_p^2(\Omega)\times H_p^2(\Omega)$, i.e.,*

$$\begin{aligned}|a(\psi,\varphi)| &\le Q^*||\psi||_{2,\Omega}||\varphi||_{2,\Omega},\\ a(\psi,\psi) &\ge Q_*||\psi||_{2,\Omega}^2,\end{aligned}$$

where $Q^* = \max\{1,\alpha^*,\beta^*\}$, $Q_* = \frac{1}{2}\min\{1,\beta_*\}$.

Proof. Employing Lemma 1, we obtain that

$$\begin{aligned}\int_\Omega \Delta\psi\Delta\bar{\varphi} d\mathbf{x} &= \int_\Omega (\frac{\partial^2\psi}{\partial x_1^2}\frac{\partial^2\bar{\varphi}}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_1^2}\frac{\partial^2\bar{\varphi}}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_2^2}\frac{\partial^2\bar{\varphi}}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2}\frac{\partial^2\bar{\varphi}}{\partial x_2^2}) d\mathbf{x}\\ &= \int_\Omega (\frac{\partial^2\psi}{\partial x_1^2}\frac{\partial^2\bar{\varphi}}{\partial x_1^2} + 2\frac{\partial^2\psi}{\partial x_1\partial x_2}\frac{\partial^2\bar{\varphi}}{\partial x_1\partial x_2} + \frac{\partial^2\psi}{\partial x_2^2}\frac{\partial^2\bar{\varphi}}{\partial x_2^2}) d\mathbf{x}.\end{aligned}$$

Then, using Cauchy-Schwarz inequality, we can derive that

$$\begin{aligned}|a(\psi,\varphi)| &= |\int_\Omega \Delta\psi\Delta\bar{\varphi} d\mathbf{x} + \int_\Omega \alpha\nabla\psi\nabla\bar{\varphi} d\mathbf{x} + \int_\Omega \beta\psi\bar{\varphi} d\mathbf{x}|\\ &\le \int_\Omega (|\frac{\partial^2\psi}{\partial x_1^2}\frac{\partial^2\varphi}{\partial x_1^2}| + 2|\frac{\partial^2\psi}{\partial x_1\partial x_2}\frac{\partial^2\varphi}{\partial x_1\partial x_2}| + |\frac{\partial^2\psi}{\partial x_2^2}\frac{\partial^2\varphi}{\partial x_2^2}|) d\mathbf{x}\\ &\quad + \alpha^*\int_\Omega (|\frac{\partial\psi}{\partial x_1}\frac{\partial\varphi}{\partial x_1}| + |\frac{\partial\psi}{\partial x_2}\frac{\partial\varphi}{\partial x_2}|) d\mathbf{x} + \beta^*\int_\Omega |\psi\varphi| d\mathbf{x}\\ &\le \max\{1,\alpha^*,\beta^*\}||\psi||_{2,\Omega}||\varphi||_{2,\Omega}.\end{aligned}$$

On the other hand, using Lemma 2, we can derive that

$$a(\psi,\psi) = \int_\Omega (|\frac{\partial^2\psi}{\partial x_1^2}|^2 + 2|\frac{\partial^2\psi}{\partial x_1\partial x_2}|^2 + |\frac{\partial^2\psi}{\partial x_2^2}|^2) d\mathbf{x}$$

$$
\begin{aligned}
&+\int_\Omega \alpha(|\frac{\partial\psi}{\partial x_1}|^2+|\frac{\partial\psi}{\partial x_2}|^2)d\mathbf{x}+\int_\Omega \beta|\psi|^2 d\mathbf{x}\\
&\geq \int_\Omega (|\frac{\partial^2\psi}{\partial x_1^2}|^2+2|\frac{\partial^2\psi}{\partial x_1\partial x_2}|^2+|\frac{\partial^2\psi}{\partial x_2^2}|^2)d\mathbf{x}+\beta_*\int_\Omega |\psi|^2 d\mathbf{x}\\
&\geq \min\{1,\beta_*\}\int_\Omega (|\frac{\partial^2\psi}{\partial x_1^2}|^2+2|\frac{\partial^2\psi}{\partial x_1\partial x_2}|^2+|\frac{\partial^2\psi}{\partial x_2^2}|^2+|\psi|^2)d\mathbf{x}\\
&\geq \frac{1}{2}\min\{1,\beta_*\}||\psi||_{2,\Omega}^2.
\end{aligned}
$$

This finishes our proof. □

Lemma 4. *If $f(\mathbf{x})\in L^2(\Omega)$, then $F(\varphi)$ is a bounded linear functions on $H_p^2(\Omega)$, i.e.,*

$$|F(\varphi)|\lesssim ||\varphi||_{2,\Omega}.$$

Proof. In light of definition of $F(\varphi)$ and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
|F(\varphi)|=&\Big|\int_\Omega f\bar{\varphi}d\mathbf{x}\Big|\\
\leq&\Big(\int_\Omega |f|^2 d\mathbf{x}\Big)^{\frac{1}{2}}\Big(\int_\Omega |\varphi|^2 d\mathbf{x}\Big)^{\frac{1}{2}}\\
\lesssim&||\varphi||_{2,\Omega}.
\end{aligned}
$$

The proof is completed. □

From Lemma 3, Lemma 4 and Lax-Milgram theorem, we have following theorem:

Theorem 1. *If $f(\mathbf{x})\in L^2(\Omega)$, then problems* (2.1) *and* (2.2) *have unique solutions $\psi(\mathbf{x})$ and $\psi_M(\mathbf{x})$, respectively.*

3.2. Error estimation

Theorem 2. *Let $\psi(\mathbf{x})$ and $\psi_M(\mathbf{x})$ be the solutions of the variational form* (2.1) *and discrete scheme* (2.2)*, respectively. Then it holds that*

$$||\psi-\psi_M||_{2,\Omega}\lesssim \inf_{\varphi\in X_M}||\psi-\varphi_M||_{2,\Omega}.$$

Proof. We obtain from (2.1) and (2.2) that

$$
\begin{aligned}
a(\psi,\varphi_M)&=F(\varphi_M),\ \forall\varphi_M\in X_M(\Omega),\\
a(\psi_M,\varphi_M)&=F(\varphi_M),\ \forall\varphi_M\in X_M(\Omega).
\end{aligned}
$$

Then we have

$$a(\psi-\psi_M,\varphi_M)=0,\ \forall\varphi_M\in X_M(\Omega). \tag{3.4}$$

Form Lemma 3 and (3.4), we arrive at

$$\begin{aligned}||\psi-\psi_M||^2_{2,\Omega} &\lesssim a(\psi-\psi_M,\psi-\psi_M)\\ &= a(\psi-\psi_M,\psi-\varphi_M+\varphi_M-\psi_M)\\ &= a(\psi-\psi_M,\varphi-\varphi_M)+a(\psi-\psi_M,\varphi_M-\psi_M)\\ &\lesssim ||\psi-\psi_M||_{2,\Omega}||\psi-\varphi_M||_{2,\Omega},\end{aligned}$$

which is equivalent to the following form

$$||\psi-\psi_M||_{2,\Omega} \lesssim ||\psi-\varphi_M||_{2,\Omega},\ \forall\varphi_M\in X_M(\Omega). \tag{3.5}$$

From (3.5) and the arbitrariness of φ_M, the desired result follows. □

Let $\Pi_M : L^2(\Omega)\to X_M(\Omega)$ be a L^2-orthogonal projection:

$$(\Pi_M\psi-\psi,\varphi)=0,\quad \forall\varphi\in X_M(\Omega).$$

Theorem 3. *For any $\psi(\mathbf{x})\in H_p^m(\Omega)$ and $0\le\mu\le m$, there is a constant C such that the following inequality holds:*

$$||\Pi_M\psi-\psi||_{\mu,\Omega}\le CM^{\mu-m}|\psi|_{m,\Omega}.$$

Proof. We first derive that

$$D^\alpha(\psi-\Pi_M\psi)=D^\alpha\Big(\sum_{|t|>M,|q|>M}\psi_{tq}e^{itx_1+iqx_2}\Big)=\sum_{|t|>M,|q|>M}(it)^{\alpha_1}(iq)^{\alpha_2}\psi_{tq}e^{itx_1+iqx_2}.$$

For any $|\alpha| : 0\leqslant|\alpha|\leqslant\mu\le m$, taking $\alpha_1\le m_1,\ \alpha_2\le m-m_1$, we have

$$\begin{aligned}||D^\alpha(\psi-\Pi_M\psi)||^2 &= (2\pi)^2\sum_{|t|>M,|q|>M}t^{2\alpha_1}q^{2\alpha_2}|\psi_{tq}|^2\\ &= (2\pi)^2\sum_{|t|>M,|q|>M}t^{2(\alpha_1-m_1)}q^{2[\alpha_2-(m-m_1)]}|\psi_{tq}|^2t^{2m_1}q^{2(m-m_1)}\\ &\le (2\pi)^2M^{2(\alpha_1-m_1)}M^{2[\alpha_2-(m-m_1)]}\sum_{|t|>M,|q|>M}|\psi_{tq}|^2t^{2m_1}q^{2(m-m_1)}\\ &\le M^{2(|\alpha|-m)}\sum_{|t|\ge0,|q|\ge0}(2\pi)^2|\psi_{tq}|^2t^{2m_1}q^{2(m-m_1)}\\ &\le M^{2(\mu-m)}|\psi|^2_{m,\Omega},\end{aligned}$$

by making a summation for $|\alpha|$ from 0 to μ, we can obtain the expected results. □

Theorem 4. *Let $\psi_M(\mathbf{x})$ be the approximation solution of $\psi(\mathbf{x})$. If $\psi(\mathbf{x})\in H_p^m(\Omega)$, the following inequality holds*

$$||\psi-\psi_M||_{2,\Omega}\lesssim M^{2-m}|\psi|_{m,\Omega}.$$

Proof. According to Theorem 2, we have

$$||\psi-\psi_M||_{2,\Omega}\lesssim\inf_{\varphi\in X_M}||\psi-\varphi_M||_{2,\Omega}.$$

We derive form Theorem 3 that

$$||\psi-\psi_M||_{2,\Omega}\lesssim||\psi-\Pi_M\psi||_{2,\Omega}\lesssim M^{2-m}|\psi|_{m,\Omega}.$$

The proof is completed. □

4. Efficient implementation of the algorithm

In this section, we will describe the implementation process of the algorithm in detail, and give a brief pseudo code. To solve (2.2) by Fourier spectral method, we shall look for

$$\psi_M = \sum_{|t|=0}^{M}\sum_{|q|=0}^{M}\psi_{tq}e^{itx_1}e^{iqx_2}. \tag{4.1}$$

Let

$$\Psi = \begin{pmatrix} \psi_{-M,-M} & \cdots & \psi_{-M,0} & \cdots & \psi_{-M,M} \\ \cdots & \ddots & \cdots & \ddots & \cdots \\ \psi_{0,-M} & \cdots & \psi_{0,0} & \cdots & \psi_{0,M} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi_{M,-M} & \cdots & \psi_{M,0} & \cdots & \psi_{M,M} \end{pmatrix}.$$

We denote by $\overline{\Psi}$ a column vectors with $(2M+1)^2$ elements, which consist of $2M+1$ columns of Ψ. Let $\overline{\varphi}_M(\mathbf{x}) = e^{-ikx_1}e^{-ilx_2}, (|k|, |l| = 0, 1, \cdots, M)$, then we have

$$\begin{aligned}
&\int_\Omega \Delta\psi_M \Delta\overline{\varphi}_M d\mathbf{x} \\
=&\sum_{|t|=0}^{M}\sum_{|q|=0}^{M}\psi_{tq}\int_0^{2\pi}\int_0^{2\pi}\Delta(e^{itx_1}e^{iqx_2})\Delta(e^{-ikx_1}e^{-ilx_2})d\mathbf{x} \\
=&\sum_{|t|=0}^{M}\sum_{|q|=0}^{M}\psi_{tq}(s_{kt}m_{lq} + o_{kt}g_{lq} + g_{kt}o_{lq} + m_{kt}s_{lq}) \\
=&S(k,:)UM(l,:)^T + O(k,:)UG(l,:)^T + G(k,:)UO(l,:)^T + M(k,:)US(l,:)^T \\
=&[M(l,:)\otimes S(k,:) + G(l,:)\otimes O(k,:) + O(l,:)\otimes G(k,:) + S(l,:)\otimes M(k,:)]\overline{\Psi},
\end{aligned}$$

where

$$\begin{aligned}
s_{kt} = 2\pi k^2 t^2 \delta_{kt},\ S = (s_{kt})_{|k|,|t|=0}^{2M+1},\ m_{kt} = 2\pi\delta_{kt},\ M = (m_{kt})_{|k|,|t|=0}^{2M+1}, \\
o_{kt} = 2\pi k^2\delta_{kt},\ O = (o_{kt})_{|k|,|t|=0}^{2M+1},\ g_{kt} = 2\pi t^2\delta_{kt},\ G = (d_{kt})_{|k|,|t|=0}^{2M+1},
\end{aligned}$$

$S(k,:)$ indicates the k-th row of the matrix S, $M(k,:)$, $O(k,:)$ and $G(k,:)$ are similar to $S(k,:)$.
$\otimes$ represents the tensor product of matrix, i.e. $M\otimes S = (m_{kt}S)_{|k|,|t|=0}^{2M+1}$.

$$\begin{aligned}
\int_\Omega \alpha(\mathbf{x})\nabla\psi_M\nabla\overline{\varphi}_M d\mathbf{x} =& \sum_{|t|=0}^{M}\sum_{|q|=0}^{M}\psi_{tq}\int_\Omega \alpha(\mathbf{x})\nabla(e^{itx_1}e^{iqx_2})\nabla(e^{-ikx_1}e^{-ilx_2})d\mathbf{x} \\
=& \sum_{|t|=0}^{M}\sum_{|q|=0}^{M}\psi_{tq}(tk+ql)\int_\Omega \alpha(\mathbf{x})e^{itx_1}e^{iqx_2}e^{-ikx_1}e^{-ilx_2}d\mathbf{x} \\
=& A((l+M+1)+(2M+1)(k+M),:)\overline{\Psi},
\end{aligned}$$

$$\int_\Omega \beta(\mathbf{x})\psi_M\overline{\varphi}_M d\mathbf{x} = \sum_{|t|=0}^{M}\sum_{|q|=0}^{M}\psi_{tq}\int_\Omega \beta(\mathbf{x})e^{itx_1}e^{iqx_2}e^{-ikx_1}e^{-ilx_2}d\mathbf{x}$$
$$=B((l+M+1)+(2M+1)(k+M),:)\overline{\Psi},$$

where

$$A = (a_{ktlq})_{|k|,|t|,|l|,|q|=0}^{2M+1}, a_{ktlq} = (tk+ql)\int_\Omega \alpha(\mathbf{x})e^{itx_1}e^{iqx_2}e^{-ikx_1}e^{-ilx_2}d\mathbf{x},$$
$$B = (b_{ktlq})_{|k|,|t|,|l|,|q|=0}^{2M+1}, b_{ktlq} = \int_\Omega \beta(\mathbf{x})e^{itx_1}e^{iqx_2}e^{-ikx_1}e^{-ilx_2}d\mathbf{x}.$$

Then the equivalent matrix form based on tensor product for the discrete scheme (2.2) is as follows:

$$(M\otimes S + G\otimes O + O\otimes G + S\otimes M + A + B)\overline{\Psi} = F, \tag{4.2}$$

where

$$F = (f_{kl})_{|k|,|l|=0}^{2M+1}, f_{kl} = \int_\Omega f(\mathbf{x})e^{-ikx_1}e^{-ilx_2}d\mathbf{x}.$$

Note that when α,β are constants, we know from the orthogonal property of Fourier basis functions that the stiffness matrix and mass matrix in (4.2) are all sparse, so we can solve (4.2) efficiently. However, for general variable coefficients α,β, the stiffness matrix and mass matrix are usually full. In that case, we can use the preconditioned iteration method or Schur-complement approach, i.e., block Gaussian elimination to solve (4.2).

5. Extension to three-dimensional case

In this section, we shall extend our algorithm to three-dimensional case. As a model, we consider the following three dimensional fourth-order problem with periodic boundary conditions:

$$\Delta^2\psi - \nabla(\alpha\nabla\psi) + \beta\psi = f,\ \mathbf{x}\in\Omega, \tag{5.1}$$
$$\psi(\mathbf{x}) = \psi(x_1+L_{x_1},x_2,x_3),\ \frac{\partial\psi(\mathbf{x})}{\partial x_1} = \frac{\partial\psi(x_1+L_{x_1},x_2,x_3)}{\partial x_1}, \tag{5.2}$$
$$\psi(\mathbf{x}) = \psi(x_1,x_2+L_{x_2},x_3),\ \frac{\partial\psi(\mathbf{x})}{\partial x_2} = \frac{\partial\psi(x_1,x_2+L_{x_2},x_3)}{\partial x_2}, \tag{5.3}$$
$$\psi(\mathbf{x}) = \psi(x_1,x_2,x_3+L_{x_3}),\ \frac{\partial\psi(\mathbf{x})}{\partial x_3} = \frac{\partial\psi(x_1,x_2,x_3+L_{x_3})}{\partial x_3}, \tag{5.4}$$

where α and β are constant coefficients, $L_{x_1} = x_{1R}-x_{1L}$, $L_{x_2} = x_{2R}-x_{2L}$, $L_{x_3} = x_{3R}-x_{3L}$, $\mathbf{x} = (x_1,x_2,x_3)$, $\Omega = (x_{1L},x_{1R})\times(x_{2L},x_{2R})\times(x_{3L},x_{3R})$.

Similar to two-dimensional case, we can derive the weak form and discrete scheme for the three-dimensional case. Define a Sobolev space:

$$H_p^2(\Omega) = \Big\{\psi\in H^2(\Omega) : \psi \text{ satisfies the periodic boundary conditions (5.2)} - (5.4)\Big\}.$$

Then a weak form of (5.1)–(5.4) is to find $\psi \in H_p^2(\Omega)$, such that

$$a(\psi, \varphi) = F(\varphi), \quad \forall \varphi \in H_p^2(\Omega). \tag{5.5}$$

Define an approximation space:

$$X_M(\Omega) = \text{span}\{e^{i2\pi t \frac{x_1 - x_{1L}}{L_{x_1}}} e^{i2\pi q \frac{x_2 - x_{2L}}{L_{x_2}}} e^{i2\pi j \frac{x_3 - x_{3L}}{L_{x_3}}} : |t|, |q|, |j| = 0, 1, \cdots, M\}.$$

Then the corresponding discrete scheme for the weak form (5.5) is to find $\psi_M \in X_M(\Omega)$, such that

$$a(\psi_M, \varphi_M) = F(\varphi_M), \quad \forall \varphi_M \in X_M(\Omega). \tag{5.6}$$

We shall derive the equivalent matrix form based on tensor product for the discrete scheme (5.6). Let

$$\psi_M = \sum_{|t|=0}^{M} \sum_{|q|=0}^{M} \sum_{|j|=0}^{M} \psi_{tq}^j e^{itx_1} e^{iqx_2} e^{ijx_3}, \tag{5.7}$$

$$\Psi^j = \begin{pmatrix} \psi_{-M,-M}^j & \cdots & \psi_{-M,0}^j & \cdots & \psi_{-M,M}^j \\ \cdots & \ddots & \cdots & \ddots & \cdots \\ \psi_{0,-M}^j & \cdots & \psi_{0,0}^j & \cdots & \psi_{0,M}^j \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi_{M,-M}^j & \cdots & \psi_{M,0}^j & \cdots & \psi_{M,M}^j \end{pmatrix}.$$

We denote by $\tilde{\Psi}^j$ a column vectors with $(2M+1)^2$ elements consisting of $2M+1$ columns of Ψ^j. Let $\Psi = (\tilde{\Psi}^{-M}, \tilde{\Psi}^{-M+1}, \cdots, \tilde{\Psi}^{M})$, and denote by $\tilde{\Psi}$ a column vectors with $(2M+1)^3$ elements consisting of $2M+1$ columns of Ψ. Taking $\overline{\varphi}_M(\mathbf{x}) = e^{-ikx_1} e^{-ilx_2} e^{-ipx_3}$, $(|k|, |l|, |p| = 0, 1, \cdots, M)$, then we have

$$\begin{aligned}
\int_\Omega \Delta\psi_M \Delta\overline{\varphi}_M d\mathbf{x} &= \sum_{|t|=0}^{M} \sum_{|q|=0}^{M} \sum_{|j|=0}^{M} \psi_{tq}^j \int_\Omega \Delta(e^{itx_1} e^{iqx_2} e^{ijx_3}) \Delta(e^{-ikx_1} e^{-ilx_2} e^{-ipx_3}) d\mathbf{x} \\
&= \sum_{|t|=0}^{M} \sum_{|q|=0}^{M} \sum_{|j|=0}^{M} \psi_{tq}^j (s_{kt} m_{lq} m_{pj} + o_{kt} o_{lq} m_{pj} + o_{kt} m_{lq} o_{pj} + o_{kt} o_{lq} m_{pj} \\
&\qquad + m_{kt} s_{lq} m_{pj} + m_{kt} o_{lq} o_{pj} + o_{kt} m_{lq} o_{pj} + m_{kt} o_{lq} o_{pj} \\
&\qquad + m_{kt} m_{lq} s_{pj}) \\
&= [M(p,:) \otimes M(l,:) \otimes S(k,:) + M(p,:) \otimes O(l,:) \otimes O(k,:) \\
&\quad + O(p,:) \otimes M(l,:) \otimes O(k,:) + M(p,:) \otimes O(l,:) \otimes O(k,:) \\
&\quad + M(p,:) \otimes S(l,:) \otimes M(k,:) + O(p,:) \otimes O(l,:) \otimes M(k,:) \\
&\quad + O(p,:) \otimes M(l,:) \otimes O(k,:) + O(p,:) \otimes O(l,:) \otimes M(k,:) \\
&\quad + S(p,:) \otimes M(l,:) \otimes M(k,:)] \tilde{\Psi}, \\
\int_\Omega \nabla\psi_M \nabla\overline{\varphi}_M d\mathbf{x} &= \sum_{|t|=0}^{M} \sum_{|q|=0}^{M} \sum_{|j|=0}^{M} \psi_{tq}^j \int_\Omega \nabla(e^{itx_1} e^{iqx_2} e^{ijx_3}) \nabla(e^{-ikx_1} e^{-ilx_2} e^{-ipx_3}) d\mathbf{x}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{|t|=0}^{M}\sum_{|q|=0}^{M}\sum_{|j|=0}^{M}\psi_{tq}^{j}(o_{kt}m_{lq}m_{pj} + o_{kt}m_{lq}o_{pj} + m_{kt}m_{lq}o_{pj})\\
&=[M(p,:)\otimes M(l,:)\otimes O(k,:) + O(p,:)\otimes M(l,:)\otimes O(k,:)\\
&\quad + M(p,:)\otimes M(l,:)\otimes O(k,:)]\tilde{\Psi},\\
\int_{\Omega}\psi_M\overline{\varphi}_M d\mathbf{x} &= \sum_{|t|=0}^{M}\sum_{|q|=0}^{M}\sum_{|j|=0}^{M}\psi_{tq}^{j}\int_{\Omega}e^{itx_1}e^{iqx_2}e^{ijx_3}e^{-ikx_1}e^{-ilx_2}e^{-ipx_3}d\mathbf{x}\\
&= \sum_{|t|=0}^{M}\sum_{|q|=0}^{M}\sum_{|j|=0}^{M}\psi_{tq}^{j}m_{kt}m_{lq}m_{pj}\\
&=[M(p,:)\otimes M(l,:)\otimes M(k,:)]\tilde{\Psi}.
\end{aligned}
$$

Then the equivalent matrix form based on tensor product for the discrete scheme (5.6) is as follows:

$$(\mathcal{A}+\mathcal{B}+C)\tilde{\Psi}=\mathcal{F},$$

where

$$
\begin{aligned}
&\mathcal{A} = M\otimes M\otimes S + M\otimes O\otimes O + O\otimes M\otimes O + M\otimes O\otimes O\\
&\ + M\otimes S\otimes M + O\otimes O\otimes M + O\otimes M\otimes O + O\otimes O\otimes M + S\otimes M\otimes M,\\
&\mathcal{B} = M\otimes M\otimes O + O\otimes M\otimes O + M\otimes M\otimes O,\ C = M\otimes M\otimes M,\\
&\mathcal{F} = (f_{klp})_{|k|,|l|,|p|=0}^{2M+1},\ f_{klp} = \int_{\Omega} f(\mathbf{x})e^{-ikx_1}e^{-ilx_2}e^{-ipx_3}d\mathbf{x}.
\end{aligned}
$$

6. Numerical experiments

In this section, we shall perform some numerical experiments to confirm the correctness of theoretical analysis and the effectiveness of our algorithm. The programs are compiled and operated in MATLAB 2018b.

6.1. Two dimensional case

Example 1. We take $\alpha = 1$, $\beta = 10$ and choose the exact solution $\psi = \sin 4x_1 \sin 8x_2$. Then f can be obtained by plugging ψ into the Eq (1.1). We shall solve (1.1)–(1.3) by using the algorithm proposed in section 4. We list in Table 1 the errors between the exact solution and the approximate solution under H^2 norm, H^2 seminorm and L^2 norm respectively for different M. In addition, we also present their comparison figures and absolute error figures for different M in Figures 1 and 2.

We observe from Table 1 that the approximate solution $\psi_M(\mathbf{x})$ reaches about 10^{-12} accuracy when $M \geq 8$. Besides, we also see from Figures 1 and 2 that the approximation solution converges to the exact solution.

Remark 1. *Though the proof of well-posedness of weak solution requires α to be a nonnegative bounded periodic function and β to be a positive bounded function, our algorithm is still valid for some large and negative α and β, and the corresponding numerical results are listed in Table 2.*

Table 1. The error between exact solution and approximation solution.

M	$M = 4$	$M = 6$	$M = 8$	$M = 10$
$\|\|\psi - \psi_M\|\|_2$	153.4243	153.4243	8.8628e-13	8.9344e-13
$\|\psi - \psi_M\|_2$	150.7964	150.7964	8.6829e-13	8.7548e-13
$\|\|\psi - \psi_M\|\|$	3.1416	3.1416	12.9165e-14	2.9192e-14

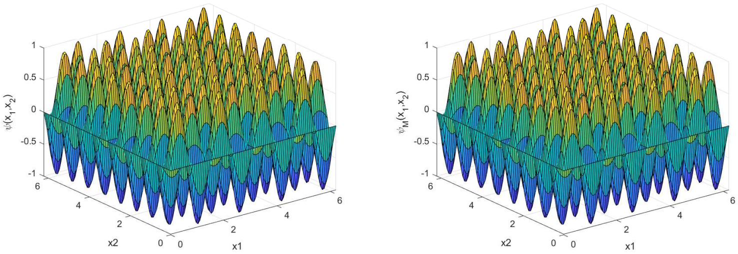

Figure 1. Comparison figures between exact solution (left) and approximation solution (right) with $M = 10$.

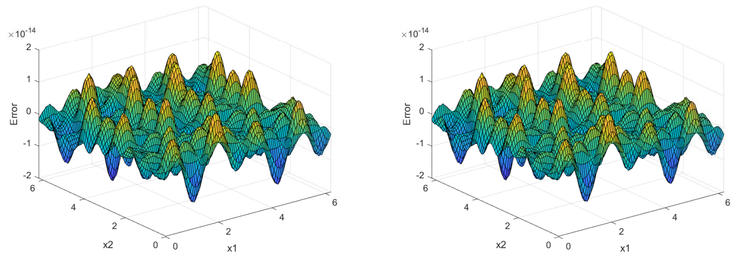

Figure 2. Error figures between exact solution and approximation solution with $M = 10$ (left) and $M = 15$ (right).

Table 2. The error $||\psi - \psi_M||_2$ between exact solution and approximation solution for different α and β.

α, β	$M = 4$	$M = 6$	$M = 8$	$M = 10$
$\alpha = 1, \beta = -10$	153.4243	153.4243	1.2239e-12	1.2306e-12
$\alpha = 1, \beta = -100$	153.4243	153.4243	1.7578e-12	1.7625e-12
$\alpha = 1, \beta = -1000$	153.4243	153.4243	6.2198e-12	6.2216e-12
$\alpha = -10, \beta = 1$	153.4243	153.4243	2.7402e-12	2.7431e-12
$\alpha = -100, \beta = 1$	153.4243	153.4243	6.1361e-10	6.1371e-10
$\alpha = -1000, \beta = 1$	153.4243	153.4243	9.3114e-13	9.6030e-13
$\alpha = -10, \beta = -10$	153.4243	153.4243	2.7778e-12	2.7800e-12
$\alpha = -100, \beta = -100$	153.4243	153.4243	6.7819e-12	1.4656e-11
$\alpha = -1000, \beta = -1000$	153.4243	153.4243	3.3564e-13	4.1513e-13

Example 2. We take $\alpha = \sin(x_1 + x_2) + 2$, $\beta = e^{\cos(x_1+x_2)}$, and choose the exact solution $\psi = e^{\sin(x_1+x_2)}$. We list in Table 3 the errors between the exact solution and the approximate solution under H^2 norm, H^2 seminorm and L^2 norm respectively for different M. Similarly, we also present their comparison figures and absolute error figures for different M in Figures 3 and 4. In order to further show the spectral accuracy of our algorithm, we present in Figure 5 error figures between exact solution and approximation solutions under L^2 and H^2 norms for different M.

We observe from Table 3 that the approximate solution $\psi_M(\mathbf{x})$ reach about 10^{-11} accuracy when $M \geq 12$. We see from Figures 3–5 that the approximation solution exponentially converges to the exact solution.

Table 3. The error between exact solution and approximation solution.

M	$M = 8$	$M = 10$	$M = 12$	$M = 14$
$\|\|\psi - \psi_M\|\|_2$	6.9210e-06	2.3352e-08	5.2053e-11	3.7072e-14
$\|\psi - \psi_M\|_2$	6.8925e-06	2.3288e-08	5.1950e-11	3.1910e-14
$\|\|\psi - \psi_M\|\|$	4.9097e-08	1.1108e-10	1.7815e-13	9.4959e-15

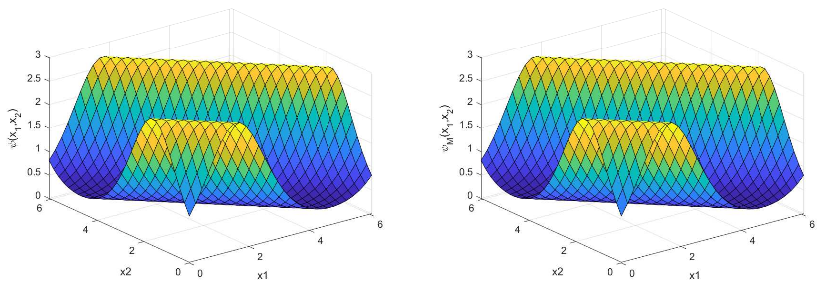

Figure 3. Comparison figures between exact solution (left) and approximation solution (right) with $M = 14$.

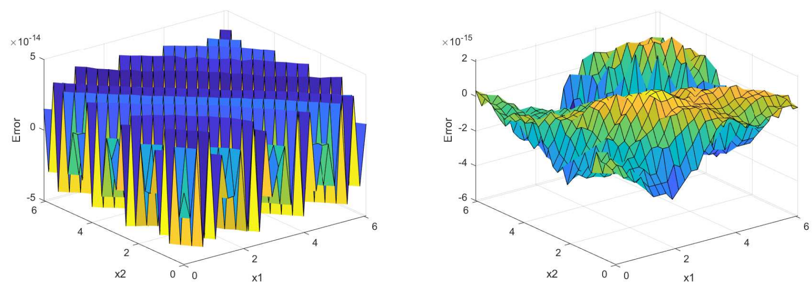

Figure 4. The error figures between exact solution and approximation solution with $M = 12$ (left) and $M = 14$ (right).

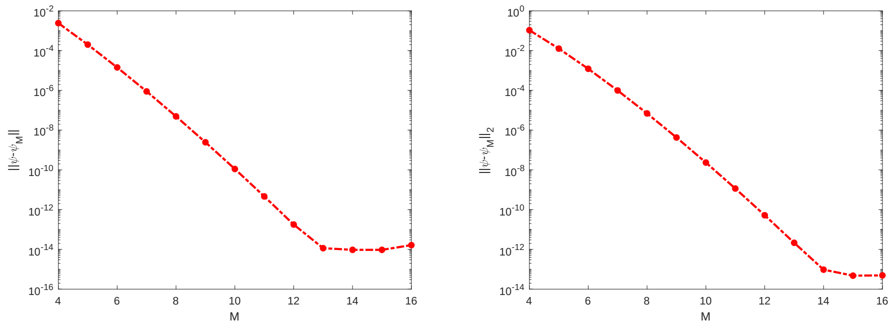

Figure 5. The error figures between exact solution and approximation solutions under L^2 (left) and H^2 (right) norms for different M.

6.2. Three dimensional case

Example 3. We take $\alpha = 1$, $\beta = 1$ and choose the exact solution $\psi = \cos 3x_1 \cos 4x_2 \cos 5x_3$. We list in Table 4 the errors between the exact solution and the approximate solution under H^2 norm, H^2 seminorm and L^2 norm respectively for different M.

We observe from Table 4 that the approximate solution reach about 10^{-13} accuracy when $M \geq 8$. That is to say, even in the three-dimensional case, our algorithm still has spectral accuracy.

Table 4. The error between exact solution and approximation solution.

M	$M = 4$	$M = 6$	$M = 8$	$M = 10$
$\|\|\psi - \psi_M\|\|_2$	281.2419	7.5429e-13	7.8632e-13	7.9945e-13
$\|\psi - \psi_M\|_2$	278.4164	7.4226e-13	7.7431e-13	7.8757e-13
$\|\|\psi - \psi_M\|\|$	0.0512	2.2170e-16	2.2428e-16	2.2440e-16

Example 4. We take $\alpha = 1$, $\beta = 1$ and choose the exact solution $\psi = e^{\cos x_1 + \cos x_2 + \cos x_3}$. We list in Table 5 the errors between the exact solution and the approximate solution under H^2 norm, H^2 seminorm and L^2 norm respectively for different M.

We observe from Table 5 that the approximate solution reach about 10^{-10} accuracy when $M \geq 12$. Again, our algorithm has spectral accuracy.

Table 5. The error between exact solution and approximation solution.

M	$M = 6$	$M = 8$	$M = 10$	$M = 12$
$\|\|\psi - \psi_M\|\|_2$	0.0071	3.9971e-05	1.3439e-07	2.9897e-10
$\|\psi - \psi_M\|_2$	0.0070	3.9726e-05	1.3384e-07	2.9809e-10
$\|\|\psi - \psi_M\|\|$	1.2946e-06	4.4641e-09	1.0100e-11	1.6372e-14

7. Conclusions

We have developed an efficient Fourier spectral-Galerkin method to solve the fourth-order elliptic equation with periodic boundary conditions and variable coefficients. Firstly, we prove the error estimations between the weak solutions and approximation solutions. Then we derive the equivalent matrix form based on tensor product for the discrete scheme. Numerical experiments validate the theoretical analysis and algorithm. Besides, the method proposed in this paper can be extended to some more complex linear and nonlinear equations, such as fourth-order parabolic equation [31], Cahn-Hiliard equation, Gross Pitaevskii equation, which is our future research goal.

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Conflict of interest

The authors declare that they have no competing interests.

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