Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(4): 9535–9556.
DOI:10.3934/math.2023481
Received: 23 September 2022
Revised: 03 January 2023
Accepted: 09 January 2023
Published: 20 February 2023

Research article

A novel numerical method for solving the Caputo-Fabrizio fractional differential equation

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Abstract: In this paper, a unique and novel numerical approach—the fractional-order Caputo-Fabrizio derivative in the Caputo sense—is developed for the solution of fractional differential equations with a non-singular kernel. After converting the differential equation into its corresponding fractional integral equation, we used Simpson's $1/3$ rule to estimate the fractional integral equation. A thorough study is then conducted to determine the convergence and stability of the suggested method. We undertake numerical experiments to corroborate our theoretical findings.

Keywords: fractional differential equation; non-singular operator; numerical approximation; stability analysis; convergence analysis
Mathematics Subject Classification: 26A33, 26C10

1. Introduction

Fractional calculus is primarily concerned with fractional integration and differentiation operations. It is an outstanding approach to situations where existing local operators are incapable of producing effective results, as it has been observed that the fractional order models are better matched with the real data than the classical integer-order derivatives [10, 25]. The theory of fractional-order calculus was initially studied and further explored in the 18*th* and 19*th* centuries. One of the distinct features of fractional derivatives is their capacity to provide a pertinent and practical choice to model important physical problems. Many physical applications are not correctly modeled using the local differential operators. Therefore, the theory of the fractional-order derivative has attracted the attention of applied mathematicians to use fractional differential equations (FDEs) as a powerful tool in various areas, particularly in the fields of physics and engineering [11, 47]. Fractional-order differential equations hold a strong foothold in some major domains, particularly in control theory [35], diffusion problems [55], control relaxation processes and viscous fluid flow [47], signal processing [45], dynamics ([34, 56]) and bio-engineering [39]. In addition, fractional-order models applied in micro-grids [14] are used in wireless networks [67]. Similarly, in fractional calculus, fractional-order models provide unprecedented significance in studying the dynamics of biological systems [7, 60]. Kilbas et al. addressed the theory of fractional differential equations and their applications [33].

Fractional derivative operators (FDOs) are significantly relevant to real data analysis, which has drawn great attention from various mathematicians and modelers in the applied sciences. A variety of fractional operators are widely used in the literature, although few of them are comparatively more common, including Riemann-Liouville, Hadamard, Weyl, ([50, 53]), Caputo [22], and Jumarie ([28, 29]). The kernel of the most commonly used fractional operators namely Caputo and Riemann-Liouville contains singularity, and hence, they may not always be able to express the non-locality of real-world situations properly. Thus, new fractional derivatives with nonsingular kernels have been defined in order to accurately model the nonlocal systems. Although fractional derivatives with singularity have many advantages, they are still not applicable in many situations. For example, the Caputo derivative requires higher regularity conditions for differentiation, and this is only classified for distinguishable functions. In the case of the Riemann-Liouville derivative, the constant refers to a non-zero value. The fractional-order Jumarie derivative will not exist if the function is discontinuous at the origin [8]. Researchers are investigating some more efficient fractional operators. To overcome the challenge of singularity and to find efficient and robust modeling results in recent years, a more efficient fractional-order Caputo-Fabrizio derivative has been introduced with the non-singular kernel by Caputo and Fabrizio. It is considered a constructive approach with the fact of transforming to integer power using Laplace transformation, therefore for some cases, we can easily calculate the exact solution [13]. The analysis of fractional operators and some of their new properties have been discussed in [51]. Major properties of the Caputo-Fabrizio derivative are presented in [36] and the progress of this newly introduced fractional operator is discussed in ([12, 57]). To study more applications, we refer to ([21, 58, 59, 61–66]). In contrast to ordinary integer derivatives, fractional-order derivatives have gained popularity in recent years as effective solutions to difficult problems. [26] proclaim that uncertain fractional-order Caputo derivatives produce outstanding outcomes in real financial markets. As experimental investigations and the relevant sensitivity analyses justified the author's claims. In another study [37], a novel uncertain fractional currency model is successfully developed and the Mittag-Leffler function is used to calculate the solution of the fractional differential equation using the Caputo derivative. The monotonicity theorem for UFDEs in the sense of Caputo is the central subject of the research [27]. A unique uncertain fractional order mean-reverting model is provided with a floating interest rate that better reflects the actual uncertain financial market. In order to determine the optimal solutions for a class of nonlinear programming problems, a fractional order mathematical model with the steepest descent path is given in [18]. A research [42] has developed an efficient numerical method, the iterative reproducing kernel algorithm, to generate numerical solutions to the fractional order Bernoulli and Riccati differential equations while taking the Caputo-Fabrizio derivative into account. Besides this, a method for solving fractional Volterra partial integro-differential equations utilizing the Caputo-Fabrizio fractional derivative is presented in [15] as a new, effective approach. Using the Legendre-Gauss-Lobatto quadrature rule in combination with unique operational matrices, an effective technique is produced. Similarly, the exponential Euler difference form for fractional differential equations in the Caputo-Fabrizio sense with variable lags is established in [65]. The study demonstrates that the developed difference form falls within the umbrella of implicit Euler differences, and to resolve this implicit difference, the fractional PECE technique is then presented. Furthermore, to solve a group of cubic and quadratic logistic equations with Caputo-Fabrizio fractional order derivatives in Hilbert space, a modified replicating kernel approach is presented in [16]. While analyzing the population growth model, the effects of the fractional order Caputo-Fabrizio are investigated in comparison to the conventional Caputo derivatives. The findings demonstrate that utilizing the novel Caputo Fabrizio derivative, the proposed sophisticated technique has numerous benefits in terms of stability. In another study [52], cylindrical geometry is used to explore the unsteady fractional advection-diffusion equation. The fractional order model of advection-diffusion employs the Caputo-Fabrizio time-fractional derivative. In order to deal with a class of fuzzy fractional differential equations containing the Caputo-Fabrizio derivative, an improved numerical-analytical method is proposed in [23]. The outcomes demonstrate the precision and high caliber of the suggested approach, particularly in nonlinear situations. With the help of the Caputo-Fabrizio operator, the analytic-approximate solutions for the fractional Volterra integro-differential equations are examined in [64]. The analysis of the derived solutions shows that the methodology is suitable to handle numerous physical problems in the Caputo-Fabrizio sense. In [54] fractional order derivatives such as Captuo-Fabrizio, Caputo, and conformable fractions have been employed with the cubic B-spline approach to approximate polynomial solutions for fractional Painleve and Bagley-Torvik equations. Moreover, in [19], study has investigated the complex behaviour of the COVID-19 Omicron version using Caputo-Fabrizio fractional operators. The existence and singularity of the model's system of solutions have been discussed. Further, a numerical technique is developed by incorporating an exponential law kernel for the study and dynamical transmission of the virus. The outcome of the research showed that the fractional model of COVID-19 is reliable.

In further fractional literature, multiple major studies have been made by incorporating Caputo-Fabrizio models. In this context, a fractional-order reaction–diffusion model named the Allen Cahn model has been studied under Atangana-Baleanu and Caputo-Fabrizio fractional derivatives [46]. The effectiveness of the proposed numerical solutions of these modified models has been extended by utilizing the Crank-Nicholson scheme. Moreover, the Caputo-Fabrizio advection-diffusion equation is investigated in the half-plane using the Laplace transform in [40]. The particular solution of the diffusion process is computed against the Dirichlet problem. Similarly, basic and efficient solutions to the Cauchy and Dirichlet problems with the Caputo-Fabrizio derivative for a heat conduction equation are investigated on a line segment in [9]. A nonlinear differential equation with the Caputo-Fabrizio operator in Banach spaces is investigated in [30] using the Banach contraction mapping principle, and further favorable conditions for its solution have been presented. Likewise, a fractional-order schistosomiasis disease model is analyzed with the help of an exponential law kernel as well as the Mittag-Leffler kernel in Liouville-Caputo sense [61]. Numerical simulations of the fractional SIR model with the Caputo-Fabrizio derivative are formulated in [62] using an iterative scheme with Laplace transform. Besides this, with the help of fractional operators, mainly Caputo-Fabrizio and Atangana-Baleanu a malarian model has been investigated in [3]. In [49], a fractional mathematical model is investigated numerically through the Genocchi collocation technique using the Caputo-Fabrizio fractional derivative. A nonlinear fractional order model of COVID-19 has been investigated by incorporating the Caputo-Fabrizio fractional derivative due to the significance of modeling and regulating the COVID-19 pandemic in [44]. [31] considers the Mittag-Leffler kernel and the exponential decay kernel to investigate the Korteweg-de Vries-Burgers (KdV-B) partial differential equation (PDE) involving nonlocal operators. The existence of the solution of the KdV-B PDE for both fractional operators is proven using fixed point theorems of -type z contraction; further, the modified double Laplace transform is used to construct a series solution. In [32], the modified coupled Korteweg-de Vries equation is studied along with the fractional order derivatives of Caputo and Caputo-Fabrizio with a time variable. The proposed model explains the nonlinear evolution of the waves caused by modest dispersion effects. Additionally, it has been found that the coupled system generates a wave solution that reveals the evolution of shock waves due to the steeping influence on temporal evolution. A viscous thermal Maxwell model utilizing the Caputo-Fabrizio derivative is studied in [1]. More specifically, they worked on fluidic-thermal transport via micro-tube when magnetic and electric fields were present. In this study, it has been claimed that the model's efficiency could be controlled by adjusting the fractional-order of the Caputo-Fabrizio operator.

Constructing numerical and analytical solutions for FDEs is a difficult task for many mathematicians. Many physical problems involving fractional models do not have existing exact solutions, so numerous researchers developed a keen interest in developing numerical solutions for fractional-order differential equations. A new scheme called, Adams-Bashforth with the Caputo-Fabrizio operator is constructed in [48] which consists of three steps to solve both fractional nonlinear and linear differential equations. Also, it has various applications in solving fractional-order chaotic systems. A fractional-order Caputo-Fabrizio derivative is used to analyze the surface waver model [5]. Researchers have proven the validity of the fixed point theorem for the solution of the modified system, and further, a particular solution for this system is derived with the help of the iterative method. The Caputo-Fabrizio fractional-order Fokker-Planck equation is numerically solved using the "Ritz method" in [20]. Further, the coefficients of basis functions are obtained by solving a nonlinear algebraic system. [51] also derives the Legendre operational matrix based on Caputo-Fabrizio using the Tau method.

The study [4] presents the fractional mass-spring-damper system with Caputo and Caputo-Fabrizio derivatives. [41] discusses the fractional dynamics of oxygen diffusion using the Laplace homotopy method, and the Caputo-Fabrizio-Caputo and Liouville-Caputo operators are used to investigate the equation of oxygen dispersion across tissues. In addition, to visualize the useful application of the proposed scheme, they compared it with conventional methods. The Modified-Caputo-Fabrizio derivative is proposed in [63] to calculate the solution of fractional-order differential equations. Analytical solutions based on the homotopy analysis method (HAM) and the multi-step homotopy analysis method (MHAM) are explored. In [24] authors developed the fractional Euler method with the order of convergence one, to solve differential equations with fractional-order Caputo-Fabrizio operator and solve the HIV model using the proposed method. In [25] trapezoidal scheme is devised to find an efficient solution to the fractional differential equation using the Caputo-Fabrizio operator with convergence order two further, the stability and convergence of the derived method are analyzed. Motivated by this study, we developed Simpson's 1/3 method for solving fractional order differential equations with the Caputo-Fabrizio derivative with an order of accuracy of four in this paper. According to the authors' knowledge, there is no work on solving Caputo-Fabrizio fractional differential equations with Simpson's 1/3 method. The advantage of the proposed fractional Simpson's 1/3 method is that it has greater accuracy than existing methods and is easy to implement.

The following structure describes how this article is set up. In order to solve differential equations of fractional order, Section 2 presents a novel numerical method based on Simpson's 1/3 rule and incorporates the Caputo-Fabrizio derivative. Section 3 discusses the constructed scheme's convergence and stability. Numerical tests in Section 4 are provided to confirm the reliability of our generated method.

2. Simpson's 1/3 rule for Caputo-Fabrizio fractional derivative

In 2015, Caputo and Fabrizio succeeded in introducing the fractional order derivative Caputo-Fabrizio by replacing the singular kernel $(z-\zeta)^{-\alpha}$ with $e^{(\frac{-\alpha(z-\zeta)}{1-\alpha})}$ in the Caputo derivative [13]. For $z \in \mathbb{H}^1(a,b)$, $0 < \alpha < 1$, the α th–order Caputo-Fabrizio fractional derivative and its corresponding integral of $z(\zeta)$, represented by ${}^{CF}D^{\alpha}z(\zeta)$ and ${}^{CF}I^{\alpha}z(\zeta)$ respectively are defined by

$$ {}^{CF}D^{\alpha}z(\zeta) := \frac{1}{1-\alpha}\int_a^t \exp[-\frac{\alpha}{1-\alpha}(\zeta-\delta)]\dot{z}(\delta)d\delta. \tag{2.1}$$

$$ {}^{CF}I^{\alpha}z(\zeta) := (1-\alpha)z(\zeta) + \alpha\int_a^{\zeta} z(\delta)d\delta. \tag{2.2}$$

In this section, we construct the fractional Simpson's 1/3 scheme to find out the solution to Caputo-Fabrizio fractional differential equations. The 1/3 Simpson rule is a closed scheme of numerical integration that approximates integration function with a quadratic polynomial. Taking into account, the following α order fractional differential equation

$$\begin{cases} {}^{CF}D^{\alpha}z(\zeta) = g(z(\zeta)), & a < \zeta < b < \infty, \\ z(a) = z_0, \end{cases} \tag{2.3}$$

whereas g refers to a continuous vector function that successfully fulfills the Lipschitz condition

$$|g(z(\zeta_1)) - g(z(\zeta_2))| \leq L\|z(\zeta_1) - z(\zeta_2)\|, \quad L > 0. \tag{2.4}$$

Applying Caputo-Fabrizio fractional integral operator on Eq (2.3) and using Proposition 3 in [2], we get

$$z(\zeta) = z_0 + {}^{CF}I^{\alpha}g(z(\zeta)), \quad a < \zeta < b < \infty,$$

Equation (2.2) yields

$$z(\zeta) = z_0 + (1-\alpha)g(z(\zeta)) + \alpha\int_a^{\zeta} g(z(s))ds. \tag{2.5}$$

Theorem 2.1. *[36] Let us suppose we have a continuous function* $g : [0, T] \times R \to R$ *with* $0 < \alpha < 1$, $T > 0$ *that meets the Lipschitz condition (2.4). The initial value problem (2.3) has a unique solution on* $C[0, T]$ *for any given* $(\gamma_\alpha + \lambda_\alpha T)L < 1$, *where*$\gamma_\alpha = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}$, $\lambda_\alpha = \frac{2\alpha}{(2-\alpha)M(\alpha)}$.

First, we will use a second-degree polynomial P_2 to approximate the integral function g in Eq (2.5), and the function will be evaluated at ζ_0, ζ_1, and ζ_2, with $\zeta_0 < \zeta_1 < \zeta_2$. The interval is divided into two subintervals such as $\zeta_1 - \zeta_0 = \zeta_2 - \zeta_1 = \tau$, for a total width of 2τ.The formula for an interpolant with a second-degree polynomial is given as follows:

$$I_2 g(z(\zeta)) = \int_a^b g(z(\zeta))d\zeta \approx \int_{\zeta_0}^{\zeta_2} P_2(z(\zeta))d\zeta,$$

this gives

$$I_2 g(z(\zeta)) = \int_{\zeta_0}^{\zeta_2} \Big[\frac{(\zeta-\zeta_1)(\zeta-\zeta_2)}{(\zeta_0-\zeta_1)(\zeta_0-\zeta_2)} g(z(\zeta_0)) + \frac{(\zeta-\zeta_0)(\zeta-\zeta_2)}{(\zeta_1-\zeta_0)(\zeta_1-\zeta_2)} g(z(\zeta_1)) + \frac{(\zeta-\zeta_0)(\zeta-\zeta_1)}{(\zeta_2-\zeta_0)(\zeta_2-\zeta_1)} g(z(\zeta_2))\Big] d\zeta.$$

Integrating the interpolant's first term, as we have, $\tau = \frac{\zeta_2-\zeta_0}{2}$ and substituting "$\zeta = s + \zeta_0$", gives us

$$\begin{aligned}
\int_{\zeta_0}^{\zeta_2} \frac{(\zeta-\zeta_1)(\zeta-\zeta_2)}{(\zeta_0-\zeta_1)(\zeta_0-\zeta_2)} d\zeta &= \frac{1}{2\tau^2}\int_{\zeta_0}^{\zeta_0+2\tau} (\zeta-\zeta_1)(\zeta-\zeta_2) d\zeta \\
&= \frac{1}{2\tau^2}\int_0^{2\tau} (s+\zeta_0-\zeta_2)(s+\zeta_0-\zeta_1) ds \\
&= \frac{1}{2\tau^2}\int_0^{2\tau} (s-2\tau)(s-\tau) ds \\
&= \frac{1}{2\tau^2}\int_0^{2\tau} [s^2 - 3\tau s + 2\tau^2] ds \\
&= \Big(\frac{1}{2\tau^2}\Big)\Big(\frac{2\tau^3}{3}\Big) = \frac{\tau}{3}.
\end{aligned}$$

After simplifying the rest of the terms, we have

$$I_2(g) = \frac{\tau}{3}\Big[g(z(\zeta_0)) + 4g(z(\zeta_1)) + g(z(\zeta_2))\Big]. \tag{2.6}$$

Using Eq (2.5), we get

$$z(\zeta_n) = z_0 + (1-\alpha)g(z(\zeta_n)) + \frac{\tau}{3}\Big[g(z(\zeta_0)) + 4g(z(\zeta_1)) + g(z(\zeta_2))\Big], \quad n = 0, 1, 2.$$

The accuracy of the numerical integration can be considerably enhanced by subdividing the integration interval into smaller intervals and applying the quadrature rule to each pair of sub-intervals. For this, we divide $[a, b]$ interval into n sub-intervals. This method is applicable when we have two adjacent sub-intervals at the same time. So, for even integer $n \geq 2$, we define

$$\tau = \frac{b-a}{n}, \ t_k = a + k\tau, \quad k = 0, 1, 2, ..., n.$$

Applying Simpson's 1/3 rule to the sub-intervals $[\zeta_{2k}, \zeta_{2(k+1)}]$, $\quad k = 0, 1, 2, ..., \frac{n-2}{2}$ gives

$$I_n(g) = \int_{\zeta_0}^{\zeta_2} g(z(\zeta))d\zeta + \int_{\zeta_2}^{\zeta_4} g(z(\zeta))d\zeta + ... + \int_{\zeta_{n-2}}^{\zeta_n} g(z(\zeta))d\zeta$$

$$= \sum_{k=0}^{\frac{n-2}{2}} \int_{\zeta_{2k}}^{\zeta_{2k+2}} g(z(\zeta))d\zeta$$

$$= \sum_{k=0}^{\frac{n-2}{2}} \left(\frac{\tau}{3} [g(z(\zeta_{2k})) + 4g(z(\zeta_{2k+1})) + g(z(\zeta_{2k+2}))] \right).$$

Using Eq (2.5), we get the following Simpson's 1/3 rule for the Caputo-Fabrizio fractional differential equation (2.3):

$$z_{n+1} = \zeta_0 + (1-\alpha)g(z_{n+1}) + \alpha\frac{\tau}{3}\Big[g(z(\zeta_0)) + 4\sum_{i=2,4,6}^{n} g(z(\zeta_i)) + 2\sum_{j=1,3,5}^{n-1} g(z(\zeta_j)) + g(z(\zeta_{n+1}))\Big], \quad n = 0, 1, 2, 3, ..., N-1. \quad (2.7)$$

We refer z_n as the approximate solution of $z(\zeta_n)$, then scheme (2.7) can be written as

$$z_{n+1} = \zeta_0 + (1-\alpha)g(z_{n+1}) + \alpha\tau \sum_{k=0}^{n+1} a_k g(z_k), \quad n = 0, 1, 2, 3, ..., N-1, \quad (2.8)$$

where the weights a_k of fractional Simpson's 1/3 rule are given as follows:

$$a_k = \begin{cases} 1/3, & k = 0, \\ 2/3, & k = 1, 3, 5, ..., \\ 4/3, & k = 2, 4, 6, ... \\ 1/3, & k = n+1. \end{cases}$$

3. Critically analyzing the convergence and stability of the proposed scheme

In this section, we investigate the proposed numerical systems' convergence and stability. The symbol C denotes any general constant. To start with, we'll estimate the error.

Lemma 3.1. *Take into account $g(z(\zeta)) \in C^4([a, b])$, such that*

$$\Big| \int_{\zeta_0}^{\zeta_{n+1}} g(z(s))ds - \alpha\tau \sum_{j=0}^{n+1} a_j g(\zeta_j) \Big| \leq C\tau^4,$$

where $C = \frac{(b-a)f^{(4)}(\delta)}{180}$, $\tau = \frac{b-a}{N}$, and $\zeta_k = a + \tau k$, $k = 0, 1, ..., n+1$ on $[a, b]$.

Proof. The integrand function will be expanded about the midpoint ζ_1 of the interval.

$$g(z(\zeta)) = g(z_1) + (\zeta - \zeta_1)g^{(1)}(z_1) + \frac{(\zeta - \zeta_1)^2}{2!}g^{(2)}(z_1) + \frac{(\zeta - \zeta_1)^3}{3!}g^{(3)}(z_1) + \frac{(\zeta - \zeta_1)^4}{4!}g^{(4)}(\delta(\zeta)),$$

where $\delta \in [\zeta_1, \zeta]$, integrating the above equation, we get

$$\begin{aligned}\int_a^b g(z(\zeta))d\zeta &= \int_{\zeta_1-\tau}^{\zeta_1+\tau}\Big[g(z_1)+(\zeta-\zeta_1)g^{(1)}(z_1)+\frac{(\zeta-\zeta_1)^2}{2!}g^{(2)}(z_1)+\frac{(\zeta-\zeta_1)^3}{3!}g^{(3)}(z_1)\\ &\quad +\frac{(\zeta-\zeta_1)^4}{4!}g^{(4)}(\delta(\zeta))\Big]dt\\ &= 2\tau g(z_1)+0.g^{(1)}(z_1)+\frac{2\tau^3}{2\cdot 3}g^{(2)}(z_1)+0.g^{(3)}(z_1)+\frac{2\tau^5}{5\cdot 4!}g^{(4)}(\delta_1),\end{aligned}$$

we arrive

$$\int_a^b g(z(\zeta))d\zeta = 2\tau g(z_1)+\frac{\tau^3}{3}g^{(2)}(z_1)+\frac{\tau^5}{60}g^{(4)}(\delta_1),\ \ \delta_1\in[a,b]. \tag{3.1}$$

Similarly, using Taylor's series, expand the function about t_1 at the endpoints of the interval

$$g(z_0) = g(z_1)-\tau g^{(1)}(z_1)+\frac{\tau^2}{2!}g^{(2)}(z_1)-\frac{\tau^3}{3!}g^{(3)}(z_1)+\frac{\tau^4}{4!}g^{(4)}(\delta_2),$$

$$g(z_2) = g(z_1)+\tau g^{(1)}(z_1)+\frac{\tau^2}{2!}g^{(2)}(z_1)+\frac{\tau^3}{3!}g^{(3)}(z_1)+\frac{\tau^4}{4!}g^{(4)}(\delta_3).$$

The above expressions for $g(z_0)$ and $g(z_2)$ are substituted into Eq (2.6) to get

$$I = \frac{\tau}{3}\Big[6g(z(\zeta_1))+\tau^2 g^{(2)}(z_1)+\frac{\tau^4}{4!}(g^{(4)}(\delta_2)+g^{(4)}(\delta_3))\Big]. \tag{3.2}$$

Let for $a\le\delta\le b$, $g^4(\delta)=\max|(g^{(4)}(\delta(\zeta)))|$. Subtracting Eq (3.2) from Eq (3.1), we found the following integration error

$$E\sim g^{(4)}(\delta)(\frac{\tau^5}{60}-\frac{\tau^5}{36}) = -\frac{\tau^5}{90}g^{(4)}(\delta). \tag{3.3}$$

Hence, the error for composite Simpson's 1/3 rule is

$$\begin{aligned}E_n(g)=I(g)-I_n(g) &= -\sum_{k=0}^{\frac{n-2}{2}}\frac{\tau^5}{90}g^{(4)}(\delta_k)\quad\text{for some}\quad \delta_k\in[\zeta_{2k},\zeta_{2k+2}]\\ &= -\frac{\tau^5}{90}(\frac{n}{2})(\frac{2}{n})\sum_{k=0}^{\frac{n-2}{2}}g^{(4)}(\delta_k)\\ &= -\frac{\tau^5}{180}ng^{(4)}(\delta),\quad \delta\in[a,b],\end{aligned}$$

putting $n=\frac{b-a}{\tau}$ generates

$$E_n(g) = -\Big[\frac{(b-a)g^{(4)}(\delta)}{180}\Big]\tau^4,\quad \delta\in[a,b]. \tag{3.4}$$

Hence

$$|E_n(g)|\le C\tau^4.$$

□

Theorem 3.2. *Conditional stability exists for the newly constructed fractional numerical approach (2.8).*

Proof. In order to prove the aforementioned statement, we use the technique of perturbation and perturb z_0 and z_n $(n = 0, 1, 2..., k + 1)$ by $\widetilde{z}_0$ and $\widetilde{z}_n$. This process transforms our numerical scheme in the following way:

$$z_{n+1} + \widetilde{z}_{n+1} = z_0 + \widetilde{z}_0 + (1 - \alpha)g(z_{n+1} + \widetilde{z}_{n+1}) + \alpha\tau \sum_{k=0}^{n+1} a_k g(z_k + \widetilde{z}_k). \tag{3.5}$$

From Eqs (2.8) and (3.5) we have

$$\widetilde{z}_{n+1} = \widetilde{z}_0 + (1 - \alpha)[(g(z_{n+1} + \widetilde{z}_{n+1}) - g(z_{n+1}))] + \alpha\tau \sum_{k=0}^{n+1} a_k[g(z_k + \widetilde{z}_k) - g(z_k)].$$

Using the triangle inequality and the Lipschitz condition with the fact that $|a_k| < 1$, we arrive

$$\begin{aligned}
\|\widetilde{z}_{n+1}\| &= \left\|\widetilde{z}_0 + (1 - \alpha + \alpha\tau)[g(z_{n+1} + \widetilde{z}_{n+1}) - g(z_{n+1})] + \alpha\tau \sum_{k=0}^{n} a_k[g(z_k + \widetilde{z}_k) - g(z_k)]\right\| \\
\|\widetilde{z}_{n+1}\| &\leq \|\widetilde{z}_0\| + (1 - \alpha + \alpha\tau)\|g(z_{n+1} + \widetilde{z}_{n+1}) - g(z_{n+1})\| + \alpha\tau \sum_{k=0}^{n} \|g(z_k + \widetilde{z}_k) - g(z_k)\| \\
\|\widetilde{z}_{n+1}\| &\leq \|\widetilde{z}_0\| + (1 - \alpha + \alpha\tau)L\|\widetilde{z}_{n+1}\| + \alpha\tau L \sum_{k=0}^{n} \|\widetilde{z}_k\| \\
\|\widetilde{z}_{n+1}\| - (1 - \alpha + \alpha\tau)L\|\widetilde{z}_{n+1}\| &\leq \|\widetilde{z}_0\| + \alpha\tau L \sum_{k=0}^{n} \|\widetilde{z}_k\| \\
(1 - (1 - \alpha + \alpha\tau)L)\|\widetilde{z}_{n+1}\| &\leq \|\widetilde{z}_0\| + \alpha\tau L \sum_{k=0}^{n} \|\widetilde{z}_k\|
\end{aligned}$$

For all permissible variables α, τ, L with $(1 - \alpha + \alpha\tau)L < 1$, we have

$$\|\widetilde{z}_{n+1}\| \leq g(\tau, \alpha)\|\widetilde{z}_0\| + \alpha\tau L g(\tau, \alpha) \sum_{k=0}^{n} \|\widetilde{z}_k\|,$$

where

$$g(\tau, \alpha) = \frac{1}{1 - (1 - \alpha + \alpha\tau)L}. \tag{3.6}$$

There is a constant C_τ for each sufficient τ

$$1 < g(\tau, \alpha) < C_\tau.$$

Therefore,

$$\|\widetilde{z}_{n+1}\| \leq C_\tau\|\widetilde{z}_0\| + C_\tau\alpha\tau L \sum_{k=0}^{n} \|\widetilde{z}_k\|.$$

Now using the Gröwnwall inequality [17], we have

$$\|\widetilde{z}_{n+1}\| \leq C\|\widetilde{z}_0\|.$$

□

Theorem 3.3. *The newly constructed fractional numerical approach is conditionally convergent of order* 4*, that is (2.8).*

$$\|z(\zeta_{n+1}) - z_{n+1}\| \le \widehat{C}\tau^4.$$

Proof. To demonstrate convergence, take into account the difference between the actual and approximate solution.

$$\begin{aligned}
z(\zeta_{n+1}) - z_{n+1} &= (1-\alpha)(g(z(\zeta_{n+1})) - g(z_{n+1})) + \alpha\left[\int_0^{\zeta_{n+1}} g(z(\zeta_k)d\zeta - \tau\sum_{k=0}^{n+1} a_k g(z_k)\right] \\
&= (1-\alpha)(g(z(\zeta_{n+1})) - g(z_{n+1})) + \alpha\left[\int_0^{\zeta_{n+1}} g(z(\zeta_k)d\zeta - \tau\sum_{k=0}^{n+1} a_k g(z(\zeta_k))\right] \\
&\quad + \alpha\tau\sum_{k=0}^{n+1} a_k[g(z(\zeta_k)) - g(z_k)].
\end{aligned}$$

Applying triangle inequality, Lipschitz condition, and Lemma 3.1,

$$\begin{aligned}
\|z(\zeta_{n+1}) - z_{n+1}\| &\le (1-\alpha)L\|z(\zeta_{n+1}) - z_{n+1}\| + \alpha C\tau^4 + \alpha\tau L\sum_{k=0}^{n+1} a_k\|z(\zeta_k) - z_k\| \\
&= (1-\alpha+\alpha\tau)L\|z(\zeta_{n+1}) - z_{n+1}\| + \alpha C\tau^4 + \alpha\tau L\sum_{k=0}^{n} a_k\|z(\zeta_k) - z_k\|
\end{aligned}$$

using the fact $0 \le a_k < 1$, for all the permissible parameters α, τ, L such that $(1-\alpha+\alpha\tau)L < 1$, we have

$$\|z(\zeta_{n+1}) - z_{n+1}\| \le g(\tau,\alpha)\alpha C\tau^4 + \alpha\tau L g(\tau,\alpha)\sum_{k=0}^{n}\|z(\zeta_k) - z_k\|,$$

where $g(\tau,\alpha)$ is given in Eq (3.6), by using the Gröwnwall inequality, we have

$$\|z(\zeta_{n+1}) - z_{n+1}\| \le \widehat{C}\tau^4,$$

where $\widehat{C} = \alpha C C_\tau$. □

4. Numerical experimentation

In order to demonstrate the usefulness and coherence of the suggested numerical scheme, examples are provided in this section. In these instances, $L2-norm$ is used to calculate the error.

$$\varepsilon(\tau) = \sqrt{\tau\sum_{n=1}^{N-1}|z(\zeta_n) - z_n|^2}, \tag{4.1}$$

where z_n is the numerical solution and $z(\zeta_n)$ shows the exact solution. The order is computed using

$$O = log_2(\varepsilon(2\tau)/\varepsilon(\tau)).$$

4.1. Example 1

Consider the fractional Caputo-Fabrizio differential equation shown below with $z(0) = 0$.

$$^{CF}D^{\frac{4}{5}}z(\zeta) = \frac{5}{8}e^{-4\zeta} - \frac{5}{8} + \frac{5}{2}\zeta, \quad \zeta \in [a, b].$$

The exact solution is $z(\zeta) = \zeta^2$. With the derived numerical scheme (2.8) we solve this problem for $a = 0$, $b = 1$. Comparison for $\alpha = 0.8$ and $\tau = 1/100$ between the solutions of the newly developed scheme and the exact solution is shown in Figure 1. We check the order of convergence by varying the step size from $\tau = 1/10$ to $1/10000$. In Table 1, we identify the error for $\alpha = 4/5$, and it can be seen that error decreases with lowering the step size. We can examine the convergence rate from Table 1 which is compatible with our theoretical results.

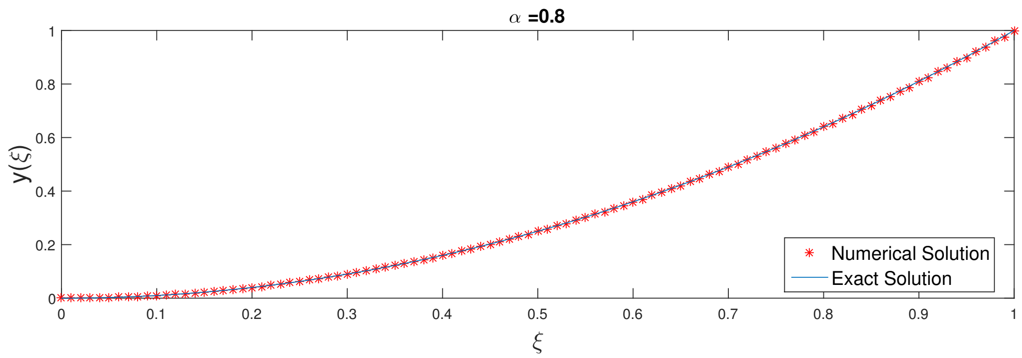

Figure 1. Comparison of exact solution and numerical solution for $\alpha = 4/5$, $\tau = 1/100$.

Table 1. The convergence order and errors when $\alpha = 4/5$.

τ	$\mathcal{E}(\tau)$	order ($\mathcal{O}$)
1/10	1.8595×10^{-2}	
1/100	1.8188×10^{-3}	3.3539
1/1000	1.8322×10^{-4}	3.3112
1/10000	1.8337×10^{-5}	3.3207

4.2. Example 2

Consider another fractional Caputo-Fabrizio fractional equation with initial condition $z(0) = 0$.

$$^{CF}D^{0.5}z(\zeta) = 6\zeta^2 - 18\zeta + 19 - 19\exp(-\zeta), \quad \zeta \in [0, 1].$$

$z(\zeta) = \zeta^3 - \frac{3}{2}\zeta^2 + \frac{1}{2}\zeta$ is the exact solution. Figure 2 shows the comparison of the exact and approximate solution profiles obtained at $\alpha = 0.5$ and $\tau = 1/100$. We calculated the error and evaluated the convergence order by decreasing the time step size from $\tau = 1/10$ to $1/10000$. Table 2 lists the error for fractional-order $\alpha = 0.5$, we can see that as the step size decreases, the error decreases. From Table 2, we can note that our numerical experiments are consistent with the theoretical analysis.

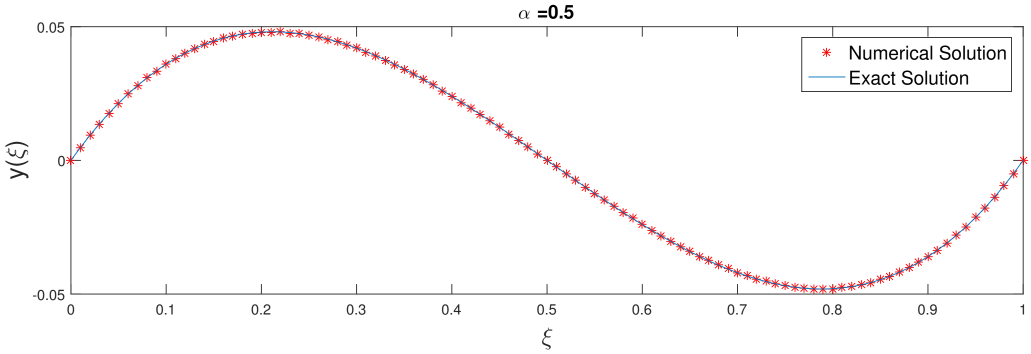

Figure 2. Comparison of exact solution and numerical solution for $\alpha = 0.5,\ \tau = 1/100$.

Table 2. The errors and convergence order when $\alpha = 0.5$.

τ	$\mathcal{E}(\tau)$	order ($\mathcal{O}$)
1/10	3.1692×10^{-3}	
1/100	8.7282×10^{-5}	5.1823
1/1000	7.800×10^{-6}	3.4839
1/10000	7.8345×10^{-7}	3.3157

4.3. Example 3

Consider another fractional Caputo-Fabrizio fractional equation with initial condition $z(0) = 1$.

$$^{CF}D^{\alpha}z(\zeta) = z(\zeta) - h(\zeta), \quad \zeta \in [0, 1],$$

where

$$h(\zeta) = \alpha \exp(\zeta)\left(-1 + \exp(\frac{\zeta}{-1+\alpha})\right).$$

$z(\zeta) = \zeta \exp(\zeta)$ is the exact solution. The comparison of the exact and approximate solution profiles is depicted in Figure 3 at $\alpha = 0.5$ and $\tau = 1/100$. The error and evaluated the convergence order by decreasing the time step size from $\tau = 1/10$ to $1/10000$ is presented in Table 3 for fractional-order $\alpha = 0.5$.

Table 3. The convergence order and errors when $\alpha = 0.5$.

τ	$\mathcal{E}(\tau)$	order ($\mathcal{O}$)
1/10	1.8307×10^{-1}	
1/100	6.1131×10^{-3}	4.9043
1/1000	5.9410×10^{-4}	3.3631
1/10000	5.9238×10^{-5}	3.3261

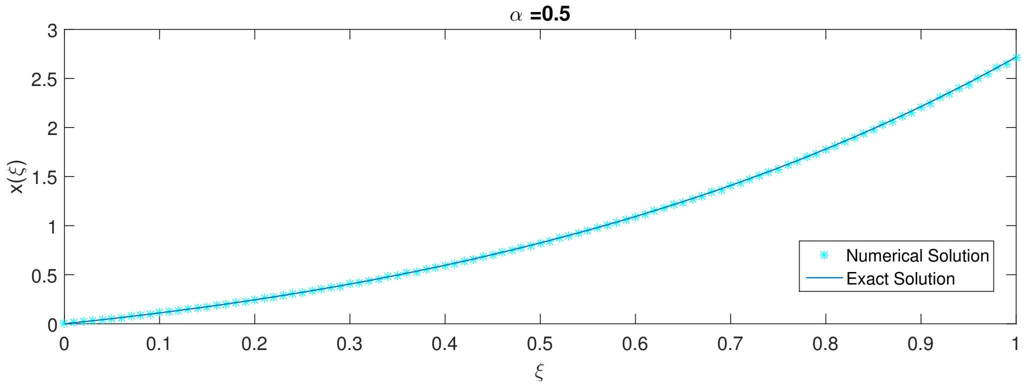

Figure 3. Comparison of exact solution and numerical solution for $\alpha = 0.5,\ \tau = 1/100$.

4.4. Example 4

Consider another fractional Caputo-Fabrizio fractional equation with initial condition $z(0) = 1$.

$$^{CF}D^{\alpha}z(\zeta) = 2z(\zeta) + g(\zeta),\quad \zeta \in [0, 1],$$

where

$$g(\zeta) = \frac{(\alpha - \alpha^2 - 1)\exp(\frac{\alpha\zeta}{\alpha-1}) - (1 + 2\alpha(-1 + \alpha))(-1 + 2\alpha\zeta) - \alpha((1 - 3\alpha + 4\alpha^2)\cos(\zeta) + \alpha\sin(\zeta))}{\alpha + 2\alpha^2(\alpha - 1)}.$$

The exact solution is $z(\zeta) = \zeta + \cos(\zeta)$. Table 4 shows the error and evaluated the convergence order by decreasing the time step size from $\tau = 1/10$ to $1/10000$ for fractional-order $\alpha = 0.9$. Figure 4 compares the exact and approximated solutions for $alpha = 0.9$ and $tau = 1/1000$.

Table 4. The convergence order and errors when $\alpha = 0.9$.

τ	$\mathcal{E}(\tau)$	order ($\mathcal{O}$)
1/10	2.6850×10^{-1}	
1/100	1.0907×10^{-2}	4.6215
1/1000	6.9686×10^{-4}	3.9682
1/10000	6.5394×10^{-7}	3.4136

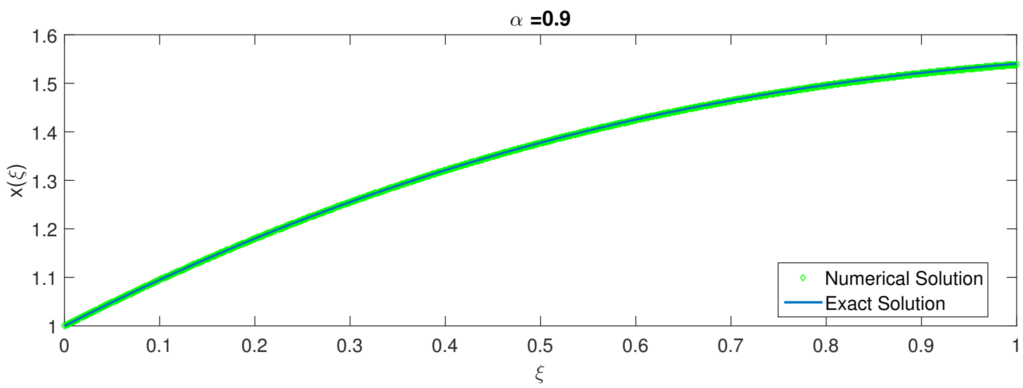

Figure 4. Comparison of exact solution and numerical solution for $\alpha = 0.9,\ \tau = 1/1000$.

4.5. Example 5: fractional order Malthusian growth model

Taking into consideration the fractional order Malthusian growth model

$$^{CF}D^{\alpha}P(\zeta) = \kappa P(\zeta), \quad 0 < \alpha \leq 1,$$

where $P(\zeta)$ denotes the population at time ζ and κ is a positive constant. The exact solution is

$$P(\zeta) = \frac{P(0)}{1+\kappa(\alpha-1)} \exp\left(\frac{\alpha\kappa\zeta}{1+\kappa(\alpha-1)}\right).$$

Figure 5 depicts the population for different values of fractional-order $0 < \alpha \leq 1$, with $P(0) = 1$ and $\kappa = 1$ for both approximate solution and exact solution. Figure 6 compares the exact and approximate solutions. Error and convergence order is given in Table 5.

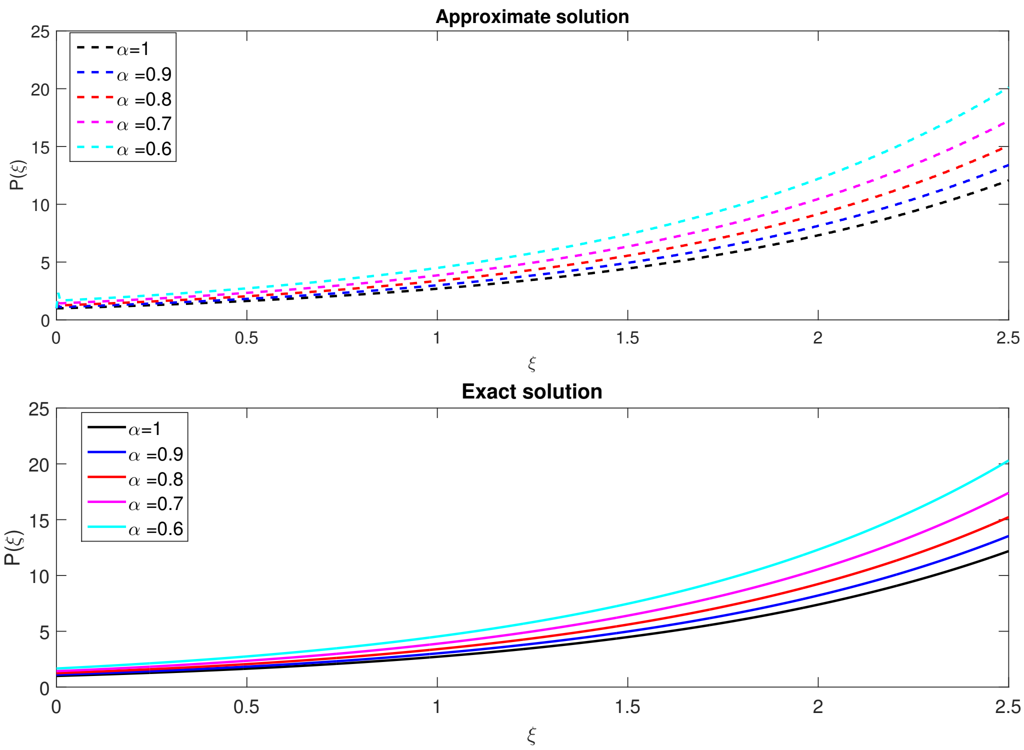

Figure 5. Comparison of the exact solution and numerical solution.

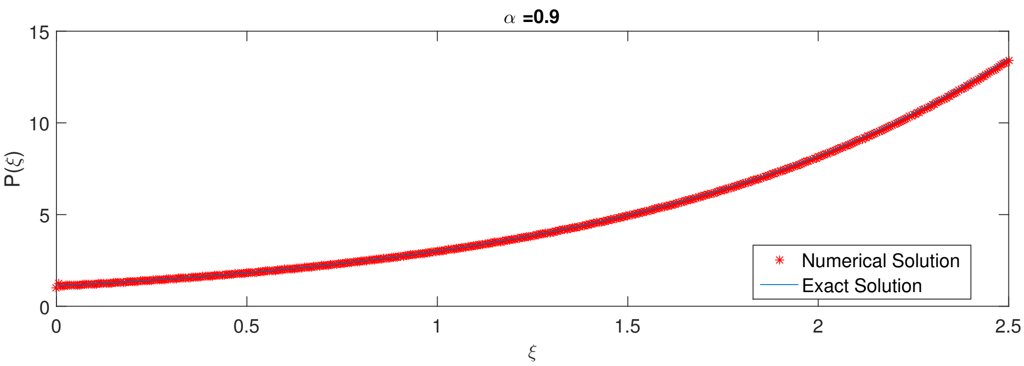

Figure 6. Comparison of the exact solution and numerical solution for $\alpha = 0.9$, $\tau = 0.005$.

Table 5. The convergence order and errors when $\alpha = 0.9$.

τ	$\mathcal{E}(\tau)$	order (O)
1/10	1.7694	
1/100	1.9222×10^{-1}	3.2024
1/1000	2.0803×10^{-2}	3.2078
1/10000	3.1529×10^{-3}	2.7220

4.6. Example 6: fractional order blood alcohol model

Taking the fractional order blood alcohol model for $0 < \alpha, \beta \leq 1$

$$\begin{cases} {}^{CF}D^{\alpha}A(\zeta) = -k_1^{\alpha}A(\zeta), \\ {}^{CF}D^{\alpha}B(\zeta) = k_1^{\beta}A(\zeta) - k_2^{\beta}B(\zeta). \end{cases}$$

where $k1$, $k2$ are constants in the positive range and $A(\zeta)$ indicates the amount of alcohol in the stomach, $B(\zeta)$ represents the amount of alcohol in the blood, The actual solution is

$$A(\zeta) = \frac{A(0)}{1 + k_1^{\alpha}(\alpha - 1)} \exp\left(\frac{-\alpha k_1^{\alpha}\zeta}{1 + k_1^{\alpha}(\alpha - 1)}\right),$$

$$B(\zeta) = \frac{A(0)k_1^{\beta}}{(k_2^{\beta}\beta - k_1^{\alpha}\alpha) + k_1^{\alpha}k_2^{\beta}(\beta - \alpha)} \left[\frac{\beta + k_1^{\alpha}(\beta - \alpha)}{1 + k_1^{\alpha}(\alpha - 1)} \exp\left(\frac{-\alpha k_1^{\alpha}\zeta}{1 + k_1^{\alpha}(\alpha - 1)}\right) - \frac{\beta}{1 + k_2^{\beta}(1 - \beta)} \exp\left(\frac{-\beta k_2^{\beta}\zeta}{1 + k_2^{\beta}(1 - \beta)}\right)\right].$$

The approximate solution of the fractional alcohol model for various fractional-order values of $0 < \alpha$, $\beta \leq 1$, with $A(0) = 245.8769$, $B(0) = 0$ and $k_1 = 0.109456$, $k_2 = 0.017727$ is given in Figure 7. Moreover, a comparison between the estimated result and the actual data is presented in Figure 8 [38].

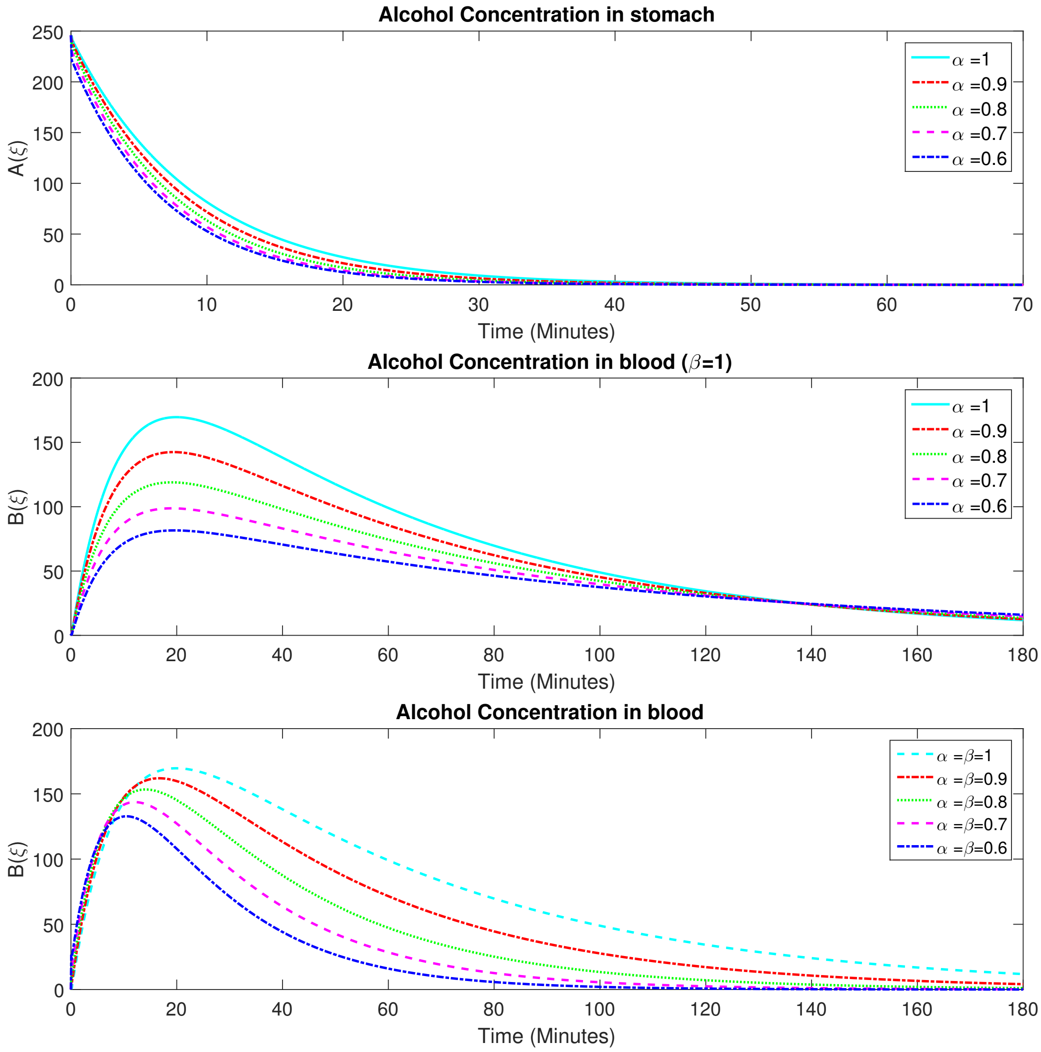

Figure 7. Comparison of the exact solution and numerical solution.

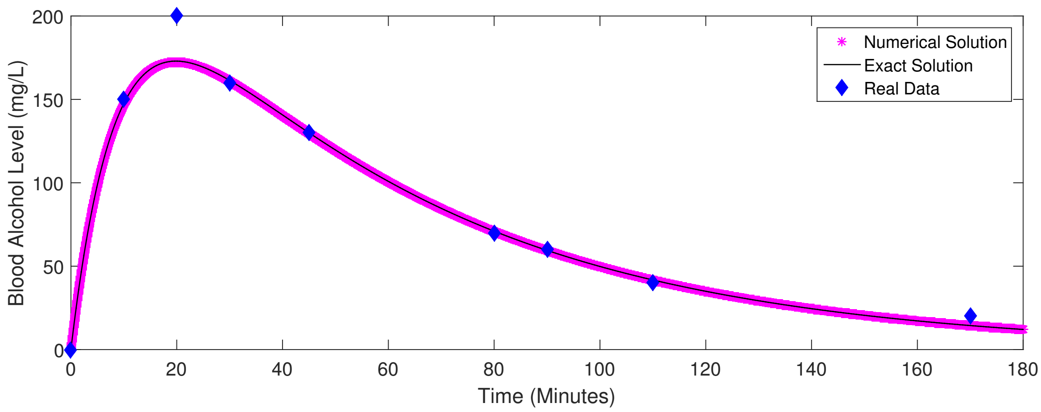

Figure 8. Blood alcohol level real data versus theoretical model.

4.7. Example 7: fractional order logistic model

Consider the fractional order logistic model [6]

$$\begin{cases} {}^{CF}D^{\alpha}y(\zeta) = y(\zeta).[1 - y(\zeta)], \quad 0 < \alpha \leq 1, \\ y(0) = y_0. \end{cases}$$

The exact solution is [43]

$$\frac{y(\zeta) - y^2(\zeta)}{(1 - y(\zeta))^{2/\alpha}} = \frac{y_0 - y_0^2}{(1 - y_0)^{2/\alpha}}.e^{\zeta}.$$

Figures 9 plots the solution of Caputo-Fabrizio and Caputo logistic function with $y(0) = 1/2$.

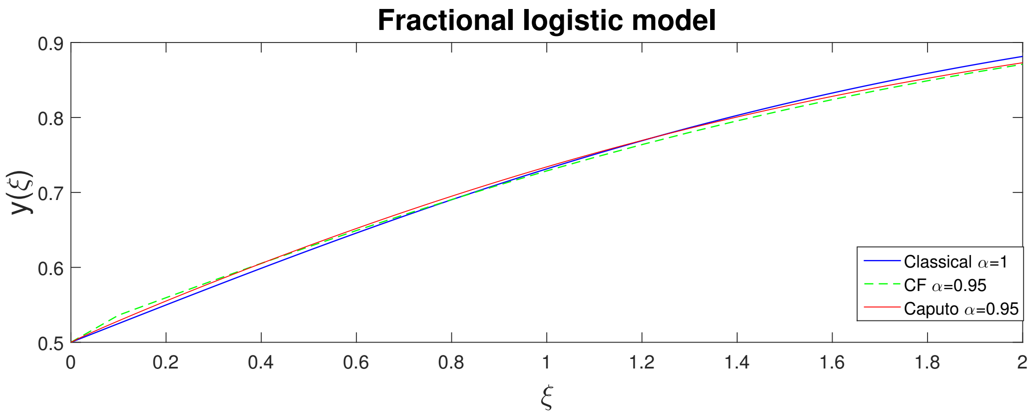

Figure 9. Comparison of solution to the Caputo-Fabrizio and Caputo logistic differential equation for $\alpha = 0.95$.

5. Conclusions

In this paper, we present a numerical technique to solve fractional differential equations precisely to the fourth order. It takes into account the Caputo-Fabrizio derivative with a non-singular kernel for fractional-order α. The fractional differential equations are approximated numerically using Simpson's 1/3 rule. Theoretical simulations show that, under the given circumstances, the suggested system is conditionally stable and convergent. Finally, we draw the conclusion that the evidence shows that the numerical and analytical outcomes are generally in good accord.

Conflict of interest

The authors declare no conflict of interest.

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