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Double total domination number of Cartesian product of paths

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Abstract: A vertex set S of a graph G is called a double total dominating set if every vertex in G has at least two adjacent vertices in S. The double total domination number $\gamma_{\times 2,t}(G)$ of G is the minimum cardinality over all the double total dominating sets in G. Let $G\Box H$ denote the Cartesian product of graphs G and H. In this paper, the double total domination number of Cartesian product of paths is discussed. We determine the values of $\gamma_{\times 2,t}(P_i\Box P_n)$ for $i = 2, 3$, and give lower and upper bounds of $\gamma_{\times 2,t}(P_i\Box P_n)$ for $i \geq 4$.

Keywords: dominating sets; total domination; double total domination; Cartesian product; paths
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1. Introduction

Throughout this article, we only deal with finite and simple graphs. For the undefined notation and terminology, one may refer to [4]. Let $G = (V(G), E(G))$ be a graph. The *open neighborhood* of a vertex $v \in V$ is denoted by $N_G(v) = \{u \in V : uv \in E(G)\}$, and the *degree* of v in G is denoted by $d(v) = |N_G(v)|$. A graph G is *k-regular* if every vertex has degree k in G. We use $\delta(G)$ and $\Delta(G)$ denote the minimum degree and the maximum degree among the vertices of G, respectively. For a subset $S \subseteq V(G)$, let $G[S]$ denote the subgraph induced by S. For simplicity, the induced subgraph $G[V(G) \setminus S]$ is denoted by $G - S$. For two disjoint subsets $X, Y \subset V(G)$, we use $e(X, Y)$ to denote the number of edges with one end in X and the other end in Y. The *Cartesian product* $G\Box H$ of two graphs G and H is the graph with vertex set $V(G\Box H) = V(G) \times V(H)$, and edge $(u_1, v_1)(u_2, v_2) \in E(G\Box H)$ if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$. In general, let P_n, C_n denote a path, a cycle of order n, respectively.

A *total dominating set* of G is a subset $S \subseteq V(G)$ such that each vertex in $V(G)$ has a neighbor in S. The *total domination number* $\gamma_t(G)$ is the cardinality of a minimum total dominating set of G. The notion of total domination in graphs was first introduced by Cockayne et al. [8]. Numerous results on this object have been obtained over the years. Reader may refer to an excellent total domination book [12] and a survey [9]. A problem on total domination appeared as Question 3 of the 40th International Mathematical Olympiad, which is equivalent to determining the total domination number of the Cartesian product of two path graphs with same even order, i.e. $\gamma_t(P_{2n}\square P_{2n})$. And further, several authors have studied the total domination number on product of graphs such as Cartesian, strong and lexicographic products (see [2, 6, 13, 14, 16, 19]).

Except the classical total domination problem, there are many different ways to generalize the total dominating set. One of them is introduced by Henning and Kazemi in [10, 11]: Given an integer k, a subset $S \subseteq V(G)$ is a *k-tuple total dominating set* (kTDS in short) of G if every vertex $v \in V(G)$ has $|N(v) \cap S| \geq k$, this is, every vertex of G has at least k neighbors in S. The *k-tuple total domination number* $\gamma_{\times k,t}(G)$ is the cardinality of a minimum kTDS of G. Henning and Kazemi [10] first studied the k-tuple total domination number, and obtained some results of the k-tuple total domination number of complete multipartite graphs. They also gave a useful observation.

Observation 1.1. *[10] Let G be graph of order n with $\delta(G) \geq k$. Then,*

(a) *$\gamma_{\times k,t}(G) \leq n$;*

(b) *if G is a spanning subgraph of graph H, then $\gamma_{\times k,t}(H) \leq \gamma_{\times k,t}(G)$;*

(c) *if v is a vertex with degree k in G and S is a kTDS in G, then $N_G(x) \subseteq S$.*

We remark that 1-tuple total domination is the well-known total domination. When $k = 2$, a k-tuple total dominating set is called a *double total dominating set*, abbreviated to DTDS, and the 2-tuple total domination number is also called the *double total domination number*, denoted by $\gamma_{\times 2,t}(G)$. This parameter was studied in [1, 3, 5, 7, 15, 17, 18]. Especially, Kazemi et al. [15] determined the value of the double total domination number of Cartesian product of some complete graphs. Bermudo et al. [3] gave the following result on the double total domination number.

Theorem 1.2. *[3] Let $j \geq 2, n \geq 3$ be two integers. Then,*

(a) *when j is odd, $\gamma_{\times 2,t}(P_j\square C_n) = \frac{(j+1)n}{2}$;*

(b) *when j is even, $\gamma_{\times 2,t}(C_j\square C_n) = \frac{jn}{2}$.*

Naturally, we may ask the following problem: What are the values of the double total domination numbers of Cartesian products of paths? In order to answer the problem, we make a step in this paper. In the next section, we give the values of $\gamma_{\times 2,t}(P_i\square P_n)$ for $i = 2, 3$, and the lower and upper bounds for $\gamma_{\times 2,t}(P_i\square P_n)$ when $i \geq 4$.

2. Double total domination number of Cartesian product of paths

In this section, we simplify the notation for $V(P_m\square P_n)$. For example, when $m = 4$, if $P_4 = abcd$, $P_n = v_1v_2 \dots v_n$, we denote simply the vertices (a, v_i) as a_i, (b, v_i) as b_i, (c, v_i) as c_i, (d, v_i) as d_i, respectively. Moreover, set $X_i = \{a_i, b_i, c_i, d_i\}$ for $1 \leq i \leq n$.

First, we give a lower bound for the double total domination number of $P_2\square P_n$.

Lemma 2.1. *Let $n \geq 2$ be an integer. Then*

$$\gamma_{\times 2,t}(P_2 \square P_n) \geq \begin{cases} 2n, & \text{if } n \leq 4; \\ (4n+6)/3, & \text{if } n \geq 5 \text{ and } n \equiv 0 \ (mod\ 3); \\ (4n+8)/3, & \text{if } n \geq 5 \text{ and } n \equiv 1 \ (mod\ 3); \\ (4n+4)/3, & \text{if } n \geq 5 \text{ and } n \equiv 2 \ (mod\ 3). \end{cases}$$

Proof. Let $P_2 = ab$, $P_n = v_1v_2 \ldots v_n$, and $X_i = \{a_i, b_i\}$ for $1 \leq i \leq n$. Denote $G = P_2 \square P_n$. Pick a minimum DTDS D of G. Since $\delta(G) = 2$ and $\Delta(G) = 3$, then $D = D_2 \cup D_3$, where $D_i = \{v \in D : |N(v) \cap D| = i\}$ for $i \in \{2, 3\}$. Note that $X_i \subseteq D$ ($i = 1, 2, n-1, n$) by Observation 1.1 (c). Then, when $n \leq 4$, we have $|D| = |V(G)| = 2n$, as desired. Next, we consider the case that $n \geq 5$.

Let $\overline{D} = V(G) \setminus D$. First, we claim that

$$2|\overline{D}| \leq e(D, \overline{D}) \leq (|D| - 4)(\Delta - 2) = |D| - 4.$$

The first inequality holds because every vertex in $\overline{D}$ is adjacent to at least two vertices in D, and the second one holds because $e(X_1 \cup X_n, \overline{D}) = 0$ and each of the remaining $|D| - 4$ vertices has at most $\Delta - 2$ neighbors in $\overline{D}$. Therefore

$$\gamma_{\times 2,t}(G) = |D| \geq \frac{2|V(G)| + 4}{3} = \frac{4n+4}{3}.$$

Since $\gamma_{\times 2,t}(G)$ is an integer, the result holds when $n \equiv 0$ $(mod\ 3)$ or $n \equiv 2$ $(mod\ 3)$.

The remaining case is $n = 3k + 1$, where $k \geq 1$ is an integer. We apply induction on k. For $k = 1$, $G = P_2 \square P_4$. As discussed above, $\gamma_{\times 2,t}(G) = 2n = \frac{4 \times 4 + 8}{3}$. Suppose that the result holds for $k - 1$, where $k \geq 2$. Our goal is to prove that it is also valid for k. Recall that $X_i \subseteq D$ ($i = 1, 2, n-1, n$). If D contains at least two vertices in $X_3 \cup X_{n-2}$, then there are at least two vertices in $X_2 \cup X_{n-1}$ that are not adjacent to any vertex in $\overline{D}$. Furthermore, there are at least six vertices in $X_1 \cup X_2 \cup X_{n-1} \cup X_n$ which are in D and not adjacent to any vertex in $\overline{D}$. Therefore, we have $e(D, \overline{D}) \leq (|D| - 6)(\Delta - 2) = |D| - 6$. On the other hand, $e(D, \overline{D}) \geq 2|\overline{D}| = 2(|V(G)| - |D|)$. Then $|D| \geq \frac{4n+6}{3} = \frac{12k+10}{3}$. So $|D| \geq \frac{12k+12}{3} = \frac{4n+8}{3}$ because $|D|$ is an integer. If D contains at most one vertex in $X_3 \cup X_{n-2}$, without loss of generality, we may assume $X_3 \cap D = \emptyset$. By definition of the double total dominating set, we know $X_4 \subseteq D$ and then $X_5 \subseteq D$. Note that, when $n = 7$, $X_5 = X_{n-2}$. This belongs to the preceding case. When $n \geq 10$, set $X = X_1 \cup X_2 \cup X_3$ and $G' = G - X$. Let $D' = D \setminus X = D \setminus (X_1 \cup X_2)$. It is easy to see that D' is a DTDS of G'. By induction,

$$|D'| \geq \gamma_{\times 2,t}(G') \geq \frac{4(3(k-1)+1)+8}{3} = \frac{4(n-3)+8}{3}.$$

Thus we have $|D| = |D'| + 4 \geq \frac{4n+8}{3}$. □

Next we determine the value of $\gamma_{\times 2,t}(P_2 \square P_n)$.

Theorem 2.2. *Let $n \geq 2$ be an integer. Then*

$$\gamma_{\times 2,t}(P_2 \square P_n) = \begin{cases} 2n, & \text{if } n \leq 4; \\ (4n+6)/3, & \text{if } n \geq 5 \text{ and } n \equiv 0 \ (mod\ 3); \\ (4n+8)/3, & \text{if } n \geq 5 \text{ and } n \equiv 1 \ (mod\ 3); \\ (4n+4)/3, & \text{if } n \geq 5 \text{ and } n \equiv 2 \ (mod\ 3). \end{cases}$$

Proof. Let X_i be defined as earlier in the proof of Lemma 2.1, where $1 \le i \le n$. Let $G = P_2 \Box P_n$. When $n \le 4$, it is as desired. Next we assume that $n \ge 5$.

We will show the upper bound through constructing a DTDS S of G. Let

$$X' = X_1 \cup X_2 \cup X_{n-1} \cup X_n.$$

By Observation 1.1 (c), X' is contained in every DTDS of G. When $n = 3k$, set $S = X' \cup (\cup_{i=1}^{k-1} X_{3i+1}) \cup (\cup_{i=1}^{k-2} X_{3i+2})$. Then $|S| = 8 + 2(k-1) + 2(k-2) = \frac{4n+6}{3}$. Therefore $\gamma_{\times 2,t}(G) \le |S| = \frac{4n+6}{3}$. When $n = 3k+1$, set $S = X' \cup (\cup_{i=1}^{k-1} X_{3i+1}) \cup (\cup_{i=1}^{k-1} X_{3i+2})$. Then,

$$\gamma_{\times 2,t}(G) \le |S| = 8 + 2(k-1) + 2(k-1) = \frac{4n+8}{3}.$$

Finally, when $n = 3k+2$, set $S = X' \cup (\cup_{i=1}^{k-1} X_{3i+1}) \cup (\cup_{i=1}^{k-1} X_{3i+2})$. Clearly,

$$\gamma_{\times 2,t}(G) \le |S| = 8 + 2(k-1) + 2(k-1) = \frac{4n+4}{3}.$$

Also, by Lemma 2.1, the proof is completed. □

For $\gamma_{\times 2,t}(P_3 \Box P_n)$, we give a lower bound firstly. Before that, we need an observation.

Observation 2.3. *Let $G = P_m \Box P_n$, $X = \cup_{i=1}^{h} X_i$ and $G' = G - X$, where h is a positive integer and $h < n$. If D is a DTDS of G, then we can obtain a "nearly" DTDS, $D \setminus X$, of G' by confining D on G'. (Each of $V(G') \setminus X_{h+1}$ has at least two neighbors in $D \setminus X$.) Extend some vertices to $D \setminus X$, and denote the resulting set by D'. If each vertex in X_{h+1} has at least two neighbors in D', then D' is a DTDS of G'.*

Lemma 2.4. *Let $n \ge 2$ be an integer. Then*

$$\gamma_{\times 2,t}(P_3 \Box P_n) \ge \begin{cases} 2n+1, & \text{if } n \equiv 1 \ (mod\ 2); \\ 2n+2, & \text{if } n \equiv 0 \ (mod\ 2). \end{cases}$$

Proof. Let $G = P_3 \Box P_n$ with $P_3 = abc$ and $P_n = v_1 v_2 \dots v_n$, and D be a minimum DTDS of G. Set $X_i = \{a_i, b_i, c_i\}$ for $1 \le i \le n$.

First, we introduce three constructions in Figure 1. (In the remaining, for the figures of the paper, a hollow dot denotes a vertex in D, a cross dot denotes a vertex not in D.) If $b_i \notin D$ $(2 \le i \le n-1)$, then $\{a_{i-1}, a_{i+1}, c_{i-1}, c_{i+1}\} \subseteq D$ because either of a_i and c_i must have at least two neighbors in D (see construction (I)). If $\{a_i, c_i\} \cap D = \emptyset$ $(3 \le i \le n-2)$, then $\{a_{i-2}, a_{i+2}, b_{i-1}, b_{i+1}, c_{i-2}, c_{i+2}\} \subseteq D$ because each of $\{a_{i-1},\ a_{i+1},\ c_{i-1},\ c_{i+1}\}$ has at least two neighbors in D (see construction (II)). If $X_i \cap D = \emptyset$ $(3 \le i \le n-2)$, then $X_{i-1} \cup X_{i+1} \cup \{a_{i-2}, a_{i+2}, c_{i-2}, c_{i+2}\} \subseteq D$ by the definition of DTDS (see construction (III)).

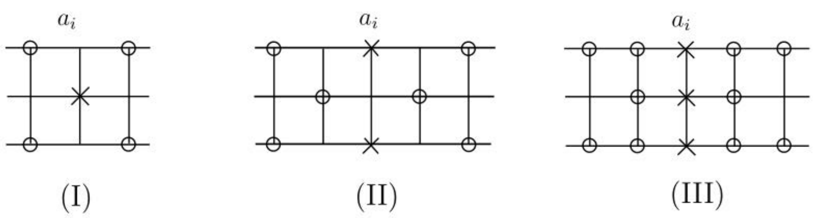

Figure 1. Three constructions in the proof of Lemma 2.4.

We prove the lemma by induction on n. When $n = 2$, the result holds by Theorem 2.2. According to Observation 1.1(c), we have the following conclusions for $3 \leq n \leq 5$. When $n = 3$, we have $\{b_1, a_2, c_2, b_3\} \subseteq D$. Focusing on vertices b_1, b_3, at least two of a_1, b_2, c_1 and at least two of a_3, b_2, c_3 are in D. It implies that $\gamma_{\times 2,t}(P_3 \square P_3) \geq 7$. We give a DTDS with 7 vertices in Figure 2(a). When $n = 4$, we have $\{a_2, a_3, b_1, b_4, c_2, c_3\} \subseteq D$ (see Figure 2(b)). On each dash curve, there are at least two vertices contained in D by the definition of DTDS. Thus, $\gamma_{\times 2,t}(P_3 \square P_4) \geq 10$. When $n = 5$, we know that $\{b_1, a_2, c_2, a_4, c_4, b_5\} \subseteq D$. Furthermore, on each dash curve, there are at least two vertices contained in D (see Figure 2(c)). Thus $|D| \geq 10$. If $\{a_3, c_3\} \cap D = \emptyset$, then D would contain all vertices of $\cup_{i=1,2,4,5} X_i$ by construction (II), which means that $|D| \geq 12$. If $\{a_3, c_3\} \cap D \neq \emptyset$, then $|D| \geq 11$. Thus $\gamma_{\times 2,t}(P_3 \square P_5) \geq 11$ in either of these cases. (We give a minimum DTDS of G with 11 vertices in Figure 2(d) when $n = 5$.)

When $n = 6$, $\{a_2, a_5, b_1, b_6, c_2, c_5\} \subseteq D$, at least 2 vertices of $N_G(b_1)$, $N_G(b_6)$ are contained in D, respectively. If $|(X_3 \cup X_4) \cap D| \geq 4$, we are done. Consider the cases that $|(X_3 \cup X_4) \cap D| \leq 3$. Without loss of generality, we may assume $|X_3 \cap D| \leq 1$. If $X_3 \cap D = \emptyset$, then $X_1 \cup X_2 \cup X_4 \subseteq D$ by construction (III), so $\gamma_{\times 2,t}(P_3 \square P_6) \geq 14$ (see Figure 2(e)). If $X_3 \cap D = \{c_3\}$ (similarly, for $X_3 \cap D = \{a_3\}$), then $\{a_2, a_4, c_2, c_4\} \subseteq D$ by construction (I). Furthermore, $\{a_1, b_2, b_4\} \subseteq D$ because $a_3 \notin D$. Thus $\gamma_{\times 2,t}(P_3 \square P_6) \geq 14$ (see Figure 2(f)). If $X_3 \cap D = \{b_3\}$, then $\{a_1, a_5, b_2, b_4, c_1, c_5\} \subseteq D$ by construction (II) (see Figure 2(g)). If $b_5 \notin D$, then $\{a_4, a_6, c_4, c_6\} \subseteq D$ by construction (I). If $b_5 \in D$, noting that a_5 (c_5) has at least two neighbors in D, at least one of a_4 and a_6 (c_4 and c_6) is in D. In either of these cases, $\gamma_{\times 2,t}(P_3 \square P_6) \geq 14$.

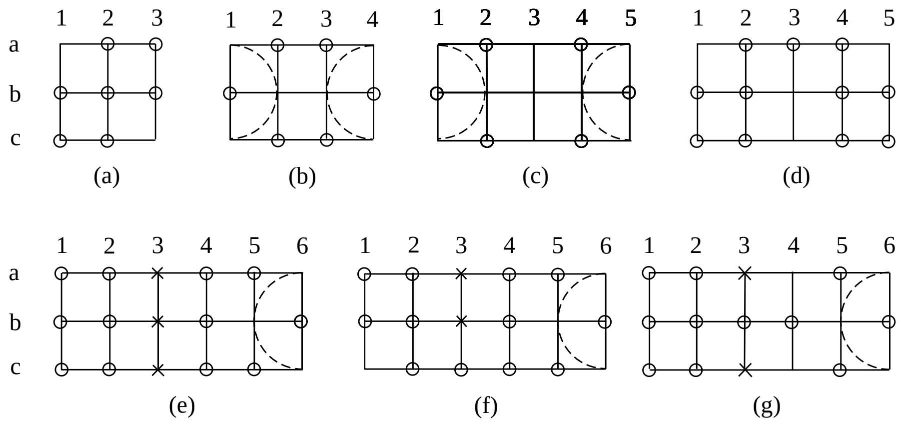

Figure 2. Some cases for $3 \leq n \leq 6$ in Lemma 2.4.

Next, we only discuss the case $n \equiv 0$ (*mod* 2), and the argument of case $n \equiv 1$ (*mod* 2) is similar by replacing $2n + 2$ with $2n + 1$. Assume that the result holds for $n - 2$ where $n \geq 8$. Suppose, to the contrary, $\gamma_{\times 2,t}(G) < 2n + 2$. Next, we will choose a vertex subset X. Let $G' = G - X$. Then, basing on a DTDS D of G, we obtain a DTDS D' of G'. Finally, we deduce contradictions according to the relation between $|D|$ and $|D'|$ and induction on index n.

By Observation 1.1 (c), we have $\{a_2, b_1, c_2\} \subseteq D$. Focusing on b_1, it is clear that at least two of a_1, c_1, b_2 are contained in D. This means that $|D \cap (X_1 \cup X_2)| \geq 5$. If $|D \cap (X_1 \cup X_2)| = 6$ and $|D \cap (X_3 \cup X_4)| \geq 4$, then we can extend the missing vertices of $X_3 \cup X_4$ (with at most two) to $D \setminus (X_1 \cup X_2)$ and obtain a vertex set $D' = (D \setminus (X_1 \cup X_2)) \cup (X_3 \cup X_4)$. Then $|D'| \leq |D| - 6 + 2 < 2n - 2$. By Observation 2.3, D' is a DTDS of $G' = G - (X_1 \cup X_2)$. By induction, $|D'| \geq \gamma_{\times 2,t}(G') \geq 2(n-2)+2 = 2n-2$, a contradiction. Similarly, if $|D \cap (X_1 \cup X_2)| = 5$ and $|D \cap (X_3 \cup X_4)| \geq 5$, then we also can obtain a contradiction. Since either of a_3, c_4 has at least two neighbors in D, we know that at least one of a_4 and b_3 (c_3 and b_4, respectively) in D. That is to say, $|D \cap (X_3 \cup X_4)| \geq 2$. Therefore, we only need discuss the following two cases to complete the proof.

Case 1. $|D \cap (X_1 \cup X_2)| = 6$ and $2 \leq |D \cap (X_3 \cup X_4)| \leq 3$.

Case 1.1. If $|D \cap (X_3 \cup X_4)| = 3$, we claim that either $D' = (D \setminus \cup_{i=1}^{4} X_i) \cup \{a_3, b_3, a_4, b_4, c_4\}$ or $D' = (D \setminus \cup_{i=1}^{4} X_i) \cup \{b_3, c_3, a_4, b_4, c_4\}$ is a DTDS of $G' = G - (X_1 \cup X_2)$. (For otherwise, focusing on vertices a_4, c_4, there would be $\{a_3, c_3, b_4\} \subseteq D$ and $a_5, c_5 \notin D$. Also, $|D \cap (X_3 \cup X_4)| = 3$ means that $b_3 \notin D$. By construction (I), $a_4, c_4 \in D$, then $|D \cap (X_3 \cup X_4)| \geq 5$, a contradiction.) There is $|D'| \geq \gamma_{\times 2,t}(G') \geq 2(n-2)+2 = 2n-2$. On the other hand, $|D'| = |D| - 9 + 5 < 2n-2$, a contradiction.

Case 1.2. If $|D \cap (X_3 \cup X_4)| = 2$, then at least one of pairs $\{a_3, a_4\}, \{b_3, b_4\}, \{c_3, c_4\}$ does not intersect with D. If $\{b_3, b_4\} \cap D = \emptyset$, then $\{a_3, c_3, a_4, c_4\} \subseteq D$ by construction (I), a contradiction. By symmetry, we discuss the case that $\{a_3, a_4\} \cap D = \emptyset$. Then there is $\{a_2, b_3, b_4, a_5\} \subseteq D$. The condition $|D \cap (X_3 \cup X_4)| = 2$ implies that $\{c_3, c_4\} \cap D = \emptyset$ and furthermore $c_5 \in D$. Focusing on vertices a_5, c_5, we know that $\{b_5, a_6, c_6\} \subseteq D$ (see Figure 3(a)). Set $X = \cup_{i=1}^{4} X_i$. Then $D' = D \setminus X$ is a DTDS of $G' = G - X$. Clearly, $|D'| = |D| - 8 < 2n-6$. By induction, $|D'| \geq \gamma_{\times 2,t}(G') \geq 2(n-4)+2 = 2n-6$, a contradiction.

Case 2. $|D \cap (X_1 \cup X_2)| = 5$ and $2 \leq |D \cap (X_3 \cup X_4)| \leq 4$.

Case 2.1. If $|D \cap (X_3 \cup X_4)| = 4$, then $D' = (D \setminus \cup_{i=1}^{4} X_i) \cup \{a_3, b_3, a_4, b_4, c_4\}$ or $D' = (D \setminus \cup_{i=1}^{4} X_i) \cup \{b_3, c_3, a_4, b_4, c_4\}$ is a DTDS of $G' = G - (X_1 \cup X_2)$. There is $|D'| \geq \gamma_{\times 2,t}(G') \geq 2(n-2)+2 = 2n-2$. On the other hand, $|D'| = |D| - 9 + 5 < 2n-2$, a contradiction.

Case 2.2. If $|D \cap (X_3 \cup X_4)| = 3$, we consider two subcases according to $b_2 \in D$ or not.

(1) $b_2 \notin D$. By construction (I), $\{a_3, c_3\} \subseteq D$. We claim that $b_3 \in D$. (For otherwise, there would be $\{a_4, c_4\} \subseteq D$ by construction (I), then $|D \cap (X_3 \cup X_4)| \geq 4$, a contradiction.) $|D \cap (X_3 \cup X_4)| = 3$ means that $X_4 \cap D = \emptyset$. By construction (III), $X_5 \cup \{a_6, c_6\} \subseteq D$ (see Figure 3(b)). Set $X = \cup_{i=1}^{4} X_i$. Then $D' = D \setminus X$ is a DTDS of $G' = G - X$. Clearly, $|D'| = |D| - 8 < 2n-6$. By induction, $|D'| \geq \gamma_{\times 2,t}(G') \geq 2(n-4)+2 = 2n-6$, a contradiction.

(2) $b_2 \in D$. By symmetry, we may assume that $a_1 \notin D$. First, we establish a claim.

Claim. $\{a_3, b_3\} \subseteq D$.

Since a_2 has at least two neighbors in D, clearly $a_3 \in D$. Focusing on vertices a_3, c_4, we know that at least one of b_3 and a_4 (c_3 and b_4, respectively) in D ($*$). If $b_3 \notin D$, then $\{a_4, c_4\} \subseteq D$ by construction (I). It implies that $|D \cap (X_3 \cup X_4)| \geq 4$, a contradiction. The claim is done.

Recall that one of c_3, b_4 is in D (see ($*$)). If $c_3 \in D$, then $X_4 \cap D = \emptyset$. By construction (III), $X_5 \cup \{a_6, c_6\} \subseteq D$ (see Figure 3(c)). Similar to (1), we can deduce a contradiction. If $b_4 \in D$, then $\{c_3, a_4, c_4\} \cap D = \emptyset$. By construction (II), there is $\{b_5, a_6, c_6\} \subseteq D$. Also, focusing on c_4, we know that $c_5 \in D$. Furthermore, focusing on a_6, we deduce that a_5 or b_6 in D (see Figure 3(d)). No matter which one of a_5, b_6 being in D, $D' = D \setminus \cup_{i=1}^{4} X_i$ is a DTDS of $G' = G - \cup_{i=1}^{4} X_i$. However, there are $|D'| = |D| - 8 < 2n-6$ and $|D'| \geq \gamma_{\times 2,t}(G') \geq 2(n-4)+2 = 2n-6$, a contradiction.

Case 2.3. If $|D \cap (X_3 \cup X_4)| = 2$, then at least one of pairs $\{a_3, a_4\}, \{b_3, b_4\}, \{c_3, c_4\}$ does not intersect with D. Similar to the proof of Case 1.2, the unique possibility is $\{a_3, a_4, c_3, c_4\} \cap D = \emptyset$. Focusing on the vertices a_2, c_2, there is $\{a_1, b_2, c_1\} \subseteq D$. This means that $|D \cap (X_1 \cup X_2)| = 6$, a contradiction.

In each of these cases, we deduce a contradiction. Therefore, we draw a conclusion that $\gamma_{\times 2,t}(P_3 \square P_n) \geq 2n + 2$ when $n \equiv 0$ (mod 2). It is analogous to verify that $\gamma_{\times 2,t}(P_3 \square P_n) \geq 2n + 1$ when $n \equiv 1$ (mod 2) by replacing $2n + 2$ with $2n + 1$ in the above proof. □

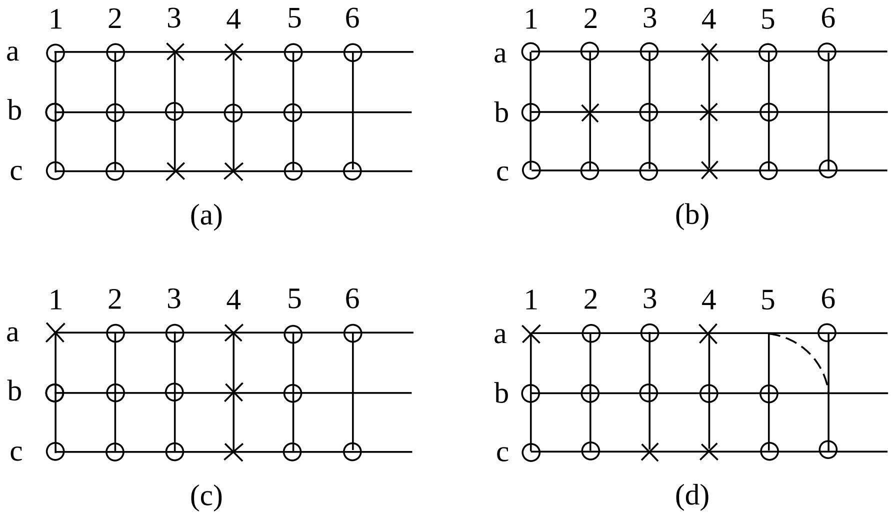

Figure 3. Illustrations for Cases 1 and 2 in Lemma 2.4.

We are ready to prove our second result.

Theorem 2.5. *Let $n \geq 2$ be an integer. Then*

$$\gamma_{\times 2,t}(P_3 \square P_n) = \begin{cases} 2n + 1, & \text{if } n \equiv 1 \ (mod\ 2); \\ 2n + 2, & \text{if } n \equiv 0 \ (mod\ 2). \end{cases}$$

Proof. When $n \equiv 0$ (mod 2), set $S = (\cup_{i=1}^{n} \{a_i, c_i\}) \cup \{b_1, b_n\}$. It is clear that S is a DTDS of graph G. Hence $\gamma_{\times 2,t}(G) \leq 2n + 2$.

When $n \equiv 3$ (mod 4), i.e. $n = 4k + 3$ for some nonnegative integer k, set

$$S = (\cup_{i=1}^{n} b_i) \bigcup (\cup_{i=0}^{k} \{a_{4i+2}, a_{4i+3}, c_{4i+1}, c_{4i+2}\})$$

(see Figure 4(a) for the case $n = 7$), then $|S| = n + 4(k + 1) = 2n + 1$. When $n \equiv 1$ (mod 4), i.e. $n = 4k + 1$ for some positive integer k, set

$$S = (\cup_{i=1}^{n} b_i \setminus \{b_{4k-1}\}) \bigcup (\cup_{i=0}^{k-1} \{a_{4i+2}, a_{4i+3}, c_{4i+1}, c_{4i+2}\}) \bigcup \{a_{4k}, c_{4k}, c_{4k+1}\}$$

(see Figure 4(b) for the case that $n = 9$), then $|S| = n - 1 + 4k + 3 = 2n + 1$. In these two cases, it is easy to check that S is a DTDS of G. Thus, $\gamma_{\times 2,t}(G) \leq |S| = 2n + 1$ when $n \equiv 1$ (mod 2).

By Lemma 2.4, the proof is completed. □

Let $G = P_m \square P_n$ with $P_m = u_1 u_2 \ldots u_m$, $P_n = v_1 v_2 \ldots v_n$, where integers $m \geq 2$, $n \geq 2$. For the vertex $u_i \in V(P_m)$ and $v_j \in V(P_n)$, we denote simply the vertex (u_i, v_j) by x_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$. For each

$1 \le i \le m$ and $1 \le j \le n$, we denote $X_j = \cup_{i=1}^{m} x_{ij}$, $Y_i = \cup_{j=1}^{n} x_{ij}$. Before moving forward, we give a useful lemma.

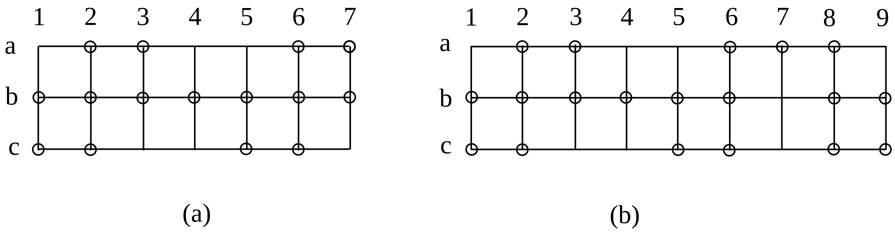

Figure 4. A DTDS S for $n = 7$, $n = 9$, respectively.

Lemma 2.6. *Let $G = P_m \Box P_n$, where integers $n \ge 4, m \ge 4$, and D be a minimum DTDS of G. Then*

$$|D \cap (X_1 \cup X_n \cup Y_1 \cup Y_m)| \ge m + n.$$

Proof. For any vertex set $W \in \{X_1, X_n, Y_1, Y_n\}$, we will show at least $\frac{|W|}{2}$ vertices of W in D. W.l.o.g., we pick $W = Y_1 = \{x_{11}, x_{12}, \cdots, x_{1n}\}$. Set $D' = V(G) \setminus D$. Since each vertex of $Y_1 \setminus \{x_{11}, x_{1n}\}$ has degree 3 and either of x_{11}, x_{1n} has degree 2, it is impossible to appear three consecutive vertices in $Y_1 \cap D'$. (For otherwise, the interior vertex is adjacent to at most one vertex in D, a contradiction.) Moreover, if $x \in Y_1 \cap D$, then x has at least one neighbor in $Y_1 \cap D$. By the above discussion and Observation 1.1 (c), we can establish the following four facts.

(F1) The length of any sequence of consecutive vertices is at most two in $Y_1 \cap D'$.
(F2) The length of any sequence of consecutive vertices is at least two in $Y_1 \cap D$.
(F3) For every four consecutive vertices in Y_1, at least two of them in D.
(F4) $\{x_{12}, x_{1(n-1)}\} \subset D$.

Concretely, we discuss the following three cases according to x_{11}, x_{1n} in D or not.

Case 1. $\{x_{11}, x_{1n}\} \cap D = \emptyset$. By (F2) and (F4), $\{x_{12}, x_{13}, x_{1(n-2)}, x_{1(n-1)}\} \subset D$. By virtue of (F3), we consider four subcases.

(1.1) $n \equiv 0 \pmod 4$.
Since $|(Y_1 \setminus \{x_{11}, x_{12}, x_{13}, x_{1n}\}) \cap D| \ge \frac{n-4}{2}$, $|Y_1 \cap D| \ge \frac{n-4}{2} + 2 = \frac{n}{2}$.
(1.2) $n \equiv 1 \pmod 4$.
Since $|(Y_1 \setminus \{x_{11}, x_{12}, x_{13}, x_{1(n-1)}, x_{1n}\}) \cap D| \ge \frac{n-5}{2}$, $|Y_1 \cap D| \ge \frac{n-5}{2} + 3 = \frac{n+1}{2}$.
(1.3) $n \equiv 2 \pmod 4$.
Since $|(Y_1 \setminus \{x_{11}, x_{12}, x_{13}, x_{1(n-2)}, x_{1(n-1)}, x_{1n}\}) \cap D| \ge \frac{n-6}{2}$, $|Y_1 \cap D| \ge \frac{n-6}{2} + 4 = \frac{n+2}{2}$.
(1.4) $n \equiv 3 \pmod 4$.
Since $|(Y_1 \setminus \{x_{11}, x_{12}, x_{13}, x_{14}, x_{1(n-2)}, x_{1(n-1)}, x_{1n}\}) \cap D| \ge \frac{n-7}{2}$, $|Y_1 \cap D| \ge \frac{n-7}{2} + 4 = \frac{n+1}{2}$.

Case 2. $|\{x_{11}, x_{1n}\} \cap D| = 1$. We may assume that $x_{11} \notin D, x_{1n} \in D$. By (F2) and (F4), $\{x_{12}, x_{13}, x_{1(n-1)}, x_{1n}\} \subset D$. Then we have the following conclusions by (F3).

(1.1) $n \equiv 0 \pmod 4$.
$|(Y_1 \setminus \{x_{11}, x_{12}, x_{13}, x_{1n}\}) \cap D| \ge \frac{n-4}{2}$, so $|Y_1 \cap D| \ge \frac{n-4}{2} + 3 = \frac{n+2}{2}$.
(1.2) $n \equiv 1 \pmod 4$.
$|(Y_1 \setminus \{x_{11}, x_{12}, x_{13}, x_{1(n-1)}, x_{1n}\}) \cap D| \ge \frac{n-5}{2}$, so $|Y_1 \cap D| \ge \frac{n-5}{2} + 4 = \frac{n+3}{2}$.

(1.3) $n \equiv 2 \pmod 4$.
$|(Y_1 \setminus \{x_{11}, x_{12}, x_{13}, x_{1(n-2)}, x_{1(n-1)}, x_{1n}\}) \cap D| \geq \frac{n-6}{2}$, then $|Y_1 \cap D| \geq \frac{n-6}{2} + 4 = \frac{n+2}{2}$.

(1.4) $n \equiv 3 \pmod 4$.
$|(Y_1 \setminus \{x_{11}, x_{12}, x_{13}, x_{14}, x_{1(n-2)}, x_{1(n-1)}, x_{1n}\}) \cap D| \geq \frac{n-7}{2}$, so $|Y_1 \cap D| \geq \frac{n-7}{2} + 4 = \frac{n+1}{2}$.

Case 3. $\{x_{11}, x_{1n}\} \subset D$. By (F2) and (F4), $\{x_{11}, x_{12}, x_{1(n-1)}, x_{1n}\} \subset D$. By (F3), the following are established.

(1.1) $n \equiv 0 \pmod 4$.
By $|(Y_1 \setminus \{x_{11}, x_{12}, x_{1(n-1)}, x_{1n}\}) \cap D| \geq \frac{n-4}{2}$, $|Y_1 \cap D| \geq \frac{n-4}{2} + 4 = \frac{n+4}{2}$.

(1.2) $n \equiv 1 \pmod 4$.
By $|(Y_1 \setminus \{x_{11}, x_{12}, x_{13}, x_{1(n-1)}, x_{1n}\}) \cap D| \geq \frac{n-5}{2}$, $|Y_1 \cap D| \geq \frac{n-5}{2} + 4 = \frac{n+3}{2}$.

(1.3) $n \equiv 2 \pmod 4$.
By $|(Y_1 \setminus \{x_{11}, x_{12}, x_{13}, x_{1(n-2)}, x_{1(n-1)}, x_{1n}\}) \cap D| \geq \frac{n-6}{2}$, $|Y_1 \cap D| \geq \frac{n-6}{2} + 4 = \frac{n+2}{2}$.

(1.4) $n \equiv 3 \pmod 4$.
Recall that $\{x_{11}, x_{12}, x_{1(n-1)}, x_{1n}\} \subset D$. Furthermore, by (F1), $|\{x_{13}, x_{14}, x_{15}\} \cap D| \geq 1$. By (F3), $|(Y_1 \setminus \{x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{1(n-1)}, x_{1n}\}) \cap D| \geq \frac{n-7}{2}$. Thus $|Y_1 \cap D| \geq \frac{n-7}{2} + 5 = \frac{n+3}{2}$.

We will complete the proof by considering all cases dependent on $x_{11}, x_{1n}, x_{m1}, x_{mn}$ in D or not.

(1) If $\{x_{11}, x_{1n}, x_{m1}, x_{mn}\} \cap D = \emptyset$, then

$$|D \cap (X_1 \cup X_n \cup Y_1 \cup Y_m)| \geq 2(\frac{n}{2} + \frac{m}{2}) = m + n$$

according to Case 1.

(2) If $|\{x_{11}, x_{1n}, x_{m1}, x_{mn}\} \cap D| = 1$, w.l.o.g., assuming that $x_{m1} \in D$, then

$$|D \cap (Y_1 \cup Y_m)| \geq \min\{\frac{n}{2} + \frac{n+2}{2}, \frac{n+1}{2} + \frac{n+3}{2}, \frac{n+2}{2} + \frac{n+2}{2}, \frac{n+1}{2} + \frac{n+1}{2}\} = n + 1$$

according to Cases 1 and 2. Similarly, $|D \cap (X_1 \cup X_n)| \geq m + 1$. Since x_{m1} is counted twice, we have $|D \cap (X_1 \cup X_n \cup Y_1 \cup Y_m)| \geq m + n + 1$.

(3) If $|\{x_{11}, x_{1n}, x_{m1}, x_{mn}\} \cap D| = 2$, then there are two possible subcases to be considered up to isomorphism. If $\{x_{m1}, x_{mn}\} \subset D$, then

$$|D \cap (Y_1 \cup Y_m)| \geq \min\{\frac{n}{2} + \frac{n+4}{2}, \frac{n+1}{2} + \frac{n+3}{2}, \frac{n+2}{2} + \frac{n+2}{2}, \frac{n+1}{2} + \frac{n+3}{2}\} = n + 2$$

according to Cases 1 and 3, and $|D \cap (X_1 \cup X_n)| \geq m + 1$ by Case 2. Noting that either of x_{m1}, x_{mn} is counted twice, $|D \cap (X_1 \cup X_n \cup Y_1 \cup Y_m)| \geq m + n + 1$. If $\{x_{m1}, x_{1n}\} \subset D$, then

$$|D \cap (Y_1 \cup Y_m)| \geq n + 1 \quad \text{and} \quad |D \cap (X_1 \cup X_n)| \geq m + 1$$

according to Case 2. Since either of x_{m1}, x_{1n} is counted twice, $|D \cap (X_1 \cup X_n \cup Y_1 \cup Y_m)| \geq m + n$.

(4) If $|\{x_{11}, x_{1n}, x_{m1}, x_{mn}\} \cap D| = 3$, then we may assume that $x_{11} \notin D$ by symmetry. By Cases 2 and 3,

$$|D \cap (Y_1 \cup Y_m)| \geq \min\{\frac{n+2}{2} + \frac{n+4}{2}, \frac{n+3}{2} + \frac{n+3}{2}, \frac{n+2}{2} + \frac{n+2}{2}, \frac{n+1}{2} + \frac{n+3}{2}\} = n + 2,$$

and similarly $|D \cap (X_1 \cup X_n)| \geq m + 2$. Noting that each of x_{1n}, x_{m1}, x_{mn} is counted twice, we have $|D \cap (X_1 \cup X_n \cup Y_1 \cup Y_m)| \geq m + n + 1$.

(5) If $\{x_{11}, x_{1n}, x_{m1}, x_{mn}\} \subset D$, then

$$|D \cap (Y_1 \cup Y_m)| \geq n + 2 \quad \text{and} \quad |D \cap (X_1 \cup X_n)| \geq m + 2$$

by Case 3. Since each of $x_{11}, x_{1n}, x_{m1}, x_{mn}$ is counted twice, $|D \cap (X_1 \cup X_n \cup Y_1 \cup Y_m)| \geq m + n$.

□

Next, we will give bounds for $\gamma_{\times 2,t}(P_m \Box P_n)$ when $m \geq 4$. When $m = 4$, it is stated as the following theorem.

Theorem 2.7. *Let $n \geq 2$ be an integer. Then*

$$\frac{9n}{4} + 1 \leq \gamma_{\times 2,t}(P_4 \Box P_n) \leq \begin{cases} \frac{12n}{5} + 2, & \text{if } n \equiv 0 \ (mod\ 5); \\ \frac{12n}{5} + \frac{18}{5}, & \text{if } n \equiv 1 \ (mod\ 5); \\ \frac{12n}{5} + \frac{16}{5}, & \text{if } n \equiv 2 \ (mod\ 5); \\ \frac{12n}{5} + \frac{14}{5}, & \text{if } n \equiv 3 \ (mod\ 5); \\ \frac{12n}{5} + \frac{12}{5}, & \text{if } n \equiv 4 \ (mod\ 5). \end{cases}$$

Proof. Let $G = P_4 \Box P_n$ with $P_4 = abcd$ and $P_n = v_1 v_2 \ldots v_n$, where $n \geq 2$. Set $X_i = \{a_i, b_i, c_i, d_i\}$ for $1 \leq i \leq n$, $Y_1 = \cup_{i=1}^{n} a_i$ and $Y_4 = \cup_{i=1}^{n} d_i$.

We firstly prove the upper bound by constructing a DTDS of G. For integer $k \geq 0$, set

$$X = \left(\cup_{i=0}^{k-1} X_{5i+1}\right) \bigcup \left(\cup_{i=0}^{k-1} X_{5i+4}\right) \bigcup \left(\cup_{i=0}^{k-1} \{a_{5i+2}, a_{5i+3}, d_{5i+2}, d_{5i+3}\}\right).$$

Clearly, $|X| = 12k$. Next, we give a DTDS, denoted by S, of G according to the value of n.

When $n = 5k$, set $S = X \cup \{b_{5k}, c_{5k}\}$. Thus $|S| = 12k + 2 = \frac{12n}{5} + 2$. Therefore $\gamma_{\times 2,t}(G) \leq |S| = \frac{12n}{5} + 2$.

When $n = 5k + 1$, set $S = X \cup X_{5k+1} \cup \{a_{5k}, d_{5k}\}$. Then $\gamma_{\times 2,t}(G) \leq |S| = 12k + 6 = \frac{12n+18}{5}$.

When $n = 5k + 2$, set $S = X \cup X_{5k+1} \cup X_{5k+2}$. So $\gamma_{\times 2,t}(G) \leq |S| = 12k + 8 = \frac{12n+16}{5}$.

When $n = 5k + 3$, set $S = X \cup X_{5k+1} \cup X_{5k+3} \cup \{a_{5k+2}, d_{5k+2}\}$. Then $\gamma_{\times 2,t}(G) \leq |S| = 12k + 10 = \frac{12n+14}{5}$.

Finally, when $n = 5k + 4$, set

$$S = X \cup X_{5k+1} \cup X_{5k+4} \cup \{a_{5k+2}, a_{5k+3}, d_{5k+2}, d_{5k+3}\}.$$

Hence

$$\gamma_{\times 2,t}(G) \leq |S| = 12k + 12 = \frac{12n + 12}{5}.$$

Next, let D be a minimum DTDS in G and $D' = V(G) \setminus D$. We will prove the lower bound by counting the edges between D and D'. Set $W = X_1 \cup X_n \cup Y_1 \cup Y_4$. By Lemma 2.6, $|D \cap W| \geq n + 4$. Note that each vertex in $D \cap W$ has at most one neighbor in D', and each one in $D \setminus W$ has at most two neighbors in D'. So

$$e(D, D') = e(D \cap W, D') + e(D \setminus W, D') \leq |D \cap W| + 2|D \setminus W| = 2|D| - |D \cap W| \leq 2|D| - (n + 4).$$

Then

$$2(4n - |D|) = 2|D'| \leq e(D, D') \leq 2|D| - (n + 4).$$

Hence we have $|D| \geq \frac{9n}{4} + 1$. □

Now, we consider the bounds of $\gamma_{\times 2,t}(P_m \Box P_n)$ for $m \geq 5$ and $n \geq 5$.

Theorem 2.8. *For integers $m \geq 5$, $n \geq 5$,*

$$\frac{mn}{2} + \frac{m+n}{4} \leq \gamma_{\times 2,t}(P_m \Box P_n) \leq \begin{cases} \frac{mn}{2} + \frac{m+n+2}{2}, & \text{if } n \equiv 0, 2 \ (mod\ 4); \\ \frac{mn}{2} + \frac{m+n+3}{2}, & \text{if } n \equiv 1 \ (mod\ 4); \\ \frac{mn}{2} + \frac{m+n+1}{2}, & \text{if } n \equiv 3 \ (mod\ 4). \end{cases}$$

Proof. Let $G = P_m \Box P_n$ with $P_m = u_1 u_2 \ldots u_m$, $P_n = v_1 v_2 \ldots v_n$, where $m \geq 5$, $n \geq 5$. For the vertex $u_i \in V(P_m)$ and $v_j \in V(P_n)$, we denote simply the vertex (u_i, v_j) by x_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$. For each $1 \leq i \leq m$ and $1 \leq j \leq n$, we denote $X_j = \cup_{i=1}^{m} x_{ij}$, $Y_i = \cup_{j=1}^{n} x_{ij}$. Let D be a minimum DTDS of G, and $D' = V(G) \setminus D$.

Let $W = X_1 \cup X_n \cup Y_1 \cup Y_m$. By Lemma 2.6, $|D \cap W| \geq m + n$. Counting the edges between D and D', we have $2(mn - |D|) = 2|D'| \leq e(D, D') = e(D \cap W, D') + e(D \setminus W, D') \leq |D \cap W| + 2|D \setminus W| = 2|D| - |D \cap W| \leq 2|D| - (m + n)$. Then $\gamma_{\times 2,t}(P_m \Box P_n) = |D| \geq \frac{mn}{2} + \frac{m+n}{4}$.

To prove the upper bounds, for integer $k \geq 1$, set

$$X = \left(\cup_{i=0}^{k-1} X_{4i+1}\right) \bigcup \left(\cup_{i=0}^{k-1} X_{4i+3}\right) \bigcup \left(\cup_{i=0}^{k-1} \{x_{1(4i+2)}, x_{m(4i+2)}\}\right).$$

Then $|X| = 2mk + 2k$. Next, we give a DTDS of G for each of the possible cases to complete the proof.

Case 1. $n = 4k$.

Let

$$A = \cup_{i=0}^{\lfloor \frac{m}{4} \rfloor - 1} \{x_{(4i+2)4k}, x_{(4i+3)4k}\},$$

set $S = X \cup A \cup \{x_{(m-2)4k}, x_{(m-1)4k}\}$. When $m \equiv 0$ $(mod\ 4)$, $\{x_{(m-2)4k}, x_{(m-1)4k}\} \subseteq A$, so $|A \cup \{x_{(m-1)4k}, x_{(m-2)4k}\}| = \frac{m}{2}$. When $m \equiv 1$ $(mod\ 4)$, $x_{(m-2)4k} \in A$, so $|A \cup \{x_{(m-1)4k}, x_{(m-2)4k}\}| = \frac{m+1}{2}$. When $m \equiv 2$ $(mod\ 4)$, $|A \cup \{x_{(m-1)4k}, x_{(m-2)4k}\}| = \frac{m+2}{2}$. When $m \equiv 3$ $(mod\ 4)$, $|A \cup \{x_{(m-1)4k}, x_{(m-2)4k}\}| = \frac{m+1}{2}$. Hence, $|A \cup \{x_{(m-1)4k}, x_{(m-2)4k}\}| \leq \frac{m+2}{2}$. Clearly, each vertex in $V(G) \setminus X_{4k}$ has at least two neighbors in S. For any vertex $x \in X_{4k}$, x has at least one neighbor in $S \cap X_{4k}$. Noting that $X_{4k-1} \subset S$, each vertex in X_{4k} has at least two neighbors in S. That is to say, S is a DTDS of G. Thus

$$\gamma_{\times 2,t}(G) \leq |S| \leq 2mk + 2k + \frac{m+2}{2} = \frac{mn}{2} + \frac{m+n+2}{2}.$$

Case 2. $n = 4k + 1$.

Set $S = X \cup X_{4k+1} \cup \{x_{1(4k)}, x_{m(4k)}\}$. Then S is a DTDS of G. So $\gamma_{\times 2,t}(G) \leq |S| = 2mk + 2k + m + 2 = \frac{mn}{2} + \frac{m+n+3}{2}$.

Case 3. $n = 4k + 2$.

Let

$$B = X_{4k+1} \cup (\cup_{i=0}^{\lfloor \frac{m}{4} \rfloor - 1} \{x_{(4i+1)(4k+2)}, x_{(4i+2)(4k+2)}\}).$$

When $m \equiv 0$ $(mod\ 4)$, set $S = X \cup B \cup \{x_{(m-1)(4k+2)}, x_{m(4k+2)}\}$. Then $|S| \leq |X| + m + \frac{m}{2} + 2 = |X| + \frac{3m+4}{2}$.
When $m \equiv 1$ $(mod\ 4)$, set $S = X \cup B \cup \{x_{(m-1)(4k+2)}, x_{m(4k+2)}\}$. Then $|S| \leq |X| + m + \frac{m-1}{2} + 2 = |X| + \frac{3m+3}{2}$.
When $m \equiv 2$ $(mod\ 4)$, set $S = X \cup B \cup \{x_{(m-1)(4k+2)}, x_{m(4k+2)}\}$. Then $|S| \leq |X| + m + \frac{m-2}{2} + 2 = |X| + \frac{3m+2}{2}$.

When $m \equiv 3\ (mod\ 4)$, set $S = X \cup B \cup (\cup_{i=m-2}^{m}\{x_{i(4k+2)}\})$. Then $|S| \leq |X| + m + \frac{m-3}{2} + 3 = |X| + \frac{3m+3}{2}$. In each of these cases, we have

$$|S| \leq |X| + \frac{3m+4}{2} = 2mk + 2k + \frac{3m+4}{2} = \frac{mn}{2} + \frac{m+n+2}{2}.$$

Next, we show that S is a DTDS of G. Clearly, for each vertex $x \in V(G) \setminus (X_{4k+1} \cup X_{4k+2})$, x has at least two neighbors in S. Noting that $X_{4k+1} \subseteq S$, each vertex in X_{4k+1} has two neighbors in X_{4k+1} except $x_{1(4k+1)}$ and $x_{m(4k+1)}$. Also, $x_{1(4k+1)}$ ($x_{m(4k+1)}$) has another neighbor $x_{1(4k+2)}$ ($x_{m(4k+2)}$) in S. For any vertex $x \in X_{4k+2}$, x has one neighbor in X_{4k+1} and at least one neighbor in $S \cap X_{4k+2}$. Then each vertex in $X_{4k+1} \cup X_{4k+2}$ has at least two neighbors in S. Therefore, S is a DTDS of G. Then

$$\gamma_{\times 2,t}(G) \leq |S| = \frac{mn}{2} + \frac{m+n+2}{2}.$$

Case 4. $n = 4k + 3$.
Set $S = X \cup X_{4k+1} \cup X_{4k+3} \cup \{x_{1(4k+2)}, x_{m(4k+2)}\}$. Clearly, S is a DTDS of G. Thus

$$\gamma_{\times 2,t}(G) \leq |S| = 2mk + 2k + 2m + 2 = \frac{mn}{2} + \frac{m+n+1}{2}.$$

□

3. Conclusions

In the paper, the values of $\gamma_{\times 2,t}(P_i \square P_n)$ for $i = 2, 3$ are determined. For $\gamma_{\times 2,t}(P_4 \square P_n)$, we give lower and upper bounds with a gap no more than $\frac{3}{20}n + \frac{13}{5}$ and, for $\gamma_{\times 2,t}(P_m \square P_n)$ with $m, n \geq 5$, we give lower and upper bounds with a gap at most $\frac{m+n}{4} + \frac{3}{2}$.

The lower bounds in Theorem 2.7 and Theorem 2.8 could be improved if one may analyze the adjacent structures of DTDSs of $P_m \square P_n$ more carefully according to definition of the double total domination. For example, it is easy to verify that $\gamma_{\times 2,t}(P_4 \square P_4) = 12$, that attains the upper bound in Theorem 2.7 for the case $n = 4$. Moreover, Figure 5(a) demonstrates that the lower bound of $\gamma_{\times 2,t}(P_5 \square P_5)$ could be improved to 18. (For an arbitrary DTDS D of $P_5 \square P_5$, each of the solid circle and the dash curves in Figure 5(a) covers at least two vertices of D.) In Figure 5(b), we give a DTDS to show that the value of $\gamma_{\times 2,t}(P_5 \square P_5)$ is exactly 18, that is greater than the lower bound in Theorem 2.8 for the case $m = n = 5$.

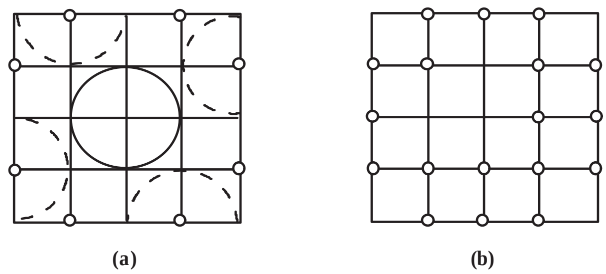

Figure 5. (a) Any DTDS of $P_5 \square P_5$ contains at least 18 vertices. (b) A DTDS with 18 vertices.

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Conflict of interest

The authors declare no conflicts of interest.

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