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Research article

Modified 5-point fractional formula with Richardson extrapolation

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Abstract: In this paper, we establish a novel fractional numerical modification of the 5-point classical central formula; called the modified 5-point fractional formula for approximating the first fractional-order derivative in the sense of the Caputo operator. Accordingly, we then introduce a new methodology for Richardson extrapolation depending on the fractional central formula in order to obtain a high accuracy for the gained approximations. We compare the efficiency of the proposed methods by using tables and figures to show their reliability.

Keywords: Richardson extrapolation; Riemann-Liouville fractional derivative and integral; Lagrange interpolating polynomial; Caputo derivative
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1. Introduction

Difference equations were used and popularized by Isaac Newton in the last quarter of the 17th century, but many of these techniques had previously been developed by Thomas Harriot (1561–1621) and Henry Briggs (1561–1630). We recommend the articles in [1, 7, 19] since they discuss several fractional-order dynamical system problems. To see more about the numerical continuation methods, see [2, 4, 6, 9, 10, 13].

Harriot made significant advances in navigation techniques, and Briggs was the person most responsible for the acceptance of logarithms as an aid to computation. Lewis Fry Richardson (1881–1953) was the first person to systematically apply mathematics to weather prediction while working in England for the Meteorological Office. As a conscientious objector during World

War I, he wrote extensively about the economic futility of warfare, using systems of differential equations to model rational interactions between countries. The extrapolation technique that bears his name was the rediscovery of a technique with roots that are at least as old as Christiaan Hugyens (1629–1695).

Thomas Simpson (1710–1761) was a self-taught mathematician who supported himself during his early years as a weaver. His primary interest was probability theory, although in 1750 he published a two-volume calculus book entitled "The doctrine and application of fluxions". Roger Cotes (1682–1716) rose from a modest background to become, in 1704, the first Plumian Professor at Cambridge University. He made advances in numerous mathematical areas including numerical methods for interpolation and integration. Newton is reputed to have said of Cotes ... if he had lived we might have known something. The Newton-Cotes formulas are generally unsuitable for use over large integration intervals. High-degree formulas would be required, and the values of the coefficients in these formulas are difficult to obtain. Also, the Newton-Cotes formulas are based on interpolatory polynomials that use equally-spaced nodes, a procedure that is inaccurate over large intervals because of the oscillatory nature of high-degree polynomials. In [3], O. Axelsson and V. A. Barker examined the Lagrange polynomials and discovered an explicit representation for them as well as the error that occurs when approximating a function on an interval, for more see [5, 15, 22].

Interpolating tabular data typically makes use of these polynomials. Remember that in situations like these, only the values of the polynomial at the predetermined locations are necessary; an explicit representation of the polynomial is not necessary. Additionally, we are unable to utilize the explicit form if the data function is unknown. Neville's Method, a variation of the Lagrange formula, was presented by E. H. Neville in his work in 1932. Due to the practical difficulty of applying the error term to Lagrange interpolation, it is frequently unclear what degree of the polynomial is needed for the application, see [17, 18, 20, 23].

The organization of this paper is arranged as follows: Section 2 aims to recall some basic facts and definitions connected with fractional calculus. Section 3 demonstrates the main results of this work so that it contains the derivation of the modified 5-point fractional formula. Section 4 develops the classical Richardson extrapolation methodology to be valid for fractional calculus. Section 5 illustrates numerical results that confirm the theoretical findings of this work, followed by the final section that summarizes the conclusion.

2. Preliminaries

In this section, basic definitions and theorems like the Riemann-Liouville integral and derivative and the Caputo derivative will be introduced [11, 21].

Definition 2.1. *The fractional Riemann-Liouville integral of a function f(t) of order* $\mu > 0$ *is initially defined by*

$$J^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t f(\tau)(t-\tau)^{\mu-1}\, d\tau, \quad t > 0,\ \mu > 0. \tag{2.1}$$

Some of the properties of the Riemann-Liouville integral are given below for completeness:

$$J^0 f(t) = f(t). \tag{2.2}$$

$$J^{\mu}t^{\gamma} = \frac{\Gamma(\gamma+1)t^{\mu+\gamma}}{\Gamma(\mu+\gamma+1)}, \quad \gamma \geq -1. \tag{2.3}$$

$$J^{\mu}J^{\beta}f(t) = J^{\beta}J^{\mu}f(t), \quad \mu,\beta \geq 0. \tag{2.4}$$

$$J^{\mu}J^{\beta}f(t) = J^{\mu+\beta}f(t), \quad \mu,\beta \geq 0. \tag{2.5}$$

Definition 2.2. *[8] Let m be the smaller number greater than α. The Caputo fractional derivative of order $\alpha > 0$, is defined as*

$$D_t^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{-\alpha+n-1}\frac{d^m f(\tau)}{d\tau^m}\,d\tau, & m-1 < \alpha < m, \\ \frac{d^m f(t)}{dt^m}, & \alpha = m, \end{cases} \tag{2.6}$$

where $m \in \mathbb{N}$, $t > 0$, and $f(t)$ is a real-valued function.

Some of the characteristics of the Caputo derivative are listed below:
(1) $D_*^{\alpha}c = 0$ where c is constant.
(2) We have

$$D_*^{\alpha}t^{\rho} = \begin{cases} \frac{\Gamma(\rho+1)}{\Gamma(\rho-\alpha+1)}t^{\rho-\alpha}, & \rho > \alpha - 1, \\ 0, & otherwise. \end{cases}$$

(3) D_*^{α} is linear, i.e.,

$$D_*^{\alpha}(\mu f(t) + \omega k(t)) = \mu D_*^{\alpha}(f(t)) + \omega D_*^{\alpha}(k(t)),$$

where μ and ω are constants.

In addition, we need to recall two basic properties. If $m-1 < \alpha \leq m$, $m \in N$, then we have

$$\begin{aligned} &D_*^{\alpha}J^{\alpha}f(t) = f(t), \\ &J^{\alpha}D_*^{\alpha}f(t) = f(t) - \sum_{i=1}^{n} f^{i}(0^+)\frac{t^i}{i!}, \quad t > 0. \end{aligned} \tag{2.7}$$

Definition 2.3. *[14] The fractional Riemann-Liouville derivative can be defined using the definition of the fractional Riemann-Liouville integral. To this end, suppose that $v = n - u$, where $0 < v < 1$ and n is the smallest integral greater than u. Then, the Riemann-Liouville fractional derivative of $f(x)$ of the order u is*

$$D^{u}f(t) = D^{n}[D^{-v}f(t)].$$

Definition 2.4. *The linear Lagrange interpolating polynomial through (x_0, y_0) and (x_1, y_1), where*

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \text{ and } L_1(x) = \frac{x - x_0}{x_1 - x_0},$$

is

$$P(x) = L_0(x)f(x_0) + L_1(X)f(x_1) = \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1).$$

Note that

$$L_0(x_0) = 1,\ L_0(x_1) = 0,\ L_1(x_0) = 0,\ L_1(x_1) = 1,$$

which implies that

$$P(x_0) = 1 \cdot f(x_0) + 0 \cdot f(x_1) = f(x_0) = y_0,$$

and

$$P(x_1) = 0 \cdot f(x_0) + 1 \cdot f(x_1) = f(x_1) = y_1.$$

So, P is the unique polynomial of degree at the most one that passes through (x_0, y_0) and (x_1, y_1).

Theorem 2.1. *Consider $x_0,x_1,...,x_n$ are $n+1$ distinct numbers. To define the nth Lagrange interpolating polynomial, where the values of the function f are provided by these numbers, of degree highest n exist with*

$$f(x_k) = p(x_k),$$

for each $k = 0, 1, ..., n$. This polynomial is given by

$$p(x) = f(x_0)L_{n,0}(x) + ... + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x), \tag{2.8}$$

where, for each $k = 0, 1, ..., n$, we have

$$L_{n,k}(x) = \frac{(x - x_0)(x - x_1)...(x - x - k - 1)(x - x_{k+1})...(x - x_n)}{(x_k - x_0)(x_k - x_1)...(x_k - x_{k-1})(x_k - x_{k+1})...(x_K - x_n)},$$

or

$$L_{n,k}(x) = \prod_{\substack{i=0\\ i\neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}.$$

We will write $L_{n,k}(x)$ simply as $L_k(x)$ when there is no confusion as to its degree.

Theorem 2.2. *Suppose that the interval $[a, b]$ contains the following distinct numbers $x_0, x_1, \cdots, x_n$ and $f \in C^{n+1}[a, b]$. Then, for each x in $[a, b]$, there exists a number $\xi(x)$ (generally unknown) between $x_0, x_1, \cdots, x_n$ and hence in (a, b), with*

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n + 1)!}(x - x_0)(x - x_1)\cdots(x - x_n),$$

where $P(x)$ is the interpolating polynomial given in (2.8).

Theorem 2.3. *Suppose that $x_0, x_1, \cdots, x_n$ are $(n + 1)$ distinct numbers in some interval $I = [a, b]$ and that $f \in c^{n+1}[a, b]$, then the function $f(x)$ can be expressed as*

$$f(x) = \sum_{k=0}^{n} f(x_k)L_k(x) + \frac{(x - x_0)\cdots(x - x_n)}{(n + 1)!} f^{(n+1)}(\xi(x)). \tag{2.9}$$

Theorem 2.4. *[12] Suppose that x_0, x_1 and x_2 are distinct points in the interval $[a, b]$ such that $a = x_0 < x_1 = x_0 + h < x_2 = x_0 + 2h = b$, with $h > 0$, and $f \in C^3[a, b]$. Then, the modified 3-point fractional formula for approximating $D^\alpha_* f(x)$ is given by*

$$\begin{aligned} D^\alpha_* f(x) =& \frac{x^{2-\alpha}}{h^2\Gamma(3 - \alpha)}\Big(f(x_0) - 2f(x_1) + f(x_2)\Big) - \frac{x^{1-\alpha}}{2h^2\Gamma(2 - \alpha)}\Big(f(x_0)(x_1 + x_2) - 2f(x_1)(x_0 + x_2) \\ &+ f(x_2)(x_0 + x_1)\Big) + \frac{f^{(3)}(\xi)}{6}\Big(\frac{6}{\Gamma(4 - \alpha)}x^{3-\alpha} - \frac{2(x_0 + x_1 + x_2)}{\Gamma(3 - \alpha)}x^{2-\alpha} + \frac{(x_0x_1 + x_0x_2 + x_1x_2)}{\Gamma(2 - \alpha)}x^{1-\alpha}\Big), \end{aligned} \tag{2.10}$$

for each $x \in [a, b]$, where $\xi \in (a, b)$.

3. Modified 5-point fractional central formula

In this part, we introduce a novel numerical formula called the modified 5-point fractional central formula to approximate the first fractional derivative in the sense of the Caputo operator.

Theorem 3.1. *Suppose that $x_0, \cdots, x_4$ are distinct points in the interval $[a, b]$ such that $a = x_0 < x_1 = x_0 + h < x_2 = x_0 + 2h < x_3 = x_0 + 3h < x_4 = x_0 + 4h = b$ with $h > 0$, and $f \in C^5[a, b]$. Then, the fractional derivative can be given by*

$$\begin{aligned} D_*^\alpha f(x) =& \left[\frac{24}{\Gamma(5-\alpha)}\sum_{k=0}^{4}\frac{f(x_k)}{A_k}x^{4-\alpha}\right] - \left[\frac{6}{\Gamma(4-\alpha)}\sum_{k=0}^{4}\frac{a_k f(x_k)}{A_k}x^{3-\alpha}\right] \\ &+ \left[\frac{2}{\Gamma(3-\alpha)}\sum_{k=0}^{4}\frac{b_k f(x_k)}{A_k}x^{2-\alpha}\right] - \left[\frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{4}\frac{c_k f(x_k)}{A_k}x^{1-\alpha}\right] \\ &+ \frac{f^{(5)}(\xi)}{5!}\left(\frac{120}{\Gamma(6-\alpha)}x^{5-\alpha} - \frac{24a_5}{\Gamma(5-\alpha)}x^{4-\alpha} + \frac{6b_5}{\Gamma(4-\alpha)}x^{3-\alpha} - \frac{2c_5}{\Gamma(3-\alpha)}x^{2-\alpha} + \frac{d_5}{\Gamma(2-\alpha)}x^{1-\alpha}\right), \end{aligned} \tag{3.1}$$

for each $x \in [a, b]$, where $\xi \in (a, b)$.

Proof. With the help of using (2.9) and assuming $n = 4$, we can have

$$f(x) = \sum_{k=0}^{4} f(x_k)L_k(x) + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{5!}f^{(5)}(\xi(x)). \tag{3.2}$$

By simplifying the summation, we get

$$\begin{aligned} f(x) =& f(x_0)L_0(x) + f(x_1)L_1 + f(x_2)L_2(x) + f(x_3)L_3(x) + f(x_4)L_4(x) \\ &+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{5!}f^{(5)}(\xi(x)). \end{aligned} \tag{3.3}$$

By using the Eq (3.2), we have the following assertion:

$$\begin{aligned} f(x) =& f(x_0)\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \\ &+ f(x_1)\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \\ &+ f(x_2)\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \\ &+ f(x_3)\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \\ &+ f(x_4)\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \\ &+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{5!}f^{(5)}(\xi(x)). \end{aligned} \tag{3.4}$$

By simplifying Eq (3.4), we consider the following assumptions:

(1) The parameters of x^3 are as follows:

$$\begin{aligned}
a_0 &= (x_1 + x_2 + x_3 + x_4),\\
a_1 &= (x_0 + x_2 + x_3 + x_4),\\
a_2 &= (x_0 + x_1 + x_3 + x_4),\\
a_3 &= (x_0 + x_1 + x_2 + x_4),\\
a_4 &= (x_0 + x_1 + x_2 + x_3).
\end{aligned}$$

(2) The parameters of x^2 are as follows:

$$\begin{aligned}
b_0 &= (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4),\\
b_1 &= (x_0x_2 + x_0x_3 + x_0x_4 + x_2x_3 + x_2x_4 + x_3x_4),\\
b_2 &= (x_0x_1 + x_0x_3 + x_0x_4 + x_1x_3 + x_1x_4 + x_3x_4),\\
b_3 &= (x_0x_1 + x_0x_2 + x_0x_4 + x_1x_2 + x_1x_4 + x_2x_4),\\
b_4 &= (x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3).
\end{aligned}$$

(3) The parameters of x are as follows:

$$\begin{aligned}
c_0 &= (x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4),\\
c_1 &= (x_0x_2x_3 + x_0x_2x_4 + x_0x_3x_4 + x_2x_3x_4),\\
c_2 &= (x_0x_1x_3 + x_0x_1x_4 + x_0x_3x_4 + x_1x_3x_4),\\
c_3 &= (x_0x_1x_2 + x_0x_1x_4 + x_0x_2x_4 + x_1x_2x_4),\\
c_4 &= (x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3).
\end{aligned}$$

(4) The constants are as follows:

$$\begin{aligned}
d_0 &= (x_1x_2x_3x_4),\\
d_1 &= (x_0x_2x_3x_4),\\
d_2 &= (x_0x_1x_3x_4),\\
d_3 &= (x_0x_1x_2x_4),\\
d_4 &= (x_0x_1x_2x_3).
\end{aligned}$$

(5) The assumptions for the term of error are as follows:

$$\begin{aligned}
a_5 =&(x_0 + x_1 + x_2 + x_3 + x_4),\\
b_5 =&(x_0x_1 + x_0x_2 + x_0x_3 + x_0x_4 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4),\\
c_5 =&(x_0x_1x_2 + x_0x_1x_3 + x_0x_1x_4 + x_0x_2x_3 + x_0x_2x_4 + x_0x_3x_4 + x_1x_2x_3\\
&+ x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4),\\
d_5 =&(x_0x_1x_2x_3 + x_0x_1x_2x_4 + x_0x_1x_3x_4 + x_0x_2x_3x_4 + x_1x_2x_3x_4),\\
k =&(x_0x_1x_2x_3x_4).
\end{aligned}$$

(6) The denominator assumptions are as follows:

$$\begin{aligned}
A_0 =&{x_0}^4 - (x_1 + x_2 + x_3 + x_4)x_0^3 + (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)x_0^2\\
&- (x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)x_0 + (x_1x_2x_3x_4),
\end{aligned}$$

$$
\begin{aligned}
A_1 =& {x_1}^4 - (x_0 + x_2 + x_3 + x_4)x_1^3 + (x_0x_2 + x_0x_3 + x_0x_4 + x_2x_3 + x_2x_4 + x_3x_4)x_1^2 \\
&- (x_0x_2x_3 + x_0x_2x_4 + x_0x_3x_4 + x_2x_3x_4)x_1 + (x_0x_2x_3x_4), \\
A_2 =& {x_2}^4 - (x_0 + x_1 + x_3 + x_4)x_2^3 + (x_0x_1 + x_0x_3 + x_0x_4 + x_1x_3 + x_1x_4 + x_3x_4)x_2^2 \\
&- (x_0x_1x_3 + x_0x_1x_4 + x_0x_3x_4 + x_1x_3x_4)x_2 + (x_0x_1x_3x_4), \\
A_3 =& {x_3}^4 - (x_0 + x_1 + x_2 + x_4)x_3^3 + (x_0x_1 + x_0x_2 + x_0x_4 + x_1x_2 + x_1x_4 + x_2x_4)x_3^2 \\
&- (x_0x_1x_2 + x_0x_1x_4 + x_0x_2x_4 + x_1x_2x_4)x_3 + (x_0x_1x_2x_4), \\
A_4 =& {x_4}^4 - (x_0 + x_1 + x_2 + x_3)x_4^3 + (x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3)x_4^2 \\
&- (x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3)x_4 + (x_0x_1x_2x_3).
\end{aligned}
$$

Then, we consequently get

$$
\begin{aligned}
f(x) =& f(x_0)\Big(\frac{x^4 - a_0x^3 + b_0x^2 - c_0x + d_0}{A_0}\Big) + f(x_1)\Big(\frac{x^4 - a_1x^3 + b_1x^2 - c_1x + d_1}{A_1}\Big) \\
&+ f(x_2)\Big(\frac{x^4 - a_2x^3 + b_2x^2 - c_2x + d_2}{A_2}\Big) + f(x_3)\Big(\frac{x^4 - a_3x^3 + b_3x^2 - c_3x + d_3}{A_3}\Big) \\
&+ f(x_4)\Big(\frac{x^4 - a_4x^3 + b_4x^2 - c_4x + d_4}{A_4}\Big) + f^{(5)}(\xi(x))\Big(\frac{x^5 - a_5x^4 + b_5x^3 - c_5x^2 + d_5x - k}{5!}\Big).
\end{aligned}
\tag{3.5}
$$

By simplifying (3.5), we get

$$
\begin{aligned}
f(x) =& \Big[\frac{f(x_0)}{A_0} + \frac{f(x_1)}{A_1} + \frac{f(x_2)}{A_2} + \frac{f(x_3)}{A_3} + \frac{f(x_4)}{A_4}\Big]x^4 \\
&- \Big[\frac{a_0f(x_0)}{A_0} + \frac{a_1f(x_1)}{A_1} + \frac{a_2f(x_2)}{A_2} + \frac{a_3f(x_3)}{A_3} + \frac{a_4f(x_4)}{A_4}\Big]x^3 \\
&+ \Big[\frac{b_0f(x_0)}{A_0} + \frac{b_1f(x_1)}{A_1} + \frac{b_2f(x_2)}{A_2} + \frac{b_3f(x_3)}{A_3} + \frac{b_4f(x_4)}{A_4}\Big]x^2 \\
&- \Big[\frac{c_0f(x_0)}{A_0} + \frac{c_1f(x_1)}{A_1} + \frac{c_2f(x_2)}{A_2} + \frac{c_3f(x_3)}{A_3} + \frac{c_4f(x_4)}{A_4}\Big]x \\
&+ \Big[\frac{d_0f(x_0)}{A_0} + \frac{d_1f(x_1)}{A_1} + \frac{d_2f(x_2)}{A_2} + \frac{d_3f(x_3)}{A_3} + \frac{d_4f(x_4)}{A_4}\Big] \\
&+ f^{(5)}(\xi(x))\Big[\frac{x^5 - a_5x^4 + b_5x^3 - c_5x^2 + d_5x - k}{5!}\Big].
\end{aligned}
\tag{3.6}
$$

By using the summation on (3.6), we get

$$
\begin{aligned}
f(x) =& \sum_{k=0}^{4}\frac{f(x_k)}{A_k}x^4 - \sum_{k=0}^{4}\frac{a_kf(x_k)}{A_k}x^3 + \sum_{k=0}^{4}\frac{b_kf(x_k)}{A_k}x^2 - \sum_{k=0}^{4}\frac{c_kf(x_k)}{A_k}x \\
&+ \sum_{k=0}^{4}\frac{d_kf(x_k)}{A_k} + f^{(5)}(\xi(x))\Big[\frac{x^5 - a_5x^4 + b_5x^3 - c_5x^2 + d_5x - k}{5!}\Big].
\end{aligned}
\tag{3.7}
$$

By applying the Caputo derivative on (3.7), we get

$$\begin{aligned}D_*^\alpha f(x) = &\sum_{k=0}^{4}\frac{f(x_k)\Gamma(5)}{A_k\Gamma(5-\alpha)}x^{4-\alpha} - \sum_{k=0}^{4}\frac{a_k f(x_k)\Gamma(4)}{A_k\Gamma(4-\alpha)}x^{3-\alpha} + \sum_{k=0}^{4}\frac{b_k f(x_k)\Gamma(3)}{A_k\Gamma(3-\alpha)}x^{2-\alpha} - \sum_{k=0}^{4}\frac{c_k f(x_k)\Gamma(2)}{A_k\Gamma(2-\alpha)}x^{1-\alpha}\\ &+ \frac{f^{(5)(\xi)}}{5!}\left[\frac{\Gamma(6)}{\Gamma(6-\alpha)}x^{5-\alpha} - \frac{a_5\Gamma(5)}{\Gamma(5-\alpha)}x^{4-\alpha} + \frac{b_5\Gamma(4)}{\Gamma(4-\alpha)}x^{3-\alpha} - \frac{c_5\Gamma(3)}{\Gamma(3-\alpha)}x^{2-\alpha} + \frac{d_5\Gamma(2)}{\Gamma(2-\alpha)}x^{1-\alpha}\right].\end{aligned} \tag{3.8}$$

By simplifying (3.8), we will get

$$\begin{aligned}D_*^\alpha f(x) = &\left[\frac{24}{\Gamma(5-\alpha)}\sum_{k=0}^{4}\frac{f(x_k)}{A_k}x^{4-\alpha}\right] - \left[\frac{6}{\Gamma(4-\alpha)}\sum_{k=0}^{4}\frac{a_k f(x_k)}{A_k}x^{3-\alpha}\right]\\ &+ \left[\frac{2}{\Gamma(3-\alpha)}\sum_{k=0}^{4}\frac{b_k f(x_k)}{A_k}x^{2-\alpha}\right] - \left[\frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{4}\frac{c_k f(x_k)}{A_k}x^{1-\alpha}\right]\\ &+ \frac{f^{(5)(\xi)}}{5!}\left[\frac{120}{\Gamma(6-\alpha)}x^{5-\alpha} - \frac{24a_5}{\Gamma(5-\alpha)}x^{4-\alpha} + \frac{6b_5}{\Gamma(4-\alpha)}x^{3-\alpha} - \frac{2c_5}{\Gamma(3-\alpha)}x^{2-\alpha} + \frac{d_5}{\Gamma(2-\alpha)}x^{1-\alpha}\right].\end{aligned} \tag{3.9}$$

□

Corollary 3.1. *Under the same assumptions of Theorem 3.1, we can obtain the modified 5-point fractional central formula, which would be as*

$$\begin{aligned}D_*^\alpha f(x_1) = &\frac{24}{\Gamma(5-\alpha)}\sum_{k=0}^{4}\frac{f(x_k)}{A_k}x_1^{4-\alpha} - \frac{6}{\Gamma(4-\alpha)}\sum_{k=0}^{4}\frac{a_k f(x_k)}{A_k}x_1^{3-\alpha}\\ &+ \frac{2}{\Gamma(3-\alpha)}\sum_{k=0}^{4}\frac{b_k f(x_k)}{A_k}x_1^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{4}\frac{c_k f(x_k)}{A_k}x_1^{1-\alpha}\\ &+ \frac{f^{(5)(\xi)}}{5!}\left[\frac{120}{\Gamma(6-\alpha)}x_1^{5-\alpha} - \frac{24a_5}{\Gamma(5-\alpha)}x_1^{4-\alpha} + \frac{6b_5}{\Gamma(4-\alpha)}x_1^{3-\alpha} - \frac{2c_5}{\Gamma(3-\alpha)}x_1^{2-\alpha} + \frac{d_5}{\Gamma(2-\alpha)}x_1^{1-\alpha}\right].\end{aligned} \tag{3.10}$$

Corollary 3.2. *From Corollary 3.1, if one lets $\alpha = 1$, then we get the classical 5-point central formula for approximating $f'(x_1)$, i.e., we get*

$$f'(x_1) = \frac{1}{12h}[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)] + \frac{h^4}{30}f^{(5)}(\xi). \tag{3.11}$$

4. Richardson extrapolation

To obtain high-accuracy results for low-order formulas, we use Richardson's extrapolation. For more about the history and applications of Richardson's extrapolation method, we recommend the article by D. C. Joyce [16]. We recommend to comment existing higher-order Richardson extrapolation formulae described and analyzed in [24]. This section introduces our methodology for Richardson extrapolation depending on our fractional central form [12]. Depending on the modified central formula [12], for $x \in [a, b]$, assume that $\alpha = 1$ and, without losing generality, we have the following:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f^{(5)}(\xi). \tag{4.1}$$

Note that we denote $f'(x)$ as Kv, $\frac{f(x+h)-f(x-h)}{2h}$ as $Yv(h)$ and $e = \frac{-f^{(5)}(\xi)}{6}$. Then, we obtain the following equation:

$$Kv = (Yv)(h) + eh^2. \tag{4.2}$$

If we substitute $\frac{h}{2}$ instead of h in the (4.2), we have the following assertion:

$$Kv = (Yv)\left(\frac{h}{2}\right) + e\left(\frac{h}{2}\right)^2.$$

By simplifying the previous equation and multiplying by (-4), we obtain

$$-4Kv = -4(Yv)\left(\frac{h}{2}\right) - eh^2. \tag{4.3}$$

By subtracting (4.3) from (4.2), we get

$$-3Kv = (Yv)(h) - (Yv)\left(\frac{h}{2}\right) - 3(Yv)\left(\frac{h}{2}\right),$$

which implies

$$Kv = (Yv)\left(\frac{h}{2}\right) + \frac{(Yv)(\frac{h}{2}) - (Yv)(h)}{3}.$$

Suppose that $Kv = F_2(h)$, and $F_1(h) = (Yv)(h)$. Then, consequently we can obtain the following equality:

$$F_2(h) = F_1\left(\frac{h}{2}\right) + \frac{F_1(\frac{h}{2}) - F_1(h)}{3}.$$

If we continue in the same way, we obtain Table 1, which illustrates how we can use Richardson's extrapolation to build a fourth-order approximation, using four first-order approximations.

Table 1. The 4th order approximation of Richardson's extrapolation.

O(h^2)	**O(h^4)**	**O(h^6)**	**O(h^8)**
$F_1(h)$	-	-	-
$F_1(\frac{h}{2})$	$F_2(h)$	-	-
$F_1(\frac{h}{4})$	$F_2(\frac{h}{2})$	$F_3(h)$	-
$F_1(\frac{h}{8})$	$F_2(\frac{h}{4})$	$F_3(\frac{h}{2})$	$F_4(h)$

In particular, for F_1, we have the following equalities:

$$F_1(h) = \frac{f(x+h) - f(x-h)}{2h},$$

$$F_1\left(\frac{h}{2}\right) = \frac{h(x+\frac{h}{2}) - f(x-\frac{h}{2})}{h},$$

$$F_1\left(\frac{h}{4}\right) = \frac{f(x+\frac{h}{4}) - f(x-\frac{h}{4})}{\frac{h}{2}},$$

$$F_1\left(\frac{h}{8}\right) = \frac{f(x+\frac{h}{8}) - f(x-\frac{h}{8})}{\frac{h}{4}}.$$

Similarly, for F_2, we have the following equalities:

$$F_2(h) = F_1\left(\frac{h}{2}\right) + \frac{F_1(\frac{h}{2}) - F_1(h)}{3},$$

$$F_2\left(\frac{h}{2}\right) = F_1\left(\frac{h}{4}\right) + \frac{F_1(\frac{h}{4}) - F_1(\frac{h}{2})}{3},$$

$$F_2\left(\frac{h}{4}\right) = F_1\left(\frac{h}{8}\right) + \frac{F_1\left(\frac{h}{8}\right) - F_1(\frac{h}{4})}{3}.$$

In addition, for F_3, we have the following assertion:

$$F_3(h) = F_2\left(\frac{h}{2}\right) + \frac{F_2(\frac{h}{2}) - F_2(h)}{15},$$

$$F_3\left(\frac{h}{2}\right) = F_2\left(\frac{h}{4}\right) + \frac{F_2(\frac{h}{4}) - F_2(\frac{h}{2})}{15}.$$

Finally, for F_4, we have

$$F_4(h) = F_3(\frac{h}{2}) + \frac{F_3(\frac{h}{2}) - F_3(h)}{63}.$$

Performing this process further results for each $i = 2, 3, \cdots$, the $O(h^{2i})$ approximation becomes

$$F_i(h) = F_{i-1}\left(\frac{h}{2}\right) + \frac{F_{i-1}(\frac{h}{2}) - F_{i-1}(h)}{4^{i-1} - 1}. \tag{4.4}$$

Remark 4.1. *One can generalize the $O(h^{2i})$ approximation for the fractional case when α is fractional and can obtain the same result as the (4.4).*

5. Numerical examples

In this section, we aim to compare the efficiency between the modified 5-point fractional central formulas with and without the Richardson extrapolation. Tables and figures are used to present and compare the outcomes. Note that, the application of the proposed fractal approach might be used in modeling non-equilibrium processes. For example, modeling the linear elasticity and viscoelastic deformation processes in biomaterials and capillary-porous materials with taking into account their fractal structure. However, we present below some general examples of the validation aims.

Example 5.1. Let the main function be $f(x) = x^3 + 4x^2 - 2$, and consider $\alpha = 1$. Then, the Caputo derivative is given by

$$f'(x_1) = \frac{6}{\Gamma(4-\alpha)}x^{3-\alpha} + \frac{8}{\Gamma(3-\alpha)}x^{2-\alpha}.$$

By Taking the point of evaluation $x_0 = 1$, then $f'(1) = 11$. In other words, the exact value of the derivatives equals the Caputo derivative. Now, by taking the step initial value $h = 0.2$ and the order of extrapolation $N = 7$, we get the results shown in Table 2.

Table 2. The numerical result using the proposed fractional central formula with Richardson extrapolation.

h	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^{14})$
0.2000	13.9200	0	0	0	0	0	0
0.1000	12.4300	11.9333	0	0	0	0	0
0.0500	11.7075	11.4667	11.4356	0	0	0	0
0.0250	11.3519	11.2333	11.2178	11.2143	0	0	0
0.0125	11.1755	11.1167	11.1089	11.1071	11.1067	0	0
0.0063	11.0878	11.0585	11.0546	11.0538	11.0536	11.0535	0
0.0031	11.0396	11.0235	11.0212	11.0207	11.0205	11.0205	11.0205

Now we consider the fractional case when $\alpha = 0.90$ and for $x_0 = 1$. This yields

$$D_*^{0.90} f(1) = \frac{6}{\Gamma(3.1)}x^{2.1} + \frac{8}{\Gamma(2.1)}x^{1.1} = 10.3749.$$

Accordingly, by using the Richardson extrapolation in view of the modified 5-point fractional central formula, we can gain numerical results shown in Table 3.

Table 3. The numerical result using the proposed fractional central formula with Richardson extrapolation.

h	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^{14})$
0.2000	13.3462	0	0	0	0	0	0
0.1000	11.8250	11.8248	0	0	0	0	0
0.0500	11.0912	11.0910	11.0909	0	0	0	0
0.0250	10.7315	10.7315	10.7312	10.7310	0	0	0
0.0125	10.6745	10.6745	10.6745	10.6745	10.6744	0	0
0.0063	10.4536	10.4536	10.4536	10.4536	10.4535	10.4533	0
0.0031	14.1713	14.4055	14.4055	14.4055	14.4055	14.4050	10.4010

In what follows, we plot some graphical comparisons in Figure 1 between the results generated by using analytical and fractional central formulas.

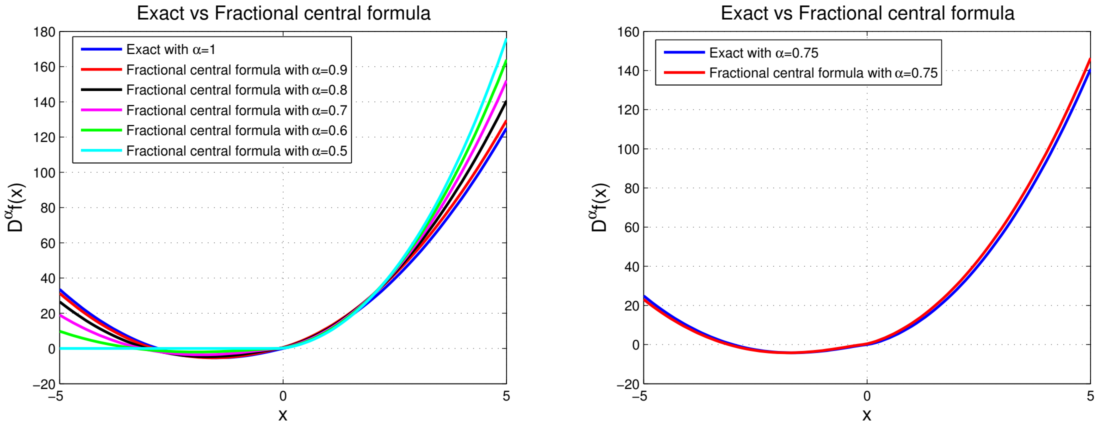

Figure 1. Comparison results between the analytical and fractional central formula.

Example 5.2. Let the main function be $f(x) = \frac{1}{3}x^{\frac{7}{2}} - \frac{1}{2}x^{\frac{5}{2}}$, and consider $\alpha = 1$. Then, the Caputo derivative is given by

$$f'(x) = \frac{1}{3}\frac{\Gamma(\frac{9}{2})}{\Gamma(\frac{9}{2}-\alpha)}x^{\frac{7}{2}-\alpha} - \frac{1}{2}\frac{\Gamma(\frac{7}{2})}{\Gamma(\frac{7}{2}-\alpha)}x^{\frac{5}{2}-\alpha}.$$

By taking the point of evaluation $x_0 = 1$; the $f'(1) = -0.0833$. In other words, the exact value of the derivatives equal the Caputo derivative. Now, by taking the step initial value $h = 0.2$ and the order of extrapolation $N = 7$, we get the results shown in Table 4.

Table 4. The numerical result using the proposed fractional central formula with Richardson extrapolation.

h	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^{14})$
0.2000	0.1971	0	0	0	0	0	0
0.1000	0.0384	-0.0144	0	0	0	0	0
0.0500	-0.0269	-0.0487	-0.0510	0	0	0	0
0.0250	-0.0562	-0.0660	-0.0671	-0.0674	0	0	0
0.0125	-0.0700	-0.0747	-0.0752	-0.0754	-0.0754	0	0
0.0063	-0.0768	-0.0790	-0.0793	-0.0794	-0.0794	-0.0794	0
0.0031	-0.0798	-0.0809	-0.0810	-0.0810	-0.0810	-0.0810	-0.0810

Now, we consider the fractional case when $\alpha = 0.75$, and for $x_0 = 1$ we get

$$D_*^{0.75}f(1) = \frac{1}{3}\frac{\Gamma(\frac{9}{2})}{\Gamma(3.75)}x^{2.75} - \frac{1}{2}\frac{\Gamma(\frac{7}{2})}{\Gamma(2.75)}x^{1.75} = -0.1565.$$

Accordingly, by using the Richardson extrapolation in view of the modified 5-point fractional central formula, we can gain numerical results shown in Table 5.

Table 5. The numerical result using the proposed fractional central formula with Richardson extrapolation.

h	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^{14})$
0.2000	0.0439	0	0	0	0	0	0
0.1000	-0.0690	-0.0688	0	0	0	0	0
0.0500	-0.1131	-0.1130	-0.1126	0	0	0	0
0.0250	-0.1317	-0.1317	-0.1317	-0.1316	0	0	0
0.0125	-0.1562	-0.1562	-0.1562	-0.1562	-0.1561	0	0
0.0063	-0.1562	-0.1562	-0.1562	-0.1562	-0.1562	-0.1562	0
0.0031	-0.1564	-0.1563	-0.1563	-0.1563	-0.1563	-0.1563	-0.1564

In what follows, we plot some graphical comparisons in Figure 2 between the results generated by using analytical and fractional central formulas.

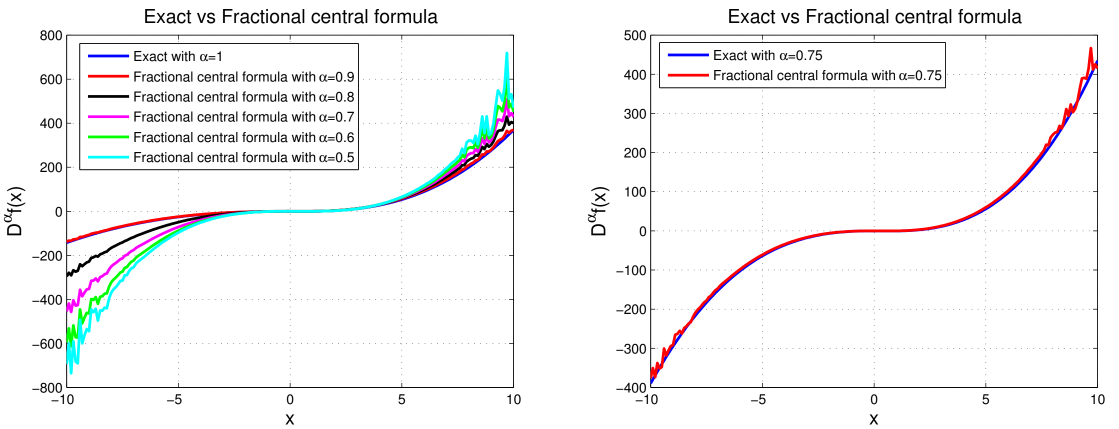

Figure 2. Comparison results between the analytical and fractional central formula.

6. Conclusions

The modified 5-point fractional formula has been introduced in this work by generalizing the classical central formula. Additionally, we have developed a new Richardson extrapolation methodology to obtain more accuracy for the proposed fractional formula in an approximation of the first derivative in the sense of the Caputo derivative. From several numerical results, we can confirm the validity of our proposed fractional formula in approximating the Caputo derivative $D_*^\alpha f$, where $0 \le \alpha \le 1$.

Conflict of interest

The authors declare no conflicts of interest.

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