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Research article

Abundant solitary wave solutions of Gardner's equation using three effective integration techniques

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Abstract: Gardner's equation has been discussed in the article for finding new solitary wave solutions. Three efficient integration techniques, namely, the Kudryashov's R function method, the generalized projective Ricatti method and $\frac{G'}{G^2}$-expansion method are implemented to obtain new dark soliton, bright soliton, singular soliton, and combo soliton solutions. Moreover, some of the obtained solutions are graphically depicted by using 3D-surface plots and the corresponding 2D-contour graphs.

Keywords: Gardner's equation; Kudryashov's R function method; generalized projective Riccati equations method; $\frac{G'}{G^2}$-expansion method; solitary wave solution

1. Introduction and preliminary results

Soliton theory has become a topic of great interest for mathematicians since its emergence with the discovery of solitary waves by Russel in 1834. Afterward, Kruskal and Zabusky introduced the concept of solitons triggering the worldwide study of solitons along with the development in mathematics and other fields of science and engineering. Solitons have many applications in different mathematical-oriented disciplines of science due to their shape-preserving properties and other useful aspects. Solitary waves are more stable over long distances because they preserve their shape and velocity after the interaction. This special characteristic of solitons stimulates mathematicians and physicists to work on them, which is why soliton theory is still progressing.

Various nonlinear PDEs have been investigated during recent years for getting their solitary wave solutions of various types. For instance, soliton solutions of mKdV equation [1] and [39], nonlinear Schrödinger equation [2], KE equation [40], Date-Jimbo-Kashiwara-Miwa equation [3], Boussinesq equation [4], Lakshmanan-Porsezian-Daniel equation [5], Kwahara equation [6], ZK-Burgers equation [7], Lakshmanan-Porsezian-Daniel equation [8] and the Błaszak-Marciniak lattice equations [41] have been extracted by different researchers. Discrete nonlinear Schrödinger equations have also been extensively studied in various physical contexts, for instance, in studying the diode-like transmission properties of waves [9–13]. Soliton theory has become the popular topic of research and has attracted the attention of researchers in recent years. Soliton resolution and the asymptotic stability of N-soliton solution for the Wadati-Konno-Ichikawa equation are discussed in [26]. Soliton resolution for the Wadati-Konno-Ichikawa equation with weighted Sobolev initial data is discussed in [27]. Soliton resolution for the complex short pulse equation with weighted Sobolev initial data in space-time solitonic regions is presented in [28]. Riemann-Hilbert problem and dynamics of soliton solutions of the fifth-order nonlinear Schrödinger equation are discussed in [29]. Integrable discretizations and soliton solutions of an Eckhaus-Kundu equation are presented in [30]. Traveling wave, lump Wave, rogue wave, multi-kink solitary wave and interaction solutions in a (3+1)-dimensional Kadomtsev-Petviashvili equation using Bäcklund transformation are presented in [31]. The interpretation and understanding of natural processes require the extraction of analytical and numerical solutions of nonlinear evolution equations (NLEEs). In the quest of finding solutions of NLEEs, a number of novel and effective techniques have been reported including Riemann-Hilbert method [32, 33], Hirota method [34], Darboux transformation method [35] and Lie symmetry analysis approach [36], etc.

This paper addresses Gardner's equation [14–16] in the form

$$\eta_t + 2\alpha\eta\eta_x + 3\beta\eta^2\eta_x + \gamma\eta_{xxx} = 0. \tag{1.1}$$

In the above model, α, β and ($\gamma > 0$) are nonzero constants. The proposed model was first introduced by Gardner in 1968 during his research on KdV equation [17]. This equation has great importance in various branches of science, especially in the quantum field and plasma physics [18]. The derivation of higher-order Gardner's equation is considered very imperative for analyzing nonlinear properties of solitary waves [19]. In recent years, Gardner's equation has been studied by many researchers [42–44] for extracting soliton solutions. Moreover, there are also Jacobi elliptic solutions in the Gardner's equation [37].

The purpose of writing this article is to construct new soliton solutions of the proposed model Eq (1.1) via three effective schemes, namely, Kudryashov's R function method, generalized projective Riccati equations method and $\frac{G'}{G^2}-$expansion approach.

In 2020, Kudryashov was the first who developed Kudryashov's R function approach [20]. This method is effective and applicable to a variety of nonlinear problems. The generalized Riccati equations method (GPRE) originated from the research work of Conte and Musette [21] in which they proposed the idea of writing an NLPDE as a polynomial of two functions. The third method utilized in this paper is $\frac{G'}{G^2}-$expansion method that was introduced in 2009 [22]. Application of these methods to the considered Gardner's equation has resulted in novel soliton solutions.

The paper contains seven sections. In Section 2, the governing equation is converted into an ODE via traveling wave transformation. Sections 3–5 deal with solution of Gardner's equation using the suggested methods. Few of the obtained solutions are graphically illustrated in Section 6, whereas the concluding points are given in Section 7.

2. Governing equation

The proposed model is considered, as

$$\eta_t + 2\alpha\eta\eta_x + 3\beta\eta^2\eta_x + \gamma\eta_{xxx} = 0. \tag{2.1}$$

If the $\beta > 0$, Eq (2.1) admits two families of solitons and oscillating wave packets (called breathers), whereas if $\beta < 0$, only one category of solitons exists. Eq (2.1) is also called the combined KdV-mKdV equation.

The following wave transformation is used for extracting solitary wave solutions of (2.1), as

$$\eta(x,t) = P(a), \quad a = \mu x - \theta t, \tag{2.2}$$

where P is real-valued function and $\mu,\ \theta$ are nonzero constants.

Applying transformation (2.2) on Eq (2.1), an ordinary differential equation (ODE) is yielded, as

$$-\theta P^{'} + 2\alpha\mu PP^{'} + 3\beta\mu P^2 P^{'} + \gamma\mu^3 P^{'''} = 0. \tag{2.3}$$

Now upon integrating Eq (2.3), we have

$$-\theta P + \alpha\mu P^2 + \beta\mu P^3 + \gamma\mu^3 P^{''} = 0. \tag{2.4}$$

Further, Eq (2.4) is solved using three proposed efficient analytical methods in the following subsections to derive solitary wave solutions of the proposed model Eq (2.1).

3. Solitary wave solutions using Method 1: Kudryashov's *R* function method

In this section, Method 1 and its implementation of the proposed model have been illustrated.

3.1. Description of Method 1

According to Method 1 [24], the predicted solution of Eq (2.4) takes the following form

$$P(a) = \sum_{i=0}^{N} A_i \left(R(a)\right)^i. \tag{3.1}$$

Where N is a balancing number, determined by applying homogenous balancing principle (balancing the highest order derivative and nonlinear term) on Eq (2.4). $R(a)$ has the the following form

$$R(a) = \frac{1}{\epsilon e^{\delta a} + \varepsilon e^{-\delta a}}, \tag{3.2}$$

where $\epsilon,\ \varepsilon$ and δ are parameters. $R(a)$ obeys the following ODE

$$R_a^2 = \delta R^2\left(1 - 4\epsilon\varepsilon R^2\right).$$

Application of homogeneous balance rule on Eq (2.4) gives $N = 1$. The solution takes the form

$$P(a) = A_0 + A_1\left(\frac{1}{\epsilon e^{\delta a} + \varepsilon e^{-\delta a}}\right). \tag{3.3}$$

3.2. Mathematical analysis

Inserting Eq (3.3) into the ODE (2.4). A set of equations in A_0 and A_1 is yielded by plugging coefficients of different powers of $R(a)$ to zero. The values of A_0 and A_1 are extracted upon solving the obtained system. The extracted values have been summarized as below

$$A_0 = -\frac{\alpha}{3\beta}, \qquad A_1 = \pm\frac{2\sqrt{2\epsilon\varepsilon}}{3\beta},$$
$$\gamma = \frac{\alpha^2}{9\beta\mu^2}, \qquad \theta = -\frac{2\alpha^2\mu}{9\beta}.$$

Bright solitons have been obtained using the values of A_0 and A_1 in Eq (3.3) as

$$\eta(x,t) = -\frac{\alpha}{3\beta} \pm \frac{2\sqrt{2\epsilon\varepsilon}}{3\beta}\left[\frac{1}{\epsilon e^{\delta(\mu x+\frac{2\alpha^2\mu}{9\beta}t)} + \varepsilon e^{-\delta(\mu x+\frac{2\alpha^2\mu}{9\beta}t)}}\right]. \tag{3.4}$$

Upon choosing the values of constants, as $\beta = \epsilon = 1,\ \varepsilon = 1,\ \alpha = 1,\ \mu = \delta = 1$, the above solution becomes

$$\eta(x,t) = \frac{1}{3}\left[-1 \pm \sqrt{2}\text{sech}\left(x + \frac{2}{9}t\right)\right]. \tag{3.5}$$

4. Solitary wave solutions using Method 2: Generalized projective Ricatti equations (GPRE) method

This section deals with the general algorithm of the GPRE method and its application to the proposed equation.

4.1. Description of Method 2

According to GPRE method [25], the predicted solution of Eq (2.4) has the following form

$$P(a) = A_0 + A_1 f(a) + B_1 g(a), \tag{4.1}$$

where A_0, A_1 and B_1 are unknowns to be evaluated. The ODEs satisfied by $f(a)$ and $g(a)$ have solutions of the following types.

Type 1.

When $e = -1$, $r = -1$, $R > 0$,

$$f_1(a) = \frac{R\text{sech}(\sqrt{R}a)}{m\,\text{sech}(\sqrt{R}a) + 1}, \qquad g_1(a) = \frac{\sqrt{R}\tanh(\sqrt{R}a)}{m\,\text{sech}(\sqrt{R}a) + 1}.$$

Type 2.

When $e = -1$, $r = 1$, $R > 0$,

$$f_2(a) = \frac{R\text{csch}(\sqrt{R}a)}{m\,\text{csch}(\sqrt{R}a) + 1}, \qquad g_2(a) = \frac{\sqrt{R}\coth(\sqrt{R}a)}{m\,\text{csch}(\sqrt{R}a) + 1}.$$

4.2. Mathematical analysis

Inserting Eq (4.1) into Eq (2.4), then plugging the coefficients of $g^p(a)$, $p = 0, 1, \ldots$ to zero, a system of equations is extracted.

Upon solving the extracted system of equations, the constants A_0, A_1, B_1, m and R, are determined. The obtained results are given as SET1 below:

$$A_0 = -\frac{\sqrt{\theta}}{\sqrt{2\mu}\sqrt{-\beta}}, \qquad A_1 = \frac{\sqrt{(m^2+r)\mu^5\gamma}}{\sqrt{2\theta}\sqrt{-\beta}}, \quad B_1 = \mp\frac{\sqrt{\gamma}\mu}{\sqrt{2}\sqrt{-\beta}},$$
$$\alpha = -\frac{3\sqrt{-\beta}\sqrt{\theta}}{\sqrt{2\mu}}, \qquad R = \frac{\theta}{\gamma\mu^3},$$

provided $\beta < 0$, $\theta > 0$ and $\mu > 0$.

The obtained results are given as SET2 below:

$$A_0 = 0, \qquad A_1 = -\frac{\sqrt{2\theta(m^2+r)}}{R\sqrt{-\beta}\sqrt{\mu}}, \quad B_1 = 0,$$
$$\alpha = -\frac{3m\sqrt{-\beta}\sqrt{\theta}}{\sqrt{2\mu(m^2+r)}}, \qquad R = \frac{\theta}{\gamma\mu^3},$$

provided $\theta > 0$, $\mu > 0$ and $\beta < 0$.

According to Type 1, the following solutions have been extracted for SET 1, as

$$\eta(x,t) = -\frac{\sqrt{\theta}}{\sqrt{2\mu}\sqrt{-\beta}} + \frac{\sqrt{(m^2-1)\mu^5\gamma}}{\sqrt{2\theta}\sqrt{-\beta}}\left[\frac{R\text{sech}(\sqrt{R}a)}{m\,\text{sech}(\sqrt{R}a)+1}\right] \mp \frac{\sqrt{\gamma}\mu}{\sqrt{2}\sqrt{-\beta}}\left[\frac{\sqrt{R}\tanh(\sqrt{R}a)}{m\,\text{sech}(\sqrt{R}a)+1}\right], \tag{4.2}$$

where $a = \mu x - \theta t$.

According to Type 2, solutions relating to SET 1 is given, as

$$\eta(x,t) = -\frac{\sqrt{\theta}}{\sqrt{2\mu}\sqrt{-\beta}} + \frac{\sqrt{(m^2+1)\mu^5\gamma}}{\sqrt{2\theta}\sqrt{-\beta}}\left[\frac{R\text{csch}(\sqrt{R}a)}{m\,\text{csch}(\sqrt{R}a)+1}\right] \mp \frac{\sqrt{\gamma}\mu}{\sqrt{2}\sqrt{-\beta}}\left[\frac{\sqrt{R}\coth(\sqrt{R}a)}{m\,\text{csch}(\sqrt{R}a)+1}\right], \tag{4.3}$$

provided that $\left(m\,\text{csch}(\sqrt{R}a)+1\right) \neq 0$ and $a = \mu x - \theta t$.

According to Type 1, the following solutions have been extracted for SET 2 as

$$\eta(x,t) = -\frac{\sqrt{2\theta(m^2+r)}}{R\sqrt{-\beta}\sqrt{\mu}}\left[\frac{R\text{sech}(\sqrt{R}a)}{m\,\text{sech}(\sqrt{R}a)+1}\right], \tag{4.4}$$

where $a = \mu x - \theta t$.

According to Type 2, solutions relating to SET 2 is given, as

$$\eta(x,t) = -\frac{\sqrt{2\theta(m^2+r)}}{R\sqrt{-\beta}\sqrt{\mu}}\left[\frac{R\text{csch}(\sqrt{R}a)}{m\,\text{csch}(\sqrt{R}a)+1}\right], \tag{4.5}$$

provided that $\left(m\,\text{csch}(\sqrt{R}a)+1\right) \neq 0$ and $a = \mu x - \theta t$.

5. Soliton solutions using Method 3: $\frac{G'}{G^2}$-expansion method

In this section, solitons for the proposed model have been retrieved with the help of Method 3.

5.1. Description of $\frac{G'}{G^2}$-expansion method

According to $\frac{G'}{G^2}$-expansion method [23], the solution of (2.4) has the form

$$P(a) = A_0 + A_1\left(\frac{G'}{G^2}\right) + B_1\left(\frac{G'}{G^2}\right)^{-1}, \tag{5.1}$$

where A_0, A_1 and B_1 are constants to be determined. The differential equation satisfied by $\frac{G'}{G^2}$ has three types of solutions [25].
Type 1.
If $\varpi\pi > 0$,

$$\frac{G'}{G^2} = \sqrt{\frac{\varpi}{\pi}}\left[\frac{C\cos\sqrt{\varpi\pi}a + D\sin\sqrt{\varpi\pi}a}{C\cos\sqrt{\varpi\pi}a - D\sin\sqrt{\varpi\pi}a}\right], \tag{5.2}$$

provided that $\left(C\cos\sqrt{\varpi\pi}a - D\sin\sqrt{\varpi\pi}a\right) \neq 0$.
Type 2.
If $\varpi\pi < 0$,

$$\frac{G'}{G^2} = -\frac{\sqrt{|\varpi\pi|}}{\pi}\left[\frac{C\sinh(2\sqrt{|\varpi\pi|}a) + C\cosh(2\sqrt{|\varpi\pi|}a) + D}{C\sinh(2\sqrt{|\varpi\pi|}a) + C\cosh(2\sqrt{|\varpi\pi|}a) - D}\right]. \tag{5.3}$$

Type 3.
If $\varpi = 0, \pi \neq 0$,

$$\frac{G'}{G^2} = -\frac{C}{\pi[Ca + D]}, \tag{5.4}$$

where C and D are nonzero constants.

5.2. Mathematical analysis

Inserting Eq (5.1) into Eq (2.4), and equating the coefficients of each power of $\frac{G'}{G^2}$ to zero. A system of equations is obtained. Upon solving the system following solution sets are obtained as
Set 1.

$$A_0 = \frac{3\theta}{2\mu\alpha}, \qquad A_1 = -\frac{3\sqrt{\mu\gamma\theta\pi}}{\alpha}, \quad B_1 = -\frac{3\sqrt{\theta^3}}{16\sqrt{\mu^5}\alpha\sqrt{\gamma\pi}},$$
$$\beta = -\frac{2\mu\alpha^2}{9\theta}, \quad \varpi\pi = -\frac{\theta}{16\mu^3\gamma},$$

provided $\theta > 0$ and $\mu > 0$.
Set 2.

$$A_0 = \frac{3\theta}{2\mu\alpha}, \qquad A_1 = 0, \quad B_1 = \pm\frac{3\sqrt{\mu\gamma\theta\varpi}}{\alpha},$$

$$\beta=-\frac{2\mu\alpha^2}{9\theta},\quad \varpi\pi=-\frac{\theta}{4\mu^3\gamma},$$

provided $\theta>0$ and $\mu>0$.
Set 3.

$$A_0=\frac{3\theta}{2\mu\alpha},\qquad A_1=\pm\frac{3\sqrt{\mu\gamma\theta\pi}}{\alpha},\quad B_1=0,$$
$$\beta=-\frac{2\mu\alpha^2}{9\theta},\quad \varpi\pi=-\frac{\theta}{4\mu^3\gamma},$$

provided $\theta>0$ and $\mu>0$. According to Type 2, the solution for SET 1 is given, as

$$\begin{aligned}\eta(x,t) &= \frac{3\theta}{2\mu\alpha}-\frac{3\sqrt{\mu\gamma\theta\pi}}{\alpha}\left[-\frac{\sqrt{|\varpi\pi|}}{\pi}\left[\frac{C\sinh(2\sqrt{|\varpi\pi|}a)+C\cosh(2\sqrt{|\varpi\pi|}a)+D}{C\sinh(2\sqrt{|\varpi\pi|}a)+C\cosh(2\sqrt{|\varpi\pi|}a)-D}\right]\right]\\ &\quad -\frac{3\sqrt{\theta^3}}{16\sqrt{\mu^5}\alpha\sqrt{\gamma}\pi}\left[-\frac{\sqrt{|\varpi\pi|}}{\pi}\left[\frac{C\sinh(2\sqrt{|\varpi\pi|}a)+C\cosh(2\sqrt{|\varpi\pi|}a)+D}{C\sinh(2\sqrt{|\varpi\pi|}a)+C\cosh(2\sqrt{|\varpi\pi|}a)-D}\right]\right]^{-1},\end{aligned}$$

where $a=\mu x-\theta t$.

Choosing $C=D$, the above solution becomes

$$\eta(x,t) = \frac{3\theta\left[2+\coth\left(\frac{\sqrt{\theta}a}{4\sqrt{\mu^3\gamma}}\right)+\tanh\left(\frac{\sqrt{\theta}a}{4\sqrt{\mu^3\gamma}}\right)\right]}{4\mu\alpha}, \tag{5.5}$$

where $a=\mu x-\theta t$.

According to Type 2, the solution for SET 2 is obtained by choosing $C=D$, as

$$\eta(x,t) = \frac{3\theta\left[1\mp\tanh\left(\frac{\sqrt{\theta}a}{2\sqrt{\mu^3\gamma}}\right)\right]}{2\mu\alpha}, \tag{5.6}$$

where $a=\mu x-\theta t$.

According to Type 2, the solution for SET 3 is obtained by choosing $C=D$, as

$$\eta(x,t) = \frac{3\theta\left[1\mp\coth\left(\frac{\sqrt{\theta}a}{2\sqrt{\mu^3\gamma}}\right)\right]}{2\mu\alpha}. \tag{5.7}$$

6. Graphical illustration

In this section, the graphical descriptions of a few extracted solutions have been discussed. The graphical description is the fundamental tool used for visualizing the properties of solutions physically. The numerical simulations of the extracted solutions have been carried out using 3D-surface graphs and 2D-contour graphs.

Figure 1 shows the graphical illustration for the solution expressed by Eq (3.5) which is determined by Kudryashov's R function method. The graphs show that the solution represents a bright soliton. The corresponding contour plot is added to show the structure of the bright soliton.

The graphical simulations for three solutions retrieved by GPRE method are shown in Figures 2–4. The integral surface of the solution expressed by Eq (4.2) is presented in Figure 2 which shows that the wave profile sharply changes from one asymptotic state to another. This kind of traveling wave is termed as a kink soliton. Figure 3 presents the surface graph and contour plot for the solution given by Eq (4.4). The graphs represent a dark soliton as a localized decrease in the wave amplitude can be observed. Figure 4 shows that the solution expressed by Eq (4.5) represents a dark-bright soliton.

Figures 5–7 provide the graphical representations for three solutions retrieved by $\frac{G'}{G^2}$-expansion method. The graphs depict that a variety of solitonic behavior is evident corresponding to the obtained results.

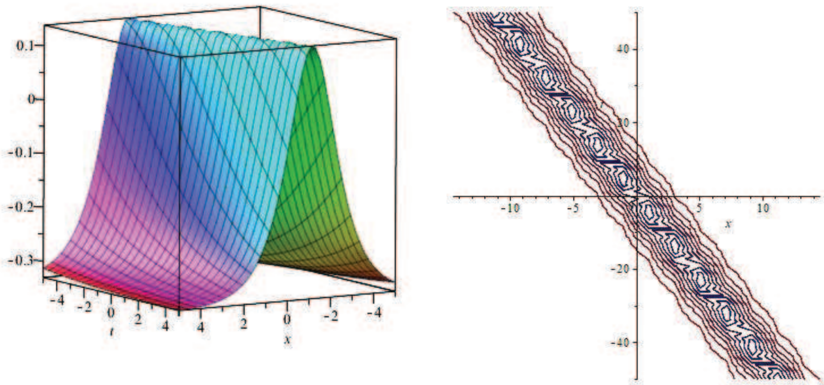

Figure 1. The 3D graph for Set 1 in Eq (3.5).

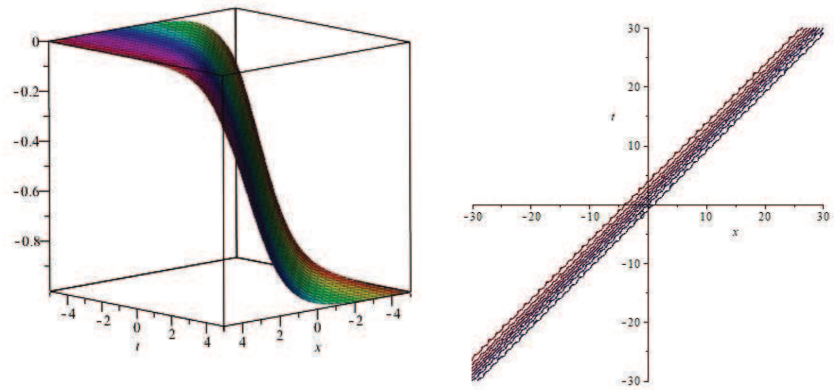

Figure 2. The 3D plot graph of Eq (4.2).

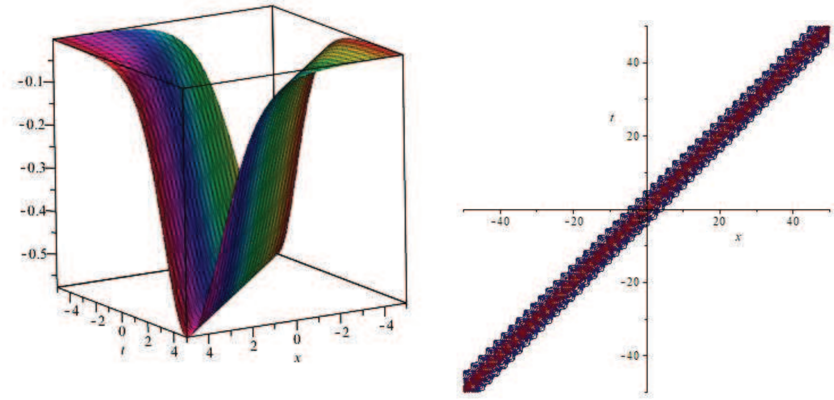

Figure 3. The 3D graph of Eq (4.4).

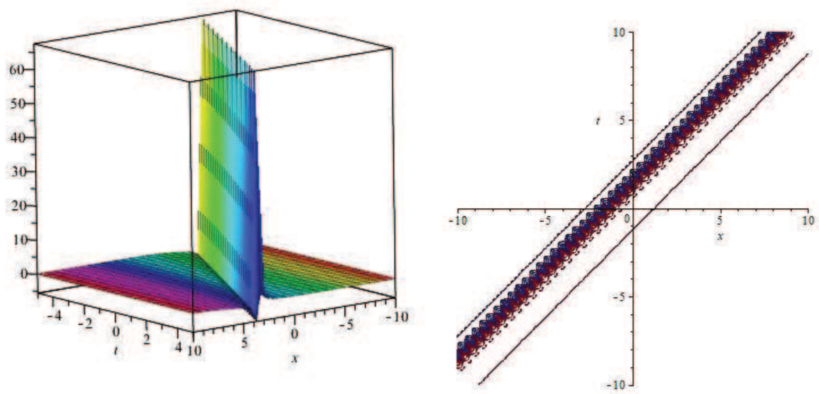

Figure 4. The 3D graph of Eq (4.5).

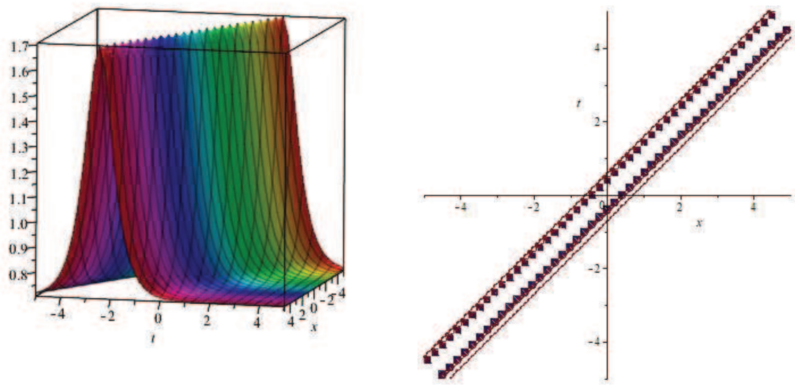

Figure 5. The 3D graph of Eq (5.5).

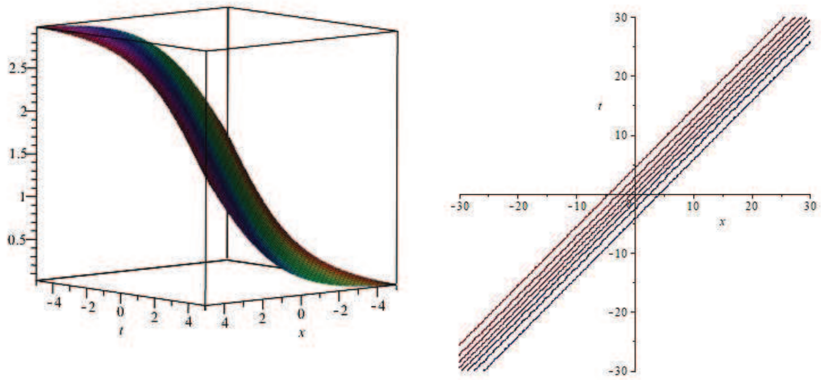

Figure 6. The 3D graph of Eq (5.6).

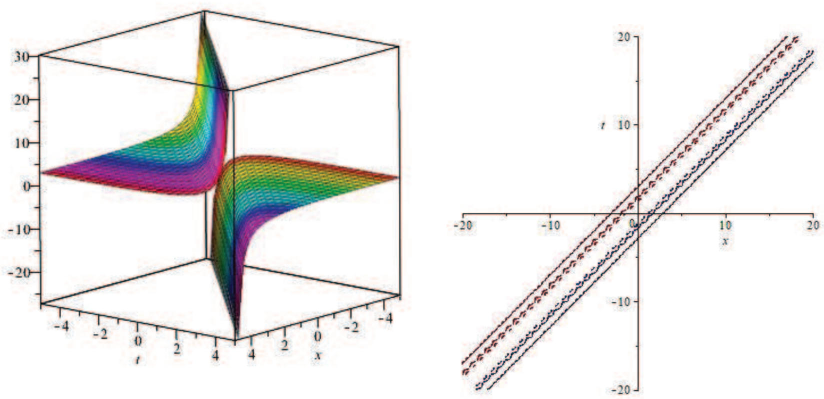

Figure 7. The 3D graph of Eq (5.7).

7. Conclusions

In this research, we have investigated the most important model termed as "Gardner's equation" of quantum field theory and plasma physics. This research aims to extract different forms of soliton solutions, which are classified as singular soliton, bright soliton, dark soliton, and dark singular combo soliton solutions. Three powerful and reliable analytical techniques have made this retrieval of soliton solutions possible. The existence criteria have been discussed for all the obtained solutions. The derived results may help describe the related physical phenomena. On comparing our derived results with the results reported in [18, 42–44], it has been observed that our obtained soliton solutions are novel and have been firstly reported in this article. In future, we can apply the same technique as explained in [38] on "Gardner's equation" for retrieving intersecting results.

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Conflict of interest

The authors declare no conflict of interest.

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