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Research article

On submodule transitivity of QTAG-modules

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Abstract: In this paper, we generalize a suitable transformation from an element-based to a submodule-based interpretation of the traditional idea of transitivity in QTAG modules. We examine QTAG modules that are transitive in the sense that the module has an automorphism that sends one isotype submodule K onto any other isotype submodule K', unless this is impossible because either the submodules or the quotient modules are not isomorphic. Additionally, the classes of strongly transitive and strongly U-transitive QTAG modules are defined using a slight adaptations of this. This work investigates the latter class in depth, demonstrating that every α- module is strongly transitive with regard to countably generated isotype submodules.

Keywords: QTAG modules; nice submodules; totally projective module; isotype submodule; Ulm-Kaplansky invariants
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1. Introduction

The idea of Abelian p-groups has been broadly generalized by a number of scholars who are interested in module theory. Many group notions, such as purity, projectivity, injectivity, height etc., have been generalized for modules. Since 1976, the concept of TAG modules and their related features have received considerable interest among several generalizations of torsion Abelian groups (see, e.g., [1, 2]).

The results for groups that are not true for modules in general are generalized for modules by applying conditions on modules or underlying rings or both. We applied the condition on modules that every finitely generated submodule of every homomorphic image of the module be a direct sum of uniserial modules, while the rings are associative with unity (called QTAG modules, see [3]). By applying this condition, several interseting group results may be established for QTAG modules, which are not true for modules in general.

The concept of transitivity for Abelian groups dates back to Kaplansky [4]. Different aspects of transitivity have been studied by many authors, and it is a hot topic in current research on Abelian groups; for more details one can go through [5–8] etc. Recently, in [5], the authors defined a new class of transitivity called as H-fully transitive using the height valuation from group G, which is an extension of Kaplansky's [4] standard definition of fully transitive groups. Several authors generalize these interseting topics to modules. In the recent developement for the same one can go through [9, 10]. In [10], authors defined two new concepts of transitivity and named them as quotient transitivity and cyclic submodule transitivity using a new approach which is based on isomorphism of quotients and makes no use of height sequences. In this paper, we focus our attention to generalize the case of subgroup transitivity to the submodule transitivity in QTAG-modules via isotype submodules. The motivation comes from the paper [11] and we generalize many results here from the same.

2. Preliminaries

One of the co-authors' earlier works, which are provided as quotes and suitably referenced here, give some of the fundamental concepts that are utilized in this study and it is prerequisite to understand the subsequence sections of this paper.

"All the rings R considered here are associative with unity, and modules M are unital QTAG-modules. An element $x \in M$ is uniform if xR is a non-zero uniform (hence uniserial) module, and for any R-module M with a unique composition series, $d(M)$ denotes its composition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\left\{d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. For an ordinal σ, a submodule N of M is said to be σ-pure, if $H_\beta(M) \cap N = H_\beta(N)$ for all $\beta \leq \sigma$ [12], and a submodule N of M is said to be an isotype in M, if it is σ-pure for every ordinal σ [13]. Let M be a module. Then, the sum of all simple submodules of M is called the socle of M and is denoted by $\mathrm{Soc}(M)$. If M and M' are QTAG modules then a homomorphism $f : M \to M'$ is an isometry if it is 1-1, onto and $H_{M'}(f(x)) = H_M(x)$, for all $x \in M$. A submodule N of a QTAG-module M is a nice submodule if every nonzero coset $a + N$ is proper with respect to N, i.e., for every nonzero $a + N$ there is an element $b \in N$ such that $H_M(a + b) = H_{M/N}(a + N)$.

A family $\mathcal{N}$ of submodules of M is called a nice system in M if

(i) $0 \in \mathcal{N}$;
(ii) $\{N_i\}_{i\in I}$ is any subset of $\mathcal{N}$, then $\Sigma_I N_i \in \mathcal{N}$;
(iii) Given any $N \in \mathcal{N}$ and any countable subset X of M, there exists $K \in \mathcal{N}$ containing $N \cup X$ such that K/N is countably generated [14].

Every submodule in a nice system is a nice submodule. An h-reduced QTAG module M is called totally projective if it has a nice system."

The notations and terminology are widely used and adhered to [15, 16].

3. Submodule transitivity in QTAG-modules

Hasan [17] extended the concept of transitivity for groups to QTAG modules with the help of U-sequences of their elements.

"A QTAG module M is fully transitive if for $x, y \in M$, $U(x) \leq U(y)$, there is an endomorphism f of M such that $f(x) = f(y)$ and it is transitive if for any two elements $x, y \in M$, with $U(x) \leq U(y)$, there is an automorphism f of M such that $f(x) = f(y)$. A QTAG-module M is strongly transitive if for $x, y \in M$, $U(x) = U(y)$, there exists an endomorphism f of M such that $f(x) = y$." Transitivity of modules have been further generalized and different notions of transitivity have been introduced by Hasan [17].

There are examples when $x, y \in M$ and there is a height-preserving isomorphism from xR to yR but no height-preserving isomorphism from $\frac{M}{xR}$ to $\frac{M}{yR}$. This motivates us to study submodule transitivity.

We start with the following:

Definition 3.1. *The submodules N, K of a QTAG module M are said to be equivalent if there exists a height-preserving automorphism f of M such that $f(N) = K$. These submodules are compatible if for any $x \in N$, $y \in K$, there exists $z \in N \cap K$ such that $H(x + z) \geq H(x + y)$.*

Remark 3.1. *If two submodules $N, K \subseteq M$ are equivalent, then $N \simeq K$, $M/N \simeq M/K$ and the isomorphisms are height preserving. For a coset $x + N$, we define the height of a coset as $sup\{H(x + y) + 1 | \ y \in N\}$.*

To proceed further we need the following:

Definition 3.2. *A QTAG module M is transitive with respect to isotype submodules if for any two isotype submodules $N, K \subseteq M$, $N \simeq K$, $\frac{M}{N} \simeq \frac{M}{K}$ with height-preserving isomorphisms.*

Although we have concentrated on transitivity up to this point, it is probably fair to argue that our key finding is the following theorem. Under plausible premises, this theorem argues that a map may be extended in such a way that it concurrently lifts a given quotient map.

Theorem 3.1. *Let M/N and M'/N' be totally projective QTAG modules with nice submodules N, N' of the modules M, M' respectively, and let K, K' be isotype submodules of M, M' containing N, N', respectively. Then, each height-preserving isomorphism $\phi : N \to N'$ extends to an isomorphism ψ from M onto M' with $\psi(K) = K'$ if and only if the Ulm-Kaplansky invariants of K relative to N are same as the Ulm-Kaplansky invariants of K' relative to N' and there is a height preserving isomorphism $\rho : M/K \to M'/K'$. If the above conditions are satisfied then ψ can be chosen to induce ρ.*

Proof. Without losing broader implications, we suppose that $M = M'$. Consider a totally projective module T and $M \oplus T$. Now, $M \oplus T \simeq M' \oplus T$, and the isomorphism between M/K and M'/K is preserved. For all ordinals α, we may consider the decomposition $Soc(H_\alpha(K)) = Soc(H_{\alpha+1}(K) \oplus S_\alpha$ and $Soc(H_\alpha(K') = Soc(H_{\alpha+1}(K') \oplus S'_\alpha)$. Now we may put $S_\alpha(N) = \{x \,|x \in S_\alpha, H(x + N) > \alpha + 1\}$ and $S'_\alpha(N') = \{x' \,|x' \in S_\alpha, H(x' + N') > \alpha + 1\}$. For each α, ϕ induces an isomorphism $\phi_\alpha : S_\alpha(N) \to S'_\alpha(N')$ where $H(x + N) > \alpha + 1$ if and only if $H(\phi_\alpha(x) + N') > \alpha + 1$. Since the Ulm-Kaplansky invariants of K

relative to N are the same as the Ulm-Kaplansky invariants of K' relative to N', we have $g\left(\frac{S_\alpha}{S_\alpha(N)}\right) = g\left(\frac{S'(\alpha)}{S'_\alpha(N)}\right)$, and we have isomorphism $\sigma_\alpha : S_\alpha \to S'_\alpha$ that extends the $\phi'_\alpha s$.

Consider the family $\mathcal{A}$ consisting of all the triples (P, Q, f), where P and Q are nice submodules of M containing N and N', respectively, satisfying the following conditions:

(i) f is a height-preserving isomorphism extending ϕ.
(ii) $f(x) + K' = P(x + K)$ for all $x \in P$
(iii) If $y \in S_\beta, x \in P$, then, $H(x + y) > \beta + 1$ if and only if $H(\sigma_\alpha(x) + f(y)) > \beta + 1$.

Thus, $(N, N', \phi) \in \mathcal{A}$. It is sufficient to show that if $P' = P + zR$ is an extension of P and $(P, Q, f) \in \mathcal{A}$, then there exists Q' and P' such that $(P', Q', f') \in \mathcal{A}$ with f' extending f. Suppose $H_(z) = \alpha + 1$ and z is proper with respect to P. We then have to find $z' \in M$ such that

(i) $H(z') = \alpha + 1$;
(ii) $u' = f(u)$ where $d\left(\frac{z'R}{u'R}\right) = d\left(\frac{zR}{uR}\right) = 1$;
(iii) z' is proper with Q;
(iv) $z' + K' = \rho(z + K)$, and
(v) if $x \in S_\alpha, y \in P$, then $H(x + z + y) > \alpha + 1$ if and only if $H(\sigma_\alpha(x) + z' + f(y)) > \alpha + 1$.

To ensure the existence of z', we have to consider two cases.

Case (i): $H(u) > \alpha + 2$ and $H(z + K) > \alpha + 1$. Since ρ is height-preserving $\rho(z + K) = v + K'$, where $H(v) > \alpha + 1$. Then, $v' - f(u) \in K' \cap H_{\alpha+2}(M)$, and therefore $f(w) = w'$, where $d\left(\frac{wR}{w'R}\right) = 1$ for some $w = v + t$, $t \in H_{\alpha+1}(K')$. We may replace $v + t$ with v, and then we have $w' = f(u)$. Since $H(z + K) > \alpha + 1$, $H(z + a) > \alpha + 1$ for some $a \in K$. Now $H(a) = \alpha + 1$ and $H(a_1) > \alpha + 2$, where $d\left(\frac{aR}{a_1R}\right) = 1$. Thus, $a_1R = a'_1R$, where $a'_1 \in H_{\alpha+1}(K)$. We have $a - a' \in Soc(H_\alpha(k))$, and there is some $b \in H_{\alpha+1}(M)$ such that $x' = z - b \in S_\alpha$. Let $z' = v + \sigma_\alpha(x')$, then (i), (ii) and (iii) hold this choice of z'. Since z is proper with respect to P and $b \in H_{\alpha+1}(M)$, x' is also proper with respect to $Q = f(P)$. Therefore, $\sigma_\alpha(x')$ is also proper with respect to $Q = f(P)$, and z' satisfies (iii), as $v \in H_{\alpha+1}(M)$. Now, z and z' ensure that (v) holds. For $x \in S_\alpha$ and $c \in P$, $H(x+z+c) > \alpha+1$, if and only if $H(x+x'+c) > \alpha+1$ and $H(\sigma_\alpha(x) + z' + f(c)) > \alpha + 1$ if and only if $H(\sigma_\alpha(x + x') + f(c)) > \alpha + 1$.

Either of the two inequalities in (v) imply that $H(y) \geq \alpha+1$ and $H(z_1) > \alpha+2$, where $d\left(\frac{(z + y)R}{z'R}\right) = 1$. Therefore, (i) and (ii) hold for $z+y$. If the first inequality in (v) holds, then, $H(z+y+k) \geq H(\sigma_\alpha(x)+z+y) > \alpha + 1$, while if the second is satisfied, then, $H(z + y + K) = H(P(z + y) + K') = H(z' + f(y) + K') \geq H(\sigma_\alpha(x) + z' + f(y)) > \alpha + 1$.

Case (ii): Now we are able to handle case (ii), when $H(u) = \alpha + 2$ or $H(z + k) \geq \alpha + 1$. Since ρ is height-preserving $\rho(z + K) = b + K'$ for some $b \in H_\alpha(M)$. Then, $b' - f(u) \in K' \cap H_{\alpha+1}(M) = H_{\alpha+1}(K')$, and therefore $H(z + K) = H(\rho(z + K)) = H(z' + K') \geq H(z')$, implying that $H(z') = \alpha + 1$. Since z' is not proper with respect to Q, this becomes the case (i). If $H(z' + f(y)) > \alpha + 1$ for some $y \in P$, then we may replace z with $z + y$. Thus, we may assume that z' satisfies (iii), and we only have to prove (v). If we replace z with $z + y$, then again by case (i) the inequalities of (v) are satisfied.

The immediate implication of Theorem 3.1 is as follow:

Theorem 3.2. *Let N and K be isotype submodules of a totally projective QTAG module M. If N, K have the same Ulm-Kaplansky invariants and there is a height-preserving isomorphism f from M/N onto M/K, then there is an automorphism $\bar{f}$ of M that maps N onto K. Moreover, if $f : M/N \to M/K$ is a height-preserving isomorphism, there is an automorphism $\bar{f}$ that induces f.*

These results suggest two different variations of the concept of transitivity. We define U-transitivity where the submodules N, K may not be isomorphic but they have same the Ulm-Kaplansky invariants.

Definition 3.3. *A QTAG module M is U-transitive if any two isotype submodules N, K having the same Ulm-Kaplansky invariants with $M/N \simeq M/K$, correspond under some automorphism of M.*

For the other variation, we need the following:

Definition 3.4. *Two equivalent submodules N, K of M are strongly equivalent if for any height preserving isomorphism $\bar{f} : M/N \to M/K$, there is an automorphism f of M that induces $\bar{f}$. A QTAG module is strongly U-transitive if every pair of isotype submodules N, K are strongly equivalent whenever they have the same Ulm-Kaplansky invariants and there exists a height-preserving isomorphism between M/N and M/K.*

The argument of the following claim demonstrates the importance of the additional strong U-transitivity criterion that is missing from U-transitivity.

Proposition 3.1. *A direct summand of a strongly U-transitive QTAG module is again strongly U-transitive.*

Proof. Suppose M is strongly U-transitive and $M = N \oplus K$. Let N_1, N_2 be isotype submodules of N such that N_1, N_2 have the same Ulm-Kaplansky invariants and there exists a height-preserving isomorphism $f : N/N_1 \to N/N_2$. Since N is the summand of M, it is an isotype submodule of M, and therefore $H_N(x + N_1) = H_M(x + N_1)$. Now, f can be extended to $\bar{f} : M/N_1 \to M/N_2$ such that $\bar{f}(x + N_1) = x + f(N_1)$ when $x \in K$, and $\bar{f}$ is again height preserving. Since M is strongly U-transitive, there is an automorphism ϕ of M that induces $\bar{f}$. Thus, $\phi(N_1) = N_2$ but $\phi(N) = N$ because $\bar{f}|N_1/N_2 = f$. Therefore, restriction of ϕ to N is an automorphism of N that induces f, and N is strongly U-transitive.

The previous demonstration does not hold up for U-transitivity, as we previously proposed, nor does it hold up if we utilize the notion of transitivity of isotype submodules based on Definition 3.2. In both instances, the strong versions are required.

Theorem 3.3. *Let N, N' be nice submodules of M and M', respectively, where M/N and M'/N' are totally projective. Let K and K' be isotype submodules of M and M', respectively, such that N and K are compatible and N' and K' are compatible. Suppose $f_\alpha(K, N \cap K) = f_\alpha(K', N' \cap K')$ for all ordinals α and $\phi : \frac{M}{K} \to \frac{M'}{K'}$ and $\rho : N \to N'$ are height preserving-isomorphism for which $\rho(a) + K' = \phi(a + K)$ $\forall a \in N$. Then, there exists an isomorphism $\psi : M \to M'$ that lifts ϕ and extends ρ.*

Proof. Let $\phi : \frac{M}{K} \to \frac{M'}{K'}$ and $\rho : N \to N'$ be height-preserving isomorphisms such that $\rho(a) + K' = \phi(a + K)$. Now, $a \in N \cap K$ if and only if $\rho(a) \in N' \cap K'$.

For each α, let

$$SocH_\alpha(K) = S_\alpha \oplus Soc(H_{\alpha+1}(K)),$$
$$SocH_\alpha(K') = S'_\alpha \oplus Soc(H_{\alpha+1}(K')).$$

We may define

$$S_\alpha(N \cap K) = \{x \in S_\alpha \,|H(x+z) > \alpha \text{ for some } z \in N \cap K\},$$
$$\text{and } S'_\alpha(N' \cap K') = \{x' \in S'_\alpha \,|H(x'+z') > \alpha \text{ for some } z' \in N' \cap K'\}.$$

The restriction of the isomorphism $\rho : N \to N'$ to an isomorphism between $N \cap K$ and $N' \cap K'$ induces an isomorphism $\phi_\alpha : S_\alpha(N \cap K) \to S'_\alpha(N' \cap K')$. If $x \in S_\alpha$ and $H(x+z) > \alpha$ with $z \in N \cap K$, then, $z_1 \in H_{\alpha+2}(K)$ where $d\left(\frac{zR}{z_1R}\right) = 1$ because $x \in Soc(M)$ and K is the isotype. If we put $z' = \rho(z)$, then, $H(z') = \alpha = H(z)$. Since $\rho(N \cap K) = N' \cap K'$, $z' \in N' \cap K'$ and $H(z'_1) = H(z_1) = H(x_1) \geq \alpha + 2$. Here, $d\left(\frac{(x+z)R}{x_1R}\right) = 1$ and $d\left(\frac{z'R}{z'_1R}\right) = 1$. Therefore, $z'_1 = u_1$, where $u' \in H_{\alpha+1}(K')$ and $d\left(\frac{u'R}{u'_1R}\right) = 1$. Hence, $z' - u' \in Soc(H_\alpha(K'))$ and $u' - z' = x' + v'$, where $x' \in S'_\alpha$ and $v' \in Soc(H_{\alpha+1}(K'))$. Now, $x' + z' \in Soc(H_\alpha(K'))$. Therefore, $H(x' + z') > \alpha$ and $x' \in (N' \cap K')$, and thus the mapping that maps x onto z' is the required isomorphism from $S_\alpha(N \cap K)$ to $S'_\alpha(N \cap K)$.

Since the Ulm-Kaplansky invariants of K with respect to $N \cap K$ are the same as those of K' with respect to $N' \cap K'$, the isomorphism from $S_\alpha(N \cap K)$ to $S'_\alpha(N' \cap K')$ can be extended to an isomorphism $\phi_\alpha : S_\alpha \to S'_\alpha$. If $x \in S_\alpha$, then $H(x+z) > \alpha$ for $z \in N \cap K$ iff $H(\phi_\alpha(x) + \rho(z)) > \alpha$. We must demonstrate that every element of N satisfies this requirement. Suppose for $x \in S_\alpha$, $b \in N$ and $H(x+b) > \alpha$. Since N and K are compatible and $x \in K, H(x+a) > \alpha$ for some $a \in N \cap K$. Based on the above, we have $H(\phi_\alpha(x) + \rho(a)) > \alpha$. Again, $H(z+b) > \alpha$ and $H(z+a) > \alpha$, therefore, $H(b-a) > \alpha$ and $H(\rho(b) - \rho(a)) > \alpha$. Hence, $H(\phi_\alpha(x) + \rho(b)) > \alpha$. We can say that $H(x+b) > \alpha$ if and only if $H(\phi_\alpha(x) + \rho(b)) > \alpha$.

Suppose P and Q are nice submodules of M and M', respectively, and $f : P \to Q$ is an isomorphism such that

(i) f is a height-preserving isomorphism that extends $\rho : N \to N'$;
(ii) $f(y) + K' = \phi(y + K)$ for all $y \in P$;
(iii) And for each α if $x \in S_\alpha$ and $y \in P$ such that $H(x+y) \geq \alpha+1$ if and only if $H(\phi_\alpha(x) + f(y) \geq \alpha+1$.

These conditions hold good if $P = N$, $Q = N'$ and $f = \rho$. Now P, Q are nice submodules of M and M', and N, N', respectively, and M/N, M'/N' are totally projective. Since nice submodules of M/N correspond to nice submodules of M and a totally projective module has a collection $\mathcal{A}$ of nice submodules, which is closed with respect to the union of modules and has the property that every countably generated submodule is contained in a countably generated module in $\mathcal{A}$. Now, it is sufficient to show that for any $z \in M$, which is proper with respect to P such that $z_1 \in P$, $d\left(\frac{zR}{z_1R}\right) = 1$, the isomorphism $f : P \to Q$ can be extended to $\bar{f} : P + zR \to Q + yR$ for some $y \in M'$, which satisfies the conditions (i), (ii) and (iii). We shall consider the two cases separately. Let $H(x) = \beta$. We shall consider the following two cases:

(i) When $H(x_1), H(x+K) > \beta + 1$, here, $d\left(\frac{xR}{x_1R}\right) = 1$;

(ii) When $H(x_1) = H(x+K) = \beta + 1$.

In both cases we have to find $z' \in M'$ with the following properties:

(a) $H(z') = \beta$;
(b) $z'_1 = f(z_1)$, here $d\left(\frac{z'R}{z'_1}\right) = 1$;
(c) z' is proper with respect to Q;
(d) $z' + K' = \phi(z + K)$; and
(e) If $x \in S_\beta, t \in P$, then $H(z + x + t) \geq \beta + 1$ if and only if $H(\phi_\alpha(x) + z' + f(t)) \geq \beta + 1$.

Then, $f : P \to Q$ can be extended to $\bar{f} : P + zR \to Q + z'R$ if we put $\bar{f}(z) = z'$ and $\bar{f}$ satisfies (i) to (iii).

Case (i): We choose $z' \in M'$, such that $\phi(z+K) = z'+K'$. Since ϕ is height preserving, $H(z'+K') > \beta + 1$, and we may assume $H(z') > \beta$. Then, by condition (ii) on f, $\phi(z_1 + K) = f(z_1) + K' = z'_1 + K'$, where $d\left(\frac{z'R}{z'_1R}\right) = 1 = d\left(\frac{zR}{z_1R}\right)$. Hence, $f(z_1) - z'_1 \in K' \cap H_{\beta+2}(M) = H_{\beta+2}(K')$. Therefore, there exists $u' \in H_{\beta+1}(K')$ such that $u'_1 \in f(z_1) - z'_1$. Now, $f(z_1) = z'_1 + u'_1$, and by replacing z' with $z' + u'$, z' satisfies (b) and (d).

Since $H(z + K) > \beta + 1$, there exists $u \in K$ such that $H(z + u) \geq \beta + 1$. Therefore, $H(u) = H(z) = \beta$ and $H(z_1 + u_1) \geq \beta + 2(d\left(\frac{uR}{u'_1R}\right) = 1)$ and $H(z + u) \geq \beta + 2$, $H(u_1) \geq \beta + 2$. Now $u_1 = u^*_1$, where $d\left(\frac{u^*R}{u^*_1R}\right) = 1, u^* \in H_{\beta+1}(K)$, and we have $u^* - u \in SocH_\beta(K)$. If $u^* - u = c + d$. If $c \in S_\beta$, $d \in SocH_{\beta+1}(K)$, then, $c = u^* - u - d = z - ((z + u) - u^* + d) = z - w$, where $w \in H_{\beta+1}(M)$. If $x' = z' + \phi_\beta(c)$, then, x' satisfies the conditions (a), (b) and (d). Since z is proper with respect to P and $w \in H_{\beta+1}(M)$, $c = z - w$ is also proper with respect to P. Now, by condition (iii) on f with $\alpha = \beta$ implies that $\phi_\beta(c)$ is proper with respect to Q, and thus z' is also proper with respect to Q because $z' \in H_{\beta+1}(M')$, and (c) also holds.

To prove (e), consider $c^* \in S_\beta$ and $y \in P$. Now, $H(c^*+z+y) \geq \beta+1$ if and only if $H(c^*+c+y) \geq \beta+1$, as $z = c + w$. By condition (iii), $H(\phi_\beta(c^* + c) + f(y)) \geq \beta + 1$ if and only if $H(\phi_\beta(c^*) + z' + f(y)) \geq \beta + 1$ because $z' = \phi_\beta(c) + z'$.

Case (ii): Let $\phi(z + K) = z' + K'$, and we may assume that $H(z') \geq \beta$. Then, by condition (ii), $f(z_1) - z'_1 \in K' \cap H_{\beta+1}(M')$, so there exists $u' \in H_\beta(K')$ such that $f(z_1) = v'$, where $d\left(\frac{(z' + u')R}{v'R}\right) = 1$. If $y = z' + u'$, then, y satisfies (b) and (d).

Since $y \in H_\beta(M')$, $H(y) = \beta$ implies $H(y_1) = H(z_1) = \beta + 1$ or $H(y) \leq H(y + K') = H(z + K) = \beta + 1$, where $d\left(\frac{yR}{y_1R}\right) = 1$. Now, (a) holds. If y is not proper with respect to Q, then $H(y + f(x)) \geq \beta + 1$ for some $x \in P$. Consider $z^* = z + x$. Now, $H(z + x) \geq \beta$ and $H(x) = H(f(x)) = H(y) = \beta$. Since x is proper with respect to P, $H(z + x) \leq \beta$, i.e., $H(z^*) = \beta$ and z^* is proper with respect to P. Also, $H(z^*_1) = H(f(z^*_1)) = H(y_1 + f(z_1)) > \beta + 1$. Therefore, $H(z^* + K) = H(\phi(z^* + K)) = H(z' + f(x) + K') = H(y + f(x) + K') > H(y + f(x)) \geq \beta + 1$.

If (c) does not hold for y, then by case (i), we can find an element y^* corresponding to z^*.

Let $a \in S_\beta$ and $x \in P$. If $H(a + x + z) \geq \beta + 1$ or $H(\phi_\alpha(a) + y + f(z)) \geq \beta + 1$, then we consider $z^* = x + z$. Now, $H(z^*) = \beta$, $H(z^*_1) > \beta + 1$, $d\left(\frac{z^*R}{z^*_1R}\right) = 1$ and $H(z^* + K) > \beta + 1$, and thus we can use case (i), since $H(a) = H(\phi_\beta(a)) = \beta$, and either $H(z + x) = \beta$ or $H(y + f(x)) = \beta$. In the latter case,

$H(y) = \beta$ implies $H(f(x)) = H(x) \geq \beta$, and so $H(z + x) \geq \beta$. Furthermore, $H(x + z) \leq H(z) = \beta$ and $H(z_1^*) = H(a + z + x) + 1 = H(\phi_\beta(a) + y + f(x)) + 1 \geq \beta + 1$, Therefore, $H(z^* + K) > \beta + 1$. Thus, we can find y with the desired properties, and the extension is established. Based on the above discussion, we conclude that there is an isomorphism from M to M^* that lifts ϕ and extends ρ.

Although the following result may be called a corollary of the above theorem, we chose to call it a lemma because it will be needed to establish two later theorems.

Lemma 3.1. *Let M be a QTAG module such that $M/H_\alpha(M)$ is totally projective for some α. Let N, N' be isotype submodules of M having the same Ulm-Kaplansky invariants and a height-preserving isomorphism from M/N to M/N'. If $H_\alpha(N)$ and $H_\alpha(N')$ are strongly equivalent in $H_\alpha(M)$, then N, N' are strongly equivalent in M.*

Proof. Let $\phi : M/N \to M/N$ be a height-preserving isomorphism. This induces an isomorphism $\phi_\alpha : \dfrac{H_\alpha(M)}{H_\alpha(N)} \to \dfrac{H_\alpha(M)}{H_\alpha(N')}$, which is again height-preserving because $H_\alpha(N) = H_\alpha(M) \cap N$ and $H_\alpha(N') = H_\alpha(M) \cap N'$. Now,

$$\frac{H_\alpha(M)}{H_\alpha(M) \cap N} \simeq \frac{H_\alpha(M) + N}{N},$$

$$\frac{H_\alpha(M)}{H_\alpha(M) \cap N'} \simeq \frac{H_\alpha(M) + N'}{N'}.$$

Since $H_\alpha(N)$ and $H_\alpha(N')$ are strongly equivalent in $H_\alpha(M)$, there is an automorphism ψ_α of $H_\alpha(M)$ that induces ϕ_α. Thus, ψ_α maps $H_\alpha(N)$ onto $H_\alpha(N')$. If $\rho = \psi_\alpha$, $M = M'$, $K = H_\alpha(M) = K'$, then, every submodule of M is compatible with $H_\alpha(M)$.

Theorem 3.4. *Let M be a QTAG module and α is an ordinal for which $M/H_\alpha(M)$ is totally projective. If $H_\alpha(M)$ is strongly U-transitive, then, M is also strongly U-transitive.*

Proof. Let N, N' be two isotype submodules of M having the same Ulm-Kaplansky invariants and a height-preserving isomorphism $\phi : M/N \to M/N'$. Now, $H_\alpha(N)$ and $H_\alpha(N')$ are isotype submodules of $H_\alpha(M)$ satisfying the same conditions. Thus, $H_\alpha(N)$ and $H_\alpha(N')$ have the same Ulm-Kaplansky invariants and there is a height-preserving isomorphism such that $\phi_\alpha : \dfrac{H_\alpha(M)}{H_\alpha(N)} \to \dfrac{H_\alpha(M)}{H_\alpha(N')}$. Since $H_\alpha(M)$ is strongly U-transitive, the submodules $H_\alpha(N)$, $H_\alpha(N')$ are equivalent in $H_\alpha(M)$. By Lemma 3.1, M is strongly U-transitive.

Now, we prove that if M is strongly U-transitive so is $H_\alpha(M)$. The total projectivity of $M/H_\alpha(M)$ is not required.

Theorem 3.5. *If M is a strongly U-transitive QTAG module, then, $H_\alpha(M)$ is also strongly U-transitive for any ordinal α.*

Proof. Suppose M is a strongly U-transitive QTAG module. Let N_1, N_2 be isotype submodules of $H_\alpha(M)$ such that N_1, N_2 have the same Ulm-Kaplansky invariants and there exists a height-preserving isomorphism $\phi_\alpha : \dfrac{H_\alpha(M)}{N_1} \to \dfrac{H_\alpha(M)}{N_2}$. Consider the submodules K_1, K_2 of M maximal with respect to the property $K_1 \cap H_\alpha(M) = N_1$, $K_2 \cap H_\alpha(M) = N_2$. Now, $H_\beta(M) \cap K_1 = H_\beta(K_1)$ for all $\beta \leq \alpha$. If $\beta > \alpha$, then, $H_\beta(M) \cap K_1 = H_\beta(K_1)$ and K_1 is isotype in M. Similarly, K_2 is isotype in M. Now, α may be a limit ordinal or not and there may be two cases.

Case (i): $\alpha = 1$. Since the submodules K_1 and K_2 are isotype which are maximal with respect to the property of containing N_1, N_2, respectively. Therefore, $\frac{Soc(K_1)}{Soc(H_1(K_1))} \simeq \frac{Soc(M)}{Soc(H_1(M))} \simeq \frac{Soc(K_2)}{Soc(H_1(K_2))}$, and thus the first Ulm-Kaplansky invariant of K_1 is same as that of K_2. Since $H_1(K_1) = N_1$ and $H_2(K_2) = N_2$, and N_1, N_2 have the same Ulm-Kaplansky invariant, K_1, K_2 also have the same Ulm-Kaplansky invariant. For $\alpha = 1$, we have an height-preserving isomorphism $\phi_1 : \frac{H_1(M)}{H_1(K_1)} \to \frac{H_1(M)}{H_1(K_2}$ that gives rise to an isomorphism $\psi_1 : H_1(M/K_1) \to H_1(M/K_2)$, and we have $\frac{H_1(M)}{H_1(K_1)} = \frac{H_1(M)}{H_1(M) \cap K} \simeq \frac{H_1(M) + K_1}{K_1} = H_1(M/K_1)$. Similarly, $\frac{H_1(M)}{H_1(K_2)} = H_1(M/K_2)$. Since ϕ_1 is height-preserving, ψ_1 is also height-preserving, and by the maximality of K_1, K_2 we have $Soc(M/K_1) = Soc(H_1(M/K_1))$. Now, the isomorphism $\psi_1 : H_1(M/K_1) \to H_1(M/K_2)$ can be extended to an isomorphism $\psi : M/K_1 \to M/K_2$, which is again height-preserving. Since M is strongly U-transitive, there is an automorphism f of M that induces ψ and $f|H_1(M)$ induces ϕ_1, thus $H_1(M)$ is strongly U-transitive. Inductively, $H_n(M)$ is also strongly U-transitive for finite ordinals n.

Case (ii): When α is a limit ordinal, we have to consider K_1, K_2 as in case (i). For all ordinals $\beta < \alpha$, $K_1 + H_\beta(M) = M$. Since K_1 is an isotype, β^{th} Ulm-Kaplansky invariant of K_1 is same as the β^{th} Ulm-Kaplansky invariant of M. Similarly, for all ordinals $\beta < \alpha$, β^{th} Ulm-Kaplansky invariant of K_2 is same as the β^{th} Ulm-invariant of M. Thus, K_1, K_2 have same Ulm-Kaplansky invariants upto α. Since $H_\alpha(K_1) = N_1$, $H_\alpha(K_2) = N_2$ and N_1, N_2 have same Ulm-Kaplansky invariants, K_1, K_2 have same Ulm-Kaplansky invariants.

Now, consider the height-preserving isomorphism $\phi : M/K_1 \to M/K_2$. Since M/K_1 and M/K_2 are h-divisible hulls of $\frac{H_\alpha(M) + K_1}{K_1}$ and $\frac{H_\alpha(M) + K_2}{K_2}$, respectively, the isomorphism ϕ can be extended to an isomorphism $\psi : M/K_1 \to M/K_2$. Also, $H(x + K_1) \geq \alpha + 1$ if and only if $x + K_1 \in \frac{H_\alpha(M) + K_1}{K_1}$ and $H(x + K_1) = \alpha$ if $x + K_1 \notin \frac{H_\alpha(M) + K_1}{K_1}$. The same is true for K_2. Therefore, the extension $\phi : M/K_1 \to M/K_2$ preserves heights.

As in case (i), let f be an automorphism of M that induces ψ. The restriction of f to $H_\alpha(M)$ is the required automorphism of $H_\alpha(M)$ that induces ϕ_α. Therefore, $H_\alpha(M)$ is strongly U-transitive.

If the submodules $N, K \subseteq M$ are countably generated, then $N \simeq K$ if and only if N, K have the same Ulm-Kaplansky invariants.

Next, we investigate a wider class of modules, which are strongly transitive with respect to countably generated isotype submodules.

Definition 3.5. *A QTAG module is $\aleph_0$-transitive if any two countably generated isotype submodules N, K are strongly equivalent in M whenever $N \simeq K$ and $M/N \simeq M/K$.*

Naji [18] defined α-modules M such that for all ordinal $\beta < \alpha$, $M/H_\beta(M)$ is totally projective.

Theorem 3.6. *All α-modules are strongly $\aleph_0$-transitive.*

Proof. Let M be an α-module and N, K be isotype submodules with a countable length. We may choose a countable ordinal β such that $N \cap H_\beta(M) = 0$, $K \cap H_\beta(M) = 0$. Since M is an α-module, $\frac{M}{H_\beta(M)}$ is totally projective. Therefore, if $N \simeq K$ and $M/N \simeq M/K$, by Lemma 3.1, N and K are strongly equivalent. Thus, for any ordinal α, α-modules are strongly $\aleph_0$-transitive.

4. Conclusions

In classical notion of transitivity a key observation due to Hill and West [11], is that the transitivity property can also be persists by the action of automorphism between two isotype subgroups of p-groups. Taking this as motivation for the current article, we generalize this case to that of QTAG-modules and the most interesting finding in this paper, is the conditions discussed in Theorem 3.1 and, it is found that for isotype submodules N, K of a totally projective QTAG-module M with same Ulm-Kaplansky invariants and a height preserving isomorphism f from M/N to M/K there exists an automorphism f' of M such that $f'(N) = K$. This leads us to define U-transitivity and strongly U-transitive QTAG-modules. It is found that if M is strongly U-transitive then $H_\alpha(M)$ is also strongly U-transitive for any ordinal α. The proposed concept of transitivity based on submodules rather than elements yields some interesting results and can be explored further by researchers of this field.

We end this article with the following discussion:

The concepts of (strong, weak, fully) transitivity for groups to QTAG-modules have been generalized using U-sequences of their elements. But here in the present manuscript, we generalize the transitivity using a new approach of submodules. We conclude this manuscript with the following open problem:

Under what conditions isotype submodules of QTAG-module M may define the properties of being strongly tranistive, weakly transitive and fully transitive for M?

Conflict of interest

The authors declare no conflicts of interest.

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