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*Research article*

# Connectedness and covering properties via infra topologies with application to fixed point theorem

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**Abstract:** A new generalization of classical topology, namely infra topology was introduced. The importance of studying this structure comes from two matters, first preserving topological properties under a weaker condition than topology, and second, the possibility of applying infra-interior and infra-closure operators to study rough-set concepts. Herein, we familiarize new concepts in this structure and establish their master properties. First, we introduce the notions of infra-connected and locally infra-connected spaces. Among some of the results we obtained, the finite product of infra-connected spaces is infra-connected, and the property of being a locally infra-connected space is an infra-open hereditary property. We successfully describe an infra-connected space using infra-open sets, which helps to study concepts given in this section under certain functions. Then, we determine the condition under which the number of infra-components is finite or countable. Second, we define the concepts of infra-compact and infra-Lindelöf spaces and study some of their basic properties. With the help of a counterexample, we elucidate that the infra-compact subset of an infra-$T_2$ space is not infra-closed, in general. We end this work by one of the interesting topics in mathematics "fixed point theorem", we show that when the infra-continuous function defined on an infra-compact space has a unique fixed point. To elucidate the topological properties that are invalid in the frame of infra topology, we provide some counterexamples.

**Keywords:** infra topology; infra-connected space; locally infra-connected space; infra-cut points; infra-compact space; infra-Lindelöf space; infra-fixed point
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## 1. Introduction and preliminaries

In [1], the author introduced and analyzed infra-topological spaces, which are derived from topological spaces. He studied the attributes of subsets of infra topological spaces such as infra derived set, infra-interior set, infra-closure set, infra-exterior set, and infra-boundary set. Witczak [30] investigated some applications of infra topologies and explained some shortcomings in [1]. Extensions of a topology are a popular area of topological studies, so there are several types of these extensions in the published literature such as supra topology [4, 24], generalized topology [14, 15], minimal structure [23], weak structures [16] and generalized weak structure [12].

Many researchers exploited topological structures and their main concepts to handle real-life issues as investigated in [2, 3, 7, 19, 21, 25–28]. Recently, some authors have applied some generalizations of topology to information systems; for example, topology [11], supra topology [10, 20] and minimal structure [13, 18]. These studies motivate us to make a contribution to constructing the frame of infra topologies because we anticipate that the environment of infra topology will be convenient to describe some practical problems. In addition, we observe that many topological properties are still valid on infra-topological spaces, which means we can dispense some topological stipulations. So, we aim through this paper to conduct an exhaustive analysis of infra-topological spaces.

We organize this manuscript as follows. In Section 2, we define the concept of connectedness via infra topologies. We show that the injective infra-continuous image of an infra-connected set is infra-connected, and prove that the property of being infra-connectedness is preserved under the finite product of spaces. Then, we give a brief overview of infra-locally connected spaces with its some basic properties. We close this section by demonstrating that all distinct infra-components forms a partition of a space and elucidating the infra-components are preserved under infra-homeomorphism functions. In Section 3, we introduce the concept of infra-compact and infra-Lindelöf spaces and analyze some properties of these two concepts. We note the validity of some properties of compact and Lindelöf spaces via infra topological structure. As an application of infra-compact spaces, we investigate some properties of fixed point theorem. In Section 4, we mention the advantages of studying topological concepts via the frame of infra topologies. Ultimately, conclusions, contributions, and upcoming work are provided in Section 5.

The family $\Theta \subseteq 2^{\mathbb{W}}$ is an infra topology on a non-empty $\mathbb{W}$ if $\emptyset \in \Theta$ and $\Theta$ is closed under finite intersection, the space $(\mathbb{W}, \Theta)$ is called an infra topological space(denoted by infra-$TS$). Every member of $\Theta$ is called an infra-open set and its complement is called an infra-closed set and the infra-clopen subset is a set which is both infra-open and infra-closed. The infra interior points of $S \subseteq \mathbb{W}$, denoted by $iInt(S)$, is the union of all infra-open sets that are contained in $S$ and the infra closure points of $S$, denoted by $iCl(S)$, is the intersection of all infra-closed sets containing $S$ [1]. The difference of infra-closure and infra-interior of a subset $S$ of $(\mathbb{W}, \Theta)$ is named infra-boundary points, symbolized by $iBd(S)$. A subset $S$ is an infra neighborhood, denoted by infra-nbd, of $x$ if there is an infra-open set $S^*$ with $x \in S^* \subseteq S$.

In the rest of this section, we recall some basic concepts and results that we need through the content of this manuscript.

**Definition 1.1.** *[1] Let $(\mathbb{W}, \Theta)$ be an infra-$TS$ and $S \subseteq \mathbb{W}$. A point $x \in \mathbb{W}$ is called an infra-limit point of $S$ if $S \bigcap (G - \{x\}) \neq \phi$ for each $G \in \Theta$ with $x \in G$.*

**Theorem 1.2.** *[30] Let $(\mathbb{W}, \Theta)$ be an infra-$TS$ and $S, G \subseteq \mathbb{W}$. Then:*

*(i) $iInt(S) \subseteq S$ and if $S \in \Theta$, then $iInt(S) = S$;*

*(ii) $x \in iInt(S)$ if there is $G \in \Theta$ with $x \in G \subseteq S$;*

*(iii) $iInt(G) \subseteq iInt(S)$ if $G \subseteq S$;*

*(iv) $iInt(iInt(S)) = iInt(S)$;*

*(v) $iInt(S \bigcap G) = iInt(S) \bigcap iInt(G)$.*

**Theorem 1.3.** *[30] Let $(\mathbb{W}, \Theta)$ be an infra-$TS$ and $S, G \subseteq \mathbb{W}$. Then:*

*(i) $S \subseteq iCl(S)$ and if $S$ is infra-closed set, then $iCl(S) = S$;*

*(ii) $x \in iCl(S)$ iff $S \bigcap G \neq \emptyset$ for each $G \in \Theta$ with $x \in G$;*

*(iii) $iCl(G) \subseteq iCl(S)$ if $G \subseteq S$;*

*(iv) $iCl(iCl(S)) = iCl(S)$;*

*(v) $iCl(S \bigcup G) = iCl(S) \bigcup iCl(G)$;*

*(vi) $\mathbb{W} - iInt(S) = iCl(\mathbb{W} - S)$ and $\mathbb{W} - iCl(S) = iInt(\mathbb{W} - S)$.*

**Definition 1.4.** *[9] An infra-$TS$ $(\mathbb{W}, \Theta)$ is called:*

*(i) Infra-$T_1$ if for each $x \neq y \in \mathbb{W}$, there are $S, S^* \in \Theta$ with $x \in S$, $y \notin S$ and $y \in S^*$, $x \notin S^*$;*

*(ii) Infra-$T_2$ (or an infra Hausdorff) if for each $x \neq y \in \mathbb{W}$, there are $S, S^* \in \Theta$ with $x \in S$, $y \in S^*$ and $S \bigcap S^* = \emptyset$;*

*(iii) Infra-regular if for each infra-closed set $G$ such that $x \notin G$, there are $S, S^* \in \Theta$ with $G \subseteq S$, $x \in S^*$ and $S \bigcap S^* = \emptyset$;*

*(iv) Infra-$T_3$ if it is infra-$T_1$ and infra-regular.*

**Definition 1.5.** *[9] Let $(\mathbb{W}, \Theta_{\mathbb{W}})$ and $(\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ be infra-$TS$s. A function $\psi : (\mathbb{W}, \Theta_{\mathbb{W}}) \to (\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ is called:*

*(i) infra-continuous if $\psi^{-1}(S^*) \in \Theta_{\mathbb{W}}$ for each $S^* \in \Theta_{\mathbb{W}^*}$;*

*(ii) infra-open (resp. infra-closed) if $\psi(S) \in \Theta_{\mathbb{W}^*}$ for each $S \in \Theta_{\mathbb{W}}$;*

*(iii) an infra-homeomorphism if $\psi$ is bijective, infra-open and infra-continuous.*

**Theorem 1.6.** *[9] Let $\{(\mathbb{W}_\delta, \Theta_\delta)\colon \delta \in \Delta\}$ be a family of infra-$TS$s. Then, $\Theta = \{\prod_{\delta \in \Delta} S_\delta\colon S_\delta \in \Theta_\delta\}$ is an infra-$TS$ on $\mathbb{W} = \prod_{\delta \in \Delta} \mathbb{W}_\delta$.*

**Proposition 1.7.** *Let $H, S$ be subsets of $\mathbb{W}$ and $H^*, S^*$ be subsets of $\mathbb{W}^*$. Then*

$$(H \times H^*) \bigcap (S \times S^*) = (H \bigcap S) \times (H^* \bigcap S^*).$$

*Proof.*

$$(x, y) \in (H \times H^*) \bigcap (S \times S^*) \Longleftrightarrow (x, y) \in (H \times H^*)$$

and

$(x, y) \in (S \times S^*) \Longleftrightarrow (x \in H \wedge y \in H^*) \wedge (x \in S \wedge y \in S^*) \Longleftrightarrow x \in (H \bigcap S) \wedge y \in (H^* \bigcap S^*) \Longleftrightarrow (x, y) \in (H \bigcap S) \times (H^* \bigcap S^*)$.

## 2. Infra-connected spaces

This section is divided into three subsection, as follows: infra-connected spaces and infra-cut points, locally infra-connected spaces and components via infra topology. We shed light on the most important properties and results related to these notions.

### *2.1. Infra-connected spaces and infra-cut points*

**Definition 2.1.** *Let $(\mathbb{W}, \Theta)$ be an infra-$TS$. The subsets $S, S^* \subseteq \mathbb{W}$ are called infra-separated if*

$$S \bigcap iCl(S^*) = iCl(S) \bigcap S^* = \emptyset.$$

Note that, if $S, S^*$ are infra-separated sets then they are disjoint but the converse is not true in general. The next example validates this fact.

**Example 2.2.** *Let*

$$\Theta = \{\mathbb{W}, G \subseteq \mathbb{W} : w_1 \notin G \text{ and } |G| \leq 3\}$$

*be an infra topology on $\mathbb{W} = \{w_1, w_2, w_3, w_4, w_5\}$. Set $S = \{w_1\}$ and $S^* = \{w_2, w_3, w_4, w_5\}$. Then, $S$ and $S^*$ are disjoint but they are not infra-separated sets because*

$$S \bigcap iCl(S^*) = S \bigcap \mathbb{W} = S.$$

It is worthy to note that an infra-$TS$ $(\mathbb{W}, \Theta)$ given above is not a supra topological space because $\{w_2, w_3\}$ and $\{w_4, w_5\}$ belong to $\Theta$ but their union does not belong to $\Theta$. Hence, $(\mathbb{W}, \Theta)$ is not a topological space.

**Proposition 2.3.** *Let $(\mathbb{W}, \Theta)$ be an infra-$TS$. If there are infra-closed sets $F_S, F_{S^*} \subseteq \mathbb{W}$ with $S \subseteq F_S$, $S^* \subseteq F_{S^*}$ and $F_S \bigcap S^* = F_{S^*} \bigcap S = \emptyset$, then $S, S^* \subseteq \mathbb{W}$ are infra-separated.*

*Proof.* Since

$$iCl(S) \subseteq F_S \subseteq \mathbb{W} - S^*,$$

and

$$iCl(S^*) \subseteq F_{S^*} \subseteq \mathbb{W} - S,$$

we obtain

$$iCl(S) \bigcap S^* = iCl(S^*) \bigcap S = \emptyset.$$

**Definition 2.4.** *Let $(\mathbb{W}, \Theta)$ be an infra-$TS$. Then, $(\mathbb{W}, \Theta)$ is called infra-disconnected if $\mathbb{W} = S \bigcup S^*$ where $S$ and $S^*$ are non-empty infra-separated. Otherwise, $(\mathbb{W}, \Theta)$ is called infra-connected.*

**Example 2.5.** *It can be noted that the infra-$TS$ $(\mathbb{W}, \Theta)$ given in Example 2.2 is infra-connected.*

**Example 2.6.** *Let $(\mathbb{N}, \Theta)$ be an infra-$TS$ with $\Theta = \{\mathbb{N}, G \subseteq \mathbb{N}$: $1 \notin G$ or $3 \notin G\}$. It is clear that $S = \{1, 2\}$ and $S^* = \{3, 4, 5, ...\}$ are infra-separated subsets of $(\mathbb{N}, \Theta)$ with $S \bigcup S^* = \mathbb{N}$. So, $(\mathbb{N}, \Theta)$ is infra-disconnected.*

**Theorem 2.7.** *Let $(\mathbb{W}, \Theta)$ be an infra-connected $TS$. Then, the following statements are hold.*

*(i) There do not exist two non-empty disjoint infra-open subsets such that their union is $\mathbb{W}$;*

*(ii) There do not exist two non-empty disjoint infra-closed subsets such that their union is $\mathbb{W}$;*

*(iii) If $S$ is infra-clopen, then $S$ is either $\emptyset$ or $\mathbb{W}$.*

*Proof.* ($i$): Assume that $\mathbb{W} = S \bigcup S^*$ where $S, S^* \in \Theta$ and $S \bigcap S^* = \emptyset$. Then, $S \subseteq \mathbb{W} - S^*$ and $S^* \subseteq \mathbb{W} - S$ and hence by Proposition 2.3, $S$ and $S^*$ are infra-separated. But $(\mathbb{W}, \Theta)$ is infra-connected; therefore, either $S = \emptyset$ or $S^* = \emptyset$.

($ii$): Assume that $\mathbb{W} = S \bigcup S^*$ where $S$,$S^*$ are infra-closed sets and $S \bigcap S^* = \emptyset$. By ($i$) either $\mathbb{W} - S = \emptyset$ or $\mathbb{W} - S^* = \emptyset$.

($iii$): Assume that $S$ is a proper non-empty infra-clopen. Then, $S \bigcup(\mathbb{W} - S) = \mathbb{W}$ and $S \bigcap(\mathbb{W} - S) = \emptyset$. By ($ii$) either $S = \emptyset$ or $\mathbb{W} - S = \emptyset$, which is a contradiction.

The converse of the above theorem is false, in general, as it is shown in the next example.

**Example 2.8.** *Let $(\mathbb{R}, \Theta)$ be an infra-TS with $\Theta = \{\mathbb{R}, G \subseteq \mathbb{R} : G$ is finite$\}$. Note that $\Theta$ does not contain two non-empty disjoint infra-open (infra-closed) sets such that their union is $\mathbb{W}$. Also, the only infra-clopen sets in $\Theta$ are the empty set and universal set. On the other hand, $iCl(\{1\}) = \{1\}$ and $iCl(\{1\}^c) = \{1\}^c$ are infra-separated subsets of $\mathbb{R}$; hence, $(\mathbb{R}, \Theta)$ is infra-disconnected.*

**Remark 2.9.** *As it is well known that infra topology is a special case of infra soft topology and they are equivalent if the set of parameters is singleton. So, according to the above theorem and example, the directions ($i$) $\Rightarrow$ ($ii$) $\Rightarrow$ ($iii$) $\Rightarrow$ ($iv$) of Proposition 3 of [8] is correct, but the converse is false, in general.*

In the next two results, we characterize infra-connected and infra-disconnected spaces with respect to infra-open sets. This matter will help us to discuss the behavior of the main concepts presented in this section under certain functions.

**Theorem 2.10.** *An infra-TS $(\mathbb{W}, \Theta)$ is infra-connected iff there does not exist a family of infra-open sets which their union is $\mathbb{W}$ and cannot be divided into two disjoint sets.*

*Proof. Necessity:* Assume that there is a collection $\mathcal{G} = \{G_\delta \colon \delta \in \Delta\}$ of infra-open sets such that $\mathbb{W} = \bigcup_{\delta\in\Delta} G_\delta$ and $\mathcal{G}$ can be divided into two disjoint sets, say

$$S = \bigcup_{j\in\Delta_1^*\subseteq\Delta} G_j,$$

and

$$S^* = \bigcup_{j\in\Delta_2^*\subseteq\Delta} G_j.$$

Since $G_j \subseteq S$ for each $j \in \Delta_1^*$, then

$$iCl(S)\bigcap S^* = \emptyset.$$

Similarly, $iCl(S^*)\bigcap S = \emptyset$. Hence, $S$ and $S^*$ are two infra-septated sets with $\mathbb{W} = S \bigcup S^*$. Therefore, $(\mathbb{W}, \Theta)$ is infra-disconnected which is a contradiction.

*Sufficiency:* Assume that $(\mathbb{W}, \Theta)$ is infra-disconnected. Then, there are $S, S^*$ with

$$S \bigcap iCl(S^*) = iCl(S) \bigcap S^* = \emptyset,$$

and $\mathbb{W} = S \bigcup S^*$. Now, for each $x \in S$ there is $x \in G_x \in \Theta$ with $G_x \bigcap S^* = \emptyset$ and for each $y \in S^*$ there is $y \in H_y \in \Theta$ with $H_y \bigcap S = \emptyset$. Therefore,

$$G_x \subseteq \mathbb{W} - S^* \subseteq S,$$

and hence $S = \bigcup G_x$. Similarly, $S^* = \bigcup H_y$ and so

$$(\bigcup G_x) \bigcup (\bigcup H_y) = \mathbb{W},$$

with $(\bigcup G_x) \bigcap (\bigcup H_y) = \emptyset$, which is a contradiction.

**Corollary 2.11.** *An infra-$TS$ $(\mathbb{W}, \Theta)$ is infra-disconnected if there exists a family of infra-open sets which their union is $\mathbb{W}$ and can be divided into two disjoint sets.*

**Proposition 2.12.** *Let $(\mathbb{W}, \Theta_{\mathbb{W}})$ and $(\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ be infra-$TS$ s.*

*(i) If $(\mathbb{W}, \Theta_{\mathbb{W}})$ is infra-connected and $\Theta_{\mathbb{W}^*} \subseteq \Theta_{\mathbb{W}}$, then $(\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ is infra-connected.*

*(ii) If $(\mathbb{W}, \Theta_{\mathbb{W}})$ is infra-disconnected and $\Theta_{\mathbb{W}} \subseteq \Theta_{\mathbb{W}^*}$, then $(\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ is infra-disconnected.*

*Proof.* The proof is obvious.

The converse of the above proposition is false, in general, as it is shown in the next example.

**Example 2.13.** *As we illustrated that the infra-$TS$ $(\mathbb{R}, \Theta)$ given in Example 2.8 is infra-disconnected. Set $\Sigma = \{\emptyset, \mathbb{R}, \{1\}\}$ as another infra topology on $\mathbb{R}$. It is clear that an infra-$TS$ $(\mathbb{R}, \Sigma)$ is infra-connected in spite of $\Sigma \subseteq \Theta$.*

**Definition 2.14.** *Let $(G, \Theta_G)$ be a subspace of an infra-$TS$ $(\mathbb{W}, \Theta)$. Then, $G \subseteq \mathbb{W}$ is called infra-disconnected if $G = S \bigcup S^*$ where $S$ and $S^*$ are non-empty infra-separated. Otherwise, $G$ is called infra-connected.*

**Proposition 2.15.** *Let $(G, \Theta_G)$ be a subspace of an infra-$TS$ $(\mathbb{W}, \Theta)$. Then,*

$$iCl_G(S) = G \bigcap iCl_{\mathbb{W}}(S)$$

*for each $S \subseteq G$ where $iCl_G$ and $iCl_{\mathbb{W}}$ are infra closure operators in $(G, \Theta_G)$ and $(\mathbb{W}, \Theta)$, respectively.*

*Proof.*
$$\begin{aligned} iCl_G(S) &= \bigcap\{F : S \subseteq F, G - F \in \Theta_G\} \\ &= \bigcap\{F^* \bigcap G : S \subseteq F^* \bigcap G, \mathbb{W} - F^* \in \Theta\} \\ &= \bigcap\{F^* \bigcap G : S \subseteq F^*, \mathbb{W} - F^* \in \Theta\} \\ &= G \bigcap(\bigcap\{F^* : S \subseteq F^*, \mathbb{W} - F^* \in \Theta\}) \\ &= G \bigcap iCl_{\mathbb{W}}(S) \end{aligned}$$

**Corollary 2.16.** *A subset $G$ of $(\mathbb{W}, \Theta)$ is infra-connected iff $(G, \Theta_G)$ is infra-connected.*

*Proof.* For

$$S, S^* \subseteq G,$$

we find

$$“(S \bigcap iCl_{\mathbb{W}}(S^*)) \bigcup (S^* \bigcap iCl_{\mathbb{W}}(S)) = \emptyset,$$

if
$$(S \bigcap iCl_G(S^*)) \bigcup (S^* \bigcap iCl_G(S)) = \emptyset\text{''} \Longrightarrow \text{``}(S \bigcap iCl_{\mathbb{W}}(S^*)) = (S^* \bigcap iCl_{\mathbb{W}}(S)) = \emptyset,$$
if
$$(S \bigcap iCl_G(S^*)) = (S^* \bigcap iCl_G(S)) = \emptyset\text{''}.$$
This ends the proof.

**Lemma 2.17.** *Let $G$ be an infra-connected subset of an infra-$TS$ $(\mathbb{W}, \Theta)$. If $G \subseteq S \bigcup S^*$ where $S$ and $S^*$ are infra-separated, then either $G \subseteq S$ or $G \subseteq S^*$.*

*Proof.* Since the infra-connected set $G$ can be written as
$$G = (G \bigcap S) \bigcup (G \bigcap S^*),$$
where $G \bigcap S$ and $G \bigcap S^*$ are infra-separated, then either $G \bigcap S = \emptyset$ or $G \bigcap S^* = \emptyset$. Therefore, either $G \subseteq S^*$ or $G \subseteq S$.

**Theorem 2.18.** *Let $(\mathbb{W}, \Theta)$ be an infra-$TS$. If $G \subseteq \mathbb{W}$ is infra-connected with $G \subseteq H \subseteq iCl(G)$, then $H$ is infra-connected.*

*Proof.* Suppose that $H = S \bigcup S^*$ where $S$ and $S^*$ are infra-separated. Then, $G \subseteq S \bigcup S^*$ and so by Lemma 2.17, either $G \subseteq S$ or $G \subseteq S^*$. If $G \subseteq S$, then
$$H \subseteq iCl(G) \subseteq iCl(S) \subseteq \mathbb{W} - S^*,$$
similarly, if $G \subseteq S^*$, then $H \subseteq \mathbb{W} - S$. Therefore, either $S^* = \emptyset$ or $S = \emptyset$, which is a contradiction. Hence, $H$ is infra-connected.

Directly from Theorem 2.18, we can prove the following corollary.

**Corollary 2.19.** *If $G \subseteq \mathbb{W}$ is infra-connected, then $iCl(G)$ is infra-connected.*

**Proposition 2.20.** *Let $(\mathbb{W}, \Theta)$ be an infra-connected $TS$. If $\emptyset \neq S \neq \mathbb{W}$, then $iBd(S) \neq \emptyset$.*

*Proof.* Assume that $iBd(S) = \emptyset$. Then, $iCl(S) = iInt(S) = S$. This implies that $iCl(S^c) = S^c$. Thus, $S$ and $S^c$ are infra-separated sets, which is a contradiction. Hence, $iBd(S) \neq \emptyset$.

**Theorem 2.21.** *A subset $G$ of an infra-$TS$ $(\mathbb{W}, \Theta)$ is infra-connected iff for each $x, y \in G$ there is an infra-connected set $H$ with $x, y \in H \subseteq G$.*

*Proof. Necessity*: Take $H = G$.
*Sufficiency*: Assume that $G = S \bigcup S^*$ where $S$ and $S^*$ are infra-separated, so there are two points $x, y$ with $x \in S$ and $y \in S^*$. Since $x, y \in G$, it follows from hypothesis that there is an infra-connected set $H$ with $x, y \in H \subseteq G = S \bigcup S^*$. By Lemma 2.17, either $H \subseteq S$ or $H \subseteq S^*$. This leads to that $S \bigcap S^* \neq \emptyset$, which is a contradiction. Hence, $G$ is infra-connected.

**Corollary 2.22.** *Let $(\mathbb{W}, \Theta)$ be an infra-$TS$. If $G = \bigcup_{\delta \in \Delta} H_\delta$ where $H_\delta$ is infra-connected and $\bigcap_{\delta \in \Delta} H_\delta \neq \emptyset$, then $G$ is infra-connected.*

*Proof.* Assume that $G = S \bigcup S^*$ where $S$ and $S^*$ are infra-separated. Since $\bigcap_{\delta \in \Delta} H_\delta \neq \emptyset$, there is $x \in H_\delta$ for each $\delta \in \Delta$. Now, either $x \in S$ or $x \in S^*$, say $x \in S$. Then, by Lemma 2.17, $H_\delta \subseteq S$ for each $\delta \in \Delta$ and so $G \subseteq S$, which is a contradiction. Hence, $G$ is infra-connected.

**Proposition 2.23.** *Let $\psi : (\mathbb{W}, \Theta_{\mathbb{W}}) \to (\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ be an injective infra-continuous function. If $G$ is infra-connected, then $\psi(G)$ is infra-connected.*

*Proof.* Assume that $\psi(G)$ is infra-disconnected. By Theorem 2.10, there is a collection $\mathcal{H} = \{H_\delta$: $\delta \in \Delta\}$ of infra-open sets such that $\psi(G) = \bigcup_{\delta\in\Delta} H_\delta$ and $\mathcal{H}$ can be divided into two disjoint sets, say

$$S = \bigcup_{j\in\Delta_1^*\subseteq\Delta} H_j,$$

and

$$S^* = \bigcup_{j\in\Delta_2^*\subseteq\Delta} H_j.$$

Since $\psi$ is injective,

$$G = \psi^{-1}(\psi(G)) = \psi^{-1}(\bigcup_{\delta\in\Delta} H_\delta) = \bigcup_{\delta\in\Delta} \psi^{-1}(H_\delta).$$

Now, the collection

$$\psi^{-1}(\mathcal{H}) = \{\psi^{-1}(H_\delta) : \delta \in \Delta\}$$

can be divided in two disjoint sets, say $\bigcup_{j\in\Delta_1^*\subseteq\Delta} \psi^{-1}(H_j)$, $\bigcup_{j\in\Delta_2^*\subseteq\Delta} \psi^{-1}(H_j)$ and by Definition 1.5, $\psi^{-1}(H_\delta) \in \Theta_{\mathbb{W}}$ for each $\delta \in \Delta$. Therefore, $G$ is infra-disconnected which is a contradiction. Hence, $\psi(G)$ is infra-connected.

**Corollary 2.24.** *An infra-connected space is an infra topological property.*

**Lemma 2.25.** *Let $H$ and $H^*$ be subsets of infra-TS s $(\mathbb{W}, \Theta_{\mathbb{W}})$ and $(\mathbb{W}^*, \Theta_{\mathbb{W}^*})$, respectively. Then,*

$$iCl(H \times H^*) = iCl(H) \times iCl(H^*).$$

*Proof.* Suppose that $(x, y) \in iCl(H \times H^*)$. For any arbitrary infra-open subsets $V$ and $V^*$ of $\mathbb{W}$ and $\mathbb{W}^*$, respectively, such that $x \in V$ and $y \in V^*$, we have $V \times V^*$ is an infra-open subset of $\mathbb{W} \times \mathbb{W}^*$ contains $(x, y)$.

By assumption,

$$(V \times V^*) \bigcap (H \times H^*) \neq \emptyset.$$

We directly obtain that $V \bigcap H \neq \emptyset$ and $V^* \bigcap H^* \neq \emptyset$. Thus, $x \in iCl(H)$ and $y \in iCl(H^*)$. Hence,

$$iCl(H \times H^*) \subseteq iCl(H) \times iCl(H^*).$$

Following similar arguments, we prove that $iCl(H) \times iCl(H^*) \subseteq iCl(H \times H^*)$.

**Theorem 2.26.** *Let $(\mathbb{W}, \Theta_{\mathbb{W}})$ and $(\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ be infra-TS s. If $(\mathbb{W}, \Theta_{\mathbb{W}})$ and $(\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ are infra-connected spaces, then $(\mathbb{W} \times \mathbb{W}^*, \Theta_{\mathbb{W}} \times \Theta_{\mathbb{W}^*})$ is infra-connected.*

*Proof.* Assume that

$$\mathbb{W} \times \mathbb{W}^* = (H \times H^*) \bigcup (S \times S^*),$$

where $(H \times H^*)$ and $(S \times S^*)$ are infra-separated subsets of $\mathbb{W} \times \mathbb{W}^*$. Then, $\mathbb{W} = H \bigcup S$ and $\mathbb{W}^* = H^* \bigcup S^*$. Since

$$iCl(H \times H^*) \bigcap (S \times S^*) = \emptyset,$$

or

$$(H \times H^{*}) \bigcap iCl(S \times S^{*}) = \emptyset.$$

Say,

$$iCl(H \times H^{*}) \bigcap (S \times S^{*}) = \emptyset,$$

it follows from the above lemma that

$$(iCl(H) \times iCl(H^{*})) \bigcap (S \times S^{*}) = \emptyset.$$

According to Proposition 1.7 we obtain

$$(iCl(H) \bigcap S) \times (iCl(H^{*}) \bigcap S^{*}) = \emptyset.$$

Thus,

$$iCl(H) \bigcap S = \emptyset \text{ or } iCl(H^{*}) \bigcap S^{*} = \emptyset,$$

which means that $(\mathbb{W}, \Theta_{\mathbb{W}})$ is infra-disconnected or $(\mathbb{W}^{*}, \Theta_{\mathbb{W}^{*}})$ is infra-disconnected. But this contradicts the hypothesis. Hence, $\mathbb{W} \times \mathbb{W}^{*}$ is infra-connected.

**Definition 2.27.** *A point $x \in \mathbb{W}$ is called an infra-cut point of an infra-connected space $(\mathbb{W}, \Theta)$ if $\mathbb{W} - \{x\}$ is infra-disconnected.*

**Example 2.28.** *It is clear that an infra-TS $(\mathbb{W}, \Theta)$ given in Example 2.2 is infra-connected. Note that $\{w_2, w_3\}$ and $\{w_4, w_5\}$ are disjoint infra open subsets of $(\mathbb{W} - \{w_1\}, \Theta_{\mathbb{W}-\{w_1\}})$ such that their union is $\mathbb{W} - \{w_1\}$. It follows from (ii) of Theorem 2.7 that $(\mathbb{W} - \{w_1\}, \Theta_{\mathbb{W}-\{w_1\}})$ is infra-disconnected. According to Corollary 2.16, $\mathbb{W} - \{w_1\}$ is an infra-disconnected set. Hence, $\{w_1\}$ is an infra-cut point of $\mathbb{W}$.*

**Theorem 2.29.** *Let $(\mathbb{W}, \Theta)$ be infra-TS. If $(\mathbb{W}, \Theta)$ is infra-connected with an infra-cut point $x$ and $\mathbb{W} - \{x\} = S \bigcup S^{*}$, where $S$ and $S^{*}$ are infra-separated, then $S \bigcup \{x\}$ and $S^{*} \bigcup \{x\}$ are infra-connected.*

*Proof.* Assume that $S \bigcup \{x\}$ is infra-disconnected. Then,

$$S \bigcup \{x\} = G \bigcup H,$$

where $G$ and $H$ are infra-separated. If $x \in G$, then $H \subseteq S$ and so

$$iCl(S^{*} \bigcup G) \bigcap H = (iCl(S^{*}) \bigcap H) \bigcup (iCl(G) \bigcap H) = (iCl(S^{*}) \bigcap H) \subseteq (iCl(S^{*}) \bigcap S) = \phi.$$

Similarly

$$(S^{*} \bigcup G) \bigcap iCl(H) = \phi.$$

Therefore, $\mathbb{W} = (S^{*} \bigcup G) \bigcup H$, which is a contradiction. Hence, $S \bigcup \{x\}$ is infra-connected.

Following similar arguments, one can prove that $S^{*} \bigcup \{x\}$ is infra-connected.

**Theorem 2.30.** *Let $(\mathbb{W}, \Theta)$ be infra-TS. If $(\mathbb{W}, \Theta)$ is infra-connected with infra-cut points $x, y$ and $\mathbb{W} - \{x\} = S \bigcup S^{*}$, $\mathbb{W} - \{y\} = H \bigcup H^{*}$, where $S$ with $S^{*}$ and $H$ with $H^{*}$ are infra-separated. If $x \in H$ and $y \in S$, then $H^{*} \subseteq S$ and $S^{*} \subseteq H$.*

*Proof.* Since
$$H^* \bigcup \{y\} \subseteq \mathbb{W} - \{x\} = S \bigcup S^*,$$
and by Theorem 2.29, $H^* \bigcup \{y\}$ is infra-connected. Then, by Lemma 2.17, $H^* \bigcup \{y\} \subseteq S$ or $H^* \bigcup \{y\} \subseteq S^*$. But $y \in S$ and hence $H^* \subseteq S$. Similarly, we prove $S^* \subseteq H$.

**Theorem 2.31.** *Let $\psi : (\mathbb{W}, \Theta_{\mathbb{W}}) \to (\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ be an infra homeomorphism function. If $x$ is an infra-cut point of $\mathbb{W}$, then $\psi(x)$ is an infra-cut point of $\mathbb{W}^*$.*

*Proof.* Assume that $\psi(x)$ is not an infra-cut point of $\mathbb{W}^*$, then $\mathbb{W}^* - \{\psi(x)\}$ is infra-connected. By Proposition 2.23, $\psi^{-1}(\mathbb{W}^* - \{\psi(x)\}) = \mathbb{W} - \{x\}$ is infra-connected, which is a contradiction. Hence, $\psi(x)$ is an infra-cut point of $\mathbb{W}^*$.

**Corollary 2.32.** *An infra-cut point is an infra topological property.*

### *2.2. Locally infra-connected spaces*

**Definition 2.33.** *Let $(\mathbb{W}, \Theta)$ be an infra-$TS$. Then, the two points $x, y$ are called infra-connected in $\mathbb{W}$ if there is an infra-connected set $G \subseteq \mathbb{W}$ with $x, y \in G$.*

**Proposition 2.34.** *Let $(\mathbb{W}, \Theta)$ be an infra-$TS$. If for each two points are infra-connected in $\mathbb{W}$, then $(\mathbb{W}, \Theta)$ is infra-connected.*

*Proof.* Assume that $x \in \mathbb{W}$. Then, for each $y \in \mathbb{W}$ different than $x$, there is infra-connected $G_y \subseteq \mathbb{W}$ with $x, y \in G_y$. Since $x \in \bigcap G_y$, it comes from Corollary 2.22 that $\mathbb{W} = \bigcup G_y$ is infra-connected.

**Definition 2.35.** *Let $(\mathbb{W}, \Theta)$ be an infra-$TS$. Then, $(\mathbb{W}, \Theta)$ is called:*

*(i) Locally infra-connected at $x \in \mathbb{W}$, if for each infra-nbd $G$ of $x$ there is an infra-nbd $H \subseteq G$ of $x$ such that each two points in $H$ are infra-connected in $G$.*

*(ii) Locally infra-connected if $(\mathbb{W}, \Theta)$ is locally infra-connected for each $x \in \mathbb{W}$. Otherwise, $(\mathbb{W}, \Theta)$ is called locally infra-disconnected.*

The next examples illustrate that infra-connected and locally infra-connected spaces are independent of each other.

**Example 2.36.** *The topologist's sine curve is an example of an infra-connected space which is not locally infra-connected.*

**Example 2.37.** *We elucidated that the infra-$TS$ $(\mathbb{R}, \Theta)$ given in Example 2.8 is infra-disconnected. On the other hand, this infra-$TS$ is locally infra-connected because every singleton set is an infra-open and infra-connected set.*

**Theorem 2.38.** *Let $(\mathbb{W}, \Theta)$ be an infra-$TS$. Then, $(\mathbb{W}, \Theta)$ is locally infra-connected at $x \in \mathbb{W}$ iff every infra-nbd of $x$ containing an infra-connected neighborhood (i.e. infra-nbd is infra-connected) of it.*

*Proof. Necessity*: Assume that $(\mathbb{W}, \Theta)$ is locally infra-connected at $x \in \mathbb{W}$. If $G$ is an infra-nbd of $x$, then there is an infra-nbd $H \subseteq G$ of $x$ such that each two points in $H$ are infra-connected in $G$. This leads to that there is an infra-connected set $S^*_y$ with $x, y \in S^*_y \subseteq G$ for each point $y \neq x \in H$. Set
$$S = \bigcup_{y \in H} S^*_y,$$

by Corollary 2.22 and since $H \subseteq S \subseteq G$, we obtain $S$ is an infra-connected neighborhood of $x$.
*Sufficiency*: It follows from Definition 2.35.

**Theorem 2.39.** *The property of being a locally infra-connected space is an infra-open hereditary property.*

*Proof.* Let $(\mathbb{W}, \Theta)$ be locally infra-connected space with an infra-open subspace $(S, \Theta_S)$. If $G$ is an infra-nbd of $x$ in $(S, \Theta_S)$, then there is $S^* \in \Theta$ with $S \bigcap S^* \subseteq G$. Since $S \in \Theta$, then $S \bigcap S^* \in \Theta$; consequently, $G$ is an infra-nbd of $x$ in $(\mathbb{W}, \Theta)$.

Therefore, there is an infra-nbd $H$ of $x$ in $(\mathbb{W}, \Theta)$ such that each two points in $H$ are infra-connected in $G$. Now, $H \bigcap G$ is an infra-nbd of $x$ in $(S, \Theta_S)$ such that each two points of $H \bigcap G$ are infra-connected in $G$. Thus, $(S, \Theta_S)$ is locally infra-connected.

**Proposition 2.40.** *A locally infra-connected space is an infra topological property.*

*Proof.* Let $\psi : (\mathbb{W}, \Theta_{\mathbb{W}}) \to (\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ be an infra-homeomorphism function where $(\mathbb{W}, \Theta_{\mathbb{W}})$ and $(\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ are infra-$TS$ s. If $(\mathbb{W}, \Theta_{\mathbb{W}})$ is locally infra-connected space and $G$ is an infra-nbd of $y \in \mathbb{W}^*$, then $\psi^{-1}(y) = x$ and $\psi^{-1}(G)$ is an infra-nbd of $x$ and so there is an infra-connected neighborhood $H \subseteq \psi^{-1}(G)$ of $\mathbb{W}$.

By Proposition 2.23, $\psi(H)$ is an infra-connected neighborhood of $y$ with $\psi(H) \subseteq G$. Therefore, $(\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ is locally infra-connected.

**Theorem 2.41.** *If $(\mathbb{W}, \Theta_{\mathbb{W}})$ and $(\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ are locally infra-connected spaces, then $(\mathbb{W} \times \mathbb{W}^*, \Theta_{\mathbb{W}} \times \Theta_{\mathbb{W}^*})$ is locally infra-connected.*

*Proof.* Let $S \times S^*$ be an infra-nbd of $(x, y) \in \mathbb{W} \times \mathbb{W}^*$, where $S$ and $S^*$ are infra-nbds of $x \in \mathbb{W}$ and $y \in \mathbb{W}^*$, respectively. Then, there are infra-connected neighborhoods $G$ of $x$ and $H$ of $y$ with $G \subseteq S$ and $H \subseteq S^*$. By Theorem 2.26, $G \times H$ is an infra-connected neighborhood of $(x, y)$ with $G \times H \subseteq S \times S^*$. Therefore, $\mathbb{W} \times \mathbb{W}^*$ is locally infra-connected.

### *2.3. Components via infra topology*

**Definition 2.42.** *Let $(\mathbb{W}, \Theta)$ be infra-$TS$ and $x \in \mathbb{W}$. The infra-component of $\mathbb{W}$ containing $x$, denoted by $iC(x)$, is defined by:*

$$iC(x) = \bigcup \{S \subseteq \mathbb{W}\text{: } x \in S \text{ and } S \text{ is infra-connected}\}.$$

**Remark 2.43.** *(i) For each $x \in \mathbb{W}$, from Corollary 2.22, we can conclude $iC(x)$ is the largest infra-connected of $\mathbb{W}$ containing $x$.*

*(ii) If $(\mathbb{W}, \Theta)$ is infra-connected, then $iC(x) = \mathbb{W}$ for each $x \in \mathbb{W}$.*

*(iii) $\mathbb{W} = \bigcup_{x \in \mathbb{W}} iC(x)$.*

**Theorem 2.44.** *Let $(\mathbb{W}, \Theta)$ be an infra-$TS$. Then, the collection of all distinct infra-components forms a partition of $\mathbb{W}$.*

*Proof.* Let $\{iC(x)\text{: } x \in \mathbb{W}\}$ be the collection of all infra-components of infra-$TS$ $(\mathbb{W}, \Theta)$. Assume that

$$iC(x) \bigcap iC(y) \neq \emptyset,$$

for $x \neq y \in \mathbb{W}$. By Corollary 2.22, $iC(x) \bigcup iC(y)$ is an infra-connected set with

$$iC(x) \subseteq iC(x) \bigcup iC(y)$$

and

$$iC(y) \subseteq iC(x) \bigcup iC(y).$$

Therefore, $iC(x)$ and $iC(y)$ are not the largest infra-connected sets, which is a contradiction. Hence $iC(x) \bigcap iC(y) = \emptyset$ and so by Remark 2.43, the infra-components is a partition of $\mathbb{W}$.

**Corollary 2.45.** *Let* $(\mathbb{W}, \Theta)$ *be an infra-TS. If the collection of infra-components consists of finite number of infra-closed sets, then the infra-components are infra-open set.*

*Proof.* Assume that for $x_1, x_2, ..., x_n \in \mathbb{W}$, we have the finite collection of infra-components $\{iC(x_1), iC(x_2), ...iC(x_k)\}$. By Theorem 2.44 we can express

$$iC(x_1) = \mathbb{W} - \bigcup_{n=2}^{k} iC(x_k).$$

Since the infra-components are infra-closed then $iC(x_1)$ is an infra-open set.

**Theorem 2.46.** *Let* $(\mathbb{W}, \Theta)$ *be an infra-TS. Then:*

*(i) Each infra-connected* $G \subseteq \mathbb{W}$ *is contained in exactly one infra-component of* $\mathbb{W}$*;*

*(ii) A non-empty infra-clopen set* $G \subseteq \mathbb{W}$ *which is infra-connected is infra-component;*

*(iii)* $iC(x) = iCl(iC(x))$ *for each* $x \in \mathbb{W}$*.*

*Proof.* $(i)$ Assume that $G$ in not an infra-component of $\mathbb{W}$ and $G \subseteq S \bigcap S^*$ where $S, S^*$ are infra-components of $\mathbb{W}$. By Corollary 2.22, $S \bigcup S^*$ is infra-connected which contains $S$ and $S^*$, this contradicts with $S, S^*$ are infra-components.
$(ii)$ By $(i)$, there is an infra-connected set $H$ with $G \subseteq H$. If $G$ is a proper subset of $H$, then

$$H = G \bigcup ((\mathbb{W} - G) \bigcap H).$$

Since $G$ is a non-empty infra-clopen set, then

$$iCl(G) \bigcap ((\mathbb{W} - G) \bigcap H) = G \bigcap iCl((\mathbb{W} - G) \bigcap H) = \emptyset.$$

Therefore, $G$ and $(\mathbb{W} - G) \bigcap H$ are infra-separated, which is a contradiction. Hence, $G$ is an infra-component.
$(iii)$ By Corollary 2.19, $iCl(iC(x))$ is an infra-connected set containing $x$ and since $iC(x)$ is the largest infra-connected containing $x$, then $iCl(iC(x)) \subseteq iC(x)$. Therefore, $iC(x) = iCl(iC(x))$.

**Proposition 2.47.** *Let* $\psi : (\mathbb{W}, \Theta_{\mathbb{W}}) \to (\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ *be an infra homeomorphism function. If* $G \subseteq \mathbb{W}$ *is an infra-component, then* $\psi(G)$ *is also an infra-component.*

*Proof.* Assume that $\psi(G)$ is not an infra-component of $(\mathbb{W}^*, \Theta_{\mathbb{W}^*})$. So, there is an infra-connected $H \subseteq \mathbb{W}^*$ with $\psi(G) \subseteq H$.

By Proposition 2.23, $\psi^{-1}(H)$ is an infra-connected with $G \subseteq \psi^{-1}(H)$, which is a contradiction. Hence, $\psi(G)$ is an infra-component.

## 3. Compactness and Lindelöfness vis infra topology

We allocate this section to present the concepts of infra-compact and infra-Lindelöf spaces and study some basic properties of them. Then, we define an infra-fixed point and show when the infra-continuous function defined on infra-compact space has a unique fixed point.

### 3.1. Infra-compact and infra-Lindelöf spaces

**Definition 3.1.** *Let* $(\mathbb{W}, \Theta)$ *be an infra-TS and* $S \subseteq \mathbb{W}$. *Then:*

*(i) A collection* $\{G_\delta : \delta \in \Delta\}$ *is an infra-open cover of* $(\mathbb{W}, \Theta)$ *if* $\mathbb{W} = \bigcup_{\delta\in\Delta} G_\delta$ *and* $G_\delta \in \Theta$*;*

*(ii)* $(\mathbb{W}, \Theta)$ *is called infra-compact (resp. infra-Lindelöf) if every infra-open cover of* $(\mathbb{W}, \Theta)$ *has a finite (resp. countable) sub-cover;*

*(iii) A subset* $S$ *of* $(\mathbb{W}, \Theta)$ *is called infra-compact (resp. infra-Lindelöf) if for every infra-open cover* $\{G_\delta\colon \delta \in \Delta\}$ *with* $S \subseteq \bigcup_{\delta\in\Delta} G_\delta$*, there is a finite (resp. countable) set* $\Delta^* \subseteq \Delta$ *such that* $S \subseteq \bigcup_{\delta\in\Delta^*} G_\delta$.

Note that $(\mathbb{W}, \Theta)$ displayed in Example 2.2 is infra-compact, whereas $(\mathbb{R}, \Theta)$ displayed in Example 2.8 is not infra-Lindelöf.

**Proposition 3.2.** *Every infra-compact space is infra-Lindelöf.*

*Proof.* It follows directly from Definition 3.1.

The following example shows that the converse of Proposition 3.2 is false, in general.

**Example 3.3.** *An infra-TS* $(\mathbb{N}, \Theta)$ *given in Example 2.6 is infra-Lindelöf, but not infra-compact.*

**Proposition 3.4.** *A collection of infra-compact (resp. infra-Lindelöf) sets is closed under a finite (resp. countable) union.*

*Proof.* It is obvious.

**Proposition 3.5.** *Let* $(\mathbb{W}, \Theta)$ *be an infra-TS and* $S \subseteq \mathbb{W}$. *If* $(\mathbb{W}, \Theta)$ *is an infra-compact (resp. infra-Lindelöf) and* $S$ *is infra-closed, then* $S$ *is infra-compact (resp. infra-Lindelöf).*

*Proof.* Assume that $S \subseteq \bigcup_{\delta\in\Delta} G_\delta$ where $G_\delta \in \Theta$. Then, $(\bigcup_{\delta\in\Delta} G_\delta) \bigcup (\mathbb{W} - S)$ is an infra-open cover of $(\mathbb{W}, \Theta)$. By hypothesis, there is a finite (resp. countable) set $\Delta^* \subseteq \Delta$ such that

$$\mathbb{W} = (\bigcup_{\delta\in\Delta^*} G_\delta) \bigcup (\mathbb{W} - S).$$

Therefore, $S \subseteq \bigcup_{\delta\in\Delta^*} G_\delta$. Hence, the proof is complete.

**Corollary 3.6.** *The intersection of infra-closed and infra-compact (resp. infra-Lindelöf) sets is infra-compact (resp. infra-Lindelöf).*

Note that, the converse of Proposition 3.5 need not be true in general. In Example 2.8, if $S = \{1, 2\}$, then $S$ is an infra-compact set but it is not infra-closed.

**Theorem 3.7.** *Let* $(\mathbb{W}, \Theta)$ *be an infra-TS. Then,* $(\mathbb{W}, \Theta)$ *is an infra-compact (resp. infra-Lindelöf) iff* $\bigcap_{\delta\in\Delta} F_\delta \neq \emptyset$ *for every collection* $\{F_\delta\colon\ \delta \in \Delta\}$ *of infra-closed subsets of* $(\mathbb{W}, \Theta)$ *that satisfy the finite (resp. countable) intersection property.*

*Proof. Necessity*: Suppose that $\bigcap_{\delta\in\Delta} F_\delta = \emptyset$. Then,

$$\mathbb{W} = \bigcup_{\delta\in\Delta} (\mathbb{W} - F_\delta).$$

By hypothesis, there is a finite set $\Delta^* \subseteq \Delta$ such that

$$\mathbb{W} = \bigcup_{\delta\in\Delta^*} (\mathbb{W} - F_\delta).$$

Therefore, $\bigcap_{\delta\in\Delta^*} F_\delta = \emptyset$, which is a contradiction. Hence, $\bigcap_{\delta\in\Delta} F_\delta \neq \emptyset$.
*Sufficiency*: Let $\{G_\delta\colon\ \delta \in \Delta\}$ be an infra-open cover of $(\mathbb{W}, \Theta)$. If

$$\mathbb{W} - (\bigcup_{\delta\in\Delta^*} G_\delta) \neq \emptyset,$$

for any finite set $\Delta^* \subseteq \Delta$, then $\bigcap_{\delta\in\Delta^*} (\mathbb{W} - G_\delta) \neq \emptyset$. Therefore, $\{(\mathbb{W} - G_\delta)\colon\ \delta \in \Delta\}$ is a collection of infra-closed subsets of $(\mathbb{W}, \Theta)$ that has a finite intersection property. Hence, $\mathbb{W} \neq \bigcup_{\delta\in\Delta} G_\delta$, which is a contradiction. Hence, $(\mathbb{W}, \Theta)$ is infra-compact.

The case between parentheses can be proved by similar technique.

In the infra-$TS$, we now analyze under which conditions the well-known relation between infra-closed sets and infra-$T_2$ spaces is satisfied.

First, note that $S = \{1, 2\}$ in Example 2.8 is not infra-closed set even though it is an infra-compact subset of infra-$T_2$ space.

We need the following definition

**Definition 3.8.** *A subset* $H$ *of an infra-TS* $(\mathbb{W}, \Theta)$ *is said to be* $\Theta$*-infra-open if for each* $x \in H$ *there exists an infra-open set* $G$ *such that* $x \in G \subseteq H$*. The complement of a* $\Theta$*-infra-open set is called* $\Theta$*-infra-closed.*

It is clear that every infra-open set is $\Theta$-infra-open but the converse need not be true in general. It can be seen that any proper infinite subset of an infra-$TS$ $(\mathbb{W}, \Theta)$ given in Example 2.8 is $\Theta$-infra-open but not infra-open.

**Proposition 3.9.** *Let* $(\mathbb{W}, \Theta)$ *be an infra-*$T_2$ *space. Then, every infra-compact subset of* $(\mathbb{W}, \Theta)$ *is* $\Theta$*-infra-closed.*

*Proof.* Let $S$ be an infra-compact subset of $(\mathbb{W}, \Theta)$. If $x \notin S$, then $x \neq x_\delta$ for each $x_\delta \in S$. So there are $G_\delta, H_\delta \in \Theta$ with $x \in G_\delta$, $x_\delta \in H_\delta$ and $G_\delta \bigcap H_\delta = \emptyset$. Since $S \subseteq \bigcup_{\delta\in\Delta} H_\delta$, there is a finite set $\Delta^* \subseteq \Delta$ such that $S \subseteq \bigcup_{\delta\in\Delta^*} H_\delta$.

Therefore,

$$x \in \bigcap_{\delta\in\Delta^*} G_\delta \subseteq (\mathbb{W} - S).$$

Hence, $S$ is $\Theta$-infra-closed.

**Theorem 3.10.** *Let $(\mathbb{W}, \Theta)$ be an infra-$T_2$ space. If $S$ is an infra-compact subset of $(\mathbb{W}, \Theta)$, then for each $x \notin S$ there are $G, H \in \Theta$ with $x \in G$, $S \subseteq H$ and $G \bigcap H = \emptyset$ provided that $\Theta$ is closed under a finite union.*

*Proof.* Let $x \notin S$. Then, there are $G_\delta, H_\delta \in \Theta$ with $x \in G_\delta$, $x_\delta \in H_\delta$ and $G_\delta \bigcap H_\delta = \emptyset$ for each $x_\delta \in S$. Since $S \subseteq \bigcup_{\delta\in\Delta} H_\delta$, there is a finite set $\Delta^* \subseteq \Delta$ such that $S \subseteq \bigcup_{\delta\in\Delta^*} H_\delta$. Therefore, $G = \bigcap_{\delta\in\Delta^*} G_\delta$ and $H = \bigcup_{\delta\in\Delta^*} H_\delta$ are the required sets.

**Theorem 3.11.** *Let $(\mathbb{W}, \Theta)$ be an infra-$T_2$ space. If $(\mathbb{W}, \Theta)$ is infra-compact, then it is infra-regular provided that $\Theta$ is closed under a finite union.*

*Proof.* Assume that $S$ is an infra-closed set with $x \notin S$. By Proposition 3.5, $S$ is infra-compact set and so by Theorem 3.10, there are $G, H \in \Theta$ with $x \in G$, $S \subseteq H$ and $G \bigcap H = \emptyset$. Hence, $(\mathbb{W}, \Theta)$ is infra-regular.

The proof of the following corollary follows from Theorem 3.11.

**Corollary 3.12.** *Let $(\mathbb{W}, \Theta)$ be an infra-$T_2$ space. If $(\mathbb{W}, \Theta)$ is infra-compact, then it is infra-$T_3$ provided that $\Theta$ is closed under a finite union.*

**Proposition 3.13.** *Let $\psi : (\mathbb{W}, \Theta_{\mathbb{W}}) \to (\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ be an infra-continuous function. If $S$ is an infra-compact (resp. infra-Lindelöf) set, then $\psi(S)$ is infra-compact (resp. infra-Lindelöf).*

*Proof.* Assume that $\psi(S) \subseteq \bigcup_{\delta\in\Delta} G_\delta$ where $G_\delta \in \Theta_{\mathbb{W}^*}$. Then, $S \subseteq \bigcup_{\delta\in\Delta} \psi^{-1}(G_\delta)$ and $\psi^{-1}(G_\delta) \in \Theta_{\mathbb{W}}$ for each $\delta \in \Delta$. Since $S$ is infra-compact (resp. infra-Lindelöf), there is a finite (countable) set $\Delta^* \subseteq \Delta$ such that $S \subseteq \bigcup_{\delta\in\Delta^*} \psi^{-1}(G_\delta)$.

Therefore,

$$\psi(S) \subseteq \bigcup_{\delta\in\Delta^*} \psi(\psi^{-1}(G_\delta)) \subseteq \bigcup_{\delta\in\Delta^*} G_\delta.$$

This ends the proof.

**Corollary 3.14.** *An infra-compact (infra-Lindelöf) space is an infra topological property.*

**Theorem 3.15.** *Let $(\mathbb{W}, \Theta)$ be an infra-$TS$. A subset $S$ is infra-compact (resp. infra-Lindelöf) set in $(\mathbb{W}, \Theta)$ iff $(S, \Theta_S)$ is infra-compact (resp. infra-Lindelöf).*

*Proof. Necessity*: Consider $S \subseteq \bigcup_{\delta\in\Delta} G_\delta$ with $G_\delta \in \Theta_S$. Then, there is $H_\delta \in \Theta$ with $G_\delta = H_\delta \bigcap S$ for each $G_\delta \in \Theta_S$. Now, $S \subseteq \bigcup_{\delta\in\Delta} H_\delta$ so there is a finite set $\Delta^* \subseteq \Delta$ such that $S \subseteq \bigcup_{\delta\in\Delta^*} H_\delta$.

Therefore,

$$S \subseteq \bigcup_{\delta\in\Delta^*} (H_\delta \bigcap S) = \bigcup_{\delta\in\Delta^*} G_\delta.$$

Hence, $(S, \Theta_S)$ is infra-compact.
*Sufficiency*: Consider $S \subseteq \bigcup_{\delta\in\Delta} G_\delta$ with $G_\delta \in \Theta$. Since

$$S \subseteq \bigcup_{\delta\in\Delta} (G_\delta \bigcap S),$$

then there is a finite set $\Delta^* \subseteq \Delta$, such that

$$S \subseteq \bigcup_{\delta\in\Delta^*} (G_\delta \bigcap S),$$

and so

$$S \subseteq \bigcup_{\delta\in\Delta^*} G_\delta.$$

Therefore, $S$ is infra-compact in $(\mathbb{W}, \Theta)$.

The case between parentheses can be proved by similar technique.

**Theorem 3.16.** *Let $(\mathbb{W}, \Theta_{\mathbb{W}})$ and $(\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ be infra-TS. If $(\mathbb{W}, \Theta_{\mathbb{W}})$ and $(\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ are infra-compact (resp. infra-Lindelöf) spaces, then $(\mathbb{W} \times \mathbb{W}^*, \Theta_{\mathbb{W}} \times \Theta_{\mathbb{W}^*})$ is infra-compact (resp. infra-Lindelöf).*

*Proof.* We prove the theorem in case of infra-Lindelöf. The case of infra-compact can be proved using similar technique.

Assume that $\{G_\delta : \delta \in \Delta\}$ is an infra-open cover of $(\mathbb{W} \times \mathbb{W}^*, \Theta \times \Theta^*)$. Then, by Theorem 1.6,

$$\mathbb{W} \times \mathbb{W}^* = \bigcup_{\delta\in\Delta}(H_\delta \times H^*_\delta),$$

where $H_\delta \in \Theta$ and $H^*_\delta \in \Theta^*$. By hypothesis, there are two countable sets $\Delta^*_1, \Delta^*_2 \subseteq \Delta$ such that

$$\mathbb{W} = \bigcup_{\delta\in\Delta^*_1} H_\delta,$$

and

$$\mathbb{W}^* = \bigcup_{\delta\in\Delta^*_2} H^*_\delta.$$

Since $\Delta^*_1 \bigcup \Delta^*_2$ is a countable set and

$$\mathbb{W} \times \mathbb{W}^* = (\bigcup_{\delta\in\Delta^*_1} H_\delta) \times (\bigcup_{\delta\in\Delta^*_2} H^*_\delta) = \bigcup_{\delta\in\Delta^*_1 \bigcup \Delta^*_2} G_\delta,$$

we finish the proof that $(\mathbb{W} \times \mathbb{W}^*, \Theta_{\mathbb{W}} \times \Theta_{\mathbb{W}^*})$ is infra-Lindelöf.

**Theorem 3.17.** *Let $(\mathbb{W}, \Theta)$ be an infra-TS and $S \subseteq \mathbb{W}$. If $(\mathbb{W}, \Theta)$ is infra-Lindelöf (resp. infra-compact) and $S$ is uncountable (resp. infinite), then $S$ has an infra-limit point.*

*Proof.* Let $(\mathbb{W}, \Theta)$ be infra-Lindelöf with uncountable subset $S$. If there is no point of $\mathbb{W}$ is an infra-limit point of $S$. Then, for each $x \in \mathbb{W}$, there is $x \in G_x \in \Theta$ with $G_x \bigcap (S - \{x\}) = \emptyset$. Since the collection $\{G_x : x \in \mathbb{W}\}$ is an infra-open cover of $(\mathbb{W}, \Theta)$, then there is a countable set say $\{x_1, x_2, ...\}$ such that $\mathbb{W} = \bigcup_{n\in\mathbb{N}} G_{x_n}$. Therefore, $S$ is countable which is a contradiction. Hence, $S$ has an infra-limit point. The case between parentheses can be proved by similar technique.

**Theorem 3.18.** *Let $(\mathbb{W}, \Theta)$ be infra-TS. If the infra-components of an infra-compact (resp. infra-Lindelöf) space are infra-open, then the number of them is finite (resp. countable).*

*Proof.* Assume that the family of components $\{iC(x_\delta) : \delta \in \Delta\}$ is infinite. Since $\{iC(x_\delta) : \delta \in \Delta\}$ forms an infra-open cover of $\mathbb{W}$, there is a finite set $\Delta^* \subseteq \Delta$ with $\mathbb{W} = \bigcup_{\delta\in\Delta^*} iC(x_\delta)$. This implies that there are some members of the cover have a nonempty intersection, which contradicts Theorem 2.44. Hence, $(\mathbb{W}, \Theta)$ has a finite number of components.

The case between parentheses can be proved by similar technique.

### 3.2. Infra-fixed point

**Proposition 3.19.** *Let* $(\mathbb{W}, \Theta)$ *be infra-TS. If* $(\mathbb{W}, \Theta)$ *is an infra-compact with a collection of sets* $\{\mathcal{J}_n$: $n \in \mathbb{N}\}$, *then* $\bigcap_{n\in\mathbb{N}} \mathcal{J}_n \neq \emptyset$ *provided that the following conditions are satisfied for each* $n \in \mathbb{N}$:

*(i)* $\mathcal{J}_n \neq \emptyset$;

*(ii)* $\mathcal{J}_n$ *is an infra-closed set;*

*(iii)* $\mathcal{J}_{n+1}$ *is a subset of* $\mathcal{J}_n$.

*Proof.* Assume that $\bigcap_{n\in\mathbb{N}} \mathcal{J}_n = \emptyset$. By $(ii)$ the collection $\{\mathbb{W} - \mathcal{J}_n$: $n \in \mathbb{N}\}$ is an infra-open cover of $(\mathbb{W}, \Theta)$. Then, there are $n_1, n_2, ..., n_k \in \mathbb{N}$, $n_1 < n_2 < ... < n_k$ with

$$\mathbb{W} = (\mathbb{W} - \mathcal{J}_{n_1}) \bigcup (\mathbb{W} - \mathcal{J}_{n_2}) \bigcup ... \bigcup (\mathbb{W} - \mathcal{J}_{n_k}).$$

It comes from $(iii)$ that

$$\mathcal{J}_{n_k} \subseteq \mathbb{W} = (\mathbb{W} - \mathcal{J}_{n_1}) \bigcup (\mathbb{W} - \mathcal{J}_{n_2}) \bigcup ... \bigcup (\mathbb{W} - \mathcal{J}_{n_k}) = \mathbb{W} - (\mathcal{J}_{n_1} \bigcap \mathcal{J}_{n_2} \bigcap ... \bigcap \mathcal{J}_{n_k}) = \mathbb{W} - \mathcal{J}_{n_k},$$

which is a contradiction. Hence, $\bigcap_{n\in\mathbb{N}} \mathcal{J}_n \neq \emptyset$.

**Definition 3.20.** *Let* $(\mathbb{W}, \Theta)$ *be an infra-TS. If every infra-continuous function* $\psi : (\mathbb{W}, \Theta) \to (\mathbb{W}, \Theta)$ *has a fixed point, then* $(\mathbb{W}, \Theta)$ *is said to have a fixed point property.*

**Proposition 3.21.** *Let* $\psi : (\mathbb{W}, \Theta_{\mathbb{W}}) \to (\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ *be an infra-homeomorphism function. If* $(\mathbb{W}, \Theta_{\mathbb{W}})$ *has a fixed point property, then* $(\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ *has a fixed point property, which means the fixed point property is an infra topological property.*

*Proof.* Let $\psi : (\mathbb{W}, \Theta_{\mathbb{W}}) \to (\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ be an infra-homeomorphism function.

Then, $\psi$ and $\psi^{-1}$ are infra-continuous. Consider $\zeta : (\mathbb{W}^*, \Theta_{\mathbb{W}^*}) \to (\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ is infra-continuous. It is clear that

$$\zeta \circ \psi : (\mathbb{W}, \Theta_{\mathbb{W}}) \to (\mathbb{W}^*, \Theta_{\mathbb{W}^*})$$

is an infra-continuous function. Also,

$$\psi^{-1} \circ \zeta \circ \psi : (\mathbb{W}, \Theta_{\mathbb{W}}) \to (\mathbb{W}, \Theta_{\mathbb{W}})$$

is an infra-continuous. By hypothesis, $(\mathbb{W}, \Theta_{\mathbb{W}})$ has a fixed point property, so there exists $x \in \mathbb{W}$ such that

$$\psi(\psi^{-1}(\zeta(\psi(x)))) = \psi(x),$$

which means that $\zeta(\psi(x)) = \psi(x)$.

Hence, $\psi(x)$ is a fixed soft point of $\zeta$, which proves that $(\mathbb{W}^*, \Theta_{\mathbb{W}^*})$ has a fixed point property.

## 4. Discussion and analysis

The present paper discusses the well-known topological concepts via one of the generalizations of topological spaces called infra topological spaces. It is known that infra topology does not satisfy the condition of arbitrary unions imposed in topological structures. One of the celebrated infra topologies, that is not a topology, is the family consisting of an infinite set $\mathbb{W}$ and all finite subsets of $\mathbb{W}$. Herein, we discover how the topological concepts of connectedness and compactness behave under the frame of infra topology and explore which one of their classical topological properties and characterizations is still valid or evaporated.

The merits of investigation the topological concepts herein are that to exploit the obtained results to the application areas of topology like rough approximation spaces generated from information systems. To our best knowledge, it is applied the topological operators of interior and closure to initiate the lower and upper approximation operators of subsets of data under study and then measure the accuracy, which helps the decision makers to made an accurate decision. From this point of view, it can be seen that relaxing a condition of topology is preferable to create more appropriate environment to address real-life problems as illustrated in the recent two published manuscripts [10, 11] which applied the current frame "infra topology" as well as "supra topology" to handle medical problems. To well understand this line of application, we give here an example of how we can benefit from infra topological structures in this area as follows; we proved that the main four characterizations of interior operators are still valid in the frame of infra topology; especially; $iInt(A \bigcap B) = iInt(A) \bigcap iInt(B)$; it can be noted this property is missing in the other generalizations of topology like supra topology. The validity of this property in the frame of infra topology leads to keep its counterparts in generalized rough approximation spaces generated by infra topology as investigated in the manuscript [11].

As a matter of fact, the current contribution paved the road for these types of applications by encouraging us and the other researchers to debate the existing applications of topological concepts like separation axioms [5] and compactness [6] in the frame of infra topological spaces. In addition, we draw the reader's attention to that the different types of generalizations of the topology, including infra-topology, ease the way of constructing examples and counterexamples that are required to show the obtained results; especially, those related to the strong topological concepts as explained in [4].

On the other hand, we evince that some topological properties are evaporated in the infra topological structures such as the property reports that "an infra topological space is infra-connected iff the empty and universal sets are the only infra-clopen sets", see Example 2.8.

## 5. Conclusions and future work

In the last two decades, study generalizations of topology have flourished, which can be useful in computer science and its applications, see [10, 11, 17, 29]. These structures are equipped with concepts analogous to those of continuity, connectedness, and compactness. To contribute to this line of research, this manuscript is writing.

The main achievements of this work are:

(i) Introduce the concepts of infra-connected and locally infra-connected spaces as well as infra-compact and infra-Lindelöf spaces.

(ii) Study the notions of cut point, components and fixed point via the infra topological frames.

(iii) Demonstrate that the finite product of infra-connected (resp., locally infra-connected) spaces is infra-connected (resp., locally infra-connected).

(iv) Show that the properties of being a locally infra-connected and locally infra-connected spaces are an infra-open hereditary property.

(v) Prove that the property of infra-compact (resp., infra-Lindelöf) spaces is kept under the finite product of infra topological spaces.

(vi) Investigate some topological properties of these concepts that are invalid in the frame of infra topology such as the infra-compact subset of an infra-$T_2$ space need not be an infra-closed set.

(vii) Determine under which conditions the infra-continuous function has a unique fixed point.

In upcoming papers, we plan to define a generalization of infra-open sets such as semi-infra-open and $\beta$-infra-open sets, also, we will use the infra-open sets to define the concept of infra-paracompact spaces and examine the properties of functionally separation axioms [22]. Moreover, we investigate how rough-set concepts can be described using infra-interior and infra-closure operators.

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## Conflict of interest

The authors declare that they have no competing interests.

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