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Research article

Some new criteria for judging $\mathcal{H}$-tensors and their applications

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Abstract: $\mathcal{H}$-tensors play a key role in identifying the positive definiteness of even-order real symmetric tensors. Some criteria have been given since it is difficult to judge whether a given tensor is an $\mathcal{H}$-tensor, and their range of judgment has been limited. In this paper, some new criteria, from an increasing constant k to scale the elements of a given tensor can expand the range of judgment, are obtained. Moreover, as an application of those new criteria, some sufficient conditions for judging positive definiteness of even-order real symmetric tensors are proposed. In addition, some numerical examples are presented to illustrate those new results.

Keywords: $\mathcal{H}$-tensor; judging range; positive diagonal matrix; symmetric tensor; positive definiteness
Mathematics Subject Classification: 15A15, 15A48, 65F05, 65F40

1. Introduction

Let n and m be integer numbers, $N = \{1, 2, \ldots, n\}$ and $\mathbb{C}(\mathbb{R})$ be the set of all complex (real) numbers. A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is called a complex (real) order m dimension n tensor, if $a_{i_1 i_2 \cdots i_m} \in \mathbb{C}(\mathbb{R})$, where $i_j = 1, 2, \ldots, n$ for $j = 1, 2, \ldots, m$. Let $\mathbb{C}^{[m,n]}$ ($\mathbb{R}^{[m,n]}$) be the set of all complex (real) order m dimension n tensors. A tensor $\mathcal{I} = (\delta_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ $(m, n \geq 2)$ is called the unit tensor [1], if its elements satisfy

$$\delta_{i_1 i_2 \cdots i_m} = \begin{cases} 1, & i_1 = i_2 = \cdots = i_m, \\ 0, & otherwise. \end{cases}$$

A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ $(m, n \geq 2)$ is called symmetric if

$$a_{i_1 i_2 \cdots i_m} = a_{i_{\pi(1)} i_{\pi(2)} \cdots i_{\pi(m)}}, \ \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices.

At present, positive definite homogeneous polynomials play a critical role in the field of dynamics, and its positive definiteness can be transformed to identify the positive definiteness of the symmetric tensor associated with it [2]. However, for a given symmetric tensor, it is difficult to determine whether it is positive definite or not because the problem is NP-hard [3]. Thus, finding effective criteria to identify the positive definitiveness of a tensor is interesting.

$\mathcal{H}$-tensor was showed, Li et al. [3], that is a special kind of tensors in 2014 and an even-order symmetric $\mathcal{H}$-tensors with positive diagonal entries is positive definite. After that, some methods that judge the positive definiteness of a given tensor have been established [4–16]. Nevertheless, as presented by their range of judgment was fixed for the given tensor whether it was positive definite or not [14–16].

In this paper, some new criteria which only depend on elements of the given tensors are proposed to judge $\mathcal{H}$-tensors; they expand the range of judgment by an increasing constant k which scales the elements of a given tensor. In addition, these criteria are used to judge the positive definiteness for even-order real symmetric tensors.

For the convenience of discussion, we start with the following notations, definitions and lemmas. The calligraphy letters $\mathcal{A}, \mathcal{B}, \cdots$ represent the tensors; the capital letters $A, B, \cdots$ denote the matrices; the lowercase letters $x, y, \cdots$ refer to the vectors.

For a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}(m, n \geq 2)$, we denote

$r_i(\mathcal{A}) = \sum\limits_{\substack{i_2 \cdots i_m \in N^{m-1} \\ \delta_{ii_2 \cdots i_m} = 0}} |a_{ii_2 \cdots i_m}| = \sum\limits_{i_2 \cdots i_m \in N^{m-1}} |a_{ii_2 \cdots i_m}| - |a_{ii \cdots i}|,$

$N_1 = \{i \in N : |a_{ii \cdots i}| > r_i(\mathcal{A})\}$, $N_2 = \{i \in N : |a_{ii \cdots i}| \leq r_i(\mathcal{A})\}$,

$N_1^{m-1} = \{i_2 i_3 \cdots i_m : i_j \in N_1, j = 2, 3, \ldots, m\}$,

$N^{m-1} \backslash N_1^{m-1} = \{i_2 i_3 \cdots i_m : i_2 i_3 \cdots i_m \in N^{m-1} \ and \ i_2 i_3 \cdots i_m \notin N_1^{m-1}\}$,

$r_0 = 1, \, r_1 = \max\limits_{i \in N_1} \left\{ \frac{r_i(\mathcal{A})}{|a_{ii \cdots i}|} \right\}, \cdots,$

$$r_{k+1} = \max_{i \in N_1} \left\{ \frac{\sum\limits_{i_2 \cdots i_m \in N^{m-1} \backslash N_1^{m-1}} |a_{ii_2 \cdots i_m}| + r_k \sum\limits_{\substack{i_2 \cdots i_m \in N_1^{m-1} \\ \delta_{ii_2 \cdots i_m} = 0}} |a_{ii_2 \cdots i_m}|}{|a_{ii \cdots i}|} \right\}, \, k = 0, 1, 2, \ldots,$$

$$\sigma_{k+1,i} = \frac{\sum\limits_{i_2 \cdots i_m \in N^{m-1} \backslash N_1^{m-1}} |a_{ii_2 \cdots i_m}| + r_k \sum\limits_{\substack{i_2 \cdots i_m \in N_1^{m-1} \\ \delta_{ii_2 \cdots i_m} = 0}} |a_{ii_2 \cdots i_m}|}{|a_{ii \cdots i}|}, \ i \in N_1, \ k = 0, 1, 2, \ldots.$$

It is obvious that we obtain $\sigma_{k+1,i} \leq r_{k+1} \leq r_k \leq \cdots \leq r_1 < r_0, \ i \in N_1, \ k = 0, 1, 2, \ldots$.

Definition 1. [17] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}(m, n \geq 2)$. If there is a positive vector $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$ such that

$$|a_{ii \cdots i}| x_i^{m-1} > \sum_{\substack{i_2 \ldots i_m \in N^{m-1} \\ \delta_{ii_2 \cdots i_m} = 0}} |a_{ii_2 \cdots i_m}| x_{i_2} \cdots x_{i_m},$$

where $|a|$ for the modulus of $a \in \mathbb{C}$ [17], then $\mathcal{A}$ is called an $\mathcal{H}$-tensor.

Definition 2. [18] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}(m, n \geq 2)$. If

$$|a_{ii \cdots i}| > r_i(\mathcal{A}), \ i \in N,$$

then $\mathcal{A}$ is called a strictly diagonally dominant tensor.
Definition 3. [8] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}(m, n \geq 2)$ and $X = diag(x_1, x_2, \cdots, x_n)$. If

$$\mathcal{B} = (b_{i_1 i_2 \cdots i_m}) = \mathcal{A}X^{m-1},$$

where

$$b_{i_1 i_2 \cdots i_m} = a_{i_1 i_2 \cdots i_m} x_{i_2} \ldots x_{i_m},\ i_j \in N,\ j = 2, 3, \ldots, m,$$

then we call $\mathcal{B}$ as the product of the tensor $\mathcal{A}$ and the matrix X.
Definition 4. [5] The product of $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}(m, n \geq 2)$ and an n-by-n matrrix $X = (x_{ij})$ on mode-k is defined by

$$(\mathcal{A}_{\times k} X)_{i_1 \cdots j_k \cdots i_m} = \sum_{i_k=1}^{n} a_{i_1 \cdots i_k \cdots i_m} x_{i_k j_k}.$$

Definition 5. [5] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ $(m, n \geq 2)$. If there exists a $\varnothing \neq S \subset N$ such that $a_{i_1 i_2 \cdots i_m} = 0$, $\forall i_1 \in S$ and $i_2, \ldots, i_m \notin S$, then $\mathcal{A}$ is called reducible. Otherwise, $\mathcal{A}$ is called irreducible.
Definition 6. [19] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ $(m, n \geq 2)$, for $i, j \in N$ and $i \neq j$, if there exists indices $k_1, k_2, \ldots, k_l$ with

$$\sum_{\substack{i_2 \ldots i_m \in N^{m-1} \\ \delta_{k_s i_2 \cdots i_m} = 0 \\ k_{s+1} \in \{i_2, \ldots, i_m\}}} |a_{k_s i_2 \cdots i_m}| \neq 0,\ s = 0, 1, \ldots, l,$$

where $k_0 = i$, $k_{l+1} = j$, we say that there is a nonzero element chain from i to j.
Definition 7. [8] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ $(m, n \geq 2)$; if the homogeneous polynomical equations satisfy:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \lambda \in \mathbb{C}\ and\ x = (x_1, x_2, \cdots, x_n)^T \neq (0, 0, \cdots, 0)^T,$$

then λ is called an eigenvalue of $\mathcal{A}$ and x is its corresponding eigenvector, where $\mathcal{A}x^{m-1}$ and $\lambda x^{[m-1]}$ are vectors, and whose i th components are

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2 \ldots i_m \in N^{m-1}} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}$$

and

$$(x^{[m-1]})_i = x_i^{m-1}.$$

Definition 8. [20] For an mth degree homogeneous polynomial of n variables, $f(x)$ can usually be denoted as

$$f(x) = \sum_{i_1 i_2 \ldots i_m \in N^m} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m},$$

where $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$. The homogeneous polynomial $f(x)$ can be represented as the tensor product of a symmetric tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ and x^m denoted by

$$f(x) \equiv \mathcal{A}x^m = \sum_{i_1 i_2 \ldots i_m \in N^m} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m},$$

where $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$ [18]. If m is even and

$$f(x) > 0\ for\ any\ x \in \mathbb{R}^n,\ x \neq 0,$$

then we say that $f(x)$ is positive definite.

Lemma 1. [17] Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$ $(m, n \geq 2)$. $\mathcal{A}$ is an $\mathcal{H}$-tensor if $\mathcal{A}$ is a strictly diagonally dominant tensor.

Lemma 2. [3] Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$ $(m, n \geq 2)$. $\mathcal{A}$ is an $\mathcal{H}$-tensor if

- (i) $\mathcal{A}$ is irreducible;
- (ii) $|a_{ii\cdots i}| \geq r_i(\mathcal{A})$ for each $i \in N$;
- (iii) For the inequality of (ii), strict inequality holds for at least one i.

Lemma 3. [8] Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$ $(m, n \geq 2)$. $\mathcal{A}$ is an $\mathcal{H}$-tensor, if

- (i) $|a_{ii\cdots i}| \geq r_i(\mathcal{A}),\ i \in N$;
- (ii) $N_1 = \{i \in N : |a_{ii\cdots i}| > r_i(\mathcal{A})\} \neq \emptyset$;
- (iii) For any $i \in N_2$, there exists a nonzero element chain from i to j such that $j \in N_1$.

Lemma 4. [8, 10] Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$ $(m, n \geq 2)$. If there exists a positive diagonal matrix X such that $\mathcal{A}X^{m-1}$ is an $\mathcal{H}$-tensor, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

2. Some criteria for judging nonsingular $\mathcal{H}$-tensors

In this section, some new criteria for judging $\mathcal{H}$-tensors are proposed, and those new criteria only depend on the elements of the given tensors.

Theorem 1. Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$ $(m, n \geq 2)$. $\mathcal{A}$ is an $\mathcal{H}$-tensor, if there exists a number $k = 0, 1, 2, \ldots$ such that

$$|a_{ii\cdots i}| > \sum_{\substack{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m \in N_1^{m-1}} r_{k+1}|a_{ii_2\cdots i_m}|,\ \forall i \in N_2. \tag{2.1}$$

Proof. First, let

$$\xi_i = \frac{1}{\sum\limits_{i_2\cdots i_m \in N_1^{m-1}} |a_{ii_2\cdots i_m}|} \left\{ |a_{ii\cdots i}| - \sum_{\substack{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| - \sum_{i_2\cdots i_m \in N_1^{m-1}} r_{k+1}|a_{ii_2\cdots i_m}| \right\},\ i \in N_2. \tag{2.2}$$

If $\sum\limits_{i_2\cdots i_m \in N_1^{m-1}} |a_{ii_2\cdots i_m}| = 0$, we define $\xi_i = +\infty$. Obviously, it follows from Eq (2.2) that $\xi_i > 0$, $i \in N_2$, and we have $r_{k+1} < r_0 = 1$ by definition of r_{k+1}, that is, $1 - r_{k+1} > 0$. Hence, there exists a positive number $\varepsilon > 0$, such that

$$0 < \varepsilon < min\left\{\min_{i \in N_2} \xi_i, 1 - r_{k+1}\right\}. \tag{2.3}$$

Construct a diagonal matrix $X = diag\{x_1, x_2, \ldots, x_n\}$ and denote $\mathcal{B} = (b_{i_1i_2\cdots i_m}) = \mathcal{A}X^{m-1}$, where

$$x_i = \begin{cases} (\varepsilon + \sigma_{k+1,i})^{\frac{1}{m-1}} & , i \in N_1, \\ 1 & , i \in N_2. \end{cases}$$

By the inequality of (2.3), we obtain X as a positive diagonal matrix.

Next, we prove the $\sum_{i_2\cdots i_m\in N_1^{m-1}} |a_{ii_2\cdots i_m}| \neq 0$ for any $i \in N_2$. Suppose on the contrary that $\sum_{i_2\cdots i_m\in N_1^{m-1}} |a_{ii_2\cdots i_m}| = 0$ for any $i \in N_2$; thus, by the inequality of (2.1), we have

$$\begin{aligned}|a_{ii\cdots i}| &> \sum_{\substack{i_2\cdots i_m\in N^{m-1}\setminus N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m\in N_1^{m-1}} r_{k+1}|a_{ii_2\cdots i_m}|\\ &= \sum_{\substack{i_2\cdots i_m\in N^{m-1}\setminus N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|\\ &= r_i(\mathcal{A}),\end{aligned}$$

which contradicts with $|a_{ii\cdots i}| \le r_i(\mathcal{A})$, $i \in N_2$; hence, $\sum_{i_2\cdots i_m\in N_1^{m-1}} |a_{ii_2\cdots i_m}| \neq 0$ for any $i \in N_2$.

Finally, we prove that $\mathcal{B}$ is a strictly diagonally dominant tensor, and we divide it into two cases as follows:

Case 1: For any $i \in N_2$, from $\sum_{i_2\cdots i_m\in N_1^{m-1}} |a_{ii_2\cdots i_m}| \neq 0$ and the inequality of (2.1), we have

$$\begin{aligned}r_i(\mathcal{B}) &= \sum_{\substack{i_2\cdots i_m\in N^{m-1}\setminus N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |b_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m\in N_1^{m-1}} |b_{ii_2\cdots i_m}|\\ &= \sum_{\substack{i_2\cdots i_m\in N^{m-1}\setminus N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m} + \sum_{i_2\cdots i_m\in N_1^{m-1}} |a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}\\ &\le \sum_{\substack{i_2\cdots i_m\in N^{m-1}\setminus N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m\in N_1^{m-1}} |a_{ii_2\cdots i_m}|(\varepsilon+\sigma_{k+1,i_2})^{\frac{1}{m-1}}\cdots(\varepsilon+\sigma_{k+1,i_m})^{\frac{1}{m-1}}\\ &\le \sum_{\substack{i_2\cdots i_m\in N^{m-1}\setminus N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m\in N_1^{m-1}} |a_{ii_2\cdots i_m}|(\varepsilon+r_{k+1})\\ &< |a_{ii\ldots i}| = |b_{ii\ldots i}|.\end{aligned}$$

Case 2: For any $i \in N_1$, we obtain that $|a_{ii\cdots i}| > r_i(\mathcal{A})$; then, $|a_{ii\cdots i}| - \sum_{\substack{i_2\cdots i_m\in N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| > 0$, and it follows from $r_{k+1} \le r_k$ that

$$r_{k+1}\sum_{\substack{i_2\cdots i_m\in N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| - r_k\sum_{\substack{i_2\cdots i_m\in N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \le 0;$$

thus, we get

$$\varepsilon > 0 \ge \frac{1}{|a_{ii\cdots i}| - \sum\limits_{\substack{i_2\cdots i_m\in N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|}\left\{r_{k+1}\sum_{\substack{i_2\cdots i_m\in N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| - r_k\sum_{\substack{i_2\cdots i_m\in N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|\right\};$$

so, we have

$$
\begin{aligned}
|b_{ii\cdots i}| - r_i(\mathcal{B}) =& |a_{ii\cdots i}|(\varepsilon + \sigma_{k+1,i}) - \sum_{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1}} |a_{ii_2\cdots i_m}| x_{i_2}\cdots x_{i_m} - \sum_{\substack{i_2\cdots i_m \in N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| x_{i_2}\cdots x_{i_m} \\
\geq & |a_{ii\cdots i}|(\varepsilon + \sigma_{k+1,i}) - \sum_{\substack{i_2\cdots i_m \in N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|(\varepsilon + \sigma_{k+1,i_2})^{\frac{1}{m-1}}\cdots(\varepsilon + \sigma_{k+1,i_m})^{\frac{1}{m-1}} \\
& - \sum_{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1}} |a_{ii_2\cdots i_m}| \\
\geq & |a_{ii\cdots i}|(\varepsilon + \sigma_{k+1,i}) - \sum_{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1}} |a_{ii_2\cdots i_m}| - \sum_{\substack{i_2\cdots i_m \in N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|(\varepsilon + r_{k+1}) \\
=& \varepsilon(|a_{ii\cdots i}| - \sum_{\substack{i_2\cdots i_m \in N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|) + |a_{ii\cdots i}|\sigma_{k+1,i} - \sum_{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1}} |a_{ii_2\cdots i_m}| \\
& - r_{k+1} \sum_{\substack{i_2\cdots i_m \in N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \\
>& r_{k+1} \sum_{\substack{i_2\cdots i_m \in N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| - r_k \sum_{\substack{i_2\cdots i_m \in N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1}} |a_{ii_2\cdots i_m}| \\
& + r_k \sum_{\substack{i_2\cdots i_m \in N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| - \sum_{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1}} |a_{ii_2\cdots i_m}| - r_{k+1} \sum_{\substack{i_2\cdots i_m \in N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \\
=& 0.
\end{aligned}
$$

From Cases 1 and 2, we obtain that $|b_{ii\cdots i}| > r_i(\mathcal{B})$ for all $i \in N$, that is, $\mathcal{B}$ is a strictly diagonally dominant tensor; thus, from Lemmas 1 and 4, $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Theorem 2. Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$ $(m, n \geq 2)$. $\mathcal{A}$ is an $\mathcal{H}$-tensor if the following are true:

- (i) $\mathcal{A}$ is irreducible.
- (ii) There exists $k = 0, 1, 2, \ldots$ such that

$$
|a_{ii\cdots i}| \geq \sum_{\substack{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m \in N_1^{m-1}} r_{k+1}|a_{ii_2\cdots i_m}|, \ \forall i \in N_2.
$$

- (iii) For the inequality of (ii), strict inequality holds for at least one $i \in N_2$.

Proof. First, let the diagonal matrix $X = diag\{x_1, x_2, \ldots, x_n\}$ and $\mathcal{B} = (b_{i_1 i_2\cdots i_m}) = \mathcal{A}X^{m-1}$, where

$$
x_i = \begin{cases} (\sigma_{k+1,i})^{\frac{1}{m-1}} & , i \in N_1, \\ 1 & , j \in N_2. \end{cases}
$$

Obviously, X is the positive diagonal matrix.

Next, we prove that $|b_{ii\dots i}| \geq r_i(\mathcal{B})$ for all $i \in N$, and strict inequality holds for at least one $i \in N$; we have divided it into three cases as follows:

Case 1: For any $i \in N_2$, we obtain

$$\begin{aligned}
r_i(\mathcal{B}) = & \sum_{\substack{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |b_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m \in N_1^{m-1}} |b_{ii_2\cdots i_m}| \\
= & \sum_{\substack{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| x_{i_2}\cdots x_{i_m} + \sum_{i_2\cdots i_m \in N_1^{m-1}} |a_{ii_2\cdots i_m}| x_{i_2}\cdots x_{i_m} \\
\leq & \sum_{\substack{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m \in N_1^{m-1}} |a_{ii_2\cdots i_m}| (\sigma_{k+1,i_2})^{\frac{1}{m-1}}\cdots(\sigma_{k+1,i_m})^{\frac{1}{m-1}} \\
\leq & \sum_{\substack{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m \in N_1^{m-1}} |a_{ii_2\cdots i_m}| r_{k+1} \\
\leq & |a_{ii\dots i}| = |b_{ii\dots i}|.
\end{aligned}$$

Case 2: For any $i \in N_1$, we obtain

$$\begin{aligned}
|b_{ii\cdots i}| - r_i(\mathcal{B}) = & |a_{ii\cdots i}|\sigma_{k+1,i} - \sum_{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1}} |a_{ii_2\cdots i_m}| x_{i_2}\cdots x_{i_m} - \sum_{\substack{i_2\cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| x_{i_2}\cdots x_{i_m} \\
\geq & \sum_{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1}} |a_{ii_2\cdots i_m}| + r_k \sum_{\substack{i_2\cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| - \sum_{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1}} |a_{ii_2\cdots i_m}| \\
& - r_{k+1} \sum_{\substack{i_2\cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \\
\geq & 0.
\end{aligned}$$

Case 3: From the condition (iii), without loss of generality, we suppose that

$$|a_{tt\cdots t}| > \sum_{\substack{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1} \\ \delta_{ti_2\cdots i_m}=0}} |a_{ti_2\cdots i_m}| + \sum_{i_2\cdots i_m \in N_1^{m-1}} r_{k+1} |a_{ti_2\cdots i_m}|;$$

similar to the proof for Case 1 of Theorem 2, we obtain that $r_t(\mathcal{B}) < |b_{tt\cdots t}|$, $t \in N_2$.

Finally, since X is a positive diagonal matrix and $\mathcal{A}$ is irreducible, $\mathcal{B}$ is also irreducible; thus, by Lemmas 2 and 4, $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Theorem 3. Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$ $(m, n \geq 2)$. $\mathcal{A}$ is an $\mathcal{H}$-tensor, if the following are true:

- (i) There exists $k = 0, 1, 2, \dots$ such that

$$|a_{ii\cdots i}| \geq \sum_{\substack{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m \in N_1^{m-1}} r_{k+1} |a_{ii_2\cdots i_m}|, \ \forall i \in N_2.$$

• (ii) $J \neq \emptyset$, where $J = \left\{ j : |a_{jj\cdots j}| > \sum\limits_{\substack{i_2\cdots i_m \in N^{m-1}\backslash N_1^{m-1} \\ \delta_{ji_2\cdots i_m}=0}} |a_{ji_2\cdots i_m}| + \sum\limits_{i_2\cdots i_m \in N_1^{m-1}} r_{k+1}|a_{ji_2\cdots i_m}|, \quad j \in N_2 \right\}$.

• (iii) For any $i \in (N\backslash J)$, there exists a nonzero element chain from i to j such that $j \in J$.

Proof. First, construct a diagonal matrix $X = diag\{x_1, x_2, \ldots, x_n\}$ and denote $\mathcal{B} = (b_{i_1 i_2 \cdots i_m}) = \mathcal{A}X^{m-1}$, where

$$x_i = \begin{cases} (\sigma_{k+1,i})^{\frac{1}{m-1}} & , i \in N_1, \\ 1 & , j \in N_2. \end{cases}$$

Obviously, X is a positive diagonal matrix.

Second, similar to the proof of Theorem 2, we conclude that $|b_{ii\cdots i}| \geq r_i(\mathcal{B})$ for all $i \in N$. From the condition $J \neq \emptyset$, we obtain that there exists at least a $t \in N$ such that $|b_{tt\cdots t}| > r_t(\mathcal{B})$. On the other hand, if $|b_{ii\cdots i}| = r_i(\mathcal{B})$, then $i \in N\backslash J$, and from the condition that for any $i \in N\backslash J$, $\mathcal{A}$ has a nonzero element chain from i to j such that $j \in J$, we obtain that $\mathcal{B}$ has a nonzero elements chain from i to j with $|b_{jj\cdots j}| > r_j(\mathcal{B})$.

Finally, based on the above analysis, we draw a conclusion that $\mathcal{B}$ satisfies the conditions of Lemma 3; hence, by Lemmas 3 and 4, $\mathcal{A}$ is an $\mathcal{H}$-tensor.

3. Some numerical examples

In this section, based on the new criteria for judging $\mathcal{H}$-tensors in section 2, some numerical examples are presented to illustrate those new criteria.

Example 1. Let us consider the tensor $\mathcal{A} = (a_{i_1 i_2 i_3}) = [A(1,:,:), A(2,:,:), A(3,:,:)] \in \mathbb{C}^{[3,3]}$, where

$$A(1,:,:) = \begin{pmatrix} 20 & 2 & 0 \\ 2 & 5 & 0 \\ 2 & 0 & 5 \end{pmatrix}, \; A(2,:,:) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \; A(3,:,:) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 5.2 \end{pmatrix}.$$

Obviously,

$$|a_{111}| = 20, \; r_1(\mathcal{A}) = 16, \; |a_{222}| = 8, \; r_2(\mathcal{A}) = 4, \; |a_{333}| = 5.2 \; and \; r_3(\mathcal{A}) = 6,$$

so $N_1 = \{1, 2\}$ and $N_2 = \{3\}$. By simple calculation, we obtain

$$\frac{r_1(\mathcal{A})}{|a_{111}|} = 0.8, \; \frac{r_2(\mathcal{A})}{|a_{222}|} = 0.5, \; \sigma_{2,1} = 0.71, \; \sigma_{2,2} = 0.45 \; and \; r_2 = 0.71;$$

when k=1, we get

$$|a_{333}| = 5.2 > 5.13 = \sum_{\substack{i_2 i_3 \in N^2 \backslash N_1^2 \\ \delta_{3i_2i_3}=0}} |a_{3i_2i_3}| + r_2 \sum_{i_2 i_3 \in N_1^2} |a_{3i_2i_3}|;$$

hence, $\mathcal{A}$ satisfies the conditions of Theorem 1 and $k = 1$; it follows from Theorem 1 that $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Example 2. Let us consider the irreducible tensor $\mathcal{A} = (a_{i_1 i_2 i_3}) = [A(1, :, :), A(2, :, :), A(3, :, :)] \in \mathbb{C}^{[3,3]}$, where

$$A(1, :, :) = \begin{pmatrix} 13 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \ A(2, :, :) = \begin{pmatrix} 13 & 0 & 0 \\ 0 & 10 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \ A(3, :, :) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{pmatrix}.$$

Obviously,

$$|a_{111}| = 13, \ r_1(\mathcal{A}) = 6, \ |a_{222}| = 10, \ r_2(\mathcal{A}) = 15, \ |a_{333}| = 16 \ \textit{and} \ r_3(\mathcal{A}) = 16,$$

so $N_1 = \{1\}$ and $N_2 = \{2, 3\}$. By simple calculation, we obtain

$$\frac{r_1(\mathcal{A})}{|a_{111}|} = r_1 = 0.46;$$

when k=0, we get

$$|a_{222}| = 10 > 8 = \sum_{\substack{i_2 i_3 \in N^2 \setminus N_1^2 \\ \delta_{2 i_2 i_3} = 0}} |a_{2 i_2 i_3}| + r_1 \sum_{i_2 i_3 \in N_1^2} |a_{2 i_2 i_3}|$$

and

$$|a_{333}| = 16 > 7.38 = \sum_{\substack{i_2 i_3 \in N^2 \setminus N_1^2 \\ \delta_{3 i_2 i_3} = 0}} |a_{3 i_2 i_3}| + r_1 \sum_{i_2 i_3 \in N_1^2} |a_{3 i_2 i_3}|;$$

hence, $\mathcal{A}$ satisfies the conditions of Theorem 2 and $k = 0$; it follows from Theorem 2 that $\mathcal{A}$ is an $\mathcal{H}$-tensor.

4. Application

In this section, based on the new criteria for judging $\mathcal{H}$-tensors in section 2, some new criteria for identifying the positive definiteness of an even-order real symmetric tensor are presented.

From Theorems 1–3, we get the following result.

Theorem 4. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an even-order real symmetric tensor of order m and n dimensions. If $a_{kk\cdots k} > 0$ for all $k \in N$, $\mathcal{A}$ is symmetric and satisfies one of the following conditions and $\mathcal{A}$ is positive definite:

- (i) All conditions of Theorem 1;
- (ii) All conditions of Theorem 2;
- (iii) All conditions of Theorem 3.

The following example is given to show this result.

Example 3. Consider the following 4th-degree homogeneous polynomial

$$f(x) \ = \ 20x_1^4 + 15x_2^4 + 10x_3^4 + 8x_1^3 x_2 + 4x_1^3 x_3 + 12x_2^2 x_3^2,$$

where $x = (x_1, x_2, x_3)^T$. Then we can obtain a symmetric tensor $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{[4,3]}$, where

$$A(1,1,:,:) = \begin{pmatrix} 20 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \; A(1,2,:,:) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; A(1,3,:,:) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$A(2,1,:,:) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; A(2,2,:,:) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 15 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \; A(2,3,:,:) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix},$$
$$A(3,1,:,:) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; A(3,2,:,:) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \; A(3,3,:,:) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 10 \end{pmatrix}.$$

Obviously,

$$|a_{1111}| = 20, \; r_1(\mathcal{A}) = 15, \; |a_{222}| = 15, \; r_2(\mathcal{A}) = 14, \; |a_{333}| = 10 \; \textit{and} \; r_3(\mathcal{A}) = 11,$$

so $N_1 = \{1, 2\}$ and $N_2 = \{3\}$. By simple calculation, we obtain

$$r_1 = 0.93.$$

Thus, we get

$$|a_{3333}| = 10 > 7.38 = \sum_{\substack{i_2 i_3 i_4 \in N^3 \setminus N_1^3 \\ \delta_{3 i_2 i_3 i_4} = 0}} |a_{3 i_2 i_3 i_4}| + r_1 \sum_{i_2 i_3 i_4 \in N_1^3} |a_{3 i_2 i_3 i_4}|;$$

hence, $\mathcal{A}$ satisfies the conditions of Theorem 1 and $k = 0$; thus, it also satisfies the conditions of Theorem 4. Hence, $f(x)$ is positive definite.

5. Conclusions

In this paper, some new criteria have been proposed for the judgment of $\mathcal{H}$-tensors, which they via an increasing constant k to scale the elements of a given tensor and only depend on elements of the given tensors. As an application, some sufficient conditions of the positive definiteness for even-order real symmetric tensors have been obtained. In addition, some numerical examples have been presented to illustrate those new results.

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Conflict of interest

The authors declare that they have no competing interests.

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